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Theory of Ordered Spaces

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Abstract. This is the first of a planned series of investigations on the theory of ordered spaces based upon four axioms. Two of these, the order (I.1.1) and the local structure (II.5.1) axioms provide the structure of the theory, and the other two [the identification (I.1.11) and cone (I.2.7) axioms] eliminate pathologies or excessive generality. In the present paper the axioms are supplemented by the nontriviality conditions (I.1.9) and a regularity property (II.4.2).

The starting point is a nonempty set M and a family of distinguished subsets, called *light rays*, which are totally ordered. The order axiom provides the properties of this order. Positive and negative cones at a point are defined in terms of increasing and decreasing subsets and are used to extend the total order on the light rays to a partial order over all of M. The first significant result is the *polygon lemma* (I.2.3) which provides an essential constructive tool. A non-topological definition is found for the interiors of the cones; it leads to a "more homogeneous" partial order relation on M.

In Sect. II, subsets called *D-sets* (Def. II.2.2), possessing certain desirable properties, are studied. The key concept of *perpendicularity* of light rays is isolated (Def. II.3.1) and used to derive the basic "separation properties," provided that the interiors of cones are nonempty. It is shown that, in a D-set, "good" properties of one cone can be transported along light rays, so that the structure of a D-set is homogeneous. In particular, if one cone has nonempty interior, so have all others. However, the existence of even one cone with nonempty interior does not follow from the axioms, but has to be imposed as an additional *regularity* condition. The local structure axiom now states that every point lies in a regular D-set. It is proved that the family of regular D-sets is closed under finite intersections. The *order topology* is defined as the topology which has this family as a base. This topology is Hausdorff, and coincides with the usual topology for Minkowski spaces.

0. Introduction

Ordered spaces play a central role in theoretical physics. The ordering of time, for instance, induces an ordering of the space of events by the future light cone.

Another example is the ordering of the state space of a thermodynamical system due to the irreversibility of thermodynamic processes. Orderings, in fact, are obtained both in the thermodynamics of equilibrium states, and in that of irreversible processes. The latter is of particular interest from the point of view of physics, since there appears to be a connection between the concepts of time-ordering, irreversibility, and dissipativity which has not yet been fully exploited. As a last example we may take the case of particle physics. Here the notion of positivity of energy gives rise to an order in the set of states. In this example there is also a connection between this ordering and time-ordering in the space of events (when the space of events coincides with Minkowski space), because the space-time ordering and the energy-momentum ordering are dual to each other. These two orderings in particle physics are connected with the ordering in thermodynamics via the KMS condition, that is to the ordering of the temperature scale. But one is still in the dark as to how all these orderings fit into one convincing scheme.

From these examples it would appear to be natural to study ordered spaces in some detail. However, the impetus to start a systematic investigation came only a few years ago, when Donaldson and Freedman[1] showed that $\mathbb{R}^4$ admits inequivalent differentiable structures. Since from the point of view of experimental physics it would hardly seem possible to distinguish between infinitely differentiable functions on the one hand, and continuous nowhere-differentiable functions on the other hand, we felt that the differentiable structure of the space of events (which one would like to have for facilitating the description) should be uniquely determined by concepts fundamental to physics. The concept which thrusts itself to the fore is the order structure given by the future (past) light cone.

In the standard theory of ordered spaces one usually starts with a topological space upon which an order structure has been grafted. If one is fortunate then the topology induced by the order structure (order topology = Alexandrov topology [1]) coincides with the given one. This, however, is the case only when the cone contains interior points. It seems to be clear that this case and the contrary would require completely different techniques for their treatment. In this paper we shall deal only with the situation in which the cone contains "interior points." Saying this does not mean that we start with a topological space; we prefer to start with a point-set furnished with an order structure, and define everything in terms of this structure.

The space of events in physics is the model from which we extract the axioms. It seems to us that it is the light rays, and not the light cones, which are the fundamental objects. These are totally ordered sets, and hence have an order-theoretic characterization. Using the light rays we are able to construct the light cone and the order associated with it as secondary concepts. This is done in the first chapter. Furthermore we are able to introduce, "purely algebraically," a set of points associated with each cone which would be its interior points when a "good" topology is present. In the second chapter we show how one can use the order structure defined by the light cones to furnish a Hausdorff topology.

Our discussion in the following chapter does not use any number system, except for a finite subset of positive integers. We have, therefore, at our disposal

[1] We quote references [2, 3, 5, 7, 10] for the general reader and [4, 6, 8] for the specialist

essentially two proof techniques: explicit construction, and proof by contradiction. The result may be viewed by some as a certain "loss of transparency" in part of our argument. We have tried to offset this, wherever possible, by breaking down major results into smaller lemmas and propositions. The reader may find that, at times, a two-dimensional diagram in something approximating Minkowski space is an aid to understanding.

The symbol ■ denotes end of proof. The symbol □ denotes end of statement in axioms and definitions.

I. Light Rays and Order

The ordered spaces which we shall study are those which possess a distinguished family of totally ordered subsets, and are such that the order properties of the space (and other properties which follow from the order) are wholly determined by the order properties of this distinguished family. We shall call members of this family *light rays,* and indeed Minkowski spaces provide the chief model from which we abstract our axioms. However, this terminological quirk should not be allowed to obscure the fact that our considerations apply to all situations in which cones are generated by extremal rays. The compact self-adjoint operators on a Hilbert space provide a non-finite-dimensional example which (if one looks at the subset of trace class operators) is closely connected to the state space of quantum mechanics.

I.1. Light Rays. Thus, the fundamental objects in our scheme are:
i) A nonempty set of *points* M.
ii) A distinguished family of subsets of M called *light rays.*
iii) An order relation "$<^l$" on every light ray.

Points of M will be denoted by lower-case Latin letters. Light rays will be denoted by the letter l. l_x will denote a light ray through the point x, $l_{x,y}$ a ray through x and y, etc. Distinct rays will be distinguished by superscripts, thus l, l', l^1_x, l^2_x, etc. The order relation $<^l$ will be reflexive, antisymmetric and transitive, i.e. $x<^l y,\ y<^l z \Rightarrow x<^l z$ (for definition see [11]). The statements $x<^l y$ (read: x precedes y, or y follows x) and $y\,^l\!> x$ will be identical. The statement "x and y are joined by a light ray" will be abbreviated "$\lambda(x,y)$", and its negation (no light ray passes through both x and y) by "$\sim\lambda(x,y)$".

A light ray l satisfies:

I.1.1. Axiom (The order axiom).
a) If $x,y\in l$, $x\neq y$, then either $x<^l y$ or $y<^l x$ (i.e. both $x<^l y$ and $y<^l x \Rightarrow x=y$).
b) If $x,z\in l$, $x\neq z$, $x<^l z$, then $\exists y\in l$ such that $x<^l y<^l z$, $x\neq y$, $y\neq z$.
c) If $y\in l$, then $\exists x,z\in l$ such that $x<^l y<^l z$, $x\neq y$, $y\neq z$.
d) If $x,y\in l^1\cap l^2$, then $x<^{l^1} y \Leftrightarrow x<^{l^2} y$. □

Here (b) is a "density" axiom, (c) states that light rays do not have end points (singularities are admissible if singular points are not considered as parts of the space), and (d) is a consistency condition, not required in Minkowski space but essential, for example, for photography.

