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# Quantum and Classical Pseudogroups. Part II Differential and Symplectic Pseudogroups

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**Abstract.** The category of sympletic pseudospaces (analogical to the category of pseudospaces in the sense of [2]) is introduced and used to define symplectic pseudogroups (structures analogical to pseudogroups [3] or quantum groups [4]). It is shown that symplectic pseudogroups are in one-to-one correspondence with Manin groups, also introduced in this paper. The set-theoretical part of these structures has been described in [I].

## Introduction

Symplectic pseudogroups introduced in this paper (Sect. 7) are classical (symplectic) counterparts of quantum (pseudo-) groups ([3, 4],...). They play in classical theory the same role as quantum groups in quantum theory. They also seem to be useful for constructing quantum groups.

Symplectic pseudogroups are symplectic manifolds with a structure similar to Hopf (or Kac) algebra, expressed in terms of symplectic relations (multiplication, unit, inverse, comultiplication, etc.).

If we neglect the symplectic and differential structure of the underlying manifold, our symplectic pseudogroup becomes a union pseudogroup. Union pseudogroups have been introduced in the first part of this paper which we refer to as to [I]. The study of union pseudogroups in [I] has to be considered as a first step in our study of symplectic pseudogroups, in which we have separated purely set-theoretical problems from differential- and symplectic-geometrical ones.

Our definition fits in a general scheme of enlarging the category of groups to a self-dual category. A passage to new kind of objects consists in replacing the usual space by a "noncommutative space." In the case of quantum (pseudo-) groups, "noncommutative spaces" are quantum (pseudo-) spaces, i.e. objects dual to $C^*$-algebras. In the case of symplectic pseudogroups, "noncommutative spaces" are symplectic pseudospaces, i.e. objects dual to $S^*$-*algebras* defined in Sect. 3. With morphisms defined in Sect. 4, $S^*$-algebras form a category which we consider as a classical counterpart of the category of $C^*$-algebras with morphisms defined in [2].

For the sake of clarity, in our presentation we separate also the differential-geometrical part from the symplectic one. According to this we introduce also $D^*$-algebras and differential pseudogroups. They also serve as an important source of $S^*$-algebras and symplectic pseudogroups (we obtain symplectic objects by applying the phase functor to differentiable objects).

There are interesting connections between structures introduced in this paper and such notions as differential and symplectic groupoids [5], double Lie groups [6], Main triples [4, 6], Poisson-Lie groups [4, 6] and dressing actions [6].

We show that symplectic pseudogroups are equivalent to Manin groups (introduced in Sect. 8), which should be considered as global counterparts of Manin triples. We indicate that Poisson-Lie groups on which the dressing fields are incomplete, do not have the corresponding Manin group or symplectic pseudogroup. In our opinion, this is the reason why some attempts to construct quantum deformations of such noncompact groups as "$ax+b$" or $SU(1,1)$ meet serious difficulties.

Let us point out also that examples of symplectic pseudogroups are provided by examples of Manin groups (the latter are easier to find, for instance the quantum $S_\mu U(N)$ has the symplectic counterpart given by the Manin group $Sl(N)$, described in [6]). On the other hand, a symplectic pseudogroup with its structure formulated in terms of symplectic relations (not the corresponding Manin group) seems to wait for a (geometric) quantization.

Sections 1, 2, 3, 4, 7 and 8 form a logical sequence, appropriate for introducing symplectic pseudospaces and symplectic pseudogroups. Remaining sections explain some important connections between the introduced symplectic objects and similar objects formulated in terms of Poisson geometry.

Because of the lack of space, we had to push several topics, such as representation theory of $S^*$-algebras (with applications) and examples of quantization of symplectic pseudogroups to separate publications.

## 1 Differentiable and Symplectic Relations

Throughout the paper, by a *manifold* we mean a smooth finite-dimensional differential manifold having a countable basis of neighbourhoods. By a *submanifold* we mean a nonempty embedded submanifold.

A *differentiable relation* is a triple $r=(R;\, Y, X)$ such that $X$, $Y$ are manifolds and $R$ is a submanifold of $Y\times X$. We shall use the notation introduced in [I]:

$$r : X \to Y \ , \qquad R = \mathscr{G}(r) \ .$$

Let $r=(R;\, Y, X)$ be a differentiable relation. The *tangent relation (tangent lift)* of $r$ is a differentiable relation $\mathrm{T}r\colon \mathrm{T}X\to\mathrm{T}Y$ ($\mathrm{T}X$ is the tangent bundle of $X$) such that $\mathscr{G}(\mathrm{T}r)=\mathrm{T}\mathscr{G}(r)$. The *phase relation (phase lift)* of $r$ is a differentiable relation $\mathrm{P}r\colon \mathrm{P}X\to\mathrm{P}Y$ ($\mathrm{P}X$ is the cotangent bundle of $X$ with the bundle projection $\pi_X$) such that $(\eta,\xi)\in\mathscr{G}(\mathrm{P}r)$ if and only if

$$\langle \xi, u\rangle = \langle \eta, v\rangle \quad \text{for} \quad (v,u)\in \mathrm{T}_{(y,x)}\mathscr{G}(r) \ , \qquad x=\pi_X(\xi) \ , \qquad y=\pi_Y(\eta) \ .$$

The *tangent-phase relation* of $r$ is a differentiable relation $\mathrm{S}r\colon \mathrm{S}X\to\mathrm{S}Y$ ($\mathrm{S}X$ is the

Whitney sum $\mathrm{T}X \oplus \mathrm{P}X$) such that $((v,\eta),(u,\xi)) \in \mathscr{G}(\mathrm{S}r)$ if and only if $(v,u) \in \mathscr{G}(\mathrm{T}r)$ and $(\eta,\xi) \in \mathscr{G}(\mathrm{P}r)$.

Let $(X_1,\omega_1)$, $(X_2,\omega_2)$ be symplectic manifolds. A *symplectic relation* from $(X_1,\omega_1)$ to $(X_2,\omega_2)$ is a differentiable relation $r\colon X_1 \to X_2$ such that $\mathscr{G}(r)$ is a lagrangian submanifold of $(X_2,\omega_2)\times(X_1,-\omega_1)$.

A *differentiable (symplectic) reduction* (cf. [7], [8]) is a differentiable (symplectic) relation $r\colon X \to Y$ of the form $r = f i^T$, where $i\colon C \to X$ is the inclusion map of a submanifold $C$ in $X$ and $f\colon C \to Y$ is a surjective submersion.

Differentiable (symplectic) relations do not form a category (under the composition of relations). In order to formulate axioms of union algebras based on differentiable or symplectic relations, we have to impose some conditions on the composition of relations occurring in the axioms. In this differentialgeometrical setting, the relevant conditions are given in terms of the transverse composition (which replaces the simple composition of binary relations, [I]) introduced in the next section.

## 2. Simplicity and Transversality

**Definition.** Two differentiable relations $\alpha: Y \to X$, $\beta: Z \to Y$ are said to be *locally transverse* if $\mathrm{S}\alpha \, 1 \, \mathrm{S}\beta$ and $\alpha\beta \neq \emptyset$.

*Remark.* Of course, $\mathrm{S}\alpha \, 1 \, \mathrm{S}\beta$ if and only if $\mathrm{T}\alpha \, 1 \, \mathrm{T}\beta$ and $\mathrm{P}\alpha \, 1 \, \mathrm{P}\beta$. Smooth mappings $f\colon Y \to X$, $g\colon Z \to Y$ are always locally transverse. In fact, since $\mathrm{T}g$ is a mapping, we have $\mathrm{T}f \, 1 \, \mathrm{T}g$. It is easy to check that also $\mathrm{P}f \, 1 \, \mathrm{P}g$ (it follows also from the fact that $\mathrm{P}f$ and $\mathrm{P}g$ are morphisms of $U^*$-algebras, see Sect. 4).

Let $A$ and $B$ be two submanifolds of a manifold $Z$. We say that $A$ intersects $B$ *transversally* if $A \cap B \neq \emptyset$ and $\mathrm{T}_z A + \mathrm{T}_z B = \mathrm{T}_z Z$ for $z \in A \cap B$. It is easy to see that in this case $A \cap B$ is a submanifold and $\mathrm{T}(A \cap B) = \mathrm{T}A \cap \mathrm{T}B$.

**Proposition 2.1.** *Two differentiable relations $\alpha: Y \to X, \beta: Z \to Y$ are locally transverse if and only if three following conditions are satisfied:*

(i) $\alpha \, 1 \, \beta$,

(ii) *$\mathscr{G}(\alpha) \times \mathscr{G}(\beta)$ intersects transversally $X \times \Delta_Y \times Z$, where $\Delta_Y$ is the diagonal of $Y \times Y$,*

(iii) *the projection map from $X \times Y \times Y \times Z$ to $X \times Z$, restricted to*

$$\mathscr{G}_{\alpha\beta} = (\mathscr{G}(\alpha) \times \mathscr{G}(\beta)) \cap (X \times \Delta_Y \times Z)$$

*is an immersion.*

*Proof.* Set $R = \mathscr{G}(\alpha) \times \mathscr{G}(\beta)$ and $\Delta = X \times \Delta_Y \times Z$. We can assume that $\alpha \, 1 \, \beta$. The statement $\mathrm{P}\alpha \, 1 \, \mathrm{P}\beta$ is then equivalent to each of the following statements:

1) $(0,-\eta)|_{\mathrm{T}\mathscr{G}(\alpha)} = 0$ and $(\eta,0)|_{\mathrm{T}\mathscr{G}(\beta)} = 0$ implies $\eta = 0$ (for $\eta \in \mathrm{P}Y$);
2) $(0,-\eta,\eta,0)|_{\mathrm{T}R} = 0$ implies $\eta = 0$ (for $\eta \in \mathrm{P}Y$),
3) $\lambda|_{\mathrm{T}\Delta} = 0$ and $\lambda|_{\mathrm{T}R} = 0$ implies $\lambda = 0$ (for $\lambda \in \mathrm{P}(X \times Y \times Y \times Z)$).

The last condition is equivalent to (ii). Thus $\mathrm{P}\alpha \, 1 \, \mathrm{P}\beta$ is equivalent to (i) and (ii). If we now assume that $\mathrm{P}\alpha \, 1 \, \mathrm{P}\beta$, then the statement $\mathrm{T}\alpha \, 1 \, \mathrm{T}\beta$ is equivalent to each of the following statements:

1) $(0,v) \in \mathrm{T}\mathscr{G}(\alpha)$ and $(v,0) \in \mathrm{T}\mathscr{G}(\beta)$ implies $v = 0$ (for $v \in \mathrm{T}Y$),
2) $(0,v,v,0) \in \mathrm{T}\mathscr{G}_{\alpha\beta}$ implies $v = 0$ (for $v \in \mathrm{T}Y$),
3) (iii). □

**Corollary.** *If $\alpha$ and $\beta$ are locally transverse, then $\mathscr{G}(\alpha\beta)$ is the image of an injective immersion (namely, the immersion in condition* (iii)*).*

**Definition.** Two differentiable relations $\alpha: Y \to X$, $\beta: Z \to Y$ are said to be *transverse* (we shall denote it by $\alpha \pitchfork \beta$) if $\mathrm{S}\alpha \, 1\, \mathrm{S}\beta$ and $\alpha\beta$ is a differentiable relation (i.e. if $\alpha$ and $\beta$ are locally transverse and $\mathscr{G}(\alpha\beta)$ is a submanifold, not only an immersed submanifold). In this case $\alpha$ and $\beta$ are said to *have a transverse composition.*

*Examples.*

1. If $\alpha$ and $\beta$ are smooth mappings then $\alpha \pitchfork \beta$.
2. If $\beta$ is a differentiable reduction then $\alpha \pitchfork \beta$ for all $\alpha$ (this property characterizes differentiable reductions, see [8]).

If $\alpha \pitchfork \beta$ then the projection map in (iii) is a diffeomorphism of $\mathscr{G}_{\alpha\beta}$ and $\mathscr{G}(\alpha\beta)$. It follows that the *simplicity map*

$$\mathscr{G}(\alpha\beta) \ni (x, z) \mapsto s_{\alpha\beta}(x, z) \in Y$$

such that $(x, s_{\alpha\beta}(x, z), s_{\alpha\beta}(x, z), z) \in \mathscr{G}_{\alpha\beta}$, is smooth.

**Proposition 2.2.** *If $\alpha \pitchfork \beta$ then $\mathrm{S}(\alpha\beta) = \mathrm{S}\alpha\,\mathrm{S}\beta$, $\mathrm{T}\alpha \pitchfork \mathrm{T}\beta$, $\mathrm{P}\alpha \pitchfork \mathrm{P}\beta$ and $\mathrm{S}\alpha \pitchfork \mathrm{S}\beta$.*

For the proof we refer to Appendix.

**Corollary.** *If $\alpha \pitchfork \beta$ then $\mathrm{S}^n(\alpha\beta) = \mathrm{S}^n\alpha\,\mathrm{S}^n\beta$ for any natural number $n$ (the tangent and the cotangent functor can be applied as many times as we wish, like in the case of mappings).*

Let us note that it is easier to check the transversality in the case of symplectic relations.

