Critical Limit One-Point Correlations of Monodromy Fields on $\mathbb{Z}^2$

Neil A. Watling

Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

Abstract. Monodromy fields on $\mathbb{Z}^2$ are a family of lattice fields in two dimensions which are a natural generalization of the two dimensional Ising field occurring in the C^*-algebra approach to Statistical Mechanics. A criterion for the critical limit one point correlation of the monodromy field $\sigma_a(M)$ at $a \in \mathbb{Z}^2$,

$$\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle,$$

is deduced for matrices $M \in GL(p, \mathbb{C})$ having non-negative eigenvalues.

Using this criterion non-identity 2×2 matrices are found with finite critical limit one point correlation. The general set of $p \times p$ matrices with finite critical limit one point correlations is also considered and a conjecture for the critical limit n point correlations postulated.

1. Introduction

The C^*-algebra approach to the Ising model via the transfer matrix is now wellknown, see [1, 4, 7–9, 10] for example. Monodromy fields on $\mathbb{Z}^2$, introduced in [14] are a family of lattice fields in two dimensions which are a natural generalization of the two dimensional Ising field. They were inspired by [21] and in a sense are lattice analogues of the continuum fields used in [21, IV] and also in the Federbush and massless Thirring models, see [19, 20, 6]. These lattice fields are interesting for several reasons. Firstly by controlling the scaling limit mathematically precise information on the continuum can be found and this approach was successfully used for the Ising field in [17, 18], secondly there are numerous analogues of continuum structures suggesting a discrete theory on the lattice itself. For $M \in GL(p, \mathbb{C})$ and $a \in \mathbb{Z}^2$ it is possible to define the monodromy field $\sigma_a(M)$ at a. This is a generalization of the Ising field in the sense that when M is the scalar -1 the vacuum expectation of a product $\sigma_{a_1}(-1) \ldots \sigma_{a_n}(-1)$ gives the square of an Ising correlation. The motivation for the name "monodromy field" is the fact that

it is possible to "create" monodromy M located at $a \in \mathbb{Z}^2$ in the solution to a certain linear difference equation on the lattice through a formula involving $\sigma_a(M)$.

In [14] the one point correlations when M is a scalar were calculated using an elliptic substitution. Also the asymptotics of the correlations were examined in the scaling limit, that is the limit that sends the lattice spacing to zero and the "temperature" to the critical point such that the correlation length remains fixed (massive scaling regime). In [15] the critical scaling limit was studied, that is the large scale asymptotics of the correlations at the critical point (massless regime). However a limitation of the analysis carried out in [15] was the fact that the monodromy fields had to appear in pairs, $\sigma_a(M)\sigma_b(M)^{-1}$, which was referred to as the twin problem. That is only correlations of the form

$$\langle \sigma_{a_1}(M_1)\sigma_{b_1}(M_1)^{-1} \dots \sigma_{a_n}(M_n)\sigma_{b_n}(M_n)^{-1} \rangle,$$

could be studied. Moreover the M_i had to have non-negative eigenvalues.

In order to find the large scale asymptotics at the critical point the following limit needs to be investigated:

$$\lim_{s \uparrow 1} \langle \sigma_{a_1}(M_1) \dots \sigma_{a_n}(M_n) \rangle.$$

This is non-trivial since the monodromy fields, $\sigma_a(M)$, are not defined for $s = 1$. A conjecture from [15] was that the limit exists and is finite if $M_1 \dots M_n = I$ and if $M_1 \dots M_n \neq I$ then the limit is 0 or ∞. The second half of this conjecture is now shown to be false by an analysis of the limiting one point correlation:

$$\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle.$$

A criterion for this limit is found enabling the existence of a non-identity M with finite critical limit correlation to be shown. However as is the case for the results in [15] this is only true for M having non-negative eigenvalues. As for the general n point correlations a product formula, see [15, 16], enables these to be written as the product of the individual one point correlations and a $\det_2$ term, see [22] for a definition. This suggests that the one point correlations are sufficient though a proof is not available as yet.

The restriction on M to have non-negative eigenvalues is somewhat inconvenient since the Ising field case is given by the scalar -1 so none of the results are applicable to this case and the critical asymptotics for the two dimensional Ising model remain unknown.

The format of the paper is as follows. Section 2 gives the basic structure and definition of the monodromy field $\sigma_a(M)$. Section 3 deduces a criterion for the critical limit one point correlations based on the result of [14] concerning the scalar case. Section 4 uses this criterion to find a non-trivial example of a matrix M with finite critical limit correlation. The structure of the set of such matrices is also studied. Finally Sect. 5 poses a conjecture for the general n point correlations using a product formula.

2. Monodromy Fields on $\mathbb{Z}^2$

2.1 Introduction. This section gives a brief summary of the structure required for the study of monodromy fields on $\mathbb{Z}^2$. For further details see [14, 15, 16].

2.2 Notation. Let $H = L^2(S^1, \mathbb{C}^2)$ and p a positive integer with $H^p = H \oplus \ldots \oplus H = H \otimes \mathbb{C}^p$. Let T be the multiplication operator on H defined by the 2×2 matrix:

$$Tf(\theta) = \begin{bmatrix} c^2/s - \cos\theta & s\sin\theta - i(c/s - c\cos\theta) \\ s\sin\theta + i(c/s - c\cos\theta) & c^2/s - \cos\theta \end{bmatrix} f(\theta),$$

where $c, s > 0$ and $c^2 - s^2 = 1$.

Let Q be the multiplication operator on H defined by:

$$Qf(\theta) = \begin{bmatrix} 0 & ie^{i\alpha(\theta)} \\ -ie^{-i\alpha(\theta)} & 0 \end{bmatrix} f(\theta),$$

where $\gamma(\theta) > 0$ and $\alpha(\theta)$ are determined by the following:

(1) $\alpha(0) = 0$,
(2) $\cosh\gamma(\theta) = c^2/s - \cos\theta$,
(3) $\sinh\gamma(\theta)e^{i\alpha(\theta)} = (c/s - c\cos\theta) + is\sin\theta$.

Note that

$$T(\theta) = \exp[-\gamma(\theta)Q(\theta)] = e^{-\gamma(\theta)}Q_+(\theta) + e^{\gamma(\theta)}Q_-(\theta),$$

where $Q_\pm = 1/2(1 \pm Q)$ with $Q^2 = 1$, Q self adjoint.

Also let z be the multiplication operator on H defined by:

$$zf(\theta) = e^{i\theta}f(\theta).$$

Now extend T, Q, and z to operators on H^p in the obvious manner, namely tensoring by I^p, i.e. the operator acts on each copy of H. With a slight abuse of notation call these operators T, Q, and z. Let $W^p = H^p \oplus \bar{H}^p$, where $\bar{H}$ denotes the Hilbert space conjugate to H, and define a conjugation P on W^p by $P(x \oplus y) = y \oplus x$. If Q_W is the operator on W^p defined as $Q \oplus (-Q)$, then Q_W anticommutes with P and Q_W is self adjoint with $Q_W^2 = 1$.

Hence Q_W defines a Q_W-Fock state of the Clifford algebra $C(W^p, P)$ whose associated representation lives in the alternating tensor algebra $\Lambda(W_+^p)$, where $W_\pm^p = Q_W^\pm W^p$ and $Q_W^\pm = 1/2(1 \pm Q_W)$. The generators of this representation are given by:

$$F(w) = a^*(Q_W^+ w) + a(PQ_W^- w),$$

where $a^*(\cdot)$, $a(\cdot)$ are creation and annihilation operators on $\Lambda(W_+^p)$.

Now define the restricted general linear group $GL_Q(H^p)$ as the group of bounded, invertible linear maps on H^p with bounded inverses whose matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the $H_+^p \oplus H_-^p$ decomposition of H^p derived from Q have b, c Hilbert Schmidt and a, d Fredholm of index 0. Also define $GL_Q^0(H^p)$ as the subgroup with d $1 + $ trace class and $GL_Q^1(H^p)$ as the subgroup with $b = c = 0$.

Results in [16, 14, 5], also a brief summary in [3], demonstrate the existence of a dense linear domain, $\mathscr{D} \subseteq \Lambda(W_+^p)$, together with two group homomorphisms, $\Gamma_Q : GL_Q^0(H^p) \to L(\mathscr{D})$ and $\Gamma : GL_Q^1(H^p) \to L(\mathscr{D})$, where $L(\mathscr{D})$ denotes the invertible linear maps from $\mathscr{D}$ to $\mathscr{D}$, such that:

$$\Gamma_Q(g)F(w) = F(g \oplus g^{*-1}w)\Gamma_Q(g), \quad g \in GL_Q^0(H^p),$$
$$\Gamma(g)F(w) = F(g \oplus g^{*-1}w)\Gamma(g), \quad g \in GL_Q^1(H^p),$$
$$\text{and } \Gamma_Q(hgh^{-1}) = \Gamma(h)\Gamma_Q(g)\Gamma(h)^{-1}.$$

Also if $GL_Q^0(H^p) \times GL_Q^1(H^p)$ is the semi-direct product with composition rule

$$g_1 \times h_1 \cdot g_2 \times h_2 = g_1 h_1 g_2 h_1^{-1} \times h_1 h_2,$$

then the above gives $g \times h \rightarrow \Gamma_Q(g)\Gamma(h)$ is a homomorphism with kernel

$$K = \{g \times h : gh = 1 \text{ and } \det d(g) = 1\}.$$

Define $\widehat{GL}_Q(H^p) = GL_Q^0(H^p) \times GL_Q^1(H^p)/K$, then $(g \times h)K \rightarrow gh$ is a well defined homomorphism $T: \widehat{GL}_Q(H^p) \rightarrow GL_Q(H^p)$ with kernel $\mathbb{C}^*$. Identifying $\widehat{GL}_Q(H^p)$ with its image in $L(\mathscr{D})$:

$$gF(w) = F(T(g) \oplus T(g)^{*-1}w)g, \qquad g \in \widehat{GL}_Q(H^p),$$

i.e. g is the implementer of $T(g)$. If Ω_Q is the vacuum vector of $\Lambda(W_+^p)$ define

$$\langle g \rangle_Q = \langle \Omega_Q, g\Omega_Q \rangle, \qquad g \in \widehat{GL}_Q(H^p),$$

that is if $g = (g' \times h')K$

$$\begin{aligned} \langle g \rangle_Q &= \langle \Omega_Q, \Gamma_Q(g')\Gamma(h')\Omega_Q \rangle \\ &= \langle \Omega_Q, \Gamma_Q(g')\Omega_Q \rangle \\ &= \det(d(g')). \end{aligned}$$

For more details of this, together with proofs, see [14, 5].

