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Crystal Base for the Basic Representation of $U_q(\widehat{\mathfrak{sl}}(n))$

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Abstract. We show the existence of the crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$ by giving an explicit description in terms of Young diagrams.

0. Introduction

In [5] Kashiwara introduces the notion of crystal base for integrable representations of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is any symmetrizable Kac–Moody Lie algebra. The crystal base has a simple structure at $q=0$. Let $\{e_i, f_i, t_i^{\pm}\}$ be a set of generators of $U_q(\mathfrak{g})$. Suppose M is an integrable $U_q(\mathfrak{g})$-module. Kashiwara [5] constructs certain operators $\tilde{e}_i, \tilde{f}_i$ acting on M. These operators are obtained by modifying the simple root vectors e_i and f_i. When M is an irreducible highest weight module with highest weight vecctor u, define:

$$L=\sum A\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}u\subset M \tag{0.1}$$

and

$$B=\{v=\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}u\in L/qL\,|\,v\neq 0\}, \tag{0.2}$$

where $A\subset K=\mathbf{Q}(q)$ is the ring of rational functions in q without pole at $q=0$. Kashiwara [5] conjectures that (L,B) satisfies the following crucial properties:

$$\tilde{e}_iL\subset L \quad \text{and} \quad \tilde{f}_iL\subset L, \quad \text{for all } i, \tag{0.3}$$

$$\tilde{e}_iB\subset B\cup\{0\} \quad \text{and} \quad \tilde{f}_iB\subset B\cup\{0\}, \quad \text{for all } i, \tag{0.4}$$

$$u=\tilde{e}_iv \quad \text{if and only if} \quad v=\tilde{f}_iu, \quad \text{for all } i \quad \text{and} \quad u,v\in B. \tag{0.5}$$

He proves his conjecture for $\mathfrak{g}=\mathfrak{sl}(n)$, $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$ and $\mathfrak{o}(2n)$ and calls (L,B) the crystal base.

In this paper we prove this conjecture for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$ with highest weight Λ_0 ($\Lambda_0(t_i^{\pm})=q^{\pm 1}\delta_{i,0}$). We start with the Fock space representation of $U_q(\widehat{\mathfrak{sl}}(n))$ constructed by Hayashi [3]. We identify the Fock space $\mathscr{F}$ with the space spanned by Young diagrams [4]. Then for each i, we decompose $\mathscr{F}$ with respect to $U_q(\mathfrak{sl}(2))_{(i)}$ generated by $\{e_i, f_i, t_i^{\pm}\}$ (see, Theorem 3.1). This leads to the construction of crystal base $(L(\mathscr{F}), B(\mathscr{F}))$ for $\mathscr{F}$ (Theorem 3.2). The set $B(\mathscr{F})$ has a structure of a colored oriented graph which Kashiwara [5] calls a crystal graph. Let $B(\mathscr{F})_\phi$ be the connected component of ϕ (the empty Young diagram) in the graph of $B(\mathscr{F})$. The submodule of $\mathscr{F}$ generated by ϕ is the basic $U_q(\hat{\mathfrak{sl}}(n))$-module $M(\Lambda_0)$ with highest weight Λ_0. We can identify $B(\mathscr{F})_\phi$ with the set of paths $\mathscr{P}(\Lambda_0)$ (see [2]). As shown in [1,2], the number of Λ_0-paths with weight μ is equal to $\dim M(\Lambda_0)_\mu$. Using these we prove Kashiwara's conjecture for $M(\Lambda_0)$ (Theorem 4.7).

1. Preliminaries

We follow the notations in [5]. We recall some essential facts. The q-analogue $U_q(\hat{\mathfrak{sl}}(n))$ of the enveloping algebra of the affine Lie algebra $\hat{\mathfrak{sl}}(n)$ is generated by $\{e_i, f_i, t_i^{\pm} = q^{\pm h_i} | 0 \leqq i \leqq n-1\}$. These generators satisfy the following important relations:

$$[e_i, f_j] = \delta_{ij}\left(\frac{t_i^+ - t_i^-}{q - q^{-1}}\right), \tag{1.1}$$

$$t_i^+ e_j t_i^- = q^{2(\alpha_i, \alpha_j)} e_j, \tag{1.2}$$

and

$$t_i^+ f_j t_i^- = q^{-2(\alpha_i, \alpha_j)} f_j, \tag{1.3}$$

where $(\alpha_i, \alpha_i) = 1$. We will need the algebra $U_q(\mathfrak{gl}(\infty))$ which is generated by $\{e_i^\infty, f_i^\infty, t_i^{\pm\infty} = q^{\pm h_i^\infty} | i \in \mathbf{Z}\}$ (see [3]). These generators also satisfy the corresponding relations (1.1)–(1.3).