**Proposition 2.3.** *If $\alpha$ and $\beta$ are symplectic relations then*
(i) $\mathrm{S}\alpha \, 1\, \mathrm{S}\beta \Leftrightarrow \mathrm{P}\alpha \, 1\, \mathrm{P}\beta \Leftrightarrow \mathrm{T}\alpha \, 1\, \mathrm{T}\beta$,
(ii) *if $\alpha \pitchfork \beta$ then $\alpha\beta$ is symplectic.*

*Proof.* Let us assume that $\alpha \, 1\, \beta$. Then $\mathrm{P}\alpha \, 1\, \mathrm{P}\beta$ is equivalent to each of the following statements (for $p \in R \cap \Delta$):
1) $\mathrm{T}_p R + \mathrm{T}_p \Delta = \mathrm{T}_p(X \times Y \times Y \times Z)$,
2) $(\mathrm{T}_p R)^{\S} \cap (\mathrm{T}\Delta)^{\S} = \{0\}$,
3) $\mathrm{T}_p R \cap (\{0\} \times \mathrm{T}_{(y,y)} \Delta_Y \times \{0\}) = \{0\}$ (here $p = (\mathrm{x}, y, y, z)$).

The last statement is equivalent to $\mathrm{T}\alpha \, 1\, \mathrm{T}\beta$. We have denoted by $E^{\S}$ the subspace orthogonal to $E$ with respect to the symplectic form (and we have used symbols $R$ and $\Delta$ introduced in the proof of Proposition 2.1).

The second part of the proposition follows from $\alpha \, 1\, \beta$, $\mathrm{T}(\alpha\beta) = \mathrm{T}\alpha\mathrm{T}\beta$ and the fact that linear symplectic relations form a category [9]. □

We end this section by a remark on associativity of the transversality. Let us note first that for any binary relations $\alpha, \beta, \gamma$,

$$\alpha \, 1\, \beta \quad \text{and} \quad (\alpha\beta) \, 1\, \gamma \quad \text{implies} \quad \alpha \, 1\, (\beta\gamma).$$

Applying this rule to $\mathrm{S}\alpha$, $\mathrm{S}\beta$, $\mathrm{S}\gamma$, where $\alpha, \beta, \gamma$ are differentiable relations, we obtain the following rule:

$$\alpha \pitchfork \beta, \quad (\alpha\beta) \pitchfork \gamma \quad \text{and} \quad \beta \pitchfork \gamma \quad \text{implies} \quad \alpha \pitchfork (\beta\gamma).$$

Of course, we have also

$$\beta \pitchfork \gamma\,, \quad \alpha \pitchfork (\beta\gamma) \text{ and } \alpha \pitchfork \beta \text{ implies } (\alpha\beta) \pitchfork \gamma\,. \tag{1}$$

## 3. $D^*$-Algebras, $S^*$-Algebras

**Definition.** A *$D^*$-algebra* is a U*-algebra $(X, m, e, s)$ such that $X$ is a manifold, $m, e, s$ are differentiable relations and

$$m \pitchfork (m \otimes I)\,, \tag{2}$$

$$m \pitchfork (I \otimes m)\,, \tag{3}$$

$$m \pitchfork (e \otimes I)\,, \tag{4}$$

$$m \pitchfork (I \otimes e)\,. \tag{5}$$

Note that $s$ is a diffeomorphism since it is an involutive differentiable relation. Projections $e_L$ and $e_R$ are smooth, because they are simplicity maps for (4) and (5).

**Proposition 3.1.** *If $(X, m, e, s)$ is a $D^*$-algebra then the projection map*

$$\mathscr{G}(m) \ni (z, (x, y)) \mapsto (x, y) \in X \times X$$

*is an injective immersion whose image is $m^T(X)$.*

*Proof.* Let $(v, 0) \in \mathscr{G}(\mathrm{T}m)$. There exists a curve $t \mapsto (z(t), (x(t), y(t))$ in $\mathscr{G}(m)$ such that $\dot{z}(0) = v$, $(\dot{x}(0), \dot{y}(0)) = 0$. We have

$$((sx(t), z(t)), (sx(t), x(t), y(t))) \in \mathscr{G}(I \otimes m)$$

and

$$(y(t), (sx(t), z(t))) \in \mathscr{G}(m)\,,$$

hence $(sx(t), z(t)) = s_{m, I\otimes m}(y(t), sx(t), x(t), y(t))$, where $s_{m, I\otimes m}$ is the simplicity map for (3). Since this map is smooth we have $v = \dot{z}(0) = 0$. It follows that the projection in the proposition is an immersion. Its injectivity follows from Lemma I.3.2.(iv). □

**Proposition 3.2.** *Let $(X, m, e, s)$ be a $D^*$-algebra. Two following conditions are equivalent:*

(i) *$e_L, e_R$ are submersions,*
(ii) *$m$ is a differentiable reduction.*

*Proof.* (i) ⇒ (ii). $m^T(X) = \{(x, y): e_R(x) = e_L(y)\}$ is submanifold and from Proposition 3.1 it follows that $\underline{m} = m|_{m^T(X)}$ is a smooth (surjective) map. We shall show that $\underline{m}$ is a submersion. If $\underline{m}(x, y) = z$ and $z(t)$ is a curve in $X$ such that $z(0) = z$, then there exists a curve $x(t)$ such that $x(0) = x$ and $e_L(x(t)) = e_L(z(t))$ (because $e_L$ is a submersion). If we set $y(t) = m(sx(t), z(t))$, we have $m(x(t), y(t)) = z(t)$.

(ii) ⇒ (i) Since $f = \underline{m}|_{m^T(E)}: m^T(E) \to E$ is a surjective submersion, also $e_L = f(I \otimes s)d$, $e_R = f(s \otimes I)d$ (where $d: X \to X \times X$ is the diagonal map) are surjective submersions. □

**Definition.** A $D^*$-algebra $(X, m, e, s)$ is said to be *regular* if $m$ is a differentiable reduction. By Proposition 3.2, regular $D^*$-algebras are in one-to-one correspondence with differential groupoids [5].

**Definition.** A *$S^*$-algebra* is a $D^*$-algebra $(X, m, e, s)$ such that $X$ is a symplectic manifold and $m$ is a symplectic relation.

**Proposition 3.3.** *If $(X, m, e, s)$ is a $S^*$-algebra then $m$ is a (symplectic) reduction.*

*Proof.* Let $\rho = \mathrm{T}_a m$ be the relation tangent to $m$ at a point $a \in \mathscr{G}(m)$ (see [10]). By Proposition 3.1, $\rho(0) = 0$. From the properties of linear symplectic relations ($\rho$ is such) it follows that $\rho$ is onto, hence the map $p$

$$\mathscr{G}(m) \ni (z, (x, y)) \mapsto z \in X$$

is a submersion. This implies that $q = p|_{p^{-1}(E)}: p^{-1}(E) \to E$ is a surjective submersion. On the other hand, the projection map in Prop. 3.1 defines a diffeomorphism of $p^{-1}(E)$ and $\mathscr{G}(s)$, hence

$$X \ni x \mapsto (e_L(x), (x, sx)) \in p^{-1}(E)$$

is a diffeomorphism. It follows that $e_L$ is a submersion. By Proposition 3.2, $m$ is a symplectic reduction. □

By Proposition 3.3, $S^*$-algebras are automatically "regular" and they coincide with symplectic groupoids [5]. Standard considerations (see [5]) show that if $(X, m, e, s)$ is a $S^*$-algebra then $e$ is a symplectic relation and $s$ is an anti-symplectomorphism.

Let $k_X: SX \to SX$ be a map defined by $k_X(u, \xi) = (u, -\xi)$ for $(u, \xi) \in SX$.

**Proposition 3.4.** *If $M = (X, m, e, s)$ is a regular $D^*$-algebra then $\mathrm{T}M = (\mathrm{T}X, \mathrm{T}m, \mathrm{T}e, \mathrm{T}s)$, $\mathrm{P}M = (\mathrm{P}X, \mathrm{P}m\,\mathrm{P}e, -\mathrm{P}s)$ and $\mathrm{S}M = (\mathrm{S}X, \mathrm{S}m, \mathrm{S}e, k_X \mathrm{S}s)$ are regular $D^*$-algebras.*

*Proof.* By Proposition 2.2, T$M$, P$M$ and S$M$ are union star algebras with unit satisfying the transversality conditions (2)–(5). We shall show that they satisfy the strong positivity (condition (I.8)). Using the notation introduced in Appendix we obtain, from (I.9),

$$|\mathscr{G}(s)\rangle = m^T e, \tag{6}$$

hence, applying (A3), we have

$$|\mathscr{G}((\mathrm{S}s)k_X)\rangle = (I \otimes k_X)|\mathscr{G}(\mathrm{S}s)\rangle = (\mathrm{S}m)^T \mathrm{S}e.$$

It follows that for each $(u, \xi) \in \mathrm{S}X$ there exists $(v, \eta) \in \mathrm{S}e(1)$ such that

$$(v, \eta) \in \mathrm{S}m((u, \xi), (\mathrm{S}s)k_X(u, \xi)) = \mathrm{S}m((u, \xi), (\mathrm{T}su, -\mathrm{P}s\xi)).$$

Since S$m$ is simple (as a reduction), the strong positivity condition is satisfied. □

In the sequel we shall be interested in those $D^*$-algebras which are regular.

## 4. Morphisms of Differential and Symplectic Groupoids

**Definition.** A *morphism* from a regular $D^*$-algebra $(X, m, e, s)$ to a regular $D^*$-algebra $(X', m', e', s')$ is a differentiable relation $h: X \to X'$ such that

$$hm = m'(h \otimes h), \tag{7}$$

$$hs = s'h, \tag{8}$$

$$he = e' \tag{9}$$

and $m' \pitchfork (h \otimes h)$, $h \pitchfork e$.

*Remark.* Equalities (7), (8), (9) mean that $h$ is a morphism of $U^*$-algebras. In this case we know (I. Lemma 5.2) that all compositions in (7), (8) and (9) are simple. In the above definition we assume additionally that they are transverse.

From the transversality in (9) it follows that the base map $f_0 = h_0^T : E' \to E$ is smooth, because $f_0 = s_{h,e}$. Let us note also that $\dim \mathscr{G}(h) = \dim E' + \dim X_a$ ($a$- any point of $E$) does not depend on $h$. This is seen from the following lemma.

**Lemma 4.1.** *If $h: X \to X'$ is morphism of regular $D^*$-algebras as above, then $\mathscr{G}(h)$ is a smooth section of the projection $e'_R \times I : X' \times X \to E' \times X$ over $\mathscr{G}(h_0 e_R) = \{(a', x) \in E' \times X : f_0(a') = e_R(x)\}$.*

*Proof.* By Lemma I.3.3, $\mathscr{G}(h) = \bigcup_{a' \in E'} \mathscr{G}(h_{a'})$, where $\mathscr{G}(h_{a'}) = \mathscr{G}(h) \cap (X'_{a'} \times X_{f_0(a')})$ and $h_{a'} : X_{f_0(a')} \to X'_{a'}$ is a mapping for all $a' \in E'$. It follows that $G(h)$ can be bijectively mapped onto $G(h_0 e_R)$. The latter set is a submanifold in $E \times X$ and the bijective map between $\mathscr{G}(h)$ and $\mathscr{G}(h_0 e_R)$ is given by $e'_R \times I$. It suffices to prove that $e'_R \times I|_{\mathscr{G}(h)}$ is an immersion. We have to show that $(x'(t), x(t)) \in \mathscr{G}(h)$ and $\left.\frac{d}{dt}\right|_{t=0} (e'_R(x'(t)), x(t)) = 0$ implies $\left.\frac{d}{dt}\right|_{t=0} x'(t) = 0$. We have $(e'_R(x'(t)), (s'x'(t), x'(t)) \in \mathscr{G}(m')$ and $((s'x'(t), x'(t)), (sx(t), x(t))) \in \mathscr{G}(h \otimes h)$, hence $(s'x'(t), x'(t)) = s_{m', h \otimes h}(e'_R(x'(t)), (sx(t), x(t)))$. Since the simplicity map $s_{m', h \otimes h}$ is smooth, it follows that $\left.\frac{d}{dt}\right|_{t=0} x'(t) = 0$. □

In the proof of our next proposition we shall use the following interesting fact, easily seen from the definition. If $h$ is a morphism of regular $D^*$-algebras then T$h$, P$h$ and S$h$ are morphisms of the corresponding lifts of $D^*$-algebras.