In other words, if $g \in GL_Q(H^p)$ and has a decomposition as $g_0 g_1$, where $g_i \in GL_Q^i(H^p)$ for $i = 0, 1$ then $T((g_0 \times g_1)K) = g$, thus the implementer of the automorphism induced by g on the Clifford algebra is given by $\Gamma_Q(g_0)\Gamma(g_1)$ at least on the dense domain $\mathscr{D}$ and up to scalar multiple – a choice of factorization at the $GL_Q(H^p)$ level being equivalent to a choice of normalization at the $\widehat{GL}_Q(H^p)$ level.

With the structures defined above it is now possible to define the monodromy field $\sigma(M)$, where $M \in GL(p, \mathbb{C})$. Let M act on H^p as $I \otimes M$ and define ε as the convolution operator on H whose Fourier transform acts on $l^2(\mathbb{Z}_{1/2}, \mathbb{C}^2)$ as $\hat{\varepsilon}f(k) = \text{sgn}(k)\hat{f}(k)$, for $k \in \mathbb{Z}_{1/2}$. Let $\varepsilon_{\pm} = (1 \pm \varepsilon)/2$ and define

$$s(M) = \varepsilon_- \otimes I_p + \varepsilon_+ \otimes M.$$

In [14] it is shown that $s(M) \in GL_Q(H^p)$, then $\sigma(M)$ is essentially defined such that $T(\sigma(M)) = s(M)$. So from comments made above a factorization of $s(M)$ into $g_0 g_1$, where $g_i \in GL_Q^i(H^p)$ is sufficient since $\sigma(M)$ may be defined as $\Gamma_Q(g_0)\Gamma(g_1)$. This factorization is constructed in [14] and using that notation $s(M) = \underline{s}(M)D(M)$ so that:

$$\sigma(M) = \Gamma_Q(\underline{s}(M))\Gamma(D(M)).$$

This definition can be extended to the points on a $\mathbb{Z}^2$ lattice as follows. Let $\Gamma(T)$ and $\Gamma(z)$ be the implementers of T and z then define

$$s_a(M) = T^{a_2} z^{a_1} s(M) z^{-a_1} T^{-a_2},$$

and

$$\sigma_a(M) = \Gamma(T)^{a_2} \Gamma(z)^{a_1} \sigma(M) \Gamma(z)^{-a_1} \Gamma(T)^{-a_2},$$

where $a = (a_1, a_2) \in \mathbb{Z}^2$. Call $\sigma_a(M)$ the monodromy field at a – so $\sigma(M)$ is the monodromy field at 0.

2.3 Remark. The multiplication operator T introduced at the beginning of 2.2 Notation is the same as that for the study of the Ising model in [17, 18], where $c = \cosh 2K^*$. Hence, using this connection, it is possible to consider the cases $s < 1$, $s = 1$, and $s > 1$ to correspond to below the critical temperature, at the critical temperature and above the critical temperature respectively.

2.4 Remark. One problem, which has been glossed over in the preceding comments, is the fact that $s(M) \notin GL_Q(H^p)$ when $s = 1$ (Q depends on T and thus s) and consequently a limiting argument is required. The objects of study are the n point correlations:

$$\langle \sigma_{a_1}(M_1) \ldots \sigma_{a_n}(M_n) \rangle_Q,$$

and in particular their limit as $s \uparrow 1$, that is, their behaviour as they approach the critical temperature. From now on it will be assumed that $s < 1$ and the Q subscript in the correlation will be dropped.

3. A Condition for the Existence of Limiting One Point Correlations

3.1 Introduction. This section aims to classify the critical limit one point correlations, $\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle$, for all matrices $M \in GL(p, \mathbb{C})$, $p \in \mathbb{N}$ with non-negative eigenvalues. The starting point for this classification is a result of [14] concerning the one dimensional scalar case, that is $(p = 1)$.

3.1.1 Proposition. *Let* $k = s^2$, $k'^2 = 1 - k^2$ *and*

$$K^{(l)} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + k^{(l)2} \sin^2 \theta}}.$$

Suppose $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. *If* $s(\lambda) = \underline{s}(\lambda)D(\lambda)$ *and* $\underline{d}(\lambda) = d(\underline{s}(\lambda)) = Q_- \underline{s}(\lambda)Q_-$, *then* $\underline{d}(\lambda)$ *is invertible and*

$$\langle \sigma(\lambda) \rangle = \det \underline{d}(\lambda) = \prod_{l > 0} \left[\frac{1 + \lambda^{-1}q^{2l}}{1 + q^{2l}} \frac{1 + \lambda q^{2l}}{1 + q^{2l}} \right],$$

where $l \in \mathbb{Z}_{1/2}$ *and* $q = \exp(-\pi K'/K)$.

Note. As $\underline{d}(\lambda)$ is invertible the correlation is non-zero.

Now assume $M \in GL(p, \mathbb{C})$ and has no negative eigenvalues. Hence there exists a matrix $S_M \in GL(p, \mathbb{C})$ such that $S_M M S_M^{-1} = J_M$, where J_M denotes the Jordan form of M. Thus

$$s(M) = (1 \otimes S_M^{-1})(\varepsilon_- \otimes I + \varepsilon_+ \otimes J_M)(1 \otimes S_M)$$
$$= (1 \otimes S_M^{-1})s(J_M)(1 \otimes S_M).$$

Therefore if $s(J_M)$ is factorized as $\underline{s}(J_M)D(J_M)$, $s(M)$ may be factorized as

$$((1 \otimes S_M^{-1})\underline{s}(J_M)(1 \otimes S_M))((1 \otimes S_M^{-1})D(J_M)(1 \otimes S_M)).$$

Hence

$$\langle \sigma(M) \rangle = \det d((1 \otimes S_M^{-1})\underline{s}(J_M)(1 \otimes S_M))$$
$$= \det(1 \otimes S_M^{-1})d(\underline{s}(J_M))(1 \otimes S_M)$$
$$\quad \text{as } Q_- = Q_- \otimes I \text{ commutes with } 1 \otimes S_M^{(-1)}$$
$$= \det d(\underline{s}(J_M))$$
$$= \langle \sigma(J_M) \rangle.$$

Appealing to [14], in general the factorizing terms are given by the following:

Suppose P_+, P_- are the orthogonal projections onto the subspaces of $L^2([-K, K], \mathbb{C})$ whose elements have fourier expansions in $\exp(i\pi lx/K)$ with no l negative, positive terms respectively. Then:

$$D(M) = I_+ \oplus (P_+ \otimes I_p + P_- \otimes M)_-,$$

where I_+ is the identity on H_+^p, $(P_+ \otimes I_p + P_- \otimes M)_-$ acts on $H_-^p \cong L^2([-K, K], \mathbb{C}) \otimes \mathbb{C}^p$ and

$$\underline{s}(M) = s(M) D(M)^{-1} \quad \text{with} \quad d(\underline{s}(M)) 1 + \text{trace class}.$$

So suppose the eigenvalues of M are $\lambda_1 \dots \lambda_p$ then the Jordan form

$$J_M = \begin{bmatrix} \lambda_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{p-1} \\ & & & \lambda_p \end{bmatrix},$$

where $\lambda_i \in \mathbb{C} \backslash (-\infty, 0]$ for $i = 1, \dots, p$ and $\delta_j = 0$ or 1 for $j = 1, \dots, p-1$ and

$$d(\underline{s}(J_M)) = \begin{bmatrix} \underline{d}(\lambda_1) & \cdots & \cdots & \cdots \\ & \underline{d}(\lambda_2) & \cdots & \cdots \\ & & \ddots & \vdots \\ & & & \underline{d}(\lambda_p) \end{bmatrix}.$$

Therefore $\det d(\underline{s}(J_M)) = \prod_{i=1}^{p} \det \underline{d}(\lambda_i)$, that is:

$$\langle \sigma(M) \rangle = \langle \sigma(J_M) \rangle = \det d(\underline{s}(J_M)) = \prod_{i=1}^{p} \det \underline{d}(\lambda_i)$$

$$= \prod_{i=1}^{p} \prod_{l>0} \left[\frac{1 + \lambda_i^{-1} q^{2l}}{1 + q^{2l}} \frac{1 + \lambda_i q^{2l}}{1 + q^{2l}} \right].$$

Also note that for $a = (a_1, a_2) \in \mathbb{Z}^2$,

$$\langle \sigma_a(M) \rangle = \det T^{a_2} z^{a_1} d(\underline{s}(M)) z^{-a_1} T^{-a_2}$$

$$= \langle \sigma(M) \rangle.$$

So we have shown the following lemma:

3.1.2 Lemma. *If $M \in GL(p, \mathbb{C})$ has no negative eigenvalues then*

$$\langle \sigma_a(M) \rangle = \prod_{i=1}^{p} \prod_{l>0} \left[\frac{1 + \lambda_i^{-1} q^{2l}}{1 + q^{2l}} \frac{1 + \lambda_i q^{2l}}{1 + q^{2l}} \right], \quad \forall a \in \mathbb{Z}^2.$$