Let M be any integrable $U_q(\hat{\mathfrak{sl}}(n))$-module. For each $i = 0, 1, \ldots, n-1$, let $U_q(\mathfrak{sl}(2))_{(i)}$ denote the subalgebra of $U_q(\hat{\mathfrak{sl}}(n))$ generated by e_i, f_i and $t_i^{\pm}$. Note that M is a union of finite-dimensional representations over $U_q(\mathfrak{sl}(2))_{(i)}$. Kashiwara [5] defines the following operators on M:

$$\tilde{e}_i = (qt_i^+ \Delta_i)^{-1/2} e_i, \quad \text{and} \quad \tilde{f}_i = t_i^+ (qt_i^+ \Delta_i)^{-1/2} f_i, \tag{1.4}$$

for $i = 0, 1, \ldots, n-1$, where Δ_i is certain element in the center of $U_q(\mathfrak{sl}(2))_{(i)}$. The action of the operator $(qt_i^+ \Delta_i)^{-1/2}$ is given as follows. Let v be a weight vector in an $(l+1)$-dimensional irreducible $U_q(\mathfrak{sl}(2))_{(i)}$ submodule of M. Suppose $t_i^+ v = q^{l-2k} v$. Then

$$\Delta_i v = (q^{l+1} - 2 + q^{-l-1}) v, \tag{1.5}$$

and

$$(qt_i^+ \Delta_i)^{-1/2} v = q^k (1 - q^{l+1})^{-1} v. \tag{1.6}$$

Let $K = \mathbf{Q}(q)$ and A be the ring of rational functions in q without pole at $q = 0$. Let L be a free A-module such that $K \otimes_A L \cong M$ and let B be a base of the $\mathbf{Q}$-vector space L/qL. The pair (L, B) is called a *crystal base* [5] of M if it satisfies the following conditions:

$$L = \bigoplus_\lambda L_\lambda, \tag{1.7}$$

where $L_\lambda = L \cap M_\lambda$ and M_λ is the λ – weight space of M,

$$B = \bigsqcup_{\lambda} B_\lambda \quad \text{where} \quad B_\lambda = B \cap (L_\lambda / qL_\lambda), \tag{1.8}$$

$$\tilde{e}_i L \subset L \quad \text{and} \quad \tilde{f}_i L \subset L, \quad \text{for all } i, \tag{1.9}$$

$$\tilde{e}_i B \subset B \cup \{0\} \quad \text{and} \quad \tilde{f}_i B \subset B \cup \{0\}, \quad \text{for all } i, \tag{1.10}$$

$$u = \tilde{e}_i v \quad \text{if and only if} \quad v = \tilde{f}_i u, \quad \text{for all } i \quad \text{and} \quad u, v \in B. \tag{1.11}$$

As noted in [5] B has a structure of colored oriented graph. The colors are labelled by i $(0 \leqq i \leqq n-1)$. For $u, v \in B$, $u \xrightarrow{i} v$ when $v = \tilde{f}_i u$. This is called the crystal graph of M.

2. The Fock Space Representation of $U_q(\widehat{\mathfrak{sl}}(n))$

In this section we will briefly describe the Fock space representation of $U_q(\mathfrak{gl}(\infty))$ and $U_q(\widehat{\mathfrak{sl}}(n))$ given in [3] with appropriate modifications. For more details we refer the reader to [3].

Consider the lattice on the fourth quadrant of the xy-plane with sites $\{(i,j) \in \mathbf{Z}^2 \mid i \geqq 0,\ j \leqq 0\}$. We consider edges on the lattice as oriented, starting from (i,j) and ending at $(i+1,j)$ or $(i,j+1)$, and labelled by the integer $i+j$. Any oriented path on this lattice determines uniquely a Young diagram Y and conversely. For example,

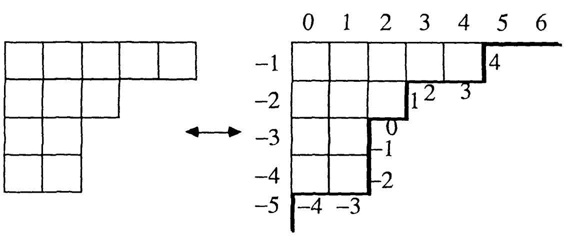


Fig. 1

In other words, given by Young diagram Y, we superimpose it on the lattice with upper left corner at the site $(0,0)$. Let $\mathscr{Y}$ be the set of all Young diagrams. Let $\mathscr{F} = \Sigma_{Y \in \mathscr{Y}} KY$ be the K-vector space having all the Young diagrams as base vectors. A Young diagram viewed as a lattice path has several corners. We say the corner is concave or convex depending on whether it is of the form ${}_{i-1}\ulcorner^{i}$ or $\lrcorner^{i}_{i-1}$.