Our first problem is to extend the order on the light rays to a partial order on all of M. This would hardly be possible if M were to consist of disconnected nontrivial pieces. However, the term "disconnected" has not yet been defined in the present context, and we proceed to fill this lacuna.

I.1.2. Definition. A subset N of M will be called *l-complete* if $x \in N$, $l \ni x \Rightarrow l \subset N$.

I.1.3. Lemma. *Let* $\{N^\alpha\}_{\alpha \in A}$ *be an indexed family of l-complete subsets of* M. *Then*
a) $M \backslash N^\alpha$ *is l-complete.*
b) $\bigcap_{\alpha \in A} N^\alpha$ *is l-complete.*
c) $\bigcup_{\alpha \in A} N^\alpha$ *is l-complete.*

Proof. a) Let $x \in M \backslash N^\alpha$, $l \ni x$. Then $x \notin N^\alpha$, and therefore, by the l-completeness of N^α, $l \cap N^\alpha = \emptyset$. Hence $y \in l \Rightarrow y \in M \backslash N^\alpha$, and therefore $l \subset M \backslash N^\alpha$.
b) $x \in \bigcap_{\alpha \in A} N^\alpha \Rightarrow x \in N^\alpha$ $\forall \alpha \in A$, therefore $l_x \subset N^\alpha$ for any l through x and any $\alpha \in A$, hence $l_x \subset \bigcap_{\alpha \in A} N^\alpha$.
c) $\bigcap_{\alpha \in A} (M \backslash N^\alpha) = M \backslash \left(\bigcup_{\alpha \in A} N^\alpha \right)$, therefore $\bigcup_{\alpha \in A} N^\alpha = M \backslash \bigcap_{\alpha \in A} (M \backslash N^\alpha)$. The result now follows from a) and b). ■

Construction of l-Complete Sets. Let A be any well-ordered set without largest element. Denote the smallest member of A by 0, and the successor of $\alpha \in A$ by $\alpha + 1$. Let

$$K_s^0 \equiv S \subset M, \quad S \neq \emptyset, \text{ otherwise arbitrary}. \tag{1a}$$

$$K_s^1 \equiv \{y; y \in l_z, z \in K_s^0\} \backslash K_s^0 . \tag{1b}$$

That is, y is restricted to lie on a light ray through z. Now define inductively

$$K_s^{\alpha+1} \equiv \{y; \lambda(y,z), z \in K_s^\alpha\} \backslash \left\{ \bigcup_{A \ni \beta \leq \alpha} K_s^\beta \right\}, \qquad \alpha \in A, \alpha > 0 . \tag{1c}$$

Note that $x \in K_s^{\alpha+1} \Rightarrow x \notin K_s^\alpha$. Finally, define

$$K_s^A \equiv \bigcup_{\alpha \in A} K_s^\alpha . \tag{2}$$

By construction, $x \in K_s^A \Rightarrow l_x \in K_s^A$ for every l through x, i.e. K_s^A is l-complete.

In particular, taking $A = \mathbb{N}$ we find that the subset $K_s^{\mathbb{N}}$ of M is l-complete. Define now the intersection of all l-complete sets containing S:

$$N_s \equiv \bigcap_{s \subset N,\, N \text{ is } l\text{-complete}} N \tag{3}$$

By Lemma I.1.3, N_s is l-complete, and therefore

$$N_s \subset K_s^{\mathbb{N}} . \tag{4}$$

However, we have:

I.1.4. Theorem.

$$N_s = K_s^{\mathbb{N}} .$$

Proof. In view of (4), it remains to prove that $N_s \supset K_s^{\mathbb{N}}$. Now $K_s^{\mathbb{N}}$ is defined by formulae (1a–c), with $\mathbb{N}$ replacing A and n replacing α. If $a_0 \in K_s^{\mathbb{N}}$, then $\exists n \in \mathbb{N}$ such that

$$a_0 \in K_s^{n-1}, \quad a_0 \notin K_s^n.$$

But $b_j \in K_s^{n-j}$ implies that $\exists b_{j+1} \in K_s^{n-j-1}$ such that $\lambda(b_j, b_{j+1})$ but $\sim\lambda(b_j, b_{j+k})$, $k=2,3,\ldots,n-j$. Thus there exists a set of points $\{a_0, a_1, \ldots, a_n\}$ such that $a_i \in K_s^{n-i}$; $\lambda(a_i, a_{i+1})$, $\sim\lambda(a_i, a_{i+k})$, $i=1,2,\ldots,n-2$ and $k=2,3,\ldots,n-i$; and $a_n \in S$. Now $a_n \in S$, $\lambda(a_{n-1}, a_n)$ imply that a_{n-1} belongs to any l-complete set which contains K_s^0, hence $a_{n-1} \in N_s$. Next, by the same argument (with K_s^1 replacing K_s^0) $a_{n-1} \in N_s \Rightarrow a_{n-2} \in N_s$, and so on, until finally $a_0 \in N_s$. Thus $K_s^{\mathbb{N}} \subset N_s$.

I.1.5. Corollary. *If $y \in N_x$, then $\exists n \in \mathbb{N}$ and $\{x_0, x_1, \ldots, x_n\} \subset M$, with $x_0 \equiv x$ and $x_n \equiv y$, such that $\lambda(x_i, x_{i+1})$ and $\sim\lambda(x_{i-k}, x_{i+1})$ for $i=0,1,\ldots,n-1$ and $k=1,2,\ldots,i$.*

Proof. Specialize Theorem I.1.4 to the case $S=\{x\}$, and take $x_i = a_{n-i}$, $i=0,1,\ldots,n$, $x_0 = x$, $x_n = y$. ■

I.1.6. Corollary.

$$y \in N_x \Leftrightarrow x \in N_y.$$

Proof. The subset $\{x_0, x_1, \ldots, x_n\}$ is equally a subset of N_y. ■

I.1.7. Theorem. *The relation $x \sim y$ iff $N_x = N_y$ is an equivalence relation.*

Proof. Follows from the properties of the equality sign in $N_x = N_y$. ■

These equivalences classes are the "connected" pieces which we have been looking for. They are important enough to be given a name.

I.1.8. Definition. M will be called *l-connected* iff $M = N_x$ $\forall x \in M$ (i.e. if M consists of a single equivalence class). □

From now on we shall make the following nontriviality assumptions:

I.1.9. Assumptions.
a) M is l-connected.
b) M does not consist of a single point.
c) M does not consist of a single light ray. □

I.1.10. Definition.

$$l(a,b) \equiv \{x;\ x, a \in l,\ a <^l x <^l b,\ a \neq x,\ x \neq b\}.$$

$$l[a,b] \equiv \{x;\ x, a, b \in l,\ a <^l x <^l b\}. \quad □$$

$l(a,b)$ and $l[a,b]$ will be called open and closed *segments* (*not* intervals!) of light rays.