3.2 Convergence Argument. The previous subsection showed that if $M \in GL(p, \mathbb{C})$ has no negative eigenvalues then

$$\langle \sigma_a(M) \rangle = \prod_{i=1}^{p} \prod_{l>0} \left[\frac{1 + \lambda_i^{-1} q^{2l}}{1 + q^{2l}} \frac{1 + \lambda_i q^{2l}}{1 + q^{2l}} \right], \quad \forall a \in \mathbb{Z}^2,$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of M, which of course can be rewritten as

$$\prod_{i=1}^{p} \langle \sigma_a(\lambda_i) \rangle.$$

Now a straightforward calculation, see also [14], shows that

$$\frac{d}{d\lambda} \{\log \langle \sigma_a(\lambda) \rangle\} = (\lambda - \lambda^{-1}) \sum_{l>0} (\lambda + q^{2l})^{-1} (\lambda + q^{-2l})^{-1},$$

and hence if C_λ denotes a contour which joins 1 to λ without crossing the negative axis or passing through 0 then

$$\log \langle \sigma_a(\lambda) \rangle = \int_{C_\lambda} (\mu - \mu^{-1}) \sum_{l>0} (\mu + q^{2l})^{-1} (\mu + q^{-2l})^{-1} d\mu.$$

3.2.1 Remark. Hence in this more general case taking logarithms as above leads to the following equation.

$$\begin{aligned}
\log \langle \sigma_a(M) \rangle &= \log \prod_{i=1}^{p} \langle \sigma_a(\lambda_i) \rangle \\
&= \sum_{i=1}^{p} \log \langle \sigma_a(\lambda_i) \rangle \\
&= \sum_{i=1}^{p} \int_{C_{\lambda_i}} (\mu_i - \mu_i^{-1}) \sum_{l>0} (\mu_i + q^{2l})^{-1} (\mu_i + q^{-2l})^{-1} d\mu_i \\
&= \sum_{i=1}^{p} \int_{C_{\lambda_i}} (\mu_i - \mu_i^{-1}) \sum_{n=0}^{\infty} (\mu_i + q^{2n+1})^{-1} (\mu_i + q^{-2n-1})^{-1} d\mu_i,
\end{aligned}$$

where C_{λ_i} denotes a contour joining 1 to λ_i without crossing the negative axis or passing through 0.

3.2.2 Remark. Using the definition of q and the Taylor series expansion for $(1-x)^{-1/2}$ it is a simple calculation to show that $q \in (0,1)$ and as $s \uparrow 1$, $q \uparrow 1$ with $K \to \infty$ and $K' \to \pi/2$. Now if the sum present in the above is replaced by the corresponding integral then the limit as $q \uparrow 1$ can be calculated as will be shown later. Consequently the difference between the sum and integral is the item of interest and it is this which will now be considered.

3.2.3 Proposition. *Suppose* $Z_{q,\mu} : [0, \infty) \to \mathbb{C}$ *is defined as*

$$Z_{q,\mu}(x) = (\mu - \mu^{-1})(\mu + q^{2x+1})^{-1}(\mu + q^{-2x-1})^{-1},$$

where $q \in (0,1)$ *and* $\mu \in C_\lambda$ *with* $\lambda \in \mathbb{C} \backslash (-\infty, 0]$. *Then*

$$\lim_{q \uparrow 1} \left(\frac{1}{2} \sum_{n=0}^{\infty} (Z_{q,\mu}(n) + Z_{q,\mu}(n+1)) - \int_0^{\infty} Z_{q,\mu}(x) dx \right) = 0.$$

The proof of this proposition follows using a series of lemmas.

3.2.4 Lemma. *Extend, in the obvious way, the function $Z_{q,\mu}$ defined above to a function on $\mathbb{C}$. Then if $|\mu| \neq 1$,*

$$
\frac{1}{2} \sum_{n=0}^{\infty} (Z_{q,\mu}(n) + Z_{q,\mu}(n+1)) - \int_0^{\infty} Z_{q,\mu}(x)\,dx
$$

$$
= \frac{1}{i} \int_0^{\infty} \frac{Z_{q,\mu}(-iy) - Z_{q,\mu}(iy)}{e^{2\pi y} - 1}\,dy + \sum_{p=1}^{\infty} \frac{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\pi i [l(\mu) + 2p\pi i]/\log q} + 1}
$$

$$
+ \sum_{p=-\infty}^{-1} \frac{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{-\pi i [l(\mu) + 2p\pi i]/\log q} + 1}
$$

$$
+ \frac{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\mathrm{sgn}[\mathrm{Im}\{l(\mu)\}]\pi i l(\mu)/\log q} + 1},
$$

where $l(\mu) = \log(-\mu)$ and log *denotes the principal value.*

Proof. The proof owes much to the Plana summation formula, see [24], of which it is essentially a variant.

Consider integrating the functions

$$
Y^{\pm}(z) = \frac{Z_{q,\mu}(z)}{e^{\pm 2\pi i z} - 1},
$$

around the indented (semi-circles of radius r round the integers, $r \to 0$) rectangles with corners 0, R, $R \mp Ri$ and $\mp Ri$ respectively, see picture below, and letting $R \to \infty$.
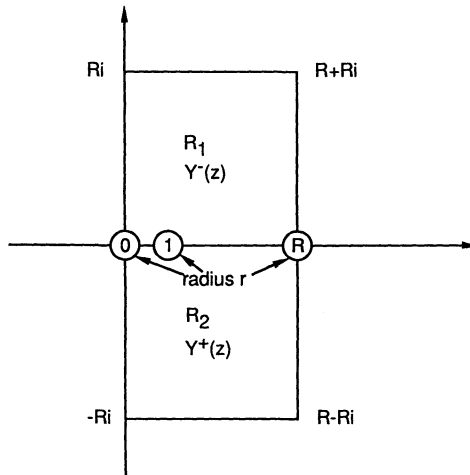

Fig. 1

Now, both $Y^{+}(z)$ and $Y^{-}(z)$ have as poles the integers, together with

$$
z_p^{\pm} = \frac{\pm [l(\mu) + 2p\pi i]}{2\log q} - \frac{1}{2}, \qquad \forall p \in \mathbb{Z}.
$$

Thus if $|\mu| > 1$ then $\mathrm{Re}\{\log(\mu)\} > 0$, hence the poles z_p^- have real part greater than zero provided q is large enough. Also it is easy to see that,

$$\mathrm{Im}\{z_p^-\} \begin{cases} >0, & \text{for } p \geq 1, \\ <0, & \text{for } p \leq -1, \end{cases}$$

with $\mathrm{sgn}[\mathrm{Im}\{z_0^-\}] = \mathrm{sgn}[\mathrm{Im}\{l(\mu)\}]$.

Similarly if $|\mu| < 1$ then the poles z_p^+ have real part greater than zero and

$$\mathrm{Im}\{z_p^+\} \begin{cases} <0, & \text{for } p \geq 1, \\ >0, & \text{for } p \leq -1, \end{cases}$$

with $\mathrm{sgn}[\mathrm{Im}\{z_0^+\}] = \mathrm{sgn}[-\mathrm{Im}\{l(\mu)\}]$.

Thus choose R large enough such that some poles are contained in the rectangles but none lie on the lines joining $\pm Ri$ to $R \pm Ri$ and $R \pm Ri$ to R. Note that $\mathrm{Im}\{l(\mu)\} \neq 0$ since $\mu \in \mathbb{C}\setminus(-\infty, 0]$, hence no $z_p^\pm$ lies on the positive real axis.

Note also that the semi-circles, with radius r, round the integers contribute a net residue of,

$$Z_{q,\mu}(n)/2, \quad 1 \leq n \leq R-1,$$
$$Z_{q,\mu}(n)/4, \quad n = 0, R$$

for each rectangle as $r \to 0$.

Hence the following equation is obtained using the residue theorem,

$$\int_{R_1} \frac{Z_{q,\mu}(z)}{e^{-2\pi i z} - 1}\, dz + \int_{R_2} \frac{Z_{q,\mu}(z)}{e^{2\pi i z} - 1}\, dz$$

$$= \frac{1}{2} \sum_{n=0}^{R-1} (Z_{q,\mu}(n) + Z_{q,\mu}(n+1))$$

$$+ 2\pi i\, \mathrm{Res}\left(\frac{Z_{q,\mu}(z)}{e^{-\mathrm{sgn}[\mathrm{Im}\{l(\mu)\}]\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]2\pi i z} - 1},\ z_0^{-\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]} \right)$$

$$+ 2\pi i \sum_{p=1}^{p(R)} \mathrm{Res}\left(\frac{Z_{q,\mu}(z)}{e^{-2\pi i z} - 1},\ z_{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]p}^{-\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]} \right)$$

$$+ 2\pi i \sum_{p=p'(R)}^{-1} \mathrm{Res}\left(\frac{Z_{q,\mu}(z)}{e^{2\pi i z} - 1},\ z_{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]p}^{-\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]} \right).$$

Now it is easy to show that the left-hand side of this equation equals

$$\int_0^R Z_{q,\mu}(x)\,dx + \frac{1}{i} \int_0^R \frac{Z_{q,\mu}(R+iy) - Z_{q,\mu}(R-iy) - Z_{q,\mu}(iy) + Z_{q,\mu}(-iy)}{e^{2\pi y} - 1}\, dy$$

$$+ \int_0^R \frac{Z_{q,\mu}(x+iR)}{e^{2\pi(R-ix)} - 1}\, dx + \int_0^R \frac{Z_{q,\mu}(x-iR)}{e^{2\pi(R+ix)} - 1}\, dx,$$

and as $R \to \infty$ this becomes

$$\int_0^\infty Z_{q,\mu}(x)\,dx + \frac{1}{i} \int_0^\infty \frac{-Z_{q,\mu}(iy) + Z_{q,\mu}(-iy)}{e^{2\pi y} - 1}\, dy,$$

the last two integrals tending to zero by taking a factor of $e^{-2\pi R}$, bounding the resultant integrand and using $\lim_{R \to \infty} \mathrm{Re}^{-2\pi R} = 0$. The parts involving $Z_{q,\mu}(R \pm iy)$ in

the other integral also tend to zero by a similar process, taking a factor of $q^{(2R+1)}$ and using $\lim_{R \to \infty} Rq^{(2R+1)} = 0$ for all $q \in (0, 1)$.