The lagebra $U_q(\mathfrak{gl}(\infty))$ acts on the Fock space $\mathscr{F}$. The actions of its generators $\{e_i^\infty, f_i^\infty, t_i^{\pm\infty} = q^{\pm h_i^\infty} \mid i \in \mathbf{Z}\}$ are given as follows. For $Y \in \mathscr{Y}$,

$$\begin{aligned} e_i^\infty Y = Y', \quad & \text{if } Y \text{ has the convex corner } \lrcorner^{i}_{i-1}, \\ & \text{then } Y' \text{ is same as } Y \text{ except this corner} \\ & \text{becomes concave } {}_{i-1}\ulcorner^{i}, \\ = 0, \quad & \text{otherwise,} \end{aligned} \tag{2.1}$$

$$f_i^\infty Y = Y'', \quad \text{if } Y \text{ has the concave corner } {}_{i-1}\overset{i}{\ulcorner},$$
then Y'' is same as Y except this corner becomes convex $\lrcorner\, i, \; i-1$

$$= 0, \quad \text{otherwise,} \tag{2.2}$$

$$t_i^{\pm\infty} Y = q^{\pm 1} Y, \quad \text{if } Y \text{ has the convcave corner } {}_{i-1}\overset{i}{\ulcorner},$$

$$= q^{\pm 1} Y, \quad \text{if } Y \text{ has the convex corner } \underset{i-1}{\lrcorner}\, i,$$

$$= Y, \quad \text{otherwise.} \tag{2.3}$$

Under the above action $\mathscr{F}$ becomes an irreducible integrable $U_q(\mathrm{gl}(\infty))$-module with highest weight Λ_0 ($\Lambda_0(h_i^\infty) = \delta_{i,0}$) and highest weight vector ϕ (the empty Young diagram).

As in [3] (with suitable normalization) $\mathscr{F}$ becomes and $U_q(\hat{\mathfrak{sl}}(n))$-module where the actions of the generators $\{e_i, f_i, t_i^\pm = q^{\pm h_i} | 0 \leqq i < n\}$ are given by the following equations:

$$e_i = \sum_{j \equiv i \bmod n} \left(\prod_{k \geqq 1} t_{j-kn}^{+\infty} \right) e_j^\infty, \tag{2.4}$$

$$f_i = \sum_{j \equiv i \bmod n} f_j^\infty \left(\prod_{k \geqq 1} t_{j+kn}^{-\infty} \right), \tag{2.5}$$

$$t_i^\pm = \prod_{j \equiv i \bmod n} t_j^{\pm\infty}. \tag{2.6}$$

Under the above action $\mathscr{F}$ is an integrable $U_q(\hat{\mathfrak{sl}}(n))$-module. However, it is not irreducible as an $U_q(\hat{\mathfrak{sl}}(n))$-module. Observe that as an $U_q(\hat{\mathfrak{sl}}(n))$-module the vector $\phi \in \mathscr{F}$ is a highest weight vector with highest weight Λ_0, ($\Lambda_0(h_i) = \delta_{i,0}$). The space $M(\Lambda_0) = U_q(\hat{\mathfrak{sl}}(n))\phi$ is the irreducible integrable highest weight $U_q(\hat{\mathfrak{sl}}(n))$-module with highest weight Λ_0.

Given any Young diagram $Y \in \mathscr{Y}$ we color the boxes in Y with n colors $i = 0, 1, \ldots, n-1$, as follows. The box with the upper left corner at site (i,j) is colored with $(i+j)'$-color where $(i+j)' = (i+j) \bmod n$. Then observe that the action of e_i (respectively f_i) on $Y \in \mathscr{Y}$ given by (2.4) (respectively (2.5)) is just removing (respectively adding) a box of color i. For $Y \in \mathscr{Y}$, the weight of Y (denoted by $wt(Y)$) is $\Lambda_0 - \sum_{i=0}^{n-1} m_i \alpha_i$ if Y contains m_i boxes of color $i, 0 \leqq i < n$.