So far we cannot rule out the possibility that two light rays meet, merge and separate again after a finite segment. However, we do *not* want this degree of generality, and therefore adopt the following axiom:

I.1.11. Axiom (The Identification Axion). *If l and l' are distinct light rays and $a \in S \equiv l \cap l'$, then there exist $p, q \in l$ such that $p <^l a <^l q$, $p \neq a$, $a \neq q$, and $l(p,q) \cap S = \{a\}$. Similarly for l'.* □

This axiom ensures that the intersection of two light rays contains no "point of accumulation."

I.1.12. Example. Let M be the cylinder $S^1 \times \mathbb{R}$, with base S^1 placed horizontally. Let the light cone through any point consist of two rays each making an angle of $\pi/4$ with the vertical at that point. These two rays intersect infinitely many times. This example fulfills the order and identification axioms.

I.1.13. Example. This example consists of the one-sheeted hyperboloid. The light rays are the families of generators. Then two light rays do not intersect more than once.

The cylinder and the one-sheeted hyperboloid are topologically indistinguishable. Clearly, their order structures are very different, and, equally clearly, this difference appears at the global and not at the local level. These examples show that the order structure may be able to distinguish between topologically identical structures.

I.2. Extension of Order. In this section we shall extend the total order on the light rays to a partial order on all of M.

I.2.1. Definition. A subset $S \subset M$ will be called *increasing* (respectively *decreasing*) *if*

$$x \in S, \quad y \,{}^l\!> x \Rightarrow y \in S$$

(respectively $x \in S$, $y <^l x \Rightarrow y \in S$). □

Increasing and decreasing subsets can be constructed by an inductive process similar to the construction of l-complete sets in the previous section. We are interested in the smallest increasing (respectively decreasing) subset containing a given point x: from Def. I.2.1 one sees that the property "increasing" (respectively "decreasing") is stable under intersections.

I.2.2. Definition.

$$C_x^+ \equiv \bigcap_{\substack{S \ni x \\ S \text{ increasing}}} S,$$

$$C_x^- \equiv \bigcap_{\substack{S \ni x \\ S \text{ decreasing}}} S,$$

and

$$C_x \equiv C_x^+ \cup C_x^- . \quad \square$$

Note that, as it is possible to construct an increasing set which excludes a given point, the intersection of *all* increasing subsets is empty.

I.2.3. Theorem (The Polygon Lemma). *If* $y \in C_x^+$, *there exist points* $x_1, x_2, \ldots, x_{n-1} \in C_x^+$ *such that* $x_i <^l x_{i+1}$, $x_i \neq x_{i+1}$, *and* $\sim \lambda(x_i, x_{i+k})$ *for* $i = 0, 1, \ldots, n-2$, $k = 2, 3, \ldots, n-i$, *with* $x_0 = x$ *and* $x_n = y$. *Similarly for* $y \in C_x^-$.

Proof. Let

$$S_x^0 \equiv \{x\},$$

$$S_x^{k+1} \equiv \{y; y \,{}^1\!> z,\ z \in S_x^k\} \setminus \Big\{ \bigcup_{\mathbb{N} \ni j \leq k} S_x^j \Big\}, \qquad k \in \mathbb{N}.$$

Finally, let

$$S_x^+ \equiv \bigcup_{k\in\mathbb{N}} S_x^k .$$

Clearly, S_x^+ is an increasing set, and if $y\in S_x^+$ then there exist points $x_1, x_2, \ldots, x_{n-1} \in S_x^+$ such that $x_i <^l x_{i+1}$, $x_i \neq x_{i+1}$, and $\sim\lambda(x_i, x_{i+k})$ for $i=0,1,\ldots,n-2$, $k=2,3,\ldots,n-i$. It is therefore enough to prove that $S_x^+ = C_x^+$. From the definition of C_x^+, $S_x^+ \supset C_x^+$. The proof that $a\in S_x^+ \Rightarrow a\in C_x^+$, i.e. $S_x^+ \subset C_x^+$, is similar to the proof of the corresponding assertion in Theorem I.1.4.

A similar proof holds for $y\in C_x^-$. ■

A finite set of points $x_0, x_1, \ldots, x_n$ satisfying the conditions

$$x_i <^l x_{i+1}, \qquad x_i \neq x_{i+1}, \qquad \sim\lambda(x_i, x_{i+k})$$

for

$$i=0,1,2,\ldots,n-2, \qquad k=2,3,\ldots,n-i$$

will be called an *ascending l-polygon* from x_0 to x_n, or a *descending l-polygon* from x_n to x_0. When there is no possibility of misunderstanding, the "l-" in the phrases above will be omitted. Theorem 1.2.3 will be called the *polygon lemma.*

I.2.4. Corollary.

$$y\in C_x^+ \Leftrightarrow x\in C_y^- .$$

Proof. Consists of the observation that an ascending polygon from x to y is equally a descending polygon from y to x. ■

The polygon lemma says that if $y\in C_x^+$, then y can be "reached" from x by traversing a *finite number* of light-ray segments. It will turn out to be an essential constructive tool. It should be noted that this is a *consequence* of our definition of N_s [Construction of l-complete sets, Eq. (3)] as the "smallest" l-complete set containing S. It would have been possible, at that stage, to define N_s as the intersection of *all* l-complete sets which are obtained by repeating the process (of joining with light rays) a *transfinite* (corresponding to a given cardinality) number of times. This would have led to transfinite polygons.

We are now ready to extend the definition of order.

I.2.5. Definition.

$$x<y \quad \text{iff} \quad y\in C_x^+ ;$$

equivalently,

$$x<y \quad \text{iff} \quad x\in C_y^- . \quad \square$$

Observe that if $\lambda(x,y)$ then $x<y \Rightarrow x<^l y$.

The basic result follows quickly:

I.2.6. Theorem.

"$<$" defines a partial order on M.

Proof. Only transitivity needs to be proven. Let $x<y$, $y<z$. Then there is an ascending polygon from x to y, and one from y to z. The concatenation of the two is an ascending polygon from x to z, hence $x<z$. ■

Even with the identification axiom, our scheme remains a little too general. Owing to axiom I.1.1 (c), a light ray cannot form a closed loop. But there is nothing to prevent the existence of closed "timelike" curves and similar pathologies, as shown by Example I.2.8. These pathologies can be eliminated rather simply, by the following axiom:

I.2.7. Axiom (The Cone Axiom).

$$C_x^+ \cap C_x^- = (x) \quad \forall x \in M. \quad \square$$

I.2.8. Example. This 2-dimensional example consists of a cylinder with cuts parallel to its axis. The configuration is shown in Fig. 1. The upward-pointing arrows delimit C_x^+ and the downward-pointing arrows delimit C_x^-. The upper edge of the rectangle is identified with the lower edge. The cuts are shown by thick lines, and $C_x^+ \cap C_x^-$ is shown by the shaded area.

This example satisfies the order axiom and the identification axiom, but violates the cone axiom.