The right-hand side of the equation is fairly straightforward. The residue of the pole at $z_p^{\pm}$ is given by

$$\frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{-1}{e^{\pm 2\pi i \cdot \{-\text{sgn}[\text{Re}\{l(\mu)\}][l(\mu) + 2p\pi i]/2 \log q\}} + 1},$$

where the choice of sign depends on which rectangle the pole is contained in. As $R \to \infty$, both $p(R) \to \infty$ and $p'(R) \to -\infty$, so the right-hand side becomes,

$$\frac{1}{2} \sum_{n=0}^{\infty} (Z_{q,\mu}(n) + Z_{q,\mu}(n+1)) - \sum_{p=1}^{\infty} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\pi i[l(\mu) + 2p\pi i]/\log q} + 1}$$

$$- \sum_{p=-\infty}^{-1} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{-\pi i[l(\mu) + 2p\pi i]/\log q} + 1}$$

$$- \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\text{sgn}[\text{Im}\{l(\mu)\}]\pi i l(\mu)/\log q} + 1}.$$

This gives the required result.

3.2.5 Remark. The case $|\mu| = 1$ is somewhat simpler. Due to the presence of the $-1/2$, in the formula for $z_p^{\pm}$ the poles $z_p^{\pm}$ all lie in the left half of the complex plane, in this case, so do not contribute any residue to the integration thus leaving only the integral as in the Plane summation formula. If $\text{sgn}(0)$ is defined to be zero then Lemma 3.2.4 gives this result and the restriction $|\mu| \neq 1$ can be lifted.

3.2.6 Lemma. *For* $\mu \in \mathbb{C}\backslash(-\infty, 0]$ *then,*

(1) $$\lim_{q \uparrow 1} \sum_{p=1}^{\infty} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\pi i[l(\mu) + 2p\pi i]/\log q} + 1} = 0,$$

(2) $$\lim_{q \uparrow 1} \sum_{p=-\infty}^{-1} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{-\pi i[l(\mu) + 2p\pi i]/\log q} + 1} = 0,$$

(3) $$\lim_{q \uparrow 1} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\text{sgn}[\text{Im}\{l(\mu)\}]\pi i l(\mu)/\log q} + 1} = 0.$$

Proof. 1):

$$\left| \sum_{p=1}^{\infty} \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\pi i[l(\mu) + 2p\pi i]/\log q} + 1} \right|$$

$$\leq \sum_{p=1}^{\infty} \left| \frac{\text{sgn}[\text{Re}\{l(\mu)\}]}{2\mu \log q} \cdot \frac{2\pi i}{e^{\pi i[l(\mu) + 2p\pi i]/\log q} + 1} \right|$$

$$\leq \frac{-\pi}{|\mu| \log q} \sum_{p=1}^{\infty} \frac{1}{e^{-\pi[\text{Im}\{l(\mu)\} + 2p\pi]/\log q} - 1}$$

using $$e^x - 1 \leq |e^{(x+iy)} + 1| \leq e^x + 1,$$

$$< \frac{-2 \log q}{\pi |\mu|} \sum_{p=1}^{\infty} \frac{1}{(\text{Im}\{l(\mu)\} + 2p\pi)^2}$$

using $\mathrm{Im}\{l(\mu)\}+2p\pi>0$ for all $p\geq 1$ and $e^x-1>x^2/2$ for $x>0$,

$$< \frac{-2\log q}{\pi|\mu|}(1/24+1/\pi^2)$$

using $\mathrm{Im}\{l(\mu)\}+2p\pi>2(p-1)\pi$ for $p\geq 2$ and $\mathrm{Im}\{l(\mu)\}+2\pi\geq\pi$.

Now it is easy to see that

$$\lim_{q\uparrow 1}\frac{-2\log q}{\pi|\mu|}(1/24+1/\pi^2)=0,$$

hence

$$\lim_{q\uparrow 1}\sum_{p=1}^{\infty}\frac{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]}{2\mu\log q}\cdot\frac{2\pi i}{e^{\pi i[l(\mu)+2p\pi i]/\log q}+1}=0,$$

as required.

2): Follows in the same manner as 1) using the transformation $p\mapsto -p$ and noting the following:

(1) $2p\pi-\mathrm{Im}\{l(\mu)\}>0$ for all $p\geq 1$,
(2) $2p\pi-\mathrm{Im}\{l(\mu)\}>2(p-1)\pi$ for $p\geq 2$,
(3) $2\pi-\mathrm{Im}\{l(\mu)\}\geq\pi$.

3):
$$\left|\frac{\mathrm{sgn}[\mathrm{Re}\{l(\mu)\}]}{2\mu\log q}\cdot\frac{2\pi i}{e^{\mathrm{sgn}[\mathrm{Im}\{l(\mu)\}]\pi il(\mu)/\log q}+1}\right|$$

$$\leqq\frac{-\pi}{|\mu|\log q}\cdot\frac{1}{e^{-\pi|\mathrm{Im}\{l(\mu)\}|/\log q}-1}$$

$$<\frac{-2\log q}{\pi|\mu|}\cdot\frac{1}{(|\mathrm{Im}\{l(\mu)\}|)^2}.$$

Note $\mathrm{Im}\{l(\mu)\}\neq 0$ as $\mu\in\mathbb{C}\backslash(-\infty,0]$. Now

$$\lim_{q\uparrow 1}\frac{-2\log q}{\pi|\mu|}\cdot\frac{1}{(|\mathrm{Im}\{l(\mu)\}|)^2}=0,$$

hence the required result is obtained.

3.2.7 Lemma. *For $\mu\in C_\lambda$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$*

$$\lim_{q\uparrow 1}\frac{1}{i}\int_0^{\infty}\frac{Z_{q,\mu}(-iy)-Z_{q,\mu}(iy)}{e^{2\pi y}-1}dy=0.$$

Proof. Substituting the formula for $Z_{q,\mu}$ and simplifying gives

$$\frac{1}{i}\int_0^{\infty}\frac{Z_{q,\mu}(-iy)-Z_{q,\mu}(iy)}{e^{2\pi y}-1}dy$$

$$=C(\mu,q)\int_0^{\infty}\frac{1}{e^{2\pi y}-1}\cdot\frac{\sin(2\pi yK'/K)}{(\mu+q^{-2iy+1})(\mu+q^{2iy-1})(\mu+q^{2iy+1})(\mu+q^{-2iy-1})}dy,$$

where

$$C(\mu,q)=\frac{2(\mu^2-1)(1-q^2)}{q}.$$

Making the substitution $x = 2\pi y K'/K$ this becomes

$$C'(\mu, q) \int_0^\infty \frac{\sin x}{(e^{Kx/K'} - 1)(\mu + qe^{ix})(\mu + q^{-1}e^{-ix})(\mu + qe^{-ix})(\mu + q^{-1}e^{ix})} dx$$

$$= C'(\mu, q) \int_0^\infty \frac{\sin x}{(e^{Kx/K'} - 1)g(x, \mu, q)} dx,$$

where

$$C'(\mu, q) = \frac{(1 - q^2)(1 - \mu^2)}{q \log q},$$

and

$$g(x, \mu, q) = (\mu^2 + q^2)(\mu^2 + q^{-2}) + 2\mu(\mu^2 + 1)(q + q^{-1})\cos x + 4\mu^2 \cos^2 x.$$

Note that the function $C'(\mu, q)$ has limit $2(\mu^2 - 1)$ as $q \uparrow 1$, hence the integral must have limit zero for the lemma to be true. To show this, split the infinite integral at the point I_{C_λ} which is defined as follows:

(1) $1 - \cos I_{C_\lambda} = \left[\min_{\mu \in C_\lambda} \{1, |\mu + 1|^2\} \right] / 3,$

(2) $\pi/3 > I_{C_\lambda} > 0.$

Note this is well defined as $\mu \in \mathbb{C} \backslash (-\infty, 0]$ so cannot be -1.
    Now

$$\left| \int_{I_{C_\lambda}}^\infty \frac{\sin x}{(e^{Kx/K'} - 1)(\mu + qe^{ix})(\mu + q^{-1}e^{-ix})(\mu + qe^{-ix})(\mu + q^{-1}e^{ix})} dx \right|$$

$$< \int_{I_{C_\lambda}}^\infty \frac{1}{(e^{Kx/K'} - 1)(1 - q)^4} dx$$

as $|\mu + q^{\pm 1}e^{\pm ix}| \geq ||\mu| - q^{\pm 1}| \geq 1 - q$, for q large enough, for all $\mu \in \mathbb{C} \backslash (-\infty, 0]$,

$$< \int_{I_{C_\lambda}}^\infty \frac{5!}{(Kx/K')^5 (1 - q)^4} dx$$

$$= \frac{5!(-\log q)^5}{4 I_{C_\lambda}^4 \pi^5 (1 - q)^4}.$$

This last expression has limit zero as $q \uparrow 1$ so

(†) $\lim\limits_{q \uparrow 1} \int_{I_{C_\lambda}}^\infty \frac{\sin x}{(e^{Kx/K'} - 1)(\mu + qe^{ix})(\mu + q^{-1}e^{-ix})(\mu + qe^{-ix})(\mu + q^{-1}e^{ix})} dx = 0.$