**Proposition 4.2.** *Let $M = (X, m, e, s)$, $M' = (X', m', e', s')$ and $M'' = (X'', m'', e'', s'')$ be regular $D^*$-algebras. If $h: X \to X'$ ($k: X' \to X''$) is a morphism from $M$ to $M'$ ($M'$ to $M''$) then $k \pitchfork h$ and $kh$ is a morphism from $M$ to $M''$.*

*Proof.* By Lemma 4.1, $\mathscr{G}(h) = \{(x', x) \in X' \times X : x' = \Psi(a', x),\ h_0^T(a') = e_R(x)\}$ and $\mathscr{G}(k) = \{(x'', x') \in X'' \times X' : x'' = \Phi(a'', x'),\ k_0^T(a'') = e'_R(x')\}$, where $\Psi : \mathscr{G}(h_0 e_R) \to X'$, $\Phi : \mathscr{G}(k_0 e'_R \to X''$ are smooth maps satisfying $e'_R(\Psi(a', x)) = a'$ and $e''_R(\Phi(a'', x')) = a''$ for $a' \in E'$, $a'' \in E''$, $x \in X$, $x' \in X'$. It follows that

$$\begin{aligned} \mathscr{G}(kh) &= \{(x'', x) \in X'' \times X : x'' = \Phi(a'', \Psi(a', x)),\ a' \\ &= e'_R(x') = k_0^T(a''),\ h_0^T(a') = e_R(x)\} \\ &= \{(x'', x) \in X'' \times X : x'' = \Phi(a'', \Psi(k_0^T(a''), x)),\ h_0^T k_0^T(a'') = e_R(x)\}\ . \end{aligned}$$

hence $\mathscr{G}(kh)$ is the image of a section of the projection $e''_R \times I$ over $\mathscr{G}(k_0 h_0 e_R)$. This shows that $kh$ is a differentiable relation. Since S$k$ and S$h$ are morphisms of $U^*$-algebras, we have S$k$1S$h$, hence $k \pitchfork h$. It remains to show that $kh \pitchfork e$ and $m'' \pitchfork (kh \otimes kh)$.

Since $he = e'$, we have $h \pitchfork e$, $k \pitchfork he$ and $k \pitchfork h$. It follows by the associativity rule (1) that $kh \pitchfork e$. Since $m$ is a reduction, $h \pitchfork m$ and $kh \pitchfork m$. Using $k \pitchfork h$, this implies $k \pitchfork hm$, hence (by associativity)

$$k \pitchfork m'(h \otimes h).$$

This result implies $km' \pitchfork (h \otimes h)$, since $k \pitchfork m'$ and $m' \pitchfork (h \otimes h)$. It follows that $m''(k \otimes k) \pitchfork (h \otimes h)$, hence $m'' \pitchfork (k \otimes k)(h \otimes h)$, because $m'' \pitchfork (k \otimes k)$ and $(k \otimes k) \pitchfork (h \otimes h)$.

**Corollary.** *Morphisms of regular $D^*$-algebras form a category.*

**Definition.** A morphism from a $S^*$-algebra $(X, m, e, s)$ to a $S^*$-algebra $(X', m', e', s')$ is a symplectic relation $h: X \to X'$ which is a morphism of the underlying regular $D^*$-algebras.

Proposition 4.2 and the corollary remain true for mosphisms of $S^*$-algebras.

*Remark.* It is striking that using differentiable (symplectic) relations which **do not** form a category (they form a **WP**-category in the sense of [11, 12]), we have defined (imposing some "algebraic" conditions) a class of a differentiable (symplectic) relations which is already a **true** category. Note that T, P and S act as true functors on this category. Functor P even acts from regular $D^*$-algebras to $S^*$-algebras, so it produces examples of $S^*$-algebras.

Basic examples of regular $D^*$-algebras $M = (X, m, e, s)$ are the following (cf. examples of $U^*$-algebras in [I]):

1. *Manifold algebra:* $M = D_X^T$(cf. [I]), where $X$ is a manifold. All $D^*$-algebras such that $m^T$ is a map are of this type.
2. *Algebra of endomorphisms of a manifold:* $M = \mathrm{End}\, Z$, where $Z$ is a manifold (see I.3).
3. *Differential group algebra* is a Lie group $(G, m, e, s)$. All $D^*$-algebras such that $m$ is a map are of this type.

Basic examples of $S^*$-algebras $M = (X, m, e, s)$ are as follows:

1. *Cotangent bundle:* $M = \mathrm{P}(D_Q^T) = \mathrm{T}^*(D_Q)$, where $Q$ is a manifold.
2. *Algebra of endomorphisms of a symplectic manifold:* $M = \mathrm{End}\, Z = Z \otimes \bar{Z}$, where $Z$ is a symplectic manifold. By $\bar{Z}$ we have denoted the manifold $Z$ equipped with the symplectic form opposite to the original symplectic form on $Z$.
3. *Symplectic group algebra* $M = \mathrm{P}(G, m, e, s)$. This algebra is useful for a study of hamiltonian actions of $(G, m, e, s)$, see Example 6.4.

If $M = (X, m, e, s)$ is a regular $D^*$-algebra then $M^T = (X, m^T, e^T, s^T)$ is said to be a *$D^*$-coalgebra* or *$D^*$-space* or *differential pseudospace. Morphisms* of $D^*$-spaces are relations transposed to morphisms of the corresponding regular $D^*$-algebras.

If $M = (X, m, e, s)$ is a $S^*$-algebra then $M^T = (X, m^T, e^T, s^T)$ is said to be a *$S^*$-coalgebra* or *$S^*$-space* or *symplectic pseudospace. Morphisms* of $S^*$-spaces are relations transposed to morphisms of the corresponding $S^*$-algebras.

*Products* of regular $D^*$-algebras and $S^*$-algebras as well as products of $D^*$-spaces and $S^*$-spaces are naturally defined (cf. [I].6).

## 5. Symplectic groupoids and Poisson Manifolds

A *Poisson manifold* (cf. [13]) is a pair $(P, \Pi)$, where $P$ is a manifold and $\Pi$ is a bi-vector field on $P$ such that the bracket

$$\{f, g\} = \Pi(df, dg)$$

defined for smooth functions on $P$, satisfies the Jacobi identity. In this case the above bracket is said to be a *Poisson bracket* and $\Pi$ is said to be a *Poisson bi-vector field* on $P$. If $(P_1, \Pi_1)$, $(P_2, \Pi_2)$ are two Poisson manifolds then a smooth map $\phi: P_1 \to P_2$ is a *Poisson map* if $\phi_* \Pi_1 = \Pi_2$ (i.e. $\phi$ preserves the Poisson bracket).

Let $(X, \omega)$ be a symplectic manifold. We denote by $\flat$ the vector bundle isomorphism from $TX$ to $T^*X$ defined by

$$\langle u^\flat, v\rangle = \omega(u, v),$$

where $u, v$ are vectors tangent to $X$ at the same point. In another notation, $u^\flat = u \rfloor \omega$. The inverse isomorphism is denoted by $\sharp$. Formula

$$\Pi_\omega(u^\flat, v^\flat) = \omega(u, v)$$

defines a bi-vector field $\Pi_\omega$ on $X$ which corresponds to the standard Poisson bracket on $X$:

$$\{f, g\} = \Pi_\omega(df, dg) = \langle dg, -(df)^\sharp\rangle = -(df)^\sharp g.$$

**Lemma 5.1.** *If $(X, m, e, s)$ is a $S^*$-algebra then foliations of $X$ defined by the left and right projections are symplectically orthogonal.*

*Proof.* If $t \mapsto a(t)$, $t \mapsto b(t)$ are curves in $X$ such that $a(0) = x = b(0)$, $e_L(a(t)) = e_L(x)$ and $e_R(b(t)) = e_R(x)$, then $(a(t), sa(t)) \in m^T(e_L(x))$ and $(b(t), e_R(x)) \in m^T(X)$. Since $m^T(e_L(x))$ coincides (locally) with a characteristic submanifold of $m^T(X)$ (because $m$ is a symplectic reduction), $(u, Ts(u))$ is symplectically orthogonal to $(v, 0)$, where $u = \left.\frac{da}{dt}\right|_{t=0}$, $v = \left.\frac{db}{dt}\right|_{t=0}$. It follows that $u$ and $v$ are symplectically orthogonal. □

From Lemma 5.1 it follows that the Poisson bracket $\{f, g\} = \Pi_\omega(df, dg)$ of two functions which are constant on right fibers $X_a$, $a \in E$, is locally constant on these fibers, because in this case

$$(dh)^\sharp \{f, g\} = \{\{f, g\}, h\} = \{\{f, h\}, g\} + \{f, \{g, h\}\} = 0$$

for each function $h$ constant on left fibers. In fact $\{f, g\}$ is globally constant on right fibers. In order to see this we shall use the following lemma.

**Lemma 5.2.** *Let $M = (X, m, e, s)$ be a regular $D^*$-algebra and let $\sigma: \{1\} \to X$ be a differentiable relation. If we set $l_\sigma = m(\sigma \otimes I)$ and $\Sigma = \sigma(1)$, then*

(i) *$m \pitchfork (\sigma \otimes I)$, $l_\sigma^T(X) = e_L^{-1} e_R(\Sigma)$ and $l_\sigma(X) = e_L^{-1} e_L(\Sigma)$,*

(ii) *if $e_R \pitchfork \sigma$ (i.e. $\Sigma$ is the image of a local section of the right projection), then $l_\sigma|_{e_L^{-1} e_R(\Sigma)}$ is a smooth map,*

(iii) *if $e_R \pitchfork \sigma$ and $e_L \pitchfork \sigma$ then $l_\sigma|_{e_L^{-1} e_R(\Sigma)}$ is a diffeomorphism from $e_L^{-1} e_R(\Sigma)$ to $e_L^{-1} e_L(\Sigma)$.*

*Proof.* Using

$$m(x,y)=z \Leftrightarrow m(z,sy)=x\,, \tag{10}$$

we obtain $\mathscr{G}(l_\sigma)=(I\otimes s)m^T(\Sigma)$, hence $l_\sigma$ is differentiable. From (10) it follows that $m \rceil (\sigma\otimes I)$. Since S$M$ is a regular $D^*$-algebra, we have also S$m \rceil$(S$\sigma \otimes I$), or S$m \rceil$S($\sigma\otimes I$). It follows that $m \pitchfork (\sigma\otimes I)$. The remaining part is easy to prove. □

Now let $f$, $g$ be constant on right fibers $X_a, a\in E$, as before. For each $x\in X$ we can find a symplectic relation $\sigma:\{1\}\to X$ such that $x\in\Sigma$ and $e_R \pitchfork \sigma, e_L \pitchfork \sigma$. By Lemma 5.2, $l_\sigma$ is a local symplectomorphism such that $e_R l_\sigma(y)=e_R(y)$ for $y\in e_L^{-1}e_R(\Sigma)$ and $l_\sigma(e_R(x))=x$, hence

$$\{f,g\}(x)=\{l_\sigma^* f, l_\sigma^* g\}(e_R(x))=\{f,g\}(e_R(x))\ .$$

The above considerations show that there is a unique Poisson bi-vector field $\Pi_R$ on $E$ such that $e_R: X\to E$ is a Poisson map. There is also a unique Poisson bi-vector field $\Pi_L$ on $E$ such that $e_L: X\to E$ is a Poisson map. We have $\Pi_R+\Pi_L=0$, since for $a\in E$,

$$\Pi_R(a)=e_{R*}(\Pi_\omega(a))=e_{L*}s_*(\Pi_\omega(a))=e_{L*}(-\Pi_\omega(a))=-\Pi_L(a)\ .$$

In order to study the connection between $S^*$-algebras and Poisson manifolds in a more detail, we consider first the linear case.

A $S^*$-algebra $(X,m,e,s)$ is said to be *linear* if $X$ is a symplectic vector space and $m,e$ are linear relations (relations with linear graphs, see [9]).

A *Poisson vector space* is a pair $(E,\Pi)$, where $E$ is a vector space and $\Pi$ is a bi-vector on $E$ ($\Pi\in E\wedge E$).

In order to relate the notions introduced above we need several lemmas concerning linear symplectic geometry.

**Lemma 5.3.** *Linear involutive antisymplectomorphisms s in a symplectic vector space X are in one-to-one correspondence with pairs* $(L_+, L_-)$ *of lagrangian subspaces in X such that* $X=L_+\oplus L_-$. *The correspondence is given by* $L_\pm=\ker(s\mp I)$.

*Proof.* If $s$ is a linear map such that $s^2=I$ and $\omega(sx,sy)=-\omega(x,y)$ for $x,y\in X$ then $L_\pm$ are isotropic and $\dim X=\dim L_+ +\dim L_-$.

**Corollary.** *If* $(X,m,e,s)$ *is a linear* $S^*$*-algebra then X is canonically isomorphic (as a symplectic space) to* $E\oplus E^*$.