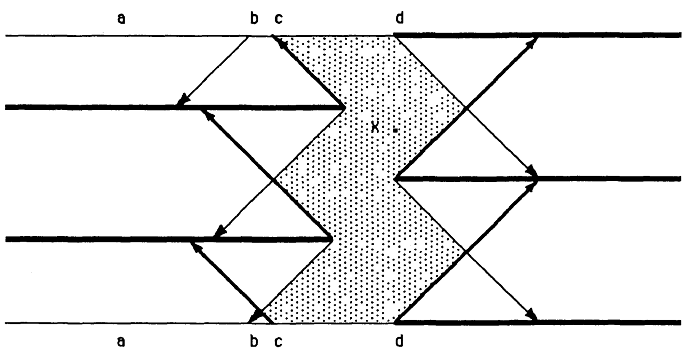

Fig. 1

We shall call C_x^+ the *forward* (or *positive*) cone at x, C_x^- the *backward* (or negative) cone at x, and C_x the *cone* at x.

Forward and backward cones have the following inclusion property:

I.2.9. Proposition.

$$x < y \Leftrightarrow C_x^+ \supset C_y^+ \Leftrightarrow C_x^- \subset C_y^- .$$

Proof. i) Let $z \in C_y^+$. Then $y < z$, therefore $x < y \Rightarrow x < z$, i.e. $z \in C_x^+$. Thus $x < y \Rightarrow C_x^+ \supset C_y^+$.

ii) If $C_x^+ \supset C_y^+$, then $y \in C_x^+$, i.e. $x < y$.

This establishes that $x < y \Leftrightarrow C_x^+ \supset C_y^+$. The equivalence $x < y \Leftrightarrow C_x^- \subset C_y^-$ is established similarly. ∎

The intersection of a light ray l_x through x with the forward cone at x will be called the *forward ray* through x and denoted thus:

$$l_x^+ \equiv l_x \cap C_x^+ .$$

Similarly, the *backward ray* through x will be defined by $l_x^- \equiv l_x \cap C_x^-$.

I.3. Timelike Points. In a Minkowski space the boundary of the light cone through x consists of the set of all light rays through x. An interior (future) point y may be characterized by the fact that descending polygons from y to x meet every light ray through x at a point "above" x. This latter property may be used to define "interior" points in our present setting. The "boundary" will then be defined as the residual set. All of this can be accomplished without defining a topology on M. However, to avoid confusion we shall not use topological terminology and notation until the topology has been defined.

It should be noted that the *finiteness* of the l-polygons plays an essential role in the following (see Theorem I.2.3 and the comments following it).

I.3.1. Definition.
a) A point $z \in C_x^+$ will be called a *timelike point* of C_x^+ if, for any ray $l \ni x$, $\exists y \in l$, $y \neq x$ such that $z \in C_y^+$. The set of all timelike points of C_x^+ will be denoted by τC_x^+.
b) The set τC_x^- is defined similarly, i.e. by interchange of order. □

I.3.2. Lemma. *$y \in \tau C_x^+$ and $z > y \Rightarrow z \in \tau C_x^+$, and the same for reversed order.*

Proof. $y \in \tau C_x^+$ implies that, given any $l \ni x$, there exists a descending polygon from y to some point $a \in l_x^+$, $a \neq x$. Next, $z > y$ implies the existence of a descending polygon from z to y. The first statement is established by concatenating the two descending polygons, and the second by reversing the order.

I.3.3. Definition.

$$\beta C_x^+ \equiv C_x^+ \backslash \tau C_x^+ , \qquad \beta C_x^- \equiv C_x^- \backslash \tau C_x^- . \quad \square$$

Note carefully that $y \in \beta C_x^+$, $y \neq x$ does *not* imply that y lies on a light ray through x; y may be the end point of a nontrivial ascending polygon from x which lies wholly in βC_x^+. Consequently, many "obvious" results have to be proven from the axioms. The remainder of this chapter is devoted to establishing those which we shall need.

I.3.4. Proposition. *Let $y > x$. Then the statements* 1 *and* 2 *(respectively* 1′ *and* 2′*) are equivalent:*

1. $y \in \beta C_x^+$;
2. $C_x^+ \cap C_y^- \subset \beta C_x^+$.

1′. $x \in \beta C_y^-$;
2′. $C_x^+ \cap C_y^- \subset \beta C_y^-$.

Proof. Owing to symmetry, it suffices to establish one of the two equivalences.

<u>1 ⇒ 2:</u> Let $y \in \beta C_x^+$, but $C_x^+ \cap C_y^- \not\subset \beta C_x^+$. Then $\exists z \in C_x^+ \cap C_y^-$ such that $z \notin \beta C_x^+$, so that necessarily $z \in \tau C_x^+$. Then, since $y > z$, Lemma I.3.2 implies that $y \in \tau C_x^+$, a contradiction.

<u>2 ⇒ 1:</u> If $C_x^+ \cap C_y^- \subset \beta C_x^+$, then $y \in C_x^+ \cap C_y^- \Rightarrow y \in \beta C_x^+$. ■

I.3.5. Definition.

$$I[a,b] \equiv C_a^+ \cap C_b^- ,$$

$$I(a,b) \equiv \tau C_a^+ \cap \tau C_b^- .$$

For the moment, sets of both kinds will be called *order intervals.* A more discriminating terminology will suggest itself later.

I.3.6. Remarks.
1. $I[a,a]=\{a\}$,
2. If $a \nless b$, $a \neq b$ then $I[a,b]=\emptyset$.

I.3.7. Proposition. *Let $y>x$. Then the statements* 1 *and* 2 *(respectively* 1′ *and* 2′*) are equivalent:*
1. $y\in\tau C_x^+$;
2. $\beta C_x^+ \cap C_y^+ = \emptyset$.

1′. $x\in\tau C_y^-$;
2′. $\beta C_y^- \cap C_x^- = \emptyset$.

Proof. Owing to symmetry, it suffices to establish one of the two equivalences:

1 ⇒ 2: Let $y\in\tau C_x^+$ and $z\in\beta C_x^+\cap C_y^+$. Then $z>y$, hence $z\in\tau C_x^+$ by Lemma I.3.2, a contradiction.

2 ⇒ 1: Since $y\in C_y^+$, $\beta C_x^+\cap C_y^+=\emptyset \Rightarrow y\notin\beta C_x^+$. But $y>x$, therefore $y\in C_x^+$, therefore $y\in\tau C_x^+$. ■

Although $y\in C_x^+ \Leftrightarrow x\in C_y^-$ (Corollary I.2.4), it does *not* follow that

$$y\in\tau C_x^+ \Leftrightarrow x\in\tau C_y^-,$$

for it is conceivable that, for some y, τC_y^- may be empty. That is, statements about τC_x^+ and τC_y^- are unrelated. However, we have the following:

I.3.8. Proposition. *The following statements are equivalent:*
1. $y\in\tau C_x^+ \Leftrightarrow x\in\tau C_y^-$;

2a. $y\in\tau C_x^+ \Rightarrow \forall l\ni y\ \exists z\in l_y^-,\ z\neq y$, *such that* $z>x$; *and*
2b. $x\in\tau C_y^- \Rightarrow \forall l\ni y\ \exists z\in l_x^+,\ z\neq x$, *such that* $z<y$.

