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Absence of Ballistic Motion

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Abstract. For large classes of Schrödinger operators and Jacobi matrices we prove that if h has only one point spectrum then for ϕ_0 of compact support

$$\lim_{t\to\infty} t^{-2}\|xe^{-ith}\phi_0\|^2 = 0\,.$$

1. Introduction

Consider a free Schrödinger particle. Then the Heisenberg position operators obeys

$$x(t) = x + tp$$

since p is a constant of the motion. Thus $|x(t)|$ grows linearly in t, indeed for any $\phi \in \mathscr{S}(\mathbb{R}^n)$:

$$\lim(\phi, x(t)^2\phi)/t^2 = (\phi, p^2\phi) > 0\,.$$

This paper had its root in a question of Joel Lebowitz asking if such ballistic motion didn't have its roots in absolutely continuous spectrum. Alas, while it is likely that Joel is correct, I have been able to obtain only partial results. Here I will prove that for Hamiltonians with pure point spectrum (think of the random case [1]), we have that for a dense set of initial ϕ that $(\phi, x(t)^2\phi)/t^2 \to 0$. Unfortunately, I have nothing to say in the singular continuous case.

For background note that it is a result of Radin-Simon [2] that when ϕ is in C_0^∞, $(\phi, x(t)^2/t^2$ is bounded at infinity in great generality.

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2. The Discrete Case

On $l^2(\mathbb{Z}^0)$, let h_0 be defined by

$$(h_0)\,(n) = \sum_{|m-n|=1} u(m)\,.$$

If v is an arbitrary real valued function and also the operator of multiplication by v on $D(v) = \{u \mid \Sigma(|v(n)|+1)^2|u(n)|^2 < \infty\}$, then $h = h_0 + v$ is self-adjoint on $D(v)$ since h_0 is bounded.

Define

$$(\mathbf{x}u)\,(n) = \mathbf{n}u(\mathbf{n})$$

and $\mathbf{p} = i[h_0, \mathbf{x}]$ formally, explicitly

$$(\mathbf{p}u)\,(n) = -\sum_{|j|=1} iju(n+\mathbf{j})\,.$$

Then $\mathbf{p}$ is bounded. Moreover, we claim that if

$$\mathbf{x}(t) = e^{itH}\mathbf{x}e^{-itH}, \qquad \mathbf{p}(t) = e^{itH}\mathbf{p}e^{-itH},$$

then

$$\mathbf{x}(t) = \mathbf{x} + \int_0^t \mathbf{p}(s)\,ds$$

as forms on $D(x)$. For it is easy to see that $\mathbf{x}(0)$ is bounded and equal to p. Thus, we have, since p is bounded:

Lemma 1.1. *For $\phi \in D(\mathbf{x})$:*

$$\lim_{t\to\infty} \frac{1}{t^2}(\phi, |\mathbf{x}(t)|^2\phi) = \lim_{t\to\infty} \frac{1}{t^2} \int_0^t ds \int_0^t du (\phi, \mathbf{p}(u)\cdot \mathbf{p}(s)\phi)\,. \tag{1}$$

With this we prove:

Theorem 1.2. *Suppose that h has only point spectrum. Then for $\phi \in D(\mathbf{x})$.*

$$\lim_{t\to\infty}(\phi, |\mathbf{x}(t)|^2\phi)/t^2 = 0\,.$$

Proof. We will show the right-hand side of (1) goes to 0 for all ϕ. The integrand in (1) is uniformly bounded, so it suffices to prove the result for a dense set of ϕ, say a finite sum of eigenfunctions of h. Let ϕ_n be a complete set of eigenfunctions of h:

$$h\phi_n = e_n\phi_n\,.$$

Thus we need only show that for all n, m:

$$\frac{1}{t^2}\int_0^t ds \int_0^t du (\phi_n, \mathbf{p}(u)\cdot \mathbf{p}(s)\phi_n) \to 0\,. \tag{2}$$