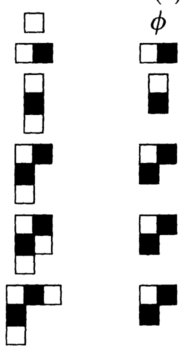
3. $U_q(\hat{\mathfrak{sl}}(2))$ Decomposition of the Fock Space

For each i ($0 \leqq i < n$), let $U_q(\mathfrak{sl}(2))_{(i)}$ denote the subalgebra of $U_q(\hat{\mathfrak{sl}}(n))$ generated by $\{e_i, f_i, t_i^\pm\}$. We say a Young diagram is *anti i-convex* if all its convex corners are non i-color. Given any Young diagram $Y \in \mathscr{Y}$ let $Y(i)$ denote its maximal subdiagram which is anti i-convex. Then Y is uniquely determined by the pair $(Y(i), \varepsilon)$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, $m = \#\{\text{concave corners in } Y(i) \text{ of color } i\}$,

and
$$\varepsilon_j = \begin{cases} 0, & \text{if the } j^{\text{th}} \text{ (counted from left to right) concave corner of color } i \text{ is vacant in } Y, \\ 1, & \text{if the } j^{\text{th}} \text{ concave corner of color } i \text{ is occupied in } Y. \end{cases}$$

For example, let $n = 2$ and let white be color 0 and black be color 1. Then by choosing $i = 0$, we have:

Y $\quad Y(0) \quad$ ε



(1)
(0)
(1)
(1, 0, 0)
(1, 1, 0)
(1, 0, 1)

For any fixed $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ we partition the set
$$\{1, 2, \ldots, m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$$
into disjoint subsets by the following inductive procedure:

(1) If there is no j such that $(\varepsilon_j, \varepsilon_{j+1}) = (0, 1)$ then define $J = \{1, 2, \ldots, m\}$.
(2) If there is some j such that $(\varepsilon_j, \varepsilon_{j+1}) = (0, 1)$ then define $K_1 = \{j, j+1\}$ and apply (1) and (2) to $\{1, 2, \ldots, m\} \backslash K_1$ to choose J or K_2. Repeat this as necessary.

For example, if $\varepsilon = (1, 0, 1, 0, 0, 1, 1, 0)$, then $m = 8$ and $J = \{1, 8\}$, $K_1 = \{2, 3\}$, $K_2 = \{5, 6\}$, $K_3 = \{4, 7\}$. Note that this partition is unique up to rearrangements of the sets K_s, $1 \leqq s \leqq t$.

Let $k = \#\{j \in J \mid \varepsilon_j = 1\}$. For any Young diagram $Y = (Y(i), \varepsilon)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$, and partition $\{1, 2, \ldots, m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$, we define
$$[Y_i]_i = \sum_{J = J_0 \sqcup J_1 \, |J_1| = k} \; \sum_{S \subset \{1, 2, \ldots, t\}} q^{\#(J_0, J_1)} (-q)^{|S|} (Y(i), \varepsilon(J_0, J_1, S)),$$
where
$$\#(J_0, J_1) = \#\{(j, j') \mid j < j', j \in J_0, j' \in J_1\}$$
$$\varepsilon(J_0, J_1, S) = (\tau_1, \tau_2, \ldots, \tau_m)$$
such that
$$\tau_j = 0 \quad \text{if} \quad j \in J_0, \; \tau_j = 1 \quad \text{if} \quad j \in J_1$$
and for $j < j'$, $\{j, j'\} \in K_s$,
$$(\tau_j, \tau_{j'}) = (1, 0) \quad \text{if} \quad s \in S$$
and
$$(\tau_j, \tau_{j'}) = (0, 1) \quad \text{if} \quad s \notin S.$$

For example, if $Y=(Y(i),(1,0,1,0))$, then

$$[Y]_i=(Y(i),(1,0,1,0))+q\{(Y(i),(0,0,1,1))-(Y(i),(1,1,0,0))\}-q^2(Y(i),(0,1,0\,1)).$$

Theorem 3.1. *Fix an anti i-convex Young diagram $Y(i)$ and a partition $\{1,2,\dots,m\}=J\bigsqcup K_1\bigsqcup\cdots\bigsqcup K_t$ such that $|J|=l$. For each $k=0,1,\dots,l$ there is a unique diagram Y_k with the data $(Y(i), J, K_1,\dots,K_t)$ such that $\#\{j\in J\,|\,\varepsilon_j=1\}=k$. Furthermore, $V_l=\bigoplus_{k=0}^{l}K[Y_k]_i$ is the $(l+1)$-dimensional irreducible integrable $U_q(\mathfrak{sl}(2))_{(i)}$-module with highest weight vector $[Y_0]_i$. Set $L_i=\bigoplus_{k=0}A[Y_k]_i$ and $B_i=\{[Y_k]_i|k=0,1,\dots,l\}$. Then (L_i,B_i) is a crystal base for the $U_q(\mathfrak{sl}(2))_{(i)}$-module V_l.*

Proof. The first assertion is clear, for if $J=\{i_1,\dots,i_l\}$ then there are precisely $l+1$ choices for $(\varepsilon_{i_1},\varepsilon_{i_2},\dots,\varepsilon_{i_l})$, namely, $(0,0,\dots,0)$, $(1,0,\dots,0)$, $(1,1,0,\dots,0),\dots,(1,1,\dots,1)$.