*Proof.* We have $L_+=E$ and $L_-$ is identified with $E^*$ using the symplectic form $\omega$ on $X$:

$$\langle\xi,a\rangle=\omega(\xi,a)\quad\text{for}\quad \xi\in L_-, a\in E$$

(see also [9]). □

If $C$ is a subspace in a vector space $E$ then the annihilator of $C$ is defined by

$$C^0=\{\xi\in E^*:\langle\xi,u\rangle=0 \text{ for } u\in C\}\ .$$

**Lemma 5.4.** *Let* $(X,\omega)$ *be a symplectic vector space. Let* $\psi: X\to E$ *be a linear map and* $\Pi=\psi_*\Pi_\omega$. *Then for any subspace* $C\subset E$,

$$\psi^{-1}(C) \text{ is coisotropic} \Leftrightarrow \Pi \rfloor C^0\subset C\ .$$

*Proof.* Let $K=\psi^{-1}(C)$. We have

$$\psi(K^{\S})=\psi(\Pi_{\omega}\lrcorner K^{0})=\psi(\Pi_{\omega}\lrcorner\psi^{*}C^{0})=\psi_{*}(\Pi_{\omega})\lrcorner C^{0}=\Pi\lrcorner C^{0},$$

hence

$$K^{\S}\subset K\Rightarrow\Pi\lrcorner C^{0}=\psi(K^{\S})\subset\psi(K)=C$$

and

$$\Pi\lrcorner C^{0}\subset C\Rightarrow K^{\S}\subset\psi^{-1}\psi(K^{\S})=\psi^{-1}(\Pi\lrcorner C^{0})\subset\psi^{-1}(C)=K.$$

**Lemma 5.5.** *Let $E$ be a vector space and let $E_1, E_2$ be its two subspaces such that $E=E_1\oplus E_2$. Let $\Pi_1\in E_1\wedge E_1$, $\Pi_2\in E_2\wedge E_2$, $\Pi=\Pi_1-\Pi_2\in E\wedge E$ and $C=\mathscr{G}(f)$, where $f: E_2\to E_1$ is a linear map. Then $\Pi\lrcorner C^0=C$ if and only if $f$ is a Poisson map.*

*Proof.* From

$$\mathscr{G}(f)^{0}=\{(\xi,\eta)\in E_1^{*}\oplus E_2^{*}:\langle\xi,f(b)\rangle+\langle\eta,b\rangle=0,\ b\in E_2\}$$

$$=\{(\xi,-f^{*}\xi):\xi\in E_1^{*}\},$$

we have $\Pi\lrcorner\mathscr{G}(f)^{0}=\{(\Pi_1\lrcorner\xi,\Pi_2\lrcorner f^{*}\xi):\xi\in E_1^{*}\}$, hence

$$\Pi\mathscr{G}(f)^{0}\subset\mathscr{G}(f)\Leftrightarrow\Pi_1\lrcorner\xi=f(\Pi_2\lrcorner f^{*}\xi)\quad\text{for}\quad\xi\in E_1^{*}\Leftrightarrow\Pi_1=f_{*}\Pi_2.\quad\square$$

Now with each linear $S^*$-algebra $(X,m,e,s)$ we associate a Poisson vector space $(E,\Pi_L)$, where $E=e(1)$, $\Pi_L=e_{L_*}\Pi_{\omega}$. By Corollary after Lemma 5.3 we can always assume that $X=E\oplus E^*$ and

$$s(a,\xi)=(a,-\xi)\quad\text{for}\quad u\in E,\xi\in E^{*}. \tag{11}$$

Let us note that the projections $e_L$ and $e_R$ are determined by the Poisson bi-vector:

$$e_L(\xi)=\tfrac{1}{2}\Pi_L\xi=-e_R(\xi)\quad\text{for}\quad\xi\in E^{*}. \tag{12}$$

In fact, if $A=e_L|_{E^*}$ then, by Lemma 5.1,

$$\mathscr{G}(-A)=\ker e_L=(\ker e_R)^{\S}=\mathscr{G}(A)^{\S}=\mathscr{G}(A^{*}),$$

hence $A^*=-A$ and

$$A\xi=\tfrac{1}{2}(A\xi-A^{*}\xi)=\tfrac{1}{2}(Aa^{k}\langle\xi,a_k\rangle-\langle\xi,Aa^{k}\rangle a_k)$$

$$=\tfrac{1}{2}(Aa^{k}\wedge a_k)\lrcorner\xi=\tfrac{1}{2}[e_{L_*}(a^{k}\wedge a_k)]\lrcorner\xi,$$

where $(a_k)_{k=1,..,n}$ is a basis in $E$ and $(a^k)_{k=1,\dots,n}$ is the dual basis in $E^*$.

**Lemma 5.6.** *Let $(X,m,e,s)$ be a linear $S^*$-algebra. We identify $X$ with $E\oplus E^*$ and $s\xi=-\xi$ for $\xi\in E^*$. For any subspace $C\subset E$ such that $\Pi_L\lrcorner C^0\subset C$ there is exactly one lagrangian subspace $\Lambda$ of $X$ such that $C\subset\Lambda\subset K$, where $K=e_L^{-1}(C)$, namely $\Lambda=C\oplus K^{\S}=C\oplus C^0$.*

*Proof.* By Lemma 5.4, $K$ is coisotropic. We shall find $K^{\S}$. We have

$$K=\{(a,\xi)\in E\oplus E^{*}:a+A\xi\in C\}=(c-A\xi,\xi):\xi\in E^{*},c\in C\},$$

hence $K^{\S}=\{(b,\eta):\langle\eta,c\rangle=\langle\xi,b+A^*\eta\rangle \text{ for } \xi\in E^*, c\in C\}=\{(A\eta,\eta):\eta\in C^0\}$. It follows that $C\cap K^{\S}=\{0\}$. We have $\dim C+\dim K^{\S}=\dim C+2\dim E-\dim K=\dim C+2\dim E-(\dim C+\dim E)=\dim E$, hence

$$\Lambda = C\oplus K^{\S}=\{(c+A\eta,\eta):c\in C,\eta\in C^0\}=C\oplus C^0 \ . \quad \square$$

Let $M=(X,m,e,s)$ be a linear $S^*$-algebra and let $l:X\to X\times\bar{X}$ be a relation such that

$$(z;x,y)\in\mathscr{G}(m)\Leftrightarrow(z,y;x)\in\mathscr{G}(l)$$

(it is easy to see that $l$ is a morphism from $M$ to $\operatorname{End}X$; $l$ is said to be the *left regular representation* of $M$). We shall use Lemma 5.6 in order to prove that $\Pi_L$ determines $l$ (hence also $m$). In fact, $\mathscr{G}(l)$ is a lagrangian subspace of $(X\times\bar{X})\times\bar{X}$ and the latter space carries the structure of a product of two linear $S^*$-algebras, $\operatorname{End}X$ and $(\bar{X},m,e,s)$. The base map of $l$ is given by $\Delta_X\ni(x,x)\mapsto f(x,x)=e_L(x)\in E$ and this is a Poisson map (on $\Delta_X$ we choose the left Poisson bi-vector). By Lemma 5.5, $C=\mathscr{G}(f^T)$ satisfies $[(\Pi_{\operatorname{End}X})_L-\Pi_L]\lrcorner C^0\subset C$. We have also

$$C\subset\mathscr{G}(l)\subset K=[(e_{\operatorname{End}X})_L\times e_L]^{-1}(C)=\{(z,y;x):e_L(z)=e_L(x)\} \ .$$

From Lemma 5.6 it follows that $\mathscr{G}(l)=C\oplus K^{\S}$. Using this we can calculate $\mathscr{G}(m)$ explicity:

$$\mathscr{G}(m)=\{(w,\xi+\eta;e_L(w,\xi),\eta,e_R(w,\eta),\xi):w\in E,\xi,\eta\in E^*\} \ . \quad (13)$$

It is easy to see that for arbitrary $\Pi\in E\wedge E$, formulae (12) and (13) (with $\Pi_L=\Pi$) define a relation $m$ satisfying the associativity and other axioms of a linear $S^*$-algebra. Thus we have proved the following lemma.

**Lemma 5.7.** *For each bi-vector $\Pi\in E\wedge E$ there is exactly one structure of a linear $S^*$-algebra on $E\oplus E^*$ such that $\Pi_L=\Pi$ (and $s$ is given by (11)).*

**Proposition 5.8.** *Let $M=(X,m,e,s)$, $M'=(X',m',e',s')$ be two linear $S^*$-algebras and let $(E,\Pi_L)$, $(E',\Pi_L')$ be the corresponding Poisson vector spaces. If $h:X\to X'$ is a linear relation which is a morphism from $M$ to $M'$, then $h_0^T$: $E'\to E$ is a Poisson map. The assignments*

$$(X,m,e,s)\mapsto(E,\Pi_L) \ , \qquad h\mapsto h_0^T$$

*define a bijective contravariant functor from the category of (linear) morphisms of linear $S^*$-algebras to the category of linear Poisson maps.*

*Proof.* Since

$$\mathscr{G}(h_0)=(e_L'\times e_L)\mathscr{G}(h)\subset\mathscr{G}(h)\subset(e_L'\times e_L)^{-1}\mathscr{G}(h_0) \ ,$$

hence $(e_L'\times e_L)^{-1}\mathscr{G}(h_0)$ is coisotropic (because it contains a lagrangian subspace) and by Lemmas 5.4 and 5.5, $h_0^T$ is a Poisson map. Conversely, if $f:E'\to E$ is a linear Poisson map then, by Lemma 5.6, there is exactly one linear symplectic relation $h:X\to X'$ such that $\mathscr{G}(f^T)\subset\mathscr{G}(h)\subset(e_L'\times e_L)^{-1}\mathscr{G}(f^T)$. It is easy to see that $h$ is a morphism of $S^*$-algebras and $f=h_0^T$ $\square$

Now let $(X,m,e,s)$ be a $S^*$-algebra. For each point $a\in E$ the tangent space $\mathrm{T}_aX$ has a structure of a linear $S^*$-algebra and $(\mathrm{T}_aE,\Pi_L(a))$, where $\Pi_L(a)=e_{L*}\Pi_\omega(a)$, is the corresponding Poisson vector space. It follows immediately from

Prop. 5.8 that the base map of a morphism of $S^*$-algebras is a Poisson map. The assignment $h \mapsto h_0^T$ is a contravariant functor from the category of $S^*$-algebras to the category of Poisson manifolds.

## 6. Symplectization and Completeness

In the preceding section we have associated with each morphism of $S^*$-algebras a Poisson map — the base map of the morphism. In this section we study the inverse problem: given a Poisson map, is it possible to construct a morphism of $S^*$-algebras whose base map is the original Poisson map?

We begin with the problem of uniqueness of such a symplectization.

**Proposition 6.1.** *Let $M=(X,m,e,s)$ and $M'=(X',m',e',s')$ be two $S^*$-algebras. If the fibers $X_a$, $a \in E$ are connected, then any morphism $h$ from $M$ to $M'$ is uniquely determined by its base map.*

*Proof.* By Lemmas 5.4 and 5.5, $K_L=(e'_L \times e_L)^{-1}\mathscr{G}(h_0)$ is a coisotropic submanifold and we have

$$\mathscr{G}(h_0) \subset \mathscr{G}(h) \subset K_L .$$

Let $z=(x',x) \in \mathscr{G}(h)$, $b'=e'_R(x')$ *an* $b=e_R(x)$. Since $(e'_R \times e_R)|_{\mathscr{G}(h)}$ has a constant rank equal to dim $\mathscr{G}(h_0)$ (by Lemma 4.1), we have

$$\mathrm{T}_z\mathscr{G}(h) \cap \mathrm{T}_z(X'_{b'} \times X_b) = \mathrm{T}_z(\mathscr{G}(h) \cap (X'_{b'} \times X_b)) = \mathrm{T}_z\mathscr{G}(h_{b'}) .$$

On the other hand, $\mathrm{T}_z K_L \supset \mathrm{T}_z\mathscr{G}(h) + \mathrm{T}_z({}_{a'}X' \times {}_aX)$, where $a'=e'_L(x')$, $a=e_l(x)$, hence

$$(\mathrm{T}_z K_l)^{\S} \subset \mathrm{T}_z\mathscr{G}(h) \cap \mathrm{T}_z(X'_{b'} \times X_b) = \mathrm{T}_z\mathscr{G}(h_{b'}) .$$

Counting dimensions yields

$$\operatorname{codim} K_L = \operatorname{codim} \mathscr{G}(h_0) = \dim E = \dim X - \dim E = \dim \mathrm{T}_z\mathscr{G}(h_{b'}) .$$

hence $(\mathrm{T}_z K_L)^{\S} = \mathrm{T}_z\mathscr{G}(h_{b'})$ is connected (as a graph of a smooth map with connected domain), it coincides with the characteristic of $K_L$ passing through $z$. This characteristic contains $(b',b) \in \mathscr{G}(h_0)$. We have proved that $\mathscr{G}(h)$ is the union of those characteristics of $K_L$ which pass through $\mathscr{G}(h_0)$. □

Now we shall formulate a condition which is essential for a Poisson map to be a base map. If $(E,\Pi)$ is a Poisson manifold, then for each (smooth) function $H$ on $E$ we denote by $\mathscr{X}_H$ the hamiltonian vector field corresponding to $H$:

$$\mathscr{X}_H = -\Pi \rfloor dH .$$

**Definition.** Let $(E,\Pi)$ and $(E',\Pi')$ be two Poisson manifolds. A Poisson map $f: E' \to E$ is said to be *complete* if $\mathscr{X}_{f^*H}$ is complete whenever $\mathscr{X}_H$ is complete, for any smooth function $H$ on $E$.

**Proposition 6.2** *Base maps of morphisms of $S^*$-algebras are complete.*

The proof will be given after the following lemma.