Let $\mathbf{p}_{nk} = (\phi_n, \mathbf{p}\phi_k)$ and

$$f_{n,m,k}(t) = t^{-2}\frac{1}{t^2}\int_0^t ds \int_0^t du\, e^{-iu(E_k-E_n)}e^{-is(E_m-E_k)}\,.$$

so

$$\text{left-hand side of (2)} = \sum_k \mathbf{p}_{nk}\cdot\mathbf{p}_{km} f_{n,m,k}(t)\,.$$

Next note that $|f| \le 1$ and

$$\sum_k |\mathbf{p}_{nk}\cdot\mathbf{p}_{km}| \le \left(\sum_k |\mathbf{p}_{km}{}^2\right)^{1/2}\left(\sum_k |\mathbf{p}_{km}|^2\right)^{1/2} = \|p\phi_n\|\,\|p\phi_m\|\,.$$

Thus by the dominated convergence theorem, it suffices to show that for each n, m, k either $\mathbf{p}_{nk}\cdot\mathbf{p}_{km} = 0$ or $f_{n,m,k}(t) \to 0$ as $t \to \infty$. The integral determining f is easy to do and one sees that $f(t) \to 0$ unless $E_n = E_k = E_m$. Thus the theorem follows from the virial theorem (Lemma 2.3) below. □

Lemma 2.3. *If $E_n = E_k$, then $\mathbf{p}_{nk} = 0$.*

Proof. Define $\mathbf{x}_M$ by

$$\begin{aligned}(\mathbf{x}_M)_i &= M & x_i &\ge M\\ &= x_i & |x_i| &\le M\\ &= -M & x_i &\le -M\end{aligned}$$

and $\mathbf{p}_M = i[h_0, \mathbf{x}_M]$. Then by a direct calculation

$$s - \lim_{M\to\infty} \mathbf{p}_M = \mathbf{p}\,,$$

so it suffices that

$$(\phi_n, \mathbf{p}_M\phi_m) = 0\,.$$

Since $\mathbf{x}_M$ is bounded, this follows by expanding the commutator. □

3. The Continuum Case

Theorem 3.1. *Let V be a multiplication operator on $L^2(\mathbb{R}^n)$ so that $H_0 + V \equiv -\Delta + V$ is bounded below on $Q(H_0)\cap Q(V)$ and let $H = H_0 + V$ be the form closure. Suppose $Q(H) \subset Q(H_0)$. (Equivalently there is a form bound $H_0 \le c(H+d)$.) Let $\phi \in D(\mathbf{x}) \cap Q(H)$. Suppose that H has only point spectrum. Then*

$$\lim_{t\to\infty} (\phi, |\mathbf{x}(t)|^2\phi)/t^2 = 0\,.$$

Proof. Except for technicalities, the same as Theorem 2.2. By Radin-Simon [2], $D(\mathbf{x}) \cap Q(H)$ is left invariant by e^{itH} and $\mathbf{x}(t) = \mathbf{x} + 2\int_0^t \mathbf{p}(s)ds$. As in Sect. 2, it suffices to show for $\phi \in Q(H)$,

$$\frac{1}{t^2}\int_0^t ds \int_0^t du (\phi, \mathbf{p}(s)\cdot\mathbf{p}(u)\phi) \to 0\,.$$

Since $\mathbf{p}(s)\,(H+i)^{-1/2}$ is uniformly bounded, we need only show this for finite sums of eigenfunctions.

As in the proof of Lemma 2.3, we define $\mathbf{x}_N$ and $\mathbf{p}_N$ but with a slightly different formula. Pick $f(x)$, C^∞ on $\mathbb{R}$ so $f' \geq 0$ and

$$f(x) = \pm 1 \quad \text{for} \quad \pm x \geq 1 = x \quad \text{for} \quad |x| \leq 1/2,$$

and define $x_N = Nf(x/N)$ and $p_N = \frac{i}{2}[H_0, x_N]$. $\mathbf{x}_N$ is bounded but $\mathbf{p}_N$ is not. However for $\phi \in Q(H_0)$ we have $\|(\mathbf{p}_N - \mathbf{p})\phi\| \to 0$ and so the argument in Lemma 2.3 extends. □

References

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