Now observe that if $Y=(Y(i),(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m))$, then the action of e_i and f_i on Y are given by the following formulas:

$$e_iY=\sum_{\substack{j\\ \varepsilon_j=1}}q^{\#\{j'|j'<j,\varepsilon_{j'}=0\}-\#\{j'|j'<j,\varepsilon_{j'}=1\}}(Y(i),(\varepsilon_1,\dots,\varepsilon_j-1,\dots,\varepsilon_m)),\tag{3.2}$$

$$f_iY=\sum_{\substack{j\\ \varepsilon_j=0}}q^{\#\{j'|j'>j,\varepsilon_{j'}=1\}-\#\{j'|j'>j,\varepsilon_{j'}=0\}}(Y(i),(\varepsilon_1,\dots,\varepsilon_j+1,\dots,\varepsilon_m)).\tag{3.3}$$

It follows from (3.2) and (3.3) that when e_i or f_i act on $[Y]_i$ (see (3.1)) the terms corresponding to $j\in\{r,r+p\}=K_s$ for any $s\in\{1,2,\dots,t\}$ cancel each other. Furthermore, for any $j\in J$ the sum of the contributions of $j'\in\{r,r+p\}=K_s$ for any $s\in\{1,2,\dots,t\}$, to the exponent of q in (3.2) or (3.3) is zero. Hence in order to compute $e_i[Y_k]_i$ or $f_i[Y_k]_i$ for any $k=0,1,\dots,l$, without loss of generality we can and do assume $t=0$, $m=l$ so $J=\{1,2,\dots,l\}$. Then by (3.1) we have

$$[Y_k]_i=\sum_{\substack{J=J_0\sqcup J_1\\ |J_0|=l-k\\ |J_1|=k}}q^{\#(J_0,J_1)}(Y(i),\varepsilon(J_0,J_1)),\tag{3.4}$$

where

$$\#(J_0,J_1)=\#\{(j,j')|j<j',j\in J_0,j'\in J_1\}$$

and

$$\varepsilon(J_0,J_1)=(\tau_1,\tau_2,\dots,\tau_l)$$

such that $\tau_j=0$ if $j\in J_0$ and $\tau_j=1$ if $j\in J_1$.

Now applying e_i (respectively f_i) to Eq. (3.4) and using formula (3.2) (respectively (3.3)) we easily get, for $0\leqq k\leqq l$,

$$e_i[Y_k]_i=q^{-(k-1)}(1+q^2+q^4+\cdots+q^{2(l-k)})[Y_{k-1}]_i,\tag{3.5}$$

$$f_i[Y_k]_i=q^{-l+k+1}(1+q^2+q^4+\cdots+q^{2k})[Y_{k+1}]_i,\tag{3.6}$$

where $[Y_{-1}]_i=0$, and $[Y_{l+1}]_i=0$.

Observe that (see [5]) by the definitions,

$$t_i^+[Y_k]_i=q^{l-2k}[Y_k]_i,\tag{3.7}$$

and

$$(qt_i^+\varDelta_i)^{-1/2}[Y_k]_i=q^k(1-q^{l+1})^{-1}[Y_k]_i.\tag{3.8}$$

Now it follows from (3.5)–(3.8) that for $1 \leqq k \leqq l$ we have

$$\begin{aligned}\tilde{e}_i[Y_k]_i &= (qt_i^+ \Delta_i)^{-1/2} e_i[Y_k]_i \\ &= (1-q^{l+1})^{-1}(1+q^2+q^4+\cdots+q^{2(l-k)})[Y_{k-1}]_i,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\tilde{f}_i[Y_k]_i &= t_i^+ (qt_i^+ \Delta_i)^{-1/2} f_i[Y_k]_i \\ &= (1-q^{l+1})^{-1}(1+q^2+q^4+\cdots+q^{2k})[Y_{k+1}]_i.\end{aligned} \tag{3.10}$$

Hence the theorem follows. ∎

Theorem 3.2. *Let $L(\mathscr{F}) = \bigoplus_{Y\in\mathscr{Y}} AY$ and $B(\mathscr{F}) = \mathscr{Y}$. Then the pair $(L(\mathscr{F}), B(\mathscr{F}))$ is a crystal base for the integrable $U_q(\widehat{\mathfrak{sl}}(n))$-module $\mathscr{F}$.*