**Lemma 6.3.** *Let $M=(X,m,e,s)$, $M'=(X',m',e',s')$ be two $S^*$-algebras and let $f:E'\to E$ be a Poisson map. Then the characteristic distribution on $K_L=(e'_L\times e_L)^{-1}(\mathscr{G}(f)^T)$ is spanned by vector fields $\mathscr{X}_{\tilde{H}}$, where*

$$\tilde{H}=e'_L{}^*f^*H-e_L{}^*H \tag{14}$$

*and $H$ is a smooth function on $E$.*

*Proof.* A covector $(\eta,\xi)\in T^*_z(X'\times X)$ annihilates $T_zK_L$ if and only if

$$\langle\eta,v\rangle+\langle\xi,u\rangle=0 \tag{15}$$

for $u\in T_xX$, $v\in T_{x'}X'$ such that $e_{L_*}u=f_*e'_{L_*}v$. Using (15) with $u,v$ such that $e_{L_*}u=0$, $e'_{L_*}v=0$ we obtain that $(\eta,\xi)\in(T_zK_L)^0$ implies $\xi=e^*_L(\alpha)$, $\eta=e'_L{}^*(\beta)$ for some covectors $\alpha\in T^*_{eL(x)}E$, $\beta\in T^*_{e'L(x')}E'$. It follows that $(\eta,\xi)\in(T_zK_L)^0$ if and only if

$$\langle\alpha,f_*e'_{L_*}v\rangle+\langle\beta,e'_{L_*}v\rangle=\langle\alpha,e_{L_*}u\rangle+\langle\beta,e'_Lv\rangle=0$$

for $v\in T_{x'}X'$, or, equivalently, if $\beta=-f^*\alpha$. It follows that $(T_zK_L)^0=\{(e'_L{}^*f^*\alpha,\ -e_L{}^*\alpha):\alpha\in T^*_{eL(x)}E\}$. If we substitute $\alpha=dH(e_L(x))$, where $H$ is a function on $E$, we obtain $(e'_L{}^*f^*\alpha,\ -e_L{}^*\alpha)=d(e'_L{}^*f^*H-e_L{}^*H)$, hence $(T_zK_L)^\S$ is spanned by $\mathscr{X}_{\tilde{H}}(z)$. □

*Proof of 6.2.* Let the Poisson map $f$ in Lemma 6.3 be the base map of a morphism $h$ from $M$ to $M'$. Let $H$ be a function on $E$. If $z=(x',x)\in\mathscr{G}(h_{b'})$ then, by Lemma 6.3, $\mathscr{X}_{\tilde{H}}(z)$ is tangent to $\mathscr{G}(h_{b'})$, hence

$$\mathscr{X}_{F'}(x')=h_{b'_*}\mathscr{X}_F(x)\ ,$$

where $F=-e_L{}^*H$, $F'=e'_L{}^*f^*H$. Therefore the integral curve of $\mathscr{X}_{F'}$ starting from $b'$ is given by $x'(t)=h_{b'}x(t)$, where $t\mapsto x(t)$ is the integral curve of $\mathscr{X}_F$ starting from $b=f(b')$. If $\mathscr{X}_H$ is complete then $\mathscr{X}_{e_L{}^*H}$ is also complete ([5], Chap. III. Sect. 1) and $t\mapsto x'(t)$ is defined for all values of $t$. But $t\mapsto e_Lx(t)$ is the integral curve of $\mathscr{X}_{f^*H}$ starting from $b'$, since $e'_{L_*}\mathscr{X}_{e'_L{}^*K}=\mathscr{X}_K$ for any function $K$ on $E'$. It follows that $\mathscr{X}_{f^*H}$ is complete.

**Proposition 6.4.** *Let $M=(X,m,e,s)$, $M'=(X',m',e',s')$ be two $S^*$-algebras and let fibers $X_b$ be connected and simply connected for $b\in E$. Then any complete Poisson map $f:E'\to E$ is a base map of a (unique) morphism $h$ from $M$ to $M'$.*

*Proof.* Define $h:X\to X'$ as a relation whose graph is the union of those characteristics of $K_L=(e'_L\times e_L)^{-1}(\mathscr{G}(f)^T)$, which pass through $\mathscr{G}(f)^T$. For $z=(x',x)\in K_L$ we have

$$T_zK_L\supset T_z({}_{a'}X'\times{}_aX)\quad\text{for}\quad a'=e'_L(x'),\, a=e_L(x)$$

and

$$(T_zK_L)^\S\subset T_z(X'_{b'}\times X_b)\quad\text{for}\quad b'=e'_R(x'),\, b_R(x)\ ,$$

hence the characteristic passing through $z$ is an immersed submanifold of $X'_{b'}\times X_b$. If $z\in\mathscr{G}(h)$, then this characteristic passes through $(b',b)$ and is equal $\mathscr{G}(h)\cap(X'_{b'}\times X_b)=\mathscr{G}(h_{b'})$. By the completeness of $f$, $\mathscr{X}_{\tilde{H}}$ is complete for each function $H$ on $E$ such that $\mathscr{X}_H$ is complete ($\tilde{H}$ is defined in (14)). By Lemma 6.3, the flows of vector fields $\mathscr{X}_{\tilde{H}}$ preserve $\mathscr{G}(h_{b'})$. This implies that the flows of vector fields $\mathscr{X}_{e_L{}^*H}$ preserve the projection of $\mathscr{G}(h_{b'})$ on $X_b$. Since subsets of $X_b$ invariant under these flows are open and $X_b$ is connected it follows that the projection of $\mathscr{G}(h_{b'})$ on $X_b$ is equal $X_b$. Since $\dim \mathscr{G}(h_{b'}) = \operatorname{codim} K_L = \dim X_b$ and $(v, 0) \in T_z\mathscr{G}(h_{b'})$ implies $v = 0$ (by Lemma 6.3), $\mathscr{G}(h_{b'})$ is a connected covering of $X_b$. Since $X_b$ is simply connected, this covering is in fact one-fold only, hence $h_{b'}: X_b \to X'_{b'}$ is a smooth map. It follows that $\mathscr{G}(h)$ is the image of the map

$$\mathscr{G}(f^T e_R) \ni (b', x) \mapsto (h_{b'}(x), x) \in X' \times X \ . \tag{16}$$

This map is a section of the projection $e'_R \times I$ over $\mathscr{G}(f^T e_R)$ (as in Lemma 4.1). We shall show that this section is smooth. Let $(b', x) \in \mathscr{G}(f^T e_R)$ (i.e. $e_R(x) = f(b')$). Then $x = \Phi(b)$, where $\Phi$ is a product of flows of complete vector fields of the form $\mathscr{X}_{e_L{}^*H}$ and $b = f(b')$. Let $H_k$, $k = 1, \ldots, n = \dim E$ be a collection of compactly supported functions on $E$ such that $dH_k$ form a basis in $T_a^*E$, $a = e_L(x)$. We set $H_\lambda = \sum_k \lambda^k H_k$ for $\lambda \in \mathbb{R}^n$ and let $\Phi_\lambda$ be the flow corresponding to $e_L{}^*H_\lambda$ (at time $t = 1$). Then

$$\mathbb{R}^n \times E' \supset \mathscr{O} \ni (\lambda, \tilde{b}') \mapsto (\Phi_\lambda \Phi(\tilde{b}'), \tilde{b}') \in \mathscr{G}(f^T e_R)$$

provides a local chart of $\mathscr{G}(F^T e_R)$ ($\mathscr{O}$ is a suitable neighbourhood of $(0, b')$ in $\mathbb{R}^n \times E'$). We have

$$h_{\tilde{b}'}(\Phi_\lambda \Phi(f(\tilde{b}'))) = \Phi'_\lambda \Phi'(\tilde{b}') \ ,$$

where $\Phi'_\lambda$ is the flow corresponding to $e'_L{}^* f^* H_\lambda$ and $\Phi'$ is the product of flows corresponding to $e'_L{}^* f^* H$ (with the same $H$'s as before). Since the right-hand side of the above equality depends smoothly on $(\lambda, \tilde{b}')$, it follows that the section in (16) is smooth, hence $h$ is a symplectic relation.

Now we shall prove (7) and (8) ((9) is rather trivial). Formula (7) is equivalent to the following equality:

$$h_{b'} m(x, y) = m'(h_{e'_L h_{b'}(y)}(x), h_{b'}(y)) \tag{17}$$

for $(x, y)$ such that $e_R(y) = f(b')$, $e_R(x) = e_L(y)$ and $b' \in E'$. To prove (17) let us fix $b'$ and $y$ such that $e_R(y) = f(b')$. We have to show that (17) holds for $x \in X_{e_L(y)}$. It holds for $x = e_L(y)$. We shall show that if it holds for $x_0$ and $t \mapsto x(t)$ is the integral curve of $\mathscr{X}_{e_L{}^*H}$ (for some $H$) such that $x(0) = x_0$, then (17) holds for all points of the curve. Indeed, if we set $l(t) = h_{b'} m(x(t), y)$, we have

$$\frac{d}{dt} l(t) = h_{b'}{}_* m_*(\mathscr{X}_{e_L{}^*H}(x(t)), 0) = h_{b'}{}_* \mathscr{X}_{e_L{}^*H}(m(x(t), y))$$

because $\mathscr{X}_{e_L{}^*H}$ is right-invariant [5]. It follows that

$$\frac{d}{dt} l(t) = \mathscr{X}_{e'_L{}^* f^* H}(l(t)) \ .$$

On the other hand, if we set $r(t) = m'(h_{e'_L h_{b'}(y)}(x(t)), h_{b'}(y))$, then

$$\begin{aligned}\frac{d}{dt} r(t) &= m'_*(h_{e'_L h_{b'}(y)*} \mathscr{X}_{e_L{}^*H}(x(t)), 0) \\ &= m'_*(\mathscr{X}_{e'_L{}^* f^* H}(h_{e'_L h_{b'}(y)}(x(t))), 0) = \mathscr{X}_{e'_L{}^* f^* H}(r(t)) \ ,\end{aligned}$$

hence $l(t)$ and $r(t)$ are integral curves of the same vector field with the same starting point. This ends the proof of (17).

Now we prove that $G(h)$ is star-invariant (formula (8)). Substituting $x=sy$ in (17) we obtain

$$b'=h_{b'}(e_R(y))= h_{b'}m(sy,y)=m'(h_{e'Lh_{b'}(y)}(sy),h_{b'}(y)) \ ,$$

hence

$$h_{e'_L h_{b'}(y)}(sy)=s'h_{b'}(y) \ . \tag{18}$$

Now, if $(h_{b'}(y),y)\in\mathcal{G}(h)$ then $e_R(sy)=e_L(y)=f(e'_L h_{b'}(y))$, hence $(h_{e'_L h_{b'}(y)}(sy),sy)\in\mathcal{G}(h)$ and from (18) we obtain $(s'h_{b'}(y),sy)\in\mathcal{G}(h)$.

Transversality conditions $h \pitchfork e$, $m' \pitchfork (h\otimes h)$ are proved as follows. To prove the first condition, let us note that $(x'(t),x(t))\in\mathcal{G}(h), \dot{x}(0)\in \mathrm{T}E$ implies $\dot{x}(0)=e_{R_*}\dot{x}(0)=f_*e'_{R_*}\dot{x}'(0)$. $\mathcal{G}(h)$ is the image of a smooth section of the projection $e'_R\times I$ over $\mathcal{G}(f^Te_R)$, hence $\mathcal{G}(\mathrm{T}h)=\mathrm{T}\mathcal{G}(h)$ is the image of a smooth section of the projection $e'_{R_*}\times I$ over $\mathcal{G}(f^T_*e_{R_*})$. Therefore, if $(u',u)\in\mathcal{G}(\mathrm{T}h)$, $(v',v)\in\mathcal{G}(\mathrm{T}h)$ and $w=\mathrm{T}m'(u'v')$then $e_{R_*}v'=e_{R_*}w$ and $(v',v)\in\mathcal{G}(\mathrm{T}h)$, hence $w$ and $v$ determine $v'$. Similarly, $w$ and $u$ determine $u'$. □

*Example 6.1.* Given a Poisson manifold $(P,\Pi)$, one can try to find a $S^*$-algebra $M=(X,m,e,s)$such that $(P,\Pi)=(E,\Pi_L)$. By Proposition 6.4, any two such $S^*$-algebras with connected and simply connected fibers are canonically isomorphic (the identity of $E$ is complete). In particular, any $S^*$-algebra $(X,m,e,s)$ with connected and simply connected fibers and such that $\Pi_L=0$ is canonically isomorphic to $\mathrm{T}^*D_E$. Any $S^*$-algebra $(X,m,e,s)$ with connected and simply connected fibers and such that $E$ is isomorphic (as a Poisson manifold) with the dual of a Lie algebra $\mathfrak{g}$ is isomorphic to the symplectic group algebra $\mathrm{P}(G)$, where $G$ is the connected and simply connected Lie group corresponding to $\mathfrak{g}$.

*Example 6.2.* Let $M=(X,m,e,s)$ be a $S^*$-algebra. Any function $f:E\to\mathbb{R}$ is a Poisson map (the real line is considered with its unique Poisson bracket equal to zero). This function is complete if and only if $\mathcal{X}_f$ is complete. If it is complete, it defines a morphism $h$ from $\mathrm{T}^*\mathbb{R}=\mathbb{R}\times\mathbb{R}^*$ to $M$. Images under $h$ of bi-sections $\{(\varepsilon,t)\in\mathbb{R}\times\mathbb{R}^*: t=\mathrm{const}\}$ form a one-parameter family of lagrangian bi-sections of $M$ (see [5] for a definition of a bi-section; from Lemma 4.1 it follows that images of smooth bi-sections under morphisms of regular $D^*$-algebras are smooth bi-sections). If $M=\mathrm{T}^*E$ then the bi-section corresponding to $t=1$ is known as the lagrangian submanifold of $\mathrm{T}^*E$ generated by $f$ (in this case each $f$ is complete).