Proof.
1 ⇒ 2: If $y\in\tau C_x^+ \Rightarrow x\in\tau C_y^-$, then (from the definition of τC_y^-) $\forall l\ni y$, $\exists z\in l_y^-$ such that $z>x$, i.e. 2a holds. Similarly, if $x\in\tau C_y^- \Rightarrow y\in\tau C_x^+$, it follows that 2b holds.

2 ⇒ 1: Assume that $y\in\tau C_x^+$, and $\forall l\ni y\ \exists z\in l_y^-, z\neq y$, such that $z>x$. Then $x\in\tau C_y^-$. The same holds with the order reversed. ■

I.3.9. Definition. An ordered space M will be said to have the *property S* (from symmetry) if

$$y\in\tau C_x^+ \Leftrightarrow x\in\tau C_y^-.$$

In this case M will be called an *S-space*, and we shall write

$$y \gg x \quad \text{iff} \quad y\in\tau C_x^+.$$

I.3.10. Theorem. *In an S-space the relation $\gg$ defines a partial order.*

Proof. Transitivity is a consequence of Lemma I.3.2. ■

I.3.11. Remark. Note that this partial order is neither reflexive nor antisymmetric. Also, if we define $x\ll y$ iff $x\in\tau C_y^-$, then, from Proposition I.3.8, $x\ll y \Leftrightarrow y\gg x$.

II. Local Structure and Topology

II.1. Preliminary Remarks. There are many procedures for defining a topology on a space M. We select two for attention. In the first case, one may be interested in a specific family of functions defined on M. It is then often reasonable to consider the coarsest (weakest) topology on M which makes each of these functions continuous. In the second case, one may wish the space M to have strong homogeneity properties. Then M should be "glued together" from copies of the same object.

We cannot quite pursue the first of these, because we are not given a family of functions on M. However, we may ask whether, on an ordered topological space, there is some sense in which order is continuous. If there is, is the coarsest topology which makes order continuous a useful one? Regarding the continuity of the order "$<$" the following definition has been given for the ordered real line [9]: The order "$<$" is continuous if, given $x<y$, $x \neq y$, there exist neighbourhoods U of x and V of y such that $u<v$ whenever $u \in U$ and $v \in V$. And indeed, on $\mathbb{R}$ it turns out that the order topology (which coincides with the usual topology) is the coarsest topology which makes the order continuous.

Unfortunately, this definition fails to provide a physically acceptable topology in higher dimensions. To see this, consider two-dimensional Minkowski space, and two distinct point x and y on any light ray, with $x<y$. With the usual topology of this space, every neighbourhood of y contains points which are spacelike to x, and vice versa; the requirement $u<v$ whenever $u \in U$ and $v \in V$, with U and V defined as above, cannot be met. In other words, with the usual topology, and with the above definition of the continuity of order, order is not continuous!

We have therefore chosen to try to define the order topology in a different manner, more reminiscent of the way in which differentiable manifolds are defined. That is, we make a distinction between the local and the global, and demand that the order structure satisfy a rather strong local, or homogeneity, condition. The only way to determine a satisfactory condition is by experimentation. The choice which we have made, and its consequences, are presented in the following.

II.2. D-Sets. We wish to embed every point in a surrounding which has all the desirable properties. The paradigm for this surrounding is the interior of a double cone $(\operatorname{Int} C_x^+ \cap C_y^-)$ in Minkowski space. The definition, arrived at after some experimentation, is given below. An important property of a D-set is that it is an S-space (Def. I.3.9). This, and a number of other results which may not hold globally, hold in D-sets. The most important of these are established in this section.

II.2.1. Definition. A subset U of M will be called a *D-set* (from the German „Durchschnitt") *iff it fulfills the following conditions:*

a) $x, y \in U \Rightarrow I[x,y] \subset U$.

b) For every $x \in U$ and every $l \ni x$ $\exists p, q \in l \cap U$ such that $p <^l x <^l q$ and $p \neq x$, $q \neq x$.

c) If $x, y \in U$, $y \in C_x^+$, and $l \ni y$, then

$$l \cap \beta C_x^+ \neq \emptyset,$$

and the same for reversed order.

d) If $x \in U$ and $l \cap \beta C_x^+$ contains two distinct points, then

$$l \cap C_x^+ \cap U \subset \beta C_x^+,$$

and the same for reversed order.

e) If $x, y \in U$ and $\lambda(x, y)$, then the ray $l_{x,y}$ is unique.
f) If $x \in U$, then there pass at least two distinct light rays through x. □

Observe that the empty set $\emptyset$ satisfies all of these conditions trivially, and is therefore a D-set.

II.2.2. Lemma. *Let U be a D-set, and let $x, y \in U$ with $y \gg x$. Let $l_y \ni y$ be a light ray through y. Then*

$$l_y \cap \beta C_x^+$$

is a singleton, and the same with order reversed.

Proof. By Proposition I.3.7, $y \gg x$ is equivalent to $\beta C_x^+ \cap C_y^+ = \emptyset$. Hence if $l_y \cap \beta C_x^+$ is nonempty then

$$l_y \cap \beta C_x^+ = l_y^- \cap \beta C_x^+ \subset I[x, y] \subset U .$$

Then, if $l_y^- \cap \beta C_x^+$ contained two distinct points, it would follow from Def. II.2.1 (d) that

$$l_y \cap C_x^+ \cap U \subset \beta C_x^+ ,$$

contradicting the assumption $y \gg x$. The same argument holds with order reversed. ■

II.2.3. Proposition. *Every D-set has the property S.*

Proof. Let $x, y \in U$ with $y \gg x$ $(y \in \tau C_x^+)$. Let $l_y \ni y$. Then, by Lemma II.2.2, $l_y^- \cap \beta C_x^+$ is a unique point, say z. If $z \neq x$ it follows that $x \ll y (x \in \tau C_y^-)$; if $z = x$ then $\exists$ a point $p \in l_y$ such that $z < p < y$, $p \neq z$, $p \neq y$. ■

The next two results show that there are plenty of D-sets.

II.2.4. Proposition. *The intersection of two D-sets is a D-set.*

Proof. Let U_1, U_2 be D-sets. If $U_1 \cap U_2 = \emptyset$ then it is trivially a D-set. Assume now that $U_1 \cap U_2 \neq \emptyset$. Property a), and properties c) to f) of Def. II.2.1 are stable under finite intersections. It remains to show that Property II.2.1 (b) continues to hold.

Let then $x \in U_1 \cap U_2$ and $l_x \ni x$. Then $\exists p_i, q_i \in U_i$, $i = 1, 2$, such that $p_i, q_i \in l_x$, $p_i <^l x <^l q_i$, $p_i \neq x$, $q_i \neq x$, $i = 1, 2$. Since l_x is linearly ordered it follows that for proper choice of indices a, b, m, n $(=1$ or $2)$, $p_a > p_b$ and $q_m < q_n$. Then $p_b < x < q_n$, $p_b \neq x$, $q_n \neq x$, and $p_b, q_n \in U_1 \cap U_2$. ■

II.2.5. Proposition. *Let U be a D-set, $x, y \in U$ and $y \gg x$. Then $I(x, y)$ is a D-set.*

Proof. We have to verify properties a)–f) of Def. II.2.1.
a) Let $p, q \in I(x, y)$. By Lemma I.3.2, $C_p^+ \subset \tau C_x^+$ and $C_q^- \subset \tau C_y^-$. Hence $I[p, q] \subset I(x, y)$.
b) Take $p \in I(x, y)$ and $l \ni p$. Then by Lemma II.2.2 the sets $l \cap \beta C_x^+ \equiv \{r\}$ and $l \cap \beta C_y^- \equiv \{s\}$ are singletons, differing from $\{p\}$. Therefore the open segment $l(r, s)$ belongs to $I(x, y)$, and condition II.2.1 (b) is fulfilled by Axiom I.1.1.
c)–f) These are obviously true of order-convex subsets of a D-set. ■

Recall that a subset S of a partially ordered set x is called *order-convex* if $a, c \in S$, $a < c \Rightarrow b \in S$ whenever $a < b < c$.