Proof. By the definition (see (3.1)), for each $0 \leqq i \leqq n-1$, $[Y]_i \in L(\mathscr{F})$ and $[Y]_i = Y + q\sum_{Y'} a_{Y'}Y', a_{Y'}\in A$. For any weight μ, the μ-weight space $\mathscr{F}_\mu$ of $\mathscr{F}$ is finite-dimensional. Suppose $\dim_K(\mathscr{F}_\mu) = n_\mu$. Choose a basis $\{Y_1, Y_2, \ldots, Y_{n_\mu}\}$ of $\mathscr{F}_\mu$. Then for each i,

$$([Y_1]_i, \ldots, [Y_{n_\mu}]_i) = (Y_1, \ldots, Y_{n_\mu})(I + qX_i), \tag{3.11}$$

where X_i is a $n_\mu \times n_\mu$ matrix with coefficients in A. Since $I + qX_i$ is invertible in A, it follows from (3.11) that for each i,

$$Y = [Y]_i + q\sum_{Y''} b_{Y''}[Y'']_i, \quad b_{Y''}\in A. \tag{3.12}$$

So by using Theorem 3.1, we have

$$\tilde{e}_i Y = \tilde{e}_i[Y]_i + q\sum_{Y''} b_{Y''}\tilde{e}_i[Y'']_i \in L(\mathscr{F}),$$

and

$$\tilde{f}_i Y = \tilde{f}_i[Y]_i + q\sum_{Y''} b_{Y''}\tilde{f}_i[Y'']_i \in L(\mathscr{F}),$$

which gives the required result. ∎

The next proposition is an immediate consequence of Theorems (3.1), (3.2) and the definition of crystal graph (see [5]).

Proposition 3.3. *Let $Y, Y' \in B(\mathscr{F})$. In the crystal graph of $B(\mathscr{F})$, $Y \xrightarrow{i} Y'$ if and only if*
i) $Y = (Y(i), (\varepsilon_1, \ldots, \varepsilon_m))$, $Y' = (Y(i), (\varepsilon'_1, \ldots, \varepsilon'_m))$,
ii) *the partition $\{1, 2, \ldots, m\} = J \bigsqcup K_1 \bigsqcup \cdots \bigsqcup K_t$ is the same for both Y and Y',*
iii) *$\varepsilon_j = \varepsilon'_j$ for $j \neq r$ for some $r\in J$, $\varepsilon_r = 0$, $\varepsilon'_r = 1$, and for $j\in J$, $\varepsilon_j = \varepsilon'_j = 1$ if $j < r$, $\varepsilon_j = \varepsilon'_j = 0$ if $j > r$.*

4. Crystal Base for $M(\Lambda_0)$

As in Sect. 2, let $M(\Lambda_0) \subset \mathscr{F}$ be the irreducible highest weight $U_q(\widehat{\mathfrak{sl}}(n))$-module with highest weight Λ_0 and highest weight vector ϕ. For $Y\in\mathscr{Y}$, let $[f_1, f_2, \ldots, f_m]$ denote the signature of Y. Let $(g_1, g_2, \ldots, g_m)$ be the largest m-tuple of nonnegative integers (in lexicographic ordering) such that:

$$(1)\quad g_1 \geqq g_2 \geqq \cdots \geqq g_m,$$

and

$$(2)\quad f_1 - ng_1 \geqq f_2 - ng_2 \geqq \cdots \geqq f_m - ng_m \geqq 0.$$

Note that they are determined by the recursive formula

$$g_j = g_{j+1} + \left[\frac{f_j - f_{j+1}}{n}\right], \quad ([x] \text{ denotes the integral part}),$$

where $g_j = 0$ for $j \gg 0$. Define $\sigma(Y)$ to be the Young diagram with signature $[ng_1, ng_2; \dots, ng_m]$. We say $\sigma(Y)$ is the σ-component of Y.

Lemma 4.1. *For $Y, Y' \in \mathscr{Y}$, $Y \xrightarrow{i} Y'$ for some i, implies that $\sigma(Y) = \sigma(Y')$.*

Proof. Recall Proposition 3.3 which gives the condition for $Y \xrightarrow{i} Y'$. Suppose that $\sigma(Y)$ has signature $[ng_1, \dots, ng_m]$ and $\sigma(Y')$ has signature $[ng'_1, \dots, ng'_m]$. If $\sigma(Y) \neq \sigma(Y')$, then $g_j \neq g'_j$ for some j. Let $[f_1, f_2, \dots, f_p]$ be the signature of $Y(i)$. Then we must have the following situation:

(1) $f_j - f_{j+1} \equiv (n-1) \bmod n$ and we have concave corners of color i at the end of the j^{th} and $(j+1)^{\text{th}}$ rows of $Y(i)$. Suppose that these are the r^{th} and $(r+1)^{\text{th}}$ corners in the decomposition of $Y = (Y(i), \varepsilon)$.