    Let $f(x) = \sin x/(e^{Kx/K'} - 1)$ then it is straightforward to show,

(1) $f(0) = K'/K,$
(2) f is decreasing, $\forall x \in [0, I_{C_\lambda}],$
(3) $f(x) > 0, \forall x \in [0, I_{C_\lambda}].$

Let $g : [0, I_{C_\lambda}] \times \mathbb{C} \times [0, 1] \to \mathbb{C}$ be defined as

$$g(x, \mu, q) = (\mu^2 + q^2)(\mu^2 + q^{-2}) + 2\mu(\mu^2 + 1)(q + q^{-1})\cos x + 4\mu^2 \cos^2 x,$$

then $\exists Q \in (0,1)$ such that,

$$|g(x, \mu, q)| > e(C_\lambda) > 0, \qquad \forall x \in [0, I_{C_\lambda}], \forall \mu \in C_\lambda, \forall q \in [Q, 1].$$

This can be seen as follows:

$$g(x, \mu, 1) = (\mu^2 + 1)(\mu^2 + 1) + 4\mu(\mu^2 + 1)\cos x + 4\mu^2 \cos^2 x$$
$$= [(\mu^2 + 1) + 2\mu \cos x]^2,$$

hence $g(x, \mu, 1) = 0$ when

$$\mu = \mu_0(x) = -\cos x \pm i\sqrt{1 - \cos^2 x}.$$

But

$$|\mu_0(x) + 1|^2 = 2(1 - \cos x) \leqq 2(1 - \cos I_{C_\lambda}) < 3(1 - \cos I_{C_\lambda}),$$

so $\mu_0(x) \notin C_\lambda$. Consequently $|g(x, \mu, 1)| > \varepsilon(C_\lambda) > 0$ for $\mu \in C_\lambda$ since $|g(x, \mu, 1)| \neq 0$ for $\mu \in C_\lambda$ and $[0, I_{C_\lambda}] \times C_\lambda \times 1$ is closed. Continuity of g then gives the result.

With the properties of f and g given above,

$$\left| \int_0^{I_{C_\lambda}} \frac{\sin x}{(e^{Kx/K'} - 1)g(x, \mu, q)} \, dx \right| < \frac{1}{\varepsilon(C_\lambda)} \cdot \frac{I_{C_\lambda} K'}{K},$$

provided q large enough. But, as above, the last expression has limit zero as $q \uparrow 1$ so

(‡) $\qquad \lim_{q \uparrow 1} \int_0^{I_{C_\lambda}} \frac{\sin x}{(e^{Kx/K'} - 1)(\mu + qe^{ix})(\mu + q^{-1}e^{-ix})(\mu + qe^{-ix})(\mu + q^{-1}e^{ix})} \, dx = 0.$

Formulae (†) and (‡) give the required result.

Proof of Proposition 3.2.3. With the information accumulated in the previous Lemmas 3.2.5, 3.2.6, and 3.2.7 it is clear that Proposition 3.2.3 is true.

3.3 Convergence Theorem. The results of Subsect. 3.2 can now be used to examine the behaviour of one point correlations as the critical temperature is approached.

3.3.1 Theorem. *Suppose $M \in GL(p, \mathbb{C})$ with its eigenvalues denoted by $\lambda_1, \ldots, \lambda_p$ and $\lambda_i \in \mathbb{C} \setminus (-\infty, 0]$ for $i = 1, \ldots, p$. Now let*

$$I = \sum_{i=1}^p \{\log(\lambda_i)\}^2,$$

then the following holds:

(1) $\qquad\qquad$ *If $\mathrm{Re}\, I > 0$ then* $\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle = +\infty$.

(2) $\qquad\qquad$ *If $\mathrm{Re}\, I < 0$ then* $\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle = 0$.

(3) $\qquad\qquad\quad$ *If $I = 0$ then* $\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle = 1$.

(4) $\qquad$ *If $\mathrm{Re}\, I = 0$, $\mathrm{Im}\, I \neq 0$ then* $\lim_{s \uparrow 1} \langle \sigma_a(M) \rangle$ *does not exist*.

3.3.2 Remark. The imaginary part of I determines the direction of rotation of the outward or inward spiral occurring in cases (1) and (2) and the direction of rotation in case (4). This will be explained further in the proof.

Proof. From Remark 3.2.1

$$\log\langle\sigma_a(M)\rangle = \sum_{i=1}^{p} \int_{C_{\lambda_i}} (\mu_i - \mu_i^{-1}) \sum_{n=0}^{\infty} (\mu_i + q^{2n+1})^{-1}(\mu_i + q^{-2n-1})^{-1} d\mu_i,$$

$$= \sum_{i=1}^{p} \int_{C_{\lambda_i}} \sum_{n=0}^{\infty} Z_{q,\mu_i}(n) d\mu_i, \quad \text{in the notation of Proposition 3.2.3}$$

$$= \sum_{i=1}^{p} \int_{C_{\lambda_i}} \left\{ \frac{Z_{q,\mu_i}(0)}{2} + \int_0^{\infty} Z_{q,\mu_i}(x) dx \right.$$

$$+ \left. \left[\frac{1}{2} \sum_{n=0}^{\infty} (Z_{q,\mu_i}(n) + Z_{q,\mu_i}(n+1)) - \int_0^{\infty} Z_{q,\mu_i}(x) dx \right] \right\} d\mu_i.$$

However from Lemma 3.2.3 and the bounds contained in Lemmas 3.2.6 and 3.2.7 dominated convergence applies to the last term above giving,

$$\lim_{q \uparrow 1} \int_{C_{\lambda_i}} \left[\frac{1}{2} \sum_{n=0}^{\infty} (Z_{q,\mu_i}(n) + Z_{q,\mu_i}(n+1)) - \int_0^{\infty} Z_{q,\mu_i}(x) dx \right] d\mu_i = 0,$$

for all $i = 1, \ldots, p$. Hence

$$\lim_{q \uparrow 1} \log\langle\sigma_a(M)\rangle = \sum_{i=1}^{p} \lim_{q \uparrow 1} \left\{ \int_{C_{\lambda_i}} \left[\frac{Z_{q,\mu_i}(0)}{2} + \int_0^{\infty} Z_{q,\mu_i}(x) dx \right] d\mu_i \right\}.$$

The behaviour of the two terms inside the sum as $q \uparrow 1$ will now be considered separately. Firstly,

$$\int_{C_{\lambda_i}} \frac{Z_{q,\mu_i}(0)}{2} d\mu_i = \int_{C_{\lambda_i}} \frac{(\mu_i - \mu_i^{-1})}{2(\mu_i + q)(\mu_i + q^{-1})} d\mu_i.$$

Therefore

$$\lim_{q \uparrow 1} \left\{ \int_{C_{\lambda_i}} \frac{Z_{q,\mu_i}(0)}{2} d\mu_i \right\} = \lim_{q \uparrow 1} \left\{ \int_{C_{\lambda_i}} \frac{(\mu_i - \mu_i^{-1})}{2(\mu_i + q)(\mu_i + q^{-1})} d\mu_i \right\}$$

$$= \int_{C_{\lambda_i}} \frac{(\mu_i - \mu_i^{-1})}{2(\mu_i + 1)^2} d\mu_i$$

$$\text{by dominated convergence,}$$

$$= \int_{C_{\lambda_i}} \left[\frac{1}{(1 + \mu_i)} - \frac{1}{2\mu_i} \right] d\mu_i$$

$$= \log(1 + \lambda_i) - \log 2 - \frac{\log \lambda_i}{2}.$$

Secondly,

$$\int_{C_{\lambda_i}} \int_0^{\infty} Z_{q,\mu_i}(x) dx \, d\mu_i$$

$$= \int_{C_{\lambda_i}} \int_0^{\infty} \frac{(\mu_i - \mu_i^{-1})}{(\mu_i + q^{2x+1})(\mu_i + q^{-2x-1})} dx \, d\mu_i$$

$$= \int_{C_{\lambda_i}} \frac{1}{\mu_i} \int_0^{\infty} \left\{ \frac{\mu_i q^{2x+1}}{1 + \mu_i q^{2x+1}} - \frac{\mu_i^{-1} q^{2x+1}}{1 + \mu_i^{-1} q^{2x+1}} \right\} dx \, d\mu_i$$

using

$$(z+r)^{-1}(z+r^{-1})^{-1} = \frac{1}{z(z-z^{-1})} \left\{ \frac{zr^{-1}}{(1+zr^{-1})} - \frac{z^{-1}r^{-1}}{(1+z^{-1}r^{-1})} \right\},$$

$$= \int_{c_{\lambda_i}} \frac{-1}{2\mu_i \log q} \log \left[\frac{1+\mu_i q}{1+\mu_i^{-1}q} \right] d\mu_i$$

$$= \int_{c_{\lambda_i}} \left\{ \frac{1}{2\mu_i} - \frac{\log(\mu_i)}{2\mu_i \log q} - \frac{1}{2\mu_i \log q} \cdot \log \left[\frac{1+\mu_i q}{1+\mu_i q^{-1}} \right] \right\} d\mu_i$$

$$= \frac{\log \lambda_i}{2} - \frac{(\log \lambda_i)^2}{4 \log q} - \int_{c_{\lambda_i}} \frac{1}{2\mu_i \log q} \cdot \log \left[\frac{1+\mu_i q}{1+\mu_i q^{-1}} \right] d\mu_i,$$

and

$$\lim_{q \uparrow 1} \int_{c_{\lambda_i}} \frac{1}{2\mu_i \log q} \cdot \log \left[\frac{1+\mu_i q}{1+\mu_i q^{-1}} \right] d\mu_i$$

$$= \int_{c_{\lambda_i}} \lim_{q \uparrow 1} \left\{ \frac{1}{2\mu_i \log q} \cdot \log \left[\frac{1+\mu_i q}{1+\mu_i q^{-1}} \right] \right\} d\mu_i$$

$$= \int_{c_{\lambda_i}} \frac{1}{(1+\mu_i)} d\mu_i$$

$$= \log(1+\lambda_i) - \log 2.$$

Combining the information contained above

$$\lim_{q \uparrow 1} \log \langle \sigma_a(M) \rangle = \sum_{i=1}^{p} \lim_{q \uparrow 1} \left\{ \frac{-(\log \lambda_i)^2}{4 \log q} \right\}$$

$$= \lim_{q \uparrow 1} \frac{-\sum_{i=1}^{p} (\log \lambda_i)^2}{4 \log q}$$

$$= \lim_{q \uparrow 1} \frac{-I}{4 \log q}.$$

But I is some complex number $\operatorname{Re} I + i \operatorname{Im} I$. So if $I \neq 0$ the $\log q$ term will dominate and the behaviour is as follows:

(1) $\operatorname{Re} I > 0$; $\operatorname{Im} I > 0$: "$\lim_{q \uparrow 1} \log \langle \sigma_a(M) \rangle = \infty + \infty i$". That is $\langle \sigma_a(M) \rangle$ spirals outwards in an anticlockwise direction as $s \uparrow 1$.

(2) $\operatorname{Re} I > 0$; $\operatorname{Im} I < 0$: "$\lim_{q \uparrow 1} \log \langle \sigma_a(M) \rangle = \infty - \infty i$". That is $\langle \sigma_a(M) \rangle$ spirals outwards in a clockwise direction as $s \uparrow 1$.

(3) $\operatorname{Re} I < 0$; $\operatorname{Im} I > 0$: "$\lim_{q \uparrow 1} \log \langle \sigma_a(M) \rangle = -\infty + \infty i$". That is $\langle \sigma_a(M) \rangle$ spirals inwards to zero in an anticlockwise direction as $s \uparrow 1$.

(4) $\operatorname{Re} I < 0$; $\operatorname{Im} I < 0$: "$\lim_{q \uparrow 1} \log \langle \sigma_a(M) \rangle = -\infty - \infty i$". That is $\langle \sigma_a(M) \rangle$ spirals inwards to zero in a clockwise direction as $s \uparrow 1$.

(5) $\operatorname{Im} I = 0$ so $I = \operatorname{Re} I$ and "$\lim\limits_{q\uparrow 1} \log\langle\sigma_a(M)\rangle = \operatorname{sgn}(I)\infty$". That is if $I > 0$ then $\lim\limits_{s\uparrow 1}\langle\sigma_a(M)\rangle = \infty$ and if $I < 0$ then $\lim\limits_{s\uparrow 1}\langle\sigma_a(M)\rangle = 0$.

(6) $\operatorname{Re} I = 0$ so $I = \operatorname{Im} I i$ and "$\lim\limits_{q\uparrow 1} \log\langle\sigma_a(M)\rangle = \operatorname{sgn}(\operatorname{Im} I)\infty i$". That is if $\operatorname{Im} I > 0$ then $\langle\sigma_a(M)\rangle$ rotates anticlockwise around the unit circle and if $\operatorname{Im} I < 0$ then $\langle\sigma_a(M)\rangle$ rotates clockwise around the unit circle, so in either case the limit does not exist.

Hence the interesting case occurs when $I = 0$ and it is a consequence of the above that,
$$\lim_{s\uparrow 1} \log\langle\sigma_a(M)\rangle = 0.$$

Therefore $\lim\limits_{s\uparrow 1}\langle\sigma_a(M)\rangle = 1$.

Theorem 3.3.1 gives a classification of the critical limit of one point correlations, which will be investigated more thoroughly in the next section. But first some remarks on the negative eigenvalue situation.

3.3.3 Remark. Firstly the special case when the eigenvalue is -1. From Proposition 3.1.1,
$$\langle\sigma_a(-1)\rangle = \prod_{l>0}\left[\frac{1-q^{2l}}{1+q^{2l}}\right]^2.$$

This is the square of the spontaneous magnetization for the Ising model, thus its critical temperature behaviour is already known, see [12, 13, 26, and 11, Chapter X] together with [2, 23, 25] for example, namely
$$\lim_{q\uparrow 1}\langle\sigma_a(-1)\rangle = 0.$$
Consequently, in some cases, the value -1 could be added as a permissible value for an eigenvalue with its treatment being separate from the others. That is, suppose the eigenvalues of M are $\lambda_1, ..., \lambda_{p-1}$ and -1 with the reduced matrix M' having eigenvalues $\lambda_1, ..., \lambda_{p-1}$. If M' satisfies Theorem 3.3.1 cases (2) or (3) then
$$\lim_{s\uparrow 1}\langle\sigma_a(M)\rangle = 0.$$
If however M' satisfies case (1) then the limit is not clear. This is due to the fact that Lemmas 3.2.6 and 3.2.7 fail if $\lambda = -1$. Consequently Proposition 3.2.3 fails and this plays a crucial role in the convergence argument.

However, having said this, the function $I(z) = (\log z)^2$, where $z \in \mathbb{C}\backslash(-\infty, 0]$ can be continuously extended to include the point $z = -1$ by defining $I(-1) = -\pi^2$. This suggests that the value -1 could be added as a permissable value using a continuity argument on the eigenvalues. That is, something like,

$$\lim_{s\uparrow 1}\log\langle\sigma_a(M[\lambda_1, ..., \lambda_p, -1])\rangle = \lim_{s\uparrow 1}\ \lim_{\substack{\lambda\to-1 \\ \lambda\in\mathbb{C}\backslash(-\infty,0]}}\ \log\langle\sigma_a(M[\lambda_1, ..., \lambda_p, \lambda])\rangle$$

$$= \lim_{q\uparrow 1}\ \lim_{\substack{\lambda\to-1 \\ \lambda\in\mathbb{C}\backslash(-\infty,0]}}\ \frac{-\left\{\sum\limits_{i=1}^{p}(\log\lambda_i)^2 + (\log\lambda)^2\right\}}{4\log q}$$

$$= \lim_{q\uparrow 1}\ \frac{-\left\{\sum\limits_{i=1}^{p}(\log\lambda_i)^2 - \pi^2\right\}}{4\log q}.$$

Unfortunately, going from step (2) to step (3) in the above is not clear.

For λ negative with $\lambda \neq -1$, defining the sequence

$$q_l = \exp\left[\frac{\text{sgn}(1+\lambda)}{2l} \cdot \log(-\lambda)\right], \quad \text{for } l \in \mathbb{Z}_{1/2}^+,$$

or equivalently

$$q_n = \exp\left[\frac{\text{sgn}(1+\lambda)}{2n-1} \cdot \log(-\lambda)\right], \quad \text{for } n \in \mathbb{N},$$

$q_n \in (0,1)$ for all n, $\lim\limits_{n \to \infty} q_n = 1$ but for each $q_n \langle \sigma_a(\lambda) \rangle_{Q(q_n)} = 0$. This suggests that for matrices with such eigenvalues the limit as $s \uparrow 1$ is zero if any such limit actually exists.

4. An Example of a Non-Trivial Limiting One Point Correlation

4.1 Introduction. The previous section gave a condition for a matrix with non-negative eigenvalues to have a monodromy field which has non-degenerate, not zero or infinity, critical limit correlation. However, as yet, the existence of any non-trivial matrix which actually satisfies this condition has not been shown. It is this matter which is considered in this section.

4.1.1 Notation. Let $\mathbb{C}_R^p$ denote the permissible values of $\lambda_1, \ldots, \lambda_p$, that is,

$$\mathbb{C}_R^p = \{(\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p : \lambda_i \in \mathbb{C} \backslash (-\infty, 0], \ \forall i = 1, \ldots, p\},$$

and define the map $I : \mathbb{C}_R^p \to \mathbb{C}$ by

$$I(\lambda_1, \ldots, \lambda_p) = \sum_{i=1}^{p} (\log \lambda_i)^2.$$

Then the object of interest is the set of points in $\mathbb{C}_R^p$ with $I(\lambda_1, \ldots, \lambda_p) = 0$ which will be denoted by $\mathscr{C}^p$.

4.2 Investigation of $\mathscr{C}^p$ for $p = 1, 2$.

4.2.1 Proposition.

(1) $(1, \ldots, 1) \in \mathscr{C}^p \quad \forall p \geq 1$.
(2) $\mathscr{C}^1 \equiv \{1\}$.

Proof. (1): $I(1, \ldots, 1) = 0$ is obvious. This is equivalent to M being the identity matrix and is the "trivial" situation referred to above.

(2):

$$\log \lambda = 0 \Leftrightarrow \lambda = 1.$$

4.2.2 Remark. This proposition may appear a bit discouraging as it says, for the scalar case ($p = 1$), there exist no non-identity complex numbers $\lambda \in \mathbb{C} \backslash (-\infty, 0]$ which have a critical limit correlation $\lim\limits_{s \uparrow 1} \langle \sigma(\lambda) \rangle$.

However the reason for this is the "lack of freedom" in the scalar case which will now be explained. From the proof of Proposition 4.2.1 to get $I(\lambda)$ zero the imaginary part of λ must be zero, thus $I(\lambda)$ is now only determined by the real part of λ and this one variable dependence is not sufficient to get a non-trivial solution. That is one variable does not provide sufficient "freedom" for a non-trivial solution to exist. However for the larger dimensional cases ($p \geq 2$) there are more variables present and hence more "freedom", so a non-trivial solution is possible. It is this that will now be shown by considering the simplest case $p = 2$.