II.2.6. Proposition. *Let U be a D-set, $a, b, c \in U$. Then*
i) $a < b \ll c \Rightarrow a \ll c$.
ii) $a \ll b < c \Rightarrow a \ll c$.

Proof. i) $a < b$ implies that there is an ascending polygon from a to b. $b \ll c$ implies that, given $l \ni c$, there is an ascending polygon from b to c which meets l at a point x, $x <^l c$, $x \neq c$. Therefore, given $l \ni c$, there is an ascending polygon from a to c which meets l at the point $x, x \neq c$. Hence $a \ll c$.
ii) By Lemma II.2.3, U has the property S. Hence $b \ll c \Leftrightarrow c \gg b$. Hence ii) is equivalent to $c \gg b > a \Rightarrow c \gg a$. The proof of this is the same as that of i) above, with order reversed. ∎

II.2.7. Lemma. *Let U be a D-set, $y \in U$, $p \in \beta C_y^- \cap U$. Then $q \gg p, q \in U \Rightarrow q \notin C_y^- \cap U$.*

Proof. There are three possibilities: (a) $q \notin C_y^- \cap U$; (b) $q \in \beta C_y^- \cap U$ and (c) $q \in \tau C_y^-$. If $q \in \beta C_y^- \cap U$ then $q < y$. But $p \ll q < y$ implies, by Proposition II.2.6, that $p \ll y$, a contradiction. If $q \in \tau C_y^- \cap U$, then $p \ll q \ll y$, i.e. $p \ll y$, the same contradiction. The only possibility which remains is $q \notin C_y^- \cap U$. ∎

II.3. Perpendicularity of Light Rays. We have now to establish that D-sets have the required features. Chief among these is that, in a D-set, "good" properties can be "transported" from one cone to another by means of l-polygons. A consequence is that D-sets enjoy the required separation properties. However, in the attempt to establish these properties we encounter technical difficulties stemming from the fact that not all points on $\beta C_x^+ \setminus \{x\}$ may be connected to x by a single light ray.

These difficulties are resolved by the careful exploitation of some of the many natural maps between segments of light rays. One of these gives rise to a particularly useful incidence relation – called *perpendicularity* in the sequel – between pairs of light rays. In the following we shall define this relation and obtain its most important consequences.

II.3.1. Definition. Let U be a D-set, $x, y \in U$, $y \gg x$, $l_x \ni x$ and $l_y \ni y$. Set

$$l_x \cap \beta C_y^- = \{p\} \quad \text{and} \quad l_y \cap \beta C_x^+ = \{q\}.$$

Let

$$r \in l_x, \quad x < r < p, \quad r \neq p;$$

$$s \in l_y, \quad q < s < y, \quad s \neq q.$$

Define the maps

$$\varrho : l_x[x, p) \to l_y$$

and

$$\sigma : l_y(q, y] \to l_x$$

by

$$\varrho(r) = l_y \cap \beta C_r^+$$

and

$$\sigma(s) = l_x \cap \beta C_s^-$$

respectively. These maps are well-defined, since the right-hand sides of the above are unique points. □

II.3.2. Proposition. *The maps ϱ and σ of Def. II.3.1 are order-preserving.*

Proof. Consider ϱ. The relation $r<r'$ implies $C_{r'}^{+}\subset C_{r}^{+}$, and therefore

$$l_y[\varrho(r),y)\supset l_y[\varrho(r'),y],$$

which implies $\varrho(r)<\varrho(r')$. Similarly for σ. ■

II.3.3. Theorem. α) *The maps ϱ and σ of Def. II.3.2 are either constant, with*

$$\varrho(r)=q \quad \textit{and} \quad \sigma(s)=p,$$

or:

β) *They are both one-to-one. In this case ϱ extends uniquely to $l_x[x,p]$, σ extends uniquely to $l_y[q,y]$, and the extended maps are inverses of each other.*

Proof. a) Let r, $r'\in l_x[x,p)$, $r\neq r'$, $r<r'$. Assume that
i) $\varrho(r)=\varrho(r')=t$,
ii) $t\neq q$.
Then $r,r'\in\beta C_t^{-}$, and since $x\in l_{r,r'}$ it follows from Def. II.2.1 (d) that $x\in\beta C_t^{-}$. Since U is an S-space, it follows that $t\in\beta C_x^{+}$. But $l_y\cap\beta C_x^{+}$ is a unique point, therefore $t=q$, a contradiction. Hence, either
i) $\varrho(r)=q$ $\forall r\in l_x[x,p)$ (in which case $t=q$), or
ii) $\varrho(r)\neq\varrho(r')$ for $r\neq r'$, i.e. ϱ is one-to-one.
b) Similarly, either $\sigma(s)=p$ $\forall s\in l_y(q,y]$, or else σ is one-to-one.
c) Let $\varrho(r)=q\forall r\in l_x[x,p)$. Then $\beta C_r^{+}\cap l_y\neq\{s\}$ $\forall s\in l_y(q,y]$, i.e.

$$\beta C_s^{-}\cap l_x\neq r \;\; \forall r\in l_x[x,p),$$

therefore $\beta C_s^{-}\cap l_x=\{p\}$, i.e. $\sigma(s)=p$ $\forall s\in l_y(q,y]$.
d) The above proves that $\varrho(r)=q\Rightarrow\sigma(s)=p$. By reversing the order of the argument we obtain $\sigma(s)=p\Rightarrow\varrho(r)=q$. Hence, if ϱ is one-to-one, so is ϱ, and conversely. The required extensions are $\varrho(p)=y$, $\sigma(q)=x$, and the extended maps are clearly inverses of each other. ■

II.3.4. Definition. The light rays l_x and l_y will be said to be *perpendicular* to each other, written $l_x\perp l_y$ or $l_y\perp l_x$, if the maps ϱ and σ of Def. II.3.1 are constant maps. □

Note that the symmetry of the perpendicularity relation is established in Theorem II.3.3.

II.3.5. Remark. Perpendicularity is a property of *pairs of light rays,* and not of pairs of points x, y on them. This may be seen as follows. Let $l_x\perp l_y$ as defined above. Let $x'\in l_x\cap U$, $x'<x$, $x'\neq x$. Then $\varrho(x')=C_{x'}^{+}\cap l_y=\{q\}$ (same argument as in proof of Theorem II.3.3). Now let $y'\in l_y\cap U$, $y'>y$, $y'\neq y$. Then $\sigma(y')=C_{y'}^{-}\cap l_x=\{p\}$. Thus ϱ maps the entire segment $(l_p^{-}\setminus\{p\})\cap U$ into $\{q\}$, and σ maps $(l_q^{+}\setminus\{q\})\cap U$ into $\{p\}$. □

The following theorem is fundamental to the succeeding considerations.