*Example 6.3. Symplectic Gelfand-Naimark duality.*

To each manifold $Q$ there corresponds a commutative $S^*$-algebra with connected and simply connected fibers, namely $\mathrm{T}^*(D_Q)$. Conversely, each commutative $S^*$-algebra $(X,m,e,s)$ with connected and simply connected fibers is canonically isomorphic to $\mathrm{T}^*(D_E)$. If $f:Q'\to Q$ is a smooth map then $f^T$ is a morphism from $D_Q$ to $D_{Q'}$ and $(\mathrm{P}f)^T=\mathrm{T}^*f$ is a morphism from $\mathrm{T}^*(D_Q)$ to $\mathrm{T}^*(D_{Q'})$. By Proposition 6.1, each morphism of contangent bundles is of this type.

*Example 6.4 A representation* of a $S^*$-algebra $M=(X,m,e,s)$ in a symplectic manifold $Z$ is a morphism $h$ from $M$ to $\mathrm{End}\, Z$. The base map $f:\Delta_Z\to E$ is said to be the *moment map* of the representation. Under the natural identification of

$\Delta_Z$ and $Z$, the moment map is a Poisson map from $Z$ to $(E, \Pi_L)$. If $M = \mathrm{P}G$ is the symplectic group algebra of a connected and simply connected Lie group $G$, then a Poisson map $f: Z \to E = \mathfrak{g}^*$ is complete (and defines a representation of $\mathrm{P}(G)$ in $Z$) if and only if the local action $u \mapsto \mathscr{X}_{\langle f,u \rangle}$ of $\mathfrak{g}$ on $Z$ gives rise to a global action of $G$ on $Z$ (see also [14]).

## 7. $D^*$-Groups and Double Lie Groups

**Definition.** A *$D^*$-group* is a $U^*$-group $K = (D, m)$ such that $D = (X, d, c, r)$ is a $D^*$-space, $m$ and $e$ are morphisms of $D^*$-spaces and $m \pitchfork (k \otimes I) d$ ($e$ and $k$ are the neutral element and the inverse in $K$).

It follows directly from the definition that the pairs $\mathrm{T}K = (\mathrm{T}D, \mathrm{T}m)$, $\mathrm{P}K = (\mathrm{P}D, \mathrm{P}m)$ and $\mathrm{S}K = (\mathrm{S}D, \mathrm{S}m)$ obtained by applying functors T, P, S to a $D^*$-group $K = (D, m)$ are again $D^*$-groups.

**Definition** (cf. [6]). A *double Lie group* is a double group $(G; A, B)$, where $G$ is a Lie group and $A$, $B$ are closed subgroups in $G$.

**Proposition 7.1.** *Let $K = (D, m)$ be a $D^*$-group and $D = (X, d, c, r)$. Then the corresponding double group $(X; C, E)$ is a double Lie group. The $U^*$-algebra of $K$, $(X, m, e, s)$, is a regular $D^*$-algebra.*

*Proof.* From $m \pitchfork (k \otimes I)\, d$ it follows that the map $(c_L \times e_L)^{-1}: C \times E \to X$ is smooth, hence $c_L \times e_L$ is a diffeomorphism. In particular, $e_R$ is a submersion. It follows that $(X, m, e, s)$ is a regular $D^*$-algebra. Since $m|_{C \times C}$ and $d^T|_{E \times E}$ are smooth, $C$ and $E$ are Lie groups and bijections $c_L \times e_R$, $c_R \times e_L$, $c_R \times e_R$ are diffeomorphisms (cf. beginning of I.10). It follows that the multiplication in $X$ is smooth, hence $X$ is a Lie group and $C$, $E$ are closed subgroups in $X$.

*Example 7.1.* Let $K = (D, m)$ be a $D^*$-group such that $D = D_X$, where $X$ is a manifold. Then $X$ with the multiplication map $m: X \times X \to X$ is a Lie group. The corresponding double Lie group is $(X; X, \{e\})$, where $e$ is the neutral element in $X$. Let us consider the tangent and the phase lift of $K$.

a) The tangent $D^*$-group of $K$, $\mathrm{T}K = (\mathrm{T}D_X, \mathrm{T}m)$, is again an ordinary Lie group, because $\mathrm{T}D_X = D_{\mathrm{T}X}$. The multiplication $\mathrm{T}m$ is a map, which in explicit terms is given by

$$\mathrm{T}m(u, v) = uh + gv \ , \tag{19}$$

where $u \in \mathrm{T}_g X$, $v \in \mathrm{T}_h X$. The corresponding double Lie group equals $(\mathrm{T}X; \mathrm{T}X, \{e\})$. We have used the following notational *convention*: we denote by $g\Omega$ ($\Omega g$) the left (right) translation by $g \in G$ of an element $\Omega$ of any tensor bundle over a Lie group $G$.

b) The phase $D^*$-group of $K$, $\mathrm{P}K = (\mathrm{P}D_X, \mathrm{P}m)$, is not an ordinary group. One can check easily that the corresponding double Lie group is $(\mathrm{P}X; X, \mathrm{T}_e^* X)$, where the cotangent bundle $\mathrm{P}X$ is viewed as a group under the following multiplication:

$$\xi \cdot \eta = \xi h + g\eta \ ,$$

where $\xi \in \mathrm{T}_g^* X$, $\eta \in \mathrm{T}_h^* X$ (cf. (19) and the *convention*).

**Proposition 7.2.** *Let* $(G; A, B)$ *be a double Lie group. Then* $\mathscr{A}=(G,\alpha,A,s_A)$ *and* $\mathscr{B}=(G,\beta,B,s_B)$ *(notation as in I.9) are regular D*-algebras and* $(\mathscr{A}^T,\beta)$ *is a D*-group.*

*Proof.* Set $\mathfrak{g}=\mathrm{T}_0G$, $\mathfrak{a}=\mathrm{T}_0A$, $\mathfrak{b}=\mathrm{T}_0B$ (here 0 is the neutral element of $G$). Since $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{b}$, we have $\mathrm{T}_gG=g\mathfrak{a}\oplus g\mathfrak{b}=\mathfrak{a}g\oplus\mathfrak{b}g$ for each $g\in G$. We have also $\mathrm{T}_gG=g\mathfrak{a}\oplus\mathfrak{b}g=\mathfrak{a}g\oplus g\mathfrak{b}$. This is true because if $u\in\mathfrak{a}$ and $gug^{-1}=Ad_gu\in\mathfrak{b}$ then $Ad_au\in\mathfrak{b}$, where $a=a_R(g)$, hence $Ad_au=0$ and $u=0$. It follows that the smooth bijection $A\times B\ni(a,b)\mapsto\Phi(a,b)=ab\in G$ is a submersion:

$$\mathrm{T}\Phi(\mathrm{T}_aA\times\mathrm{T}_bB)=\{\dot{a}b+a\dot{b}:\dot{a}\in\mathrm{T}_aA,\dot{b}\in\mathrm{T}_bB\}=\mathfrak{a}ab+ab\mathfrak{b}=\mathrm{T}_{ab}G\ .$$

This implies that $a_L\times b_R$ is a diffeomorphism. Using the map $(a,b)\mapsto ba$ we can prove that $a_R\times b_L$ is a diffeomorphism. In particular, projections $a_L$, $a_R$, $b_L$ and $b_R$ are smooth submersions. It follows that $\alpha^T(G)=\{(g,h):a_R(g)=a_L(h)\}$ is a submanifold. Since

$$\mathscr{G}(\alpha)=\{(b_1ab_2;b_1a,ab_2):a\in A,b_1,b_2\in B\}\ , \tag{20}$$

it is easy to see that $\alpha$ is a differentiable reduction. Formula (6) implies that $s_A$ is a differentiable relation. It remains to prove that $\alpha\pitchfork(A\otimes I)$ and $\beta\pitchfork(k\otimes I)\alpha^T$, where $k$ is the inverse in $G$.

To this end let us consider first the tangent group $\mathrm{T}G$ (Example 7.1). Since each $w\in\mathrm{T}_{ab}G$ has a unique decomposition

$$w=\dot{a}\dot{b}=\dot{a}b+a\dot{b}\ (\dot{a}\in\mathrm{T}_AA,\dot{b}\in\mathrm{T}_bB)\ ,$$

$(\mathrm{T}G;\mathrm{T}A,\mathrm{T}B)$ is a double Lie group. We denote the corresponding projections by $\dot{a}_L$, etc., hence we have $w=\dot{a}_L(w)\dot{b}_R(w)=\dot{b}_L(w)\dot{a}_R(w)$. Since $\mathrm{T}a_L(g\mathfrak{b})=0$, we have

$$\mathrm{T}a_L(w)=\mathrm{T}a_L(\dot{a}b+a\dot{b})=\mathrm{T}a_L(\dot{a}b)=\frac{d}{dt}\bigg|_{t=0}a_L(a(t)b)=\frac{d}{dt}\bigg|_{t=0}a(t)=\dot{a}\ ,$$

where $t\mapsto a(t)$ is a curve representing $\dot{a}$. It follows that $\mathrm{T}a_L=\dot{a}_L$ and, similarly, $\mathrm{T}a_R=\dot{a}_R$. Let $\dot{\alpha},\dot{\beta}:\mathrm{T}G\otimes\mathrm{T}G\to\mathrm{T}G$ be the differentiable reductions associated with the double Lie group $(\mathrm{T}G;\mathrm{T}A,\mathrm{T}B)$. From (20) we have

$$\begin{aligned}\mathscr{G}(\mathrm{T}\alpha)=\{&(\dot{b}_1ab_2+b_1\dot{a}b_2+b_1a\dot{b}_2;\dot{b}_1a+b_1\dot{a},\dot{a}b_2\\&+a\dot{b}_2):\dot{a}\in\mathrm{T}_aA,\dot{b}_1\in\mathrm{T}_{b_l}B,\dot{b}_2\in\mathrm{T}_{b_2}B\}\ .\end{aligned} \tag{21}$$

Since

$$\dot{b}_1ab_2+b_1\dot{a}b_2+b_1a\dot{b}_2=(\dot{b}_1a+b_1\dot{a})b_2+(b_1a)\dot{b}_2=(\dot{b}_1a+b_1\dot{a})\dot{b}_2=\dot{b}_1\dot{a}\dot{b}_2\ ,$$

$$\dot{b}_1a+b_1\dot{a}=\dot{b}_1\dot{a}$$

and

$$\dot{a}b_2+a\dot{b}_2=\dot{a}\dot{b}_2\ ,$$

it follows that $\mathrm{T}\alpha=\dot{\alpha}$. In the same way we obtain $\mathrm{T}\beta=\dot{\beta}$. It follows that $\mathrm{T}\alpha\,1(\mathrm{T}A\otimes I)$ and $\mathrm{T}\beta\,1\mathrm{T}((k\otimes I)\alpha^T)$. Note that the multiplication in $\mathrm{T}G$ defined by $\mathrm{T}\alpha$ and $\mathrm{T}\beta$ coincides with $\mathrm{T}m$, where $m$ is the multiplication in $G$.

Now let us consider the phase space $\mathrm{P}G$. Since each $\zeta\in\mathrm{P}_{ab}G$ has a unique decomposition

$$\zeta=\xi\eta=\xi b+a\eta\ \ (\xi\in(\mathrm{T}_aA)^0,\eta\in(\mathrm{T}_aB)^0)\ ,$$

$(\mathrm{P}G; (\mathrm{T}A)^0, (\mathrm{T}B)^0)$ is a double Lie group. However, as we shall see, the corresponding differentiable reductions **do not** coincide (in general) with $\mathrm{P}\alpha$ and $\mathrm{P}\beta$. In fact, (21) implies that $(\zeta;\xi,\eta)\in\mathscr{G}(\mathrm{P}\alpha)$ if and only if $\langle\zeta,\dot{b}_1ab_2+b_1\dot{a}b_2+b_1a\dot{b}_2\rangle=\langle\xi,\dot{b}_1a+b_1\dot{a}\rangle+\langle\eta,\dot{a}b_2+a\dot{b}_2\rangle$ for $\dot{a}\in\mathrm{T}_aA$, $\dot{b}_1\in\mathrm{T}_{b_1}B$, $\dot{b}_2\in\mathrm{T}_{b_2}B$. It follows that $(\zeta;\xi,\eta)\in\mathscr{G}(\mathrm{P}\alpha)$ if and only if

$$b_1^{-1}\zeta b_2^{-1}-b_1^{-1}\xi\in\mathfrak{b}^0a\ , \tag{22}$$

$$b_1^{-1}\zeta b_2^{-1}-\eta b_2^{-1}\in\mathfrak{a}\mathfrak{b}^0 \tag{23}$$

and

$$b_1^{-1}\zeta b_2^{-1}-b_1^{-1}\xi-\eta b_2^{-1}\in\mathfrak{a}^0a=a\mathfrak{a}^0\ . \tag{24}$$

In particular, if $(\zeta;\xi,\eta)\in\mathscr{G}(\mathrm{P}\alpha)$ and $\xi\in(\mathrm{T}A)^0$ then $b_1=0$ and

$$\zeta b_2^{-1}-\eta b_2^{-1}\in a\mathfrak{b}^0\cap a\mathfrak{a}^0\ ,$$

hence $\zeta=\eta$ and $\xi$ is uniquely determined by

$$\xi-\eta b_2^{-1}\in\mathfrak{b}^0a\quad\text{and}\quad\xi\in a\mathfrak{a}^0\ . \tag{25}$$