(2) (i) Either $\varepsilon_r = \varepsilon'_r = 0$, $\varepsilon_{r+1} = 0$ and $\varepsilon'_{r+1} = 1$,
(ii) or $\varepsilon_{r+1} = \varepsilon'_{r+1} = 0$, $\varepsilon_r = 0$ and $\varepsilon'_r = 1$.

In the case of (i), $r+1 \in J$ since $\varepsilon_{r+1} = 0$ in Y and $\varepsilon_{r+1} = 1$ in Y'. But in Y', $(\varepsilon_r, \varepsilon_{r+1}) = (0, 1)$, hence $\{r, r+1\} \in K_s$ for some s, which is a contradiction. Similarly, (ii) also leads to contradiction. ■

Lemma 4.2. *For any $Y \in \mathscr{Y}$, $Y = \sigma(Y)$ if and only if Y is highest in the sense of crystal graph* (i.e., *there is no $Y' \in \mathscr{Y}$ such that $Y' \xrightarrow{i} Y$*).

Proof. Let $[f_1, f_2, \dots, f_k]$ be the signature of Y. Suppose $Y = \sigma(Y)$. Then $f_j - f_{j+1} \equiv 0 \bmod n$. Hence the color of the last box of each row is the same as the color of the concave corner of the subsequent row. So for any fixed i, if $Y = (Y(i), (\varepsilon_1, \dots, \varepsilon_m))$ with the partition $\{1, 2, \dots, m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$, then $J = \{m\}$ with $\varepsilon_m = 0$ or $J = \phi$(empty). In either case Y is highest in the sense of crystal graph.

Now suppose $Y \neq \sigma(Y)$. Let $[ng_1, \dots, ng_k]$ be the signature of $\sigma(Y)$. Then $f_j \neq ng_j$ for some j. Assume j to be the largest integer such that $f_j \neq ng_j$. Let i be the color of the last box in the j^{th} row of Y. Then $Y = (Y(i), (\varepsilon_1, \dots, \varepsilon_m))$, $\{1, 2, \dots, m\} = J \sqcup K_1 \sqcup \cdots \sqcup K_t$ and $\varepsilon_r = 1$, $r \in J$, where the r^{th} corner of color i which is occupied corresponds to the last box of the j^{th} row in Y. Hence by Proposition 3.3 we can find $Y' \in \mathscr{Y}$ such that $Y' \xrightarrow{i} Y$. So Y cannot be highest. ■

Proposition 4.3. *Let $Z \in \mathscr{Y}$ such that $\sigma(Z) = Z$. Let $B(\mathscr{F})_Z$ denote the connected component of Z in the crystal graph of $B(\mathscr{F})$. Then $B(\mathscr{F})_Z = \{Y \in \mathscr{Y} \mid \sigma(Y) = Z\}$.*

Proof. It follows from Lemma 4.1 that $B(\mathscr{F})_Z \subseteqq \{Y \in \mathscr{Y} \mid \sigma(Y) = Z\}$. Now suppose $Y \in \mathscr{Y}$ and $\sigma(Y) = Z$. We want to show that $Y \in B(\mathscr{F})_Z$. If $Y = Z$ then there is nothing to prove. If $Y \neq Z$, then by Lemma 4.2, there exists $Y_1 \in \mathscr{Y}$ such that $Y_1 \to Y$. Hence using induction we get $Y \in B(\mathscr{F})_Z$ as desired. ■

Now define

$$L = \sum_{0 \leqq i_1, i_2, \ldots, i_k \leqq n-1} A \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \phi \tag{4.1}$$

and

$$B = \{v = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \phi \in L/qL \mid v \neq 0\}. \tag{4.2}$$

Let $M(\Lambda_0) \subset \mathscr{F}$ denote the irreducible integrable $U_q(\widehat{\mathfrak{sl}}(n))$-module with highest weight Λ_0 (i.e., $\Lambda_0(t_i^{\pm}) = q^{\pm 1}\delta_{i,0}$) and highest weight vector ϕ. Then $M(\Lambda_0) = U_q(\widehat{\mathfrak{sl}}(n))\phi$.

Proposition 4.4. $M(\Lambda_0) = K \otimes_A L$.