II.3.6. Theorem. *Let U be a D-set, $x, y \in U$, $y \gg x$ and $l_x \ni x$. Then there exists $l_y \ni y$ with $l_y \not\perp l_x$ (read: l_y not perpendicular to l_x).*

Proof. Let $\{p\} = \beta C_y^- \cap l_x$. For any ray l_y through y such that $l_y \perp l_x$, and any y' on l_y such that $y' \in l_y(q, y]$, where $q = l_y \cap C_x^+$, we have

$$\beta C_{y'}^- \cap l_x = \{p\}.$$

If this were true for every $l_y \ni y$, then it would follow (from the definition of τC_y^-) that $p \in \tau C_y^-$, which would contradict $p \in \beta C_y^-$. ■

We are now in a position to prove that, if there exists a pair of timelike points (in an l-connected D-set), then there exist enough of them. This is accomplished in steps in the remainder of this section.

II.3.6. Lemma. *Let U be a D-set, $x, y \in U$ and $y \gg x$. Let $l_y \ni y$. Choose $l_x \not\perp l_y$ and define $\{q\} = \beta C_x^+ \cap l_y$. Then for $w \in l_x^- \backslash \{x\}$ we have*

$$q \gg w.$$

Proof. According to Def. I.3.1 we have to establish that, for any $l_w \ni w$, $\exists u \in l_w^+ \backslash \{w\}$ such that $q \in C_u^+$.

a) Let $l_w \perp l_y$. Then, setting

$$\{u\} = l_w \cap \beta C_y^- = l_w \cap \beta C_q^-$$

we have

$$q \in C_u^+.$$

b) Let $l_w \not\perp l_y$. If $l_w = l_x$ then obviously $q \in C_x^+$. If $l_w \neq l_x$ then define v and r by

$$\{v\} = \beta C_w^+ \cap l_y \quad \text{and} \quad \{r\} = \beta C_y^- \cap l_w.$$

Now there is a (1, 1) map

$$\varrho : l[w, r] \to l[v, y].$$

Choose $s \in l(v, q]$ and set $u = \varrho^{-1}(s)$. Then $u \in l_w^+$, $u \neq w$, and $q \in C_u^+$. ■

II.3.7. Theorem. *Let U, V be D-sets, where U is l-connected and $V \subset U$.*
i) Let $x \in V$ and $y \in U$ with $y \gg x$. Then $\exists w \in V$ with $x \gg w$.
ii) The same, with order reversed.

Proof. It suffices to prove part i). Since $y \gg x$ and U is l-connected, there exists a decreasing l-polygon in U

$$y = z_0 > z_1 > z_2 > \ldots > z_n = x,$$

with

$$\lambda(z_i, z_{i+1}), \quad i = 0, 1, \ldots, n-1.$$

We shall construct in V a decreasing l-polygon

$$x = w_0 > w_1 > w_2 > \ldots > w_n,$$

with

$$\lambda(w_i, w_{i+1}), \quad i = 0, 1, \ldots, n-1$$

and

$$z_i \gg w_i, \quad i=0,1,\ldots,n.$$

The construction is by induction. For $i=0$ we have $z_0 \gg w_0$ by assumption. Suppose that we have constructed w_i, $i=1,2,\ldots,k$, as required. Since $z_k \gg w_k$, the intersection

$$l_{z_k, z_{k+1}} \cap \beta C^+_{w_k} \equiv \{q_{k+1}\}$$

is a unique point q_{k+1}. Since $z_{k+1} > x$ and, by the induction hypothesis, $x > w_k$, it follows that

$$z_{k+1} > q_{k+1}.$$

Now choose a light ray $l_{w_k} \perp l_{z_k, z_{k+1}}$, and then $w_{k+1} \in (l^-_{w_k} \setminus \{w_k\}) \cap V$. Then Lemma II.3.6 gives

$$w_{k+1} \ll q_{k+1} < z_{k+1},$$

i.e.

$$w_{k+1} \ll z_{k+1}.$$

Finally, set $w_n = w$. Then $w \in V$ and $x \gg w$. ■

After this preparation, we are able to establish the main result of this section:

II.3.8. Theorem. *Let U, V be D-sets, where U is l-connected and $V \subset U$. Let $x \in V$ and $y \in U$ with $y \gg x$. Then we can find points u, v, w with $u, v \in V$, $w \in U$ such that*

$$u \ll x \ll v \ll y \ll w.$$

Proof. a) Apply Theorem II.3.7(i) to $y \gg x$ to obtain a point $u \in V$ such that $u \ll x$.
b) Let $V = U$. Apply Theorem II.3.7(ii) (order reversed!) to $y \gg x$ to obtain a point $w \in U$ with $w \gg y$.
c) By Proposition II.2.5 and II.2.4, $I(u,y)$ and $I(u,y) \cap V \equiv W$ are D-sets. Since $x \in W$, we may apply Theorem II.3.7(ii), with $V = U = W$, to the pair $u \ll x$ to obtain a point $v \in W = V$, $v \gg x$. ■

In the next section we shall see how the above property can be transported outside C_x.

II.4. Separation Properties. The main result of this section is Theorem II.4.5, which states that distinct points in a D-set can, under certain circumstances, be separated by disjoint D-subsets. The required circumstances are that the original D-set be l-connected, and that there should exist a pair of timelike points in it. The latter does not follow from our axioms, and has to be imposed as a regularity condition. The proofs utilize a method for transporting "good" properties of one cone to other cones.

II.4.1. Proposition. *Let U be an l-connected D-set, $x, y, z \in U$ and $y \gg x$. Then there exist $p, q \in U$ with $p \ll z \ll q$.*

Proof. Since U is l-connected and $x, z \in U$, $\exists$ an l-polygon

$$x = z_0, z_1, \ldots, z_n = z$$

in U connecting x with z. That is, $\lambda(z_i, z_{i+1})$, $i=0,1,\dots,n-1$. By Theorem II.3.7 there exist $p_0, q_0 \in U$ with $p_0 \ll z_0 = x \ll q_0$. Assume that we have constructed, successively, the points $p_1, q_1, \dots, p_k, q_k \in U$, $k<n$, with $p_i \ll z_i \ll q_i$, $i=1,2,\dots,k$. Since $\lambda(z_k, z_{k+1})$, either $z_{k+1} > z_k$ or $z_{k+1} < z_k$. In the first case we have $p_k \ll z_{k+1}$. We may then choose $p_{k+1} = p_k$ and construct q_{k+1} by Theorem II.3.7. In the second case we have $q_k \gg z_k > z_{k+1}$. We may therefore choose $q_{k+1} = q_k$ and construct $p_{k+1} \ll z_{k+1}$ by Theorem II.3.7. This establishes the inductive step, and therefore the result. ■

In words, if there exists a pair of timelike points in an l-connected D-set U, then every point in U has a timelike predecessor and a timelike successor in U.