It follows that $\mathrm{P}\alpha\,\mathrm{l}(\mathrm{P}A\otimes I)$. Note that the left projection of $\eta$ associated with $\mathrm{P}\alpha$ given by $\xi$ in (25) is **different** (in general) from the left projection $\xi'$ of $\eta$ on $(\mathrm{T}A)^0$ in the double Lie group $(\mathrm{P}G; (\mathrm{T}A)^0, (TB)^0)$, which is given by

$$\xi'-\eta b_2^{-1}\in a\mathfrak{b}^0\quad\text{and}\quad\xi'\in a\mathfrak{a}^0\ .$$

We shall show now that $\mathrm{P}\beta\pitchfork(\mathrm{P}k\otimes I)\mathrm{P}\alpha^T$. From [I] we know that

$$(b;x,y)\in\mathscr{G}(\beta(s_B\otimes I))\quad\text{and}\quad(x,y;a')\in\mathscr{G}((s_A\otimes I)\alpha^T)$$

if and only if $x=y=ab=b'a'$ for some $a\in A$, $b'\in B$. In this case we have

$$(b;a^{-1}b',b'a')\in\mathscr{G}(\beta)\quad\text{and}\quad(b'^{-1}a,ab;a')\in\mathscr{G}(\alpha^T)\ . \tag{26}$$

If $(0;\xi_1,\eta)\in\mathscr{G}(\mathrm{P}\beta)$ and $(\xi_2,\eta;0)\in\mathscr{G}(\mathrm{P}\alpha^T)$, where all covectors are attached to the corresponding points in (26), then $\eta a'^{-1}\in b'\mathfrak{a}^0$ and $\eta b^{-1}\in a\mathfrak{b}^0$ (from (23)), hence

$$\eta\in b'\mathfrak{a}^0a'\cap a\mathfrak{b}^0b=b'a'\mathfrak{a}^0\cap ab\mathfrak{b}^0\ ,$$

i.e. $\eta=0$. By (22) and (24) also $\xi_1=0$ and $\xi_2=0$. □

## 8. $S^*$-Groups and Manin Groups

**Definition.** A $S^*$*-group* is a $D^*$-group $(D,m)$ such that $D$ is a $S^*$-space and $m$, $e$ are symplectic relations.

The phase lift of a $D^*$-group is a $S^*$-group.

**Definition.** A *Manin group* is a double Lie group $(G;A,B)$, where $G$ is equipped with an invariant non-degenerate scalar product, vanishing on $\mathrm{T}A$ and $\mathrm{T}B$.

The notion of a Manin group is a global version of the notion of a Manin triple ([4, 6],...). Interesting examples of Manin groups are given in [6].

**Theorem.** *There is a one-to-one correspondence between $S^*$-groups and Manin groups.*

The rest of this section is devoted to a proof of this theorem.

Let $(D,m)$ be a $S^*$-group. By Proposition 7.1 we can assume that $(D,m)$ is associated with a double Lie group $(G;A,B)$, i.e. $(D,m)=(\mathscr{A}^T,\beta)$. Let

$$I=P_1+P_2, \quad I=Q_1+Q_2$$

be the decomposition of the identity of $\mathrm{T}_g G$ on projectors, corresponding to decompositions

$$\mathrm{T}_g G=g\mathfrak{a}\oplus g\mathfrak{b}, \quad \mathrm{T}_g G=\mathfrak{a}g\oplus\mathfrak{b}g\ ,$$

for each fixed $g\in G$ (the notation here is as in the proof of Proposition 7.2). We set

$$n(u,v)=\omega((P_1-Q_2)u,v) \tag{27}$$

for $u,v\in\mathrm{T}_g G$, where $\omega$ is the symplectic form.

**Lemma 8.1.** *The bilinear form $n$ is symmetric, non-degenerate and G-invariant. Subspaces $g\mathfrak{a}$, $\mathfrak{a}g$, $g\mathfrak{b}$ and $\mathfrak{b}g$ are all isotropic with respect to $n$.*

*Proof.* Since $(g\mathfrak{a})^{\S}=\mathfrak{a}g$ and $(g\mathfrak{b})^{\S}=\mathfrak{b}g$, we have $\omega(Q_2u,v)=\omega(Q_2u,P_1v+P_2v)=\omega(Q_2u,P_1v)=\omega(u,P_1v)$, hence

$$n(u,v)=\omega(P_1u,v)+\omega(P_1v,u) \tag{28}$$

and $n$ is symmetric. We have also

$$n(u,v)=\omega(Q_1u,v)+\omega(Q_1v,u)=\omega(u,P_2v)+\omega(v,P_2u)$$
$$=\omega(u,Q_2v)+\omega(v,Q_2u)\ .$$

It follows that $\ker P_1$, $\ker P_2$, $\ker Q_1$ and $\ker Q_2$ are isotropic with respect to $n$. The non-degeneracy follows from the fact that $(P_1-Q_2)$ is invertible: if $(P_1-Q_2)u=0$ then $P_1u=Q_2u$, hence $u\in g\mathfrak{b}\cap\mathfrak{a}g$ and therefore $u=0$.

We have $n(u,v)=\omega(u,v)$ for $u\in g\mathfrak{a}$, $v\in g\mathfrak{b}$. This condition together with the isotropy of $g\mathfrak{a}$ and $g\mathfrak{b}$ fully characterizes $n$. It follows that $n$ is left-invariant if and only if

$$\omega(g\dot{a},g\dot{b})=\omega(\dot{a},\dot{b}) \quad \text{for} \quad \dot{a}\in\mathfrak{a},\ \dot{b}\in\mathfrak{b},\ g\in G\ . \tag{29}$$

For each $a\in A$, $b\in B$ and any curve $a(t)$ in $A$ and $b(t)$ in $B$ we have

$$(aba(t);ab,ba(t))\in\mathscr{G}(\beta)\ , \quad (ab,ba(t);ab,ba(t),a(t))\in\mathscr{G}(I\otimes\alpha)$$

and

$$(abb(t);abb(t),bb(t))\in\mathscr{G}(\beta)\ , \quad (abb(t),bb(t);abb(t),b,b(t))\in\mathscr{G}(I\otimes\alpha))\ ,$$

hence

$$(aba(t);ab,ba(t),a(t))\in\mathscr{G}(\beta(I\otimes\alpha))$$

and

$$(abb(t);abb(t),b,b(t))\in\mathscr{G}(\beta(I\otimes\alpha))\,.$$

This implies that vectors $(ab\dot{a};0,b\dot{a},\dot{a})$ and $(ab\dot{b};ab\dot{b},0,\dot{b})$ are symplectically orthogonal ($\beta(I\otimes\alpha)$ is a symplectic relation), hence $\omega(ab\dot{a},ab\dot{b})=\omega(\dot{a},\dot{b})$. It follows that $n$ is left-invariant. A similar argument shows that it is also right-invariant. □

It is clear (by Lemma 8.1) that $(G;A,B)$ is a Manin group, where $G$ is considered as being equipped with the scalar product $n$ defined in (27).

Now we shall show that any Manin group carries the structure of a $S^*$-group. Let $(G; A, B)$ be a Manin group and let $n$ be the scalar product on $G$. Since $n(P_1u, v) = n(P_1u, P_2v) = n(u, P_2v)$ and $n(Q_2u, v) = n(u, Q_1v)$, we have $n((P_1 - Q_2)u, v) = n(u, (P_2 - Q_1)v) = -n(u, (P_1 - Q_2)v)$, hence $P_1 - Q_2$ is antisymmetric with respect to $n$. Therefore the inverse, $(P_1 - Q_2)^{-1}$, is also antisymmetric. We set

$$\omega(u, v) = n((P_1 - Q_2)^{-1}u, v) \ .$$

Then $\omega$ is the 2-form such that

$$n(u, v) = \omega((P_1 - Q_2)u, v) \ . \tag{30}$$

If we denote again by $n$ the symmetric map from $T_gG$ to $T_g^*G$ associated with $n$, then

$$n = \flat(P_1 - Q_2)$$

($\flat$ defined by $\omega$). We have

$$P_1 - Q_2 = \tfrac{1}{2}[(P_1 - P_2) + (Q_1 - Q_2)] = \tfrac{1}{2}(gR + Rg) \ ,$$

where $R$ is the reflection in $\mathfrak{a}$ parallel to $\mathfrak{b}$ in $\mathfrak{g}$. It follows that

$$\sharp n = \tfrac{1}{2}(gR + Rg)$$

and

$$\sharp = \tfrac{1}{2}[g(Rn^{-1}) + (Rn^{-1})g] \ ,$$

where $\sharp = \flat^{-1}$. Therefore we have

$$\Pi_\omega(g) = \tfrac{1}{2}(g\Pi_0 + \Pi_0 g) \ ,$$

where $\Pi_0$ is the canonical [4] bi-vector on $\mathfrak{g}$. By the results of [15], $\Pi_\omega$ is a Poisson bi-vector field, hence $\omega$ is a symplectic form (i.e. it is closed).

It remains to prove that $\alpha$ (and $\beta$) is a symplectic relation. Since $n$ is non-degenerate and $\mathfrak{a}$, $\mathfrak{b}$ are $n$-isotorpic, then $\dim A = \dim B = \frac{1}{2}\dim G$. From (20) we have

$$\dim \mathscr{G}(\alpha) = \dim A + 2\dim B = \tfrac{3}{2}\dim G = \tfrac{1}{2}\dim(G \times G \times G) \ .$$

We have to show that $\mathscr{G}(\alpha)$ is isotropic with respect to the symplectic form. By (21), we have to show that

$$\omega(\dot{b}_1ab_2 + b_1\dot{a}b_2 + b_1a\dot{b}_2, \dot{b}_1'ab_2 + b_1\dot{a}'b_2 + b_1a\dot{b}_2')$$
$$= \omega(\dot{b}_1a + b_1\dot{a}, \dot{b}_1'a + b_1\dot{a}') + \omega(\dot{a}b_2 + a\dot{b}_2, \dot{a}'b_2 + a\dot{b}_2')$$

for $\dot{a}, \dot{a}' \in T_aA$, $\dot{b}_1, \dot{b}_1' \in T_{b_1}B$, $\dot{b}_2, \dot{b}_2' \in T_{b_2}B$, and this is equivalent to nine following conditions

1° $\omega(\dot{b}_1ab_2, \dot{b}_1'ab_2) = \omega(\dot{b}_1a, \dot{b}_1'a)$,
2° $\omega(\dot{b}_1ab_2, b_1\dot{a}'b_2) = \omega(\dot{b}_1a, b_1\dot{a}')$,
3° $\omega(\dot{b}_1ab_2, b_1a\dot{b}_2') = 0$,
4° $\omega(b_1\dot{a}b_2, \dot{b}_1'ab_2) = \omega(b_1\dot{a}, \dot{b}_1'a)$,
5° $\omega(b_1\dot{a}b_2, b_1\dot{a}'b_2) = \omega(b_1\dot{a}, b_1\dot{a}') + \omega(\dot{a}b_2, \dot{a}'b_2)$,

6° $\omega(b_1\dot{a}b_2, b_1a\dot{b}_2') = \omega(\dot{a}b_2, a\dot{b}_2')$,
7° $\omega(b_1a\dot{b}_2, \dot{b}'ab_2) = 0$,
8° $\omega(b_1a\dot{b}_2, b_1\dot{a}'b_2) = \omega(a\dot{b}_2, \dot{a}', b_2)$,
9° $\omega(b_1a\dot{b}_2, b_1a\dot{b}_2') = \omega(a\dot{b}_2, a\dot{b}_2')$.

Since $g\mathfrak{a}$, $g\mathfrak{b}$, $\mathfrak{a}g$ and $\mathfrak{b}g$ are all isotropic with respect to $n$, hence by (30) we have

$$(g\mathfrak{a})^{\S} = (P_1 - Q_2)(g\mathfrak{a}) = \mathfrak{a}g \ ,$$
$$(g\mathfrak{b})^{\S} = (P_1 - Q_2)(g\mathfrak{b}) = \mathfrak{b}g \ .$$

This implies 3° and 4°. Conditions 1°, 2° and 4° are contained in the following statement:

$$\omega\,|_{\mathfrak{b}g \times \mathrm{T}_gG} \quad \text{is right } B\text{-invariant} \ .$$

This statement is in fact true. Let $u \in \mathfrak{b}g$, $v \in \mathrm{T}_gG$ and $b \in B$. Then $u = (P_1 - Q_2)w = -Q_2w$ for some $w \in g\mathfrak{b}$. We have

$$\omega(u,v) = \omega((P_1 - Q_2)w, v) = n(w,v) = n(wb, vb) = \omega((P_1 - Q_2)(wb), vb) \ .$$

Since $Q_2$ is right-invariant and $wb \in gb\mathfrak{b}$, we have

$$\omega(u,v) = \omega(-Q_2(wb), vb) = \omega(-(Q_2w)b, vb) = \omega(ub, vb) \ .$$

Similarly, conditions 6°, 8° and 9° hold, because $\omega\,|_{g\mathfrak{b}\times \mathrm{T}_gG}$ is left $B$-invariant.
Up to now we have proved that $V \subset (\mathrm{T}\mathscr{G}(\alpha))^{\S}$, where

$$V = \{(\dot{b}_1ab_2 + b_1a\dot{b}_2; \dot{b}_1a, a\dot{b}_2) : \dot{b}_1 \in \mathrm{T}_{b_1}B, \dot{b}_2 \in \mathrm{T}_{b_2}B\} \ .$$

It remains to prove condition 5° which says that

$$W = \{(b_1\dot{a}b_2; b_1\dot{a}, \dot{a}b_2) : \dot{a} \in \mathrm{T}_aA\}$$

is isotropic. We are not able to prove it directly. However it follows immediately from the fact we have

$$\mathrm{T}\mathscr{G}(\alpha) = V \oplus W = V \oplus W_L = V \oplus W_R$$

and $W_L \subset W_R^{\S}$, where

$$W_L = \{(ub_1ab_2; ub_1a, (u^{b_1})ab_2) : u \in \mathfrak{a}\} \ ,$$
$$W_R = \{(b_1ab_2v; b_1a({}^{b_2}v), ab_2v) : v \in \mathfrak{a}\}$$

(in order to see that $W_L \subset \mathrm{T}\mathscr{G}(\alpha)$ note that $(xb_1ab_2; xb_1a, (x^{b_1})ab_2) \in \mathscr{G}(\alpha)$ for $x \in A$).