4.2.3 Proposition.

$$\mathscr{C}^2 \equiv \{(e^{\alpha + i\theta}, e^{\pm(\theta - i\alpha)}): \text{where } \alpha, \theta \in (-\pi, \pi)\}.$$

Proof. By definition, for $(\lambda_1, \lambda_2) \in \mathscr{C}^2$,

$$(\log \lambda_1)^2 + (\log \lambda_2)^2 = 0.$$

So

$$\log \lambda_1 = \pm i \log \lambda_2.$$

That is, if $\lambda_1 = e^{\alpha + i\theta}$, where $\alpha \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$ then

$$\log \lambda_2 = \pm(\theta - i\alpha),$$

and hence,

$$\lambda_2 = e^{\pm(\theta - i\alpha)}.$$

However, since log is the principal value the imaginary part only takes values in the interval $(-\pi, \pi)$. Hence α also has to be restricted to this interval for such a λ_2 to exist. The result now follows from the above.

4.2.4 Remark. Proposition 4.2.3 demonstrates the existence of non-trivial 2×2 matrices M which possess a critical limit one point correlation.

4.2.5 Corollary. *If M is a 2×2 matrix with eigenvalues λ_1 and λ_2, where for $i = 1$ and 2, $\lambda_i \in \mathbb{C} \setminus (-\infty, 0]$, and*

$$\lim_{q \uparrow 1} \langle \sigma_a(M) \rangle = 1,$$

then

$$e^{-\pi} < |\lambda_1|, |\lambda_2| < e^{\pi}.$$

Proof. Follows from definition of $\mathscr{C}^2$.

4.2.6 Remark. With reference to the comments in Remark 3.3.3, the value -1 taken as the eigenvalue λ_1 would correspond to $\alpha = 0$, $\theta = \theta'$, where $\theta' \to \pm\pi$. Consequently there would be a corresponding $\lambda_2 = e^{\theta'}$, where $\theta' \to \pm\pi$, such that $(\lambda_1, \lambda_2) \in \mathscr{C}^2$. This suggests that the points $(-1, e^{\pm\pi})$ and $(e^{\pm\pi}, -1)$ could be added to the set $\mathscr{C}^2$ given in Proposition 4.2.3.

Note that the plus or minus option in the exponential both here and in the description of $\mathscr{C}^2$ is not surprising as $\langle \sigma_a(M) \rangle$ is invariant under the transformation $\lambda \mapsto \lambda^{-1}$, where λ is one of the eigenvalues of M.

4.3 Investigation of $\mathscr{C}^p$

4.3.1 Remark. The condition I can be simplified somewhat as follows. Suppose $\log \lambda_j = x_j + iy_j$ for $j = 1, \ldots, p$ where $x_j \in \mathbb{R}$ and $y_j \in (-\pi, \pi)$. Then I becomes

$$\sum_{j=1}^{p} (x_j + iy_j)^2 = \sum_{j=1}^{p} (x_j^2 - y_j^2) + 2i \sum_{j=1}^{p} (x_j y_j).$$

So $I = 0$ if and only if,

(1) $\displaystyle \sum_{j=1}^{p} (x_j^2 - y_j^2) = 0,$

(2) $\displaystyle \sum_{j=1}^{p} (x_j y_j) = 0.$

Now introduce the points $\underline{x} = (x_1, \ldots, x_p)$ and $\underline{y} = (y_1, \ldots, y_p)$ of $\mathbb{R}^p$. Then $I = 0$ if and only if,

(a) $\|\underline{x}\|_2 = \|\underline{y}\|_2$,
(b) $\underline{x} \bullet \underline{y} = 0$, where $\bullet$ denotes the scalar or "dot" product.

This last set of conditions has a nice geometrical interpretation. If you choose a vector y in $\mathbb{R}^p$ then the vector x needs to be of the same magnitude lying in the $(p-1)$-dimensional subspace which is the hyperplane perpendicular to y.

Since $y_i = \operatorname{Im}\{\log \lambda_i\}$, $y_i \in (-\pi, \pi)$ and consequently the vector y is contained in the open subset of $\mathbb{R}^p$ given by,

$$B^p = \{(r_1, \ldots, r_p) : -\pi < r_i < \pi \text{ for } i = 1, \ldots, p\}.$$

This imposes a restriction on the magnitude of the vector x and hence on the magnitude of the eigenvalues of M as will be shown next.

4.3.2 Lemma. *Suppose $\underline{x} \in \mathbb{R}^p$ and $\underline{y} \in B^p$ and they satisfy the conditions* (a) *and* (b) *given above with $p \geq 2$. Then*

$$|x_i| < \pi \sqrt{p-1}, \quad \text{for all } i = 1, \ldots, p.$$

Proof. Assume, without loss of generality, that $y_p \neq 0$ then condition (b) gives,

$$x_p = \frac{-\displaystyle\sum_{i=1}^{p-1} x_i y_i}{y_p}.$$

Substituting this in (a) and writing as a quadratic in x_1 gives,

$$x_1^2 \left(\frac{(y_1^2 + y_p^2)}{y_p^2} \right) + x_1 \left(\frac{2y_1 \displaystyle\sum_{i=2}^{p-1} x_i y_i}{y_p^2} \right) + \frac{\left(\displaystyle\sum_{i=2}^{p-1} x_i y_i \right)^2}{y_p^2}$$

$$+ \sum_{i=2}^{p-1} x_i^2 - \sum_{i=1}^{p} y_i^2 = 0.$$

Thus

$$x_1 = \frac{-y_1 \displaystyle\sum_{i=2}^{p-1} x_i y_i \pm y_p \sqrt{(y_1^2 + y_p^2)\left(\displaystyle\sum_{i=1}^{p} y_i^2 - \displaystyle\sum_{i=2}^{p-1} x_i^2 \right) - \left(\displaystyle\sum_{i=2}^{p-1} x_i y_i \right)^2}}{(y_1^2 + y_p^2)}.$$

But x_1 is real, so

$$(y_1^2 + y_p^2)\left(\sum_{i=1}^{p} y_i^2 - \sum_{i=2}^{p1} x_i^2\right) - \left(\sum_{i=2}^{p-1} x_i y_i\right)^2 \geq 0.$$

Now, without loss of generality, consider x_2. This is a maximum in the above inequality when $x_3 = \ldots = x_{p-1} = 0$ in which case, after rearrangement

$$x_2^2 \leq \frac{(y_1^2 + y_p^2)\sum\limits_{i=1}^{p} y_i^2}{(y_1^2 + y_2^2 + y_p^2)}$$

$$= (y_1^2 + y_p^2) + \frac{\sum\limits_{i=3}^{p-1} y_i^2}{(1 + y_2^2/(y_1^2 + y_p^2))}$$

$$< 2\pi^2 + (p-3)\pi^2 = (p-1)\pi^2.$$

That is $|x_2| < \pi\sqrt{p-1}$. The symmetry present implies that the inequality holds for all x_i.

4.3.3 Remark. Note that the result above is the best possible inequality of its type. Taking, for example, $x_1 = (\pi - \varepsilon)\sqrt{p-1}$, $x_2 = \ldots = x_p = 0$ and $y_1 = 0$, $y_2 = \ldots = y_p = \pi - \varepsilon$ demonstrates that the x_i and y_i may be chosen so that the magnitude of the x_i can be arbitrarily close to $\pi\sqrt{p-1}$.

4.3.4 Corollary. *Suppose M is a $p \times p$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_p$, where $\lambda_i \in \mathbb{C}\backslash(-\infty, 0]$ for $i = 1, \ldots, p$, then if there exists a λ_i such that*

$$|\lambda_i| \notin (e^{-\pi\sqrt{p-1}}, e^{\pi\sqrt{p-1}}),$$

then

$$\lim_{q \uparrow 1} \langle \sigma_a(M) \rangle \neq 1.$$

Proof. Suppose $\lim\limits_{q \uparrow 1} \langle \sigma_a(M) \rangle = 1$ then, by the main result of Sect. 3, $I = 0$.

But from Lemma 4.3.2 if $I = 0$ then

$$|x_i| < \pi\sqrt{p-1}, \quad \forall i = 1, \ldots, p.$$

Now $\lambda_i = e^{x_i + iy_i}$, so $|\lambda_i| = e^{x_i}$, consequently if $I = 0$,

$$e^{-\pi\sqrt{p-1}} < |\lambda_i| < e^{\pi\sqrt{p-1}}, \quad \forall i = 1, \ldots, p.$$

This gives the result.

4.3.5 Remark. By considering Corollary 4.3.4 the following can be seen. Suppose any λ_i is outside the shaded annulus in the diagram below.

Then $(\lambda_1, \ldots, \lambda_p) \notin \mathscr{C}^p$.

4.3.6 Remark. The structure of $\mathscr{C}^p$ is not clear for $p > 2$, however the following properties can be seen fairly easily:

(1) $\{1\} \equiv \mathscr{C}^1 \subset \mathscr{C}^2 \subset \ldots \subset \mathscr{C}^p \subset \ldots,$
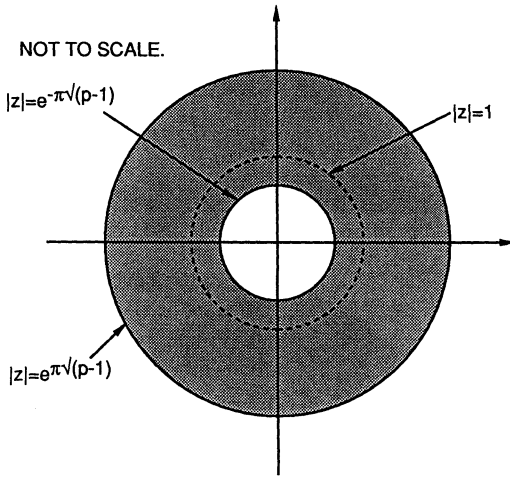
NOT TO SCALE.