However, our axioms so far do not guarantee the existence of a pair of timelike points in a D-set. Hence we make the following definition:

II.4.2. Definition. A D-set U will be called *regular* iff it satisfies the following conditions:
a) U is l-connected.
b) There exist $x, y \in U$ with $x \ll y$. □

The property of l-connectedness is not hereditary; property (b) is, for D-sets; more precisely:

II.4.3. Corollary. *Every l-connected nonempty D-subset V of a regular D-set U is regular.*

Proof. Let $x \in V$. Then $x \in U$, so that by Proposition II.4.1, $\exists z \in U$ with $z \gg x$. Then by Theorem II.3.8 $\exists p, q \in V$ such that $p \ll x \ll q$. Hence V is regular. ■

The key result of this section is the following theorem separating points by positive and negative cones:

II.4.4. Theorem. *Let U be a regular D-set, $x, y \in U$ such that $y \gg x$, and $b \in U \backslash C_y^-$. Then $\exists a \in U \backslash C_y^-$ such that $b \gg a$.*

Proof. There are three possibilities, according to the location of b. They are:
1. $b \gg y$.
2. $b \in \beta C_y^+$.
3. $b \notin C_y^+$.
We establish the existence of the point a case-by-case.

1. By Theorem II.3.8, $\exists$ a point a such that $b \gg a \gg y$. The second condition means that $a \in U \backslash C_y^-$.
2. Let $b \in \beta C_y^+$. Then either (a) $\lambda(y, b)$ or (b) $\sim\lambda(y, b)$.

a) Suppose that $\lambda(y, b)$. By Theorem II.3.6, $\exists l_x \ni x$ such that $l_x \pm l_{y,b}$. Since $x \ll y$, $x \ll b$, l_x^+ intersects βC_y^- at a unique point r and βC_b^- at a unique point s. If $r = s$ then $l_x \perp l_{y,b}$, hence $r \neq s$, and moreover $s > r$. Take $a \in l(r, s)$. Then by Lemma II.3.6, $a \ll b$, and by construction $a \in U \backslash C_y^-$.
b) Suppose now that $\sim\lambda(y, b)$. There exists an ascending l-polygon from y to b

$$y = b_0, b_1, \dots, b_n = b,$$

$$\lambda(b_i, b_{i+1}), \quad b_i < b_{i+1}, \quad b_i \in \beta C_y^+, \quad i=0,1,\dots,n-1.$$

Now there exists a ray $l_x \ni x$ such that $l_x \pm l_{b_0, b_1}$. Let

$$\{r\} = l_x^+ \cap C_y^-, \qquad \{s\} = l_x^+ \cap C_{b_1}^-.$$

Then $r < s$, $r \neq s$. Choose $a \in l(r, s)$. Then $a \ll b$ by Lemma II.3.6, and $a \in U \setminus C_y^-$ by construction.

3. Finally, let $b \notin C_y^+$. As $b \notin C_y^-$ by assumption, this means $b \notin C_y$. Since U is a regular D-set, τC_b^- is nonempty. There exists r such that

$$r \in \tau C_b^- \quad \text{but} \quad r \notin \tau C_y^-.$$

For if $r \in \tau C_b^- \Rightarrow r \in \tau C_y^-$, then $\tau C_b^- \subset \tau C_y^-$, i.e. $b \in C_y^-$, a contradiction. If $r \notin \beta C_y^-$, set $a = r$. If $r \in \beta C_y^-$, apply Lemma II.2.7 to obtain a such that $r \ll a \ll b$. ■

From the above separation we are now able to construct a separation by regular D-sets:

II.4.5. Theorem. *If U is a regular D-set, $x, y \in U$, $x \neq y$ then there exist regular D-subsets V and W of U such that $x \in V$, $y \in W$, and $V \cap W = \emptyset$.*

Proof. Since $x \neq y$ one cannot have both $y \in C_x^+$ and $y \in C_x^-$ simultaneously. We may therefore assume, without loss of generality, that $y \notin C_x^-$. By Theorem II.4.4 $\exists p \in U$ such that $p \ll y$ and $p \notin C_x^- \cap U$. But then $x \notin C_p^+$. Applying Theorem II.4.4 with order reversed to x and C_p^+, $x \notin C_p^+$, we see that $\exists q \in U$ such that $q \gg x$ and $q \notin C_p^+ \cap U$. Then $C_q^- \cap C_p^+ = \emptyset$. By Theorem II.3.7, we see that $\exists$ points $p', q' \in U$ such that $p \ll y \ll p'$, $q \gg x \gg q'$. Then $x \in I(q', q)$, $y \in I(p, p')$, $I(q', q) \cap I(p, p') = \emptyset$, and $I(q, q')$, $I(p, p')$ are regular D-subsets of U. ■

II.5. Local Structure and Topology. In the preceding sections we have defined D-sets and have established their fundamental properties. They lead us very naturally to the local structure axiom and the topology.

II.5.1. Axiom (Local Structure Axiom). *The ordered space M satisfies the following axiom: For each $x \in M \;\exists$ a regular D-set U_x such that $x \in D_x \subset M$.* □

II.5.2. Definition. The *order topology* on M is defined to be that topology which has the family of regular D-subsets as a base. □

II.5.3. Remarks. 1. It follows from Proposition II.2.4 (the intersection of two D-sets is a D-set) and Corollary II.4.3 (every l-connected nonempty D-subset of a regular D-set is itself regular) that the family of regular D-subsets is indeed a base for a topology.

2. Theorem II.4.5 now states that the order topology is Hausdorff.

II.5.4. Theorem. *In every D-set, $\tau C_x^+ = \operatorname{Int} C_x^+$ (the interior of C_x^+) and $\beta C_x^+ = \partial C_x^+$ (the boundary of C_x^+), and the same for reversed order.*

Proof. Only the second assertion needs to be proven. If $y \notin C_x^+$, then y is separated from C_x^+ by an open set, hence $y \notin \operatorname{Cl} C_x^+$ (the closure of C_x^+). Hence $\operatorname{Cl} C_x^+ = C_x^+$. Therefore $\partial C_x^+ = C_x^+ \setminus \operatorname{Int} C_x^+ = \beta C_x^+$. ■

II.5.5. Remark. The order topology introduced above clearly coincides with the standard topology on $\mathbb{R}^4$ in Minkowski space, and is therefore strictly coarser than the "fine topology" for Minkowski space introduced by Zeeman [12]. By itself, the order topology does not imply a "causal" or a linear structure on M.

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Note added in proof: After this manuscript was submitted for publication, we came across the article by J. Schröter [N 1]. We would like to take this opportunity for clarifying the relationship between our work and the analysis of the mathematical structures of the space-time of general relativity which was started by Hermann Weyl [N 2] and continued by Synge [N 3, N 4], Ehlers, Pirani, and Schild [N 5], Woodhouse [N 6] and most recently by Schröter (loc. cit.). We are attempting to put the order structure on an axiomatic basis, and hope to end where the abovementioned works [N 1, N 5, N 6] begin.

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