## 9. $S^*$-Groups and Poisson-Lie Groups

If $(D, m)$ is a $S^*$-group, then the base map of $m, m|_{C\times C}$, is both the group multiplication in $C$ and a Poisson map. Therefore $C$ is a Poisson-Lie group ([4, 6],...). With sensibly defined morphisms of $S^*$-groups (cf. the definition of a morphism of union pseudogroups in [1]), the passage from $S^*$-groups to the corresponding Poisson-Lie groups is a (covariant) functor. Working with $S^*$-groups rather than Poisson-Lie groups has the following advantages:

- The symmetry between algebraic and space structure of a pseudogroup is evident.

- The structure of a $S^*$-group is well prepared for quantization which consists in replacing certain symplectic relations by operators; we can expect that there exist distinguished "invariant" polarizations which are necessary for the quantization (in the case of ordinary groups it often happens; while passing from an ordinary group to a pseudogroup the number of symmetries does not change). A class of quantum deformations of the Heisenberg has been already obtained by this method in [16]. The quantization assigns to one classical object only one quantum object.

- A passage from $(D, m)$ to $C$ may cause a lost of information (the case disconnected fibers, union pseudogroups, multiply-connected fibers).

- Some Poisson-Lie groups **do not have** the corresponding $S^*$-group. If such $S^*$-group exists, the Manin triple corresponding to a given Poisson-Lie group is the Lie algebra of the Manin group. But in general a Manin triple need not give rise to a Manin group (like a double Lie algebra [6] need not give rise to a double Lie group). Of course, a necessary condition is the completeness of the dressing fields [6]. Another necessary condition is the completeness of the group multiplication in a Poisson-Lie group (cf. Proposition 6.2). These conditions seem to be closely related each to other (and they seem to be essentially sufficient).

What really happens if a Manin triple $(\mathfrak{g}; \mathfrak{a}, \mathfrak{b})$ does not give rise to a Manin group? Let $G$ be the connected and simply connected Lie group corresponding to $\mathfrak{g}$ and let $A$, $B$ be the subgroups corresponding to $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that $A$, $B$ are closed and $A \cap B = \{0\}$. Then the first statement of Proposition 7.2 remains true provided we replace $G$ by $P = A \cdot B \cap B \cdot A$ (cf. I.9). Moreover, arguments of Sect. 8 show that $P$ is a symplectic manifold and the algebras in Proposition 7.2 are $S^*$-algebras. But, unless $P = G$, we do not have equalities in (I.29), (I.30), (I.31).

## 10. Appendix: Proof of Proposition 2.2

Let $\alpha \pitchfork \beta$. We shall show that $\mathrm{S}(\alpha\beta) = \mathrm{S}(\alpha)\mathrm{S}(\beta)$. If $\gamma(t) = (x(t), z(t))$ is a curve in $\mathscr{G}(\alpha\beta)$, then $(x(t), s_{\alpha\beta}(\gamma(t))$ is a curve in $\mathscr{G}(\alpha)$ and $(s_{\alpha\beta}(\gamma(t)), z(t))$ is a curve in $\mathscr{G}(\beta)$. It follows that $\mathrm{T}(\alpha\beta) \subset \mathrm{T}\alpha\mathrm{T}\beta$. Conversely, if $(u, v) \in \mathscr{G}(\mathrm{T}\alpha)$, $(v, w) \in \mathscr{G}(\mathrm{T}\beta)$, then $(u, v, v, w) \in \mathrm{T}R \cap \mathrm{T}\Delta = \mathrm{T}(R \cap \Delta)$. It follows that there exists a curve of the form $(x(t), y(t), z(t))$ representing $(u, v, w)$ such that $(x(t), z(t)) \in \mathscr{G}(\alpha\beta)$, hence $(u, v) \in \mathscr{G}(\mathrm{T}(\alpha\beta))$. The equality $\mathrm{P}(\alpha\beta) = \mathrm{P}\alpha\mathrm{P}\beta$ follows from $\mathrm{T}(\alpha\beta) = \mathrm{T}\alpha\mathrm{T}\beta$ by applying the duality functor (taking into account that $\alpha \, 1 \beta$, cf. [10]).

In the proof of the first part we have used the notation introduced in the proof of Proposition 2.1 ($R$ and $\Delta$). In the second part we shall use also the following convention. If $A$ is a submanifold of $B$, we shall denote by $|A\rangle$ a differentiable relation from $\{1\}$ to $B$ whose image is $A$ (we shall use this convention only in such cases when it is clear what is $B$). We set also $\langle A| = |A\rangle^T$. We have the following lemmas.

**Lemma A.1.** *$\alpha \pitchfork \beta$ if and only if $(I \otimes \langle \Delta_Y| \otimes I) \pitchfork (|\mathscr{G}(\alpha)\rangle \otimes |\mathscr{G}(\beta)\rangle)$.*

**Lemma A.2.** *If $\rho \pitchfork \lambda$, where $\rho$ is a differential reduction and the domain of $\lambda$ is $\{1\}$, then $\mathrm{S}\rho \pitchfork \mathrm{S}\lambda$.*

These lemmas will be proved later in this section.

From the lemmas it follows that $\alpha \pitchfork \beta$ implies

$$\mathrm{S}(I\otimes\langle\Delta_Y|\otimes I)\pitchfork\mathrm{S}(|\mathscr{G}(\alpha)\rangle\otimes|\mathscr{G}(\beta)\rangle) \ . \tag{A.1}$$

We shall show that (A.1) implies

$$(I\otimes\langle\Delta_{\mathrm{S}Y}|\otimes I)\pitchfork(|\mathscr{G}(\mathrm{S}\alpha)\rangle\otimes|\mathscr{G}(\mathrm{S}\beta)\rangle) \ . \tag{A.2}$$

We have

$$\mathrm{S}\langle\Delta_Y| = \langle\Delta_{\mathrm{S}Y}|(k_Y\otimes I) \ ,$$

where $k_Y: \mathrm{S}Y\to\mathrm{S}Y$ is defined by $k_Y(v,\eta)=(v,-\eta)$. We have also

$$\begin{aligned} \mathrm{S}|\mathscr{G}(\alpha)\rangle &= (I\otimes k_Y)|\mathscr{G}(\mathrm{S}(\alpha)\rangle \ , \\ \mathrm{S}|\mathscr{G}(\beta)\rangle &= (I\otimes k_Z)|\mathscr{G}(\mathrm{S}(\beta)\rangle \ . \end{aligned} \tag{A.3}$$

It follows from (A1) that

$$(I\otimes\langle\Delta_{\mathrm{S}Y}|\otimes I)(I\otimes k_Y\otimes I\otimes I)$$

$$\pitchfork(I\otimes k_Y\otimes I\otimes k_Z)(|\mathscr{G}(\mathrm{S}\alpha)\rangle\otimes|\mathscr{G}(\mathrm{S}\beta)\rangle)$$

and we have

$$(I\otimes I\otimes I\otimes k_Z)(I\otimes\langle\Delta_{\mathrm{S}Y}|\otimes I)(I\otimes k_Y\otimes I\otimes k_Z)$$

$$\pitchfork(I\otimes k_Y\otimes I\otimes k_Z)(|\mathscr{G}(\mathrm{S}\alpha)\rangle\otimes|\mathscr{G}(\mathrm{S}\beta)\rangle) \ ,$$

hence (A2). By Lemma A.1, (A2) implies $\mathrm{S}\alpha\pitchfork\mathrm{S}\beta$. If we use functor T or P instead of S, we obtain $\mathrm{T}\alpha\pitchfork\mathrm{T}\beta$ and $\mathrm{P}\alpha\pitchfork\mathrm{P}\beta$.

*Proof of Lemma A.1.* We set $\rho = I\otimes\langle\Delta_Y|\otimes I$, $\lambda = |\mathscr{G}(\alpha)\rangle\otimes|\mathscr{G}(\beta)\rangle$. It is easy to see that $\alpha 1 \beta$ if and only if $\rho 1 \lambda$ and

$$\alpha\beta \text{ is differentiable} \Leftrightarrow \rho\lambda \text{ is differentiable} \ .$$

Assume that $\alpha 1 \beta$, then $\mathrm{P}\alpha 1 \mathrm{P}\beta$, is equivalent to each of the following statements:
1) $(0,n)\in\mathscr{G}(\mathrm{P}\alpha)$ and $(\eta,0)\in\mathscr{G}(\mathrm{P}\beta)$ implies $\eta=0$ (for $\eta\in\mathrm{P}Y$),
2) $((0,0),(0,-\eta,\eta,0))\in\mathscr{G}(\mathrm{P}\rho)$ and $((0,-\eta,\eta,0)\in\mathscr{G}(P\lambda)$ implies $\eta=0$.
3) $((0,0),\kappa)\in\mathscr{G}(\mathrm{P}\rho)$ and $(\kappa,0)\in\mathscr{G}(\mathrm{P}\lambda)$ implies $\kappa=0$ (for $\kappa\in\mathrm{P}X\times\mathrm{P}Y\times\mathrm{P}Y\times\mathrm{P}Z$),
4) $\mathrm{P}\rho 1 \mathrm{P}\lambda$.

If we assume $\mathrm{P}\alpha 1 \mathrm{P}\beta$, then $\mathrm{T}\alpha 1 \mathrm{T}\beta$ is equivalent to each of the following statements:
1) $(0,v)\in\mathscr{G}(\mathrm{T}\alpha)$ and $(v,0)\in\mathscr{G}(\mathrm{T}\beta)$ implies $v=0$ (for $v\in\mathrm{T}Y$),
2) $(0,u)\in\mathscr{G}(\mathrm{T}\rho)$ and $(u,0)\in\mathscr{G}(\mathrm{T}\lambda)$ implies $u=0$ (for $u\in\mathrm{T}(X\times Y\times Y\times Z)$),
3) $\mathrm{T}\rho 1 \mathrm{T}\lambda$. □

*Proof of Lemma A.2.* Let $\rho = f\iota^T, \iota: C\to X$, $f: C\to Y$ be the canonical decomposition of $\rho$ (by the definition of a reduction), $L=\lambda(1)$. We shall prove first the following lemma.

**Lemma A.3.** *$\rho\pitchfork\lambda$ if and only if $\rho\lambda$ is differentiable and*
(i) $\rho 1 \lambda$,
(ii) *$L$ intersects $C$ transversally,*
(iii) *$f|_{L\cap C}$ is an immersion.*

*Proof.* Assume $\rho \perp \lambda$, then $\mathrm{P}\rho \perp \mathrm{P}\lambda$ is equivalent to each of the following statements:
1) $(0,\eta) \in \mathscr{G}(\mathrm{P}\rho)$ and $(\eta,0) \in \mathscr{G}(\mathrm{P}\lambda)$ implies $\eta = 0$,
2) $\eta|_{TC} = 0$ and $\eta|_{TL} = 0$ implies $\eta = 0$,
3) (ii).
Assume $\mathrm{P}\rho \perp \mathrm{P}\lambda$, then $\mathrm{T}\rho \perp \mathrm{T}\lambda$ is equivalent to each of the following statements:
1) $(0,v) \in \mathscr{G}(\mathrm{T}\rho)$ and $(v,0) \in \mathscr{G}(\mathrm{T}\lambda)$ implies $v = 0$,
2) $v \in \ker df$ and $v \in \mathrm{T}L$ implies $v = 0$,
3) (iii). □

From Lemma A.3 it follows the following local "normal form" of $\rho$ and $\lambda$:

$$Y \cong A \otimes B \ ,$$

$$X \cong A \otimes B \otimes E \otimes D \ ,$$

$$\lambda \cong |A\rangle \otimes |\{b\}\rangle \otimes |E\rangle \otimes |\{d\}\rangle \ ,$$

$$\rho \cong I \otimes I \otimes \langle\{e\}| \otimes \langle D| \quad (\text{i.e. } C \cong A \otimes B \otimes \{e\} \otimes D) \ ,$$

where $b, e, d$ – certain points of $B, E, D$, respectively. Since the product preserves the transversality, it is sufficient to prove the transversality of the corresponding factors, for instance $\langle D| \pitchfork |\{d\}\rangle$, and this is easy.

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