Proof. By definition, $M(\Lambda_0) \supset K \otimes_A L$. For $\mu = \Lambda_0 - \sum_{i=0}^{n-1} m_i \alpha_i$ let $M(\Lambda_0)_\mu$ denote the μ weight space of $M(\Lambda_0)$. By Theorem 5.4 in [2] (also see Theorem in [1]) $\dim(M(\Lambda_0)_\mu) = \#\mathscr{P}(\Lambda_0)_\mu$, where $\mathscr{P}(\Lambda_0)_\mu$ denotes the set of Λ_0-paths of weight μ. But there is a one-to-one correspondence between $\mathscr{P}(\Lambda_0)_\mu$ and the set $\{Y \in B(\mathscr{F})_\phi \mid wt(Y) = \mu\}$. (See [2]. Young diagrams in this paper and those in [2] are transposed to each other.) For any $Y \in B(\mathscr{F})_\phi$ by Proposition 4.3 there exists some $(i_1, i_2, \ldots, i_k)$ such that $Y = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \phi$ in $L(\mathscr{F})/qL(\mathscr{F})$. Hence $\dim_K(M(\Lambda_0)_\mu) \leqq \dim_K(K \otimes_A L)_\mu$ for each weight μ. Therefore, $M(\Lambda_0) = K \otimes_A L$. ■

Lemma 4.5. $(K \otimes_A L) \cap L(\mathscr{F}) = L$.

Proof. Let $L_0 = \sum_{Y \in B(\mathscr{F})_\phi} A \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \phi$, where for each $Y \in B(\mathscr{F})_\phi$ we choose a sequence $(i_1, i_2, \ldots, i_k)$ such that $\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \phi = Y$ in $L(\mathscr{F})/qL(\mathscr{F})$. Then by an argument similar to the proof of Proposition 4.4 we get $M(\Lambda_0) = K \otimes_A L_0$. Hence $K \otimes_A L = K \otimes_A L_0$.

Clearly $L \subset (K \otimes_A L) \cap L(\mathscr{F}) = (K \otimes_A L_0) \cap L(\mathscr{F})$. Now let $v \in (K \otimes_A L_0) \cap L(\mathscr{F})$. Then $v \in L(\mathscr{F})$ and $v \in K \otimes_A L_0 = M(\Lambda_0)$. Let $v \in M(\Lambda_0)_\mu$ for some weight μ and $\dim(M(\Lambda_0))_\mu = n_\mu$. Then $v = \sum_{i=1}^{n_\mu} c_i y_i$, $c_i \in K$, $y_i \in L_0$. Also since $v \in L(\mathscr{F})$, we have $v = \sum_{i=1}^{n_\mu} a_i Y_i$, $a_i \in A$, $Y_i \in \mathscr{Y}$. Then

$$(y_1, y_2, \ldots, y_{n_\mu}) = (Y_1, Y_2, \ldots, Y_{n_\mu})(I + qX),$$

where X is an $n_\mu \times n_\mu$ matrix with coefficients in A. Hence

$$v = (Y_1, Y_2, \ldots, Y_{n_\mu}) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_\mu} \end{pmatrix}$$

$$= (y_1, y_2, \ldots, y_{n_\mu}) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_\mu} \end{pmatrix}$$

$$= (Y_1, Y_2, \ldots, Y_{n_\mu})(I + qX)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_\mu} \end{pmatrix}.$$

So

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_\mu} \end{pmatrix} = (I + qX)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_\mu} \end{pmatrix}.$$

But since $a_i \in A$, $i = 1, 2, \ldots, n_\mu$ and $I + qX$ is invertible in A, it follows that $c_i \in A$. Hence $v = \sum c_i y_i \in L$, which completes the proof. ∎

Corollary 4.6. *$L/qL \subset L(\mathscr{F})/qL(\mathscr{F})$. So B is a subset of $B(\mathscr{F})$.*

Proof. It is enough to show that $qL(\mathscr{F}) \cap L = qL$. It follows from Proposition 4.4 and Lemma 4.5 that

$$\begin{aligned} qL(\mathscr{F}) \cap L &= qL(\mathscr{F}) \cap L(\mathscr{F}) \cap M(\Lambda_0) \\ &= qL(\mathscr{F}) \cap M(\Lambda_0) \\ &= q(L(\mathscr{F}) \cap M(\Lambda_0)) = qL. \quad \blacksquare \end{aligned}$$

Theorem 4.7. *The pair (L, B) is a crystal base for the irreducible integrable highest weight $U_q(\widehat{\mathfrak{sl}}(n))$-module $M(\Lambda_0)$.*

Proof. Let $v \in L$. By the definition $\tilde{f}_i v \in L$ for all $i = 0, 1, \ldots, n-1$. For each $i, \tilde{e}_i v \in M(\Lambda_0) = K \otimes_A L$. Since $L \subseteqq L(\mathscr{F})$, by Theorem 3.2 $\tilde{e}_i v \in L(\mathscr{F})$. Hence by Lemma 4.5, $\tilde{e}_i v \in (K \otimes_A L) \cap L(\mathscr{F}) = L$. Now the result follows from Theorem 3.2, Proposition 4.4 and Corollary 4.6. ∎

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