$|z|=e^{-\pi\sqrt{(p-1)}}$

$|z|=1$

$|z|=e^{\pi\sqrt{(p-1)}}$

Fig. 2

where the inclusion $\mathscr{C}^k \to \mathscr{C}^{k+1}$ is the map which takes

$$(\lambda_1, ..., \lambda_k) \in \mathscr{C}^k \mapsto (\lambda_1, ..., \lambda_k, 1) \in \mathscr{C}^{k+1}.$$

This is equivalent to embedding the $k \times k$ matrix corresponding to the element of $\mathscr{C}^k$ in a $(k+1) \times (k+1)$ matrix by adding a 1 on the diagonal.

Hence $\mathscr{C}^p \supsetneq \{(1, ..., 1)\}$ for all $p \geq 2$, that is, for $p \geq 2$ there exist non-trivial $p \times p$ matrices M_p whose monodromy one point correlation $\langle \sigma_a(M_p) \rangle$ has a critical limit. In fact the geometrical interpretation given earlier indicates that the inclusion given above is strict.

(2) $\mathscr{C}^n \times \mathscr{C}^m \subset \mathscr{C}^{n+m}$,

where the inclusion $\mathscr{C}^n \times \mathscr{C}^m \to \mathscr{C}^{n+m}$ is the map which takes

$$(\lambda_1, ..., \lambda_n) \times (\mu_1, ..., \mu_m) \in \mathscr{C}^n \times \mathscr{C}^m \mapsto (\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_m) \in \mathscr{C}^{n+m}.$$

The map at this level is equivalent to placing the $n \times n$ mateix and the $m \times m$ matrix down the diagonal to form an $(n+m) \times (n+m)$ matrix. Note that the map given in (1) is a special case of this map when $n=k$ and $m=1$.

(3) If

$$I_{\mathbb{R}} = \{(\lambda_1, ..., \lambda_p) \in \mathbb{C}_R^p : \lambda_i \in \mathbb{R} \ \forall i\},$$

and

$$I_1 = \{(\lambda_1, ..., \lambda_p) \in \mathbb{C}_R^p : |\lambda_i| = 1 \ \forall i\},$$

then $\mathscr{C}^p \cap (I_{\mathbb{R}} \cup I_1) = \{(1, ..., 1)\}$.

(4) If for $j = 1, ..., p$ with $p \geq 2$

$$I_j = \{(\lambda_1, ..., \lambda_p) \in \mathbb{C}_R^p : \lambda_i \in \mathbb{C}_R^1 \cap \mathbb{R}, i \neq j; \text{Im}(\lambda_j) \neq 0\},$$

then $\mathscr{C}^p \cap \bigcup_{j=1}^{p} I_j = \emptyset$.

A simple analysis of the function I allows the last two comments to be generalized slightly, as follows:

(5) By considering the imaginary part of I the following can be seen. Suppose λ_i, is contained in either of the shaded regions in the diagram below for all $i = 1, ..., p$. Then the imaginary part of I is a sum of all positive or all negative terms and consequently cannot be zero.
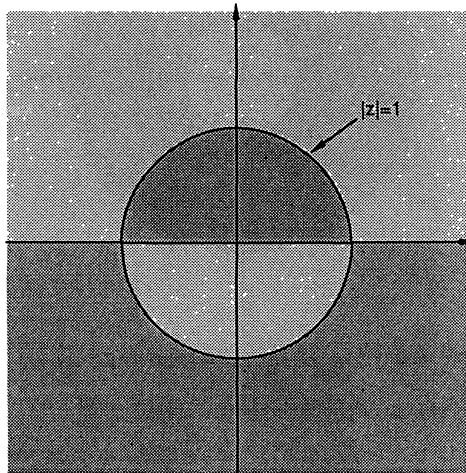

Fig. 3

Thus $(\lambda_1, ..., \lambda_p) \notin \mathscr{C}^p$.

(6) By considering the real part of I the following can be seen. Suppose λ_i is either contained in, or outside of, the shaded region in the diagram below for all $i = 1, ..., p$. Then the real part of I is a sum of all positive or all negative terms and consequently cannot be zero.
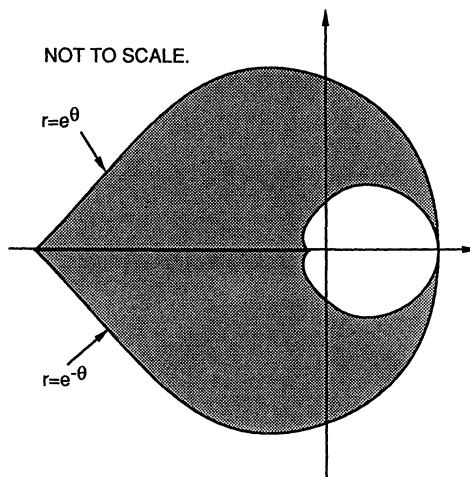

Fig. 4

Thus $(\lambda_1, ..., \lambda_p) \notin \mathscr{C}^p$.

Section 5. N Point Correlations

5.1 Introduction. The two previous sections dealt with one point correlations and their critical limits. This section will consider higher order correlations. Unfortunately, as yet, there are no concrete results concerning critical limits only a conjecture. The starting point for this analysis is the "product formula," see [16, 15], which is stated below for reference.

5.1.1 Theorem. *Suppose that* $g_k \in \widehat{GL}_Q(H)$ *for* $k=1,\ldots,N$ *with*

$$T(g_k) = G_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in GL_Q(H).$$

Suppose also that d_k *is invertible for each* $k=1,\ldots,N$. *Then*

$$\langle g_1 \ldots g_N \rangle = \prod_{k=1}^{N} \langle g_k \rangle \det{}_2(1+LR),$$

provided $\langle g_1 \ldots g_N \rangle \neq 0$, *where:*
L *denotes the* $N \times N$ *block matrix with entries for* $i < k$,

$$l_{ik} = \begin{cases} -Q_+, & k=i+1, \\ -a_{i+1}Q_+, & k=i+2, \\ -a_{i+1}\ldots a_{k-1}Q_+, & k>i+2, \end{cases}$$

for $i > k$,

$$l_{ik} = \begin{cases} Q_-, & i=k+1, \\ d_{k+1}^{-1}Q_-, & i=k+2, \\ d_{i-1}^{-1}\ldots d_{k+1}^{-1}Q_-, & i>k+2, \end{cases}$$

and for $i = k$

$$l_{ii} = 0,$$

and $R = R_1 \oplus R_2 \oplus \ldots \oplus R_N$, *where*

$$R_k = \begin{bmatrix} -b_k d_k^{-1} c_k & b_k d_k^{-1} \\ d_k^{-1} c_k & 0 \end{bmatrix}.$$

5.2 Conjecture for Limiting N Point Correlations. The "product formula" given in Theorem 5.1.1 gives rise to the following two Corollaries when applied to the specific case of monodromy fields.

5.2.1 Corollary. *Suppose* $M_j \in GL(p, \mathbb{C})$ *and has no negative eigenvalues for* $j = 1,\ldots,n$. *Then*

$$\langle \sigma_{a_1}(M_1) \ldots \sigma_{a_n}(M_n) \rangle = \prod_{k=1}^{n} \langle \sigma(M_k) \rangle \det{}_2(1+LR),$$

where L *and* R *have the structure defined above with*

$$s_{a_k}(M_k) = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}.$$

Proof. By definition $T(\sigma_a(M)) = s_a(M)$. The condition on the eigenvalues of M_j implies that d_j is invertible for all $j = 1, ..., n$. Hence Theorem 5.1.1 gives the above as

$$\langle \sigma(M) \rangle = \langle \sigma_a(M) \rangle, \qquad \forall a \in \mathbb{Z}^2.$$

5.2.2 Notation. Let the expression $(1 + LR)$ present in Corollary 5.2.1 be denoted by

$$X(M_1, ..., M_n : a_1, ..., a_n),$$

and let

$$X(M_1, ..., M_n) \overset{\text{def}}{=} X(M_1, ..., M_n : a, ..., a).$$

5.2.3 Corollary. *Suppose $M_j \in GL(p, \mathbb{C})$ and has no negative eigenvalues for $j = 1, ..., n$. Then*

$$\langle \sigma_{a_1}(M_1) ... \sigma_{a_n}(M_n) \rangle = \langle \sigma(M_1 ... M_n) \rangle \frac{\det_2 X(M_1, ..., M_n : a_1, ..., a_n)}{\det_2 X(M_1, ..., M_n)}.$$

Proof. Apply the product formula to

$$\langle \sigma(M_1 ... M_n) \rangle = \langle \sigma(M_1) ... \sigma(M_n) \rangle.$$

This together with Corollary 5.2.1 gives the result.

5.2.4 Conjecture.

If $\lim\limits_{s \uparrow 1} \langle \sigma(M_1 ... M_n) \rangle$ exists then $\lim\limits_{s \uparrow 1} \langle \sigma_{a_1}(M_1) ... \sigma_{a_n}(M_n) \rangle$ exists.

5.2.5 Remark. To prove this conjecture the existence of a limit for the determinant expression is required. This appears intractable at the present since $s_a(M) \notin GL_{Q_c}(H^p)$, where Q_c denotes the critical temperature Q, that is $Q_c = \lim\limits_{s \uparrow 1} Q$. Note the conjecture is trivially true when $a_1 = ... = a_n$ as expressions are equal.

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