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Multiple Forced Oscillations for the N-Pendulum Equation

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Abstract. We consider the periodically forced N-pendulum equation. Forced oscillations are obtained, and their multiplicity is studied in terms of the mean value of the forcing term.

Introduction

Let m_i, l_i be positive constants, $i=1,\dots,N$, and set $M_j=\sum_{i=j}^{N} m_i$, $j=1,\dots,N$. If $\theta=(\theta_1,\dots,\theta_N)$, $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$, then the Lagrangian

$$\mathscr{L}(\theta,\xi,t)=\frac{1}{2}\sum_{i,j=1}^{N} M_{\max(i,j)}\, l_i l_j \cos(\theta_i-\theta_j)\xi_i\xi_i + g\sum_{j=1}^{N} M_j l_j \cos\theta_j + \sum_{j=1}^{N} f_j(t)\theta_j$$

($g=$ constant of gravitation)

corresponds to the mechanical system of N coplanar penduli with masses m_k, length l_k subject to the forcing terms $f_k=f_k(t)$, $k=1,\dots,N$. (Here θ_k is the angle of the k-pendulum with the vertical.) As is well known, the corresponding equations of motion are:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\xi_j}(\theta,\dot\theta,t)-\frac{\partial\mathscr{L}}{\partial\theta_j}(\theta,\dot\theta,t)=0,\quad j=1,\dots,N. \tag{0.1}$$

Assuming the forcing terms are T-periodic (i.e. $f_k(t+T)=f_k(t)\ \forall t\in\mathbb{R}\ \forall k=1,\dots,N$) we are interested in finding T-periodic solutions for (0.1).

Notice that this problem admits a natural $\mathbb{Z}^N$ symmetry, in the sense that if $\theta=\theta(t)$ is a T-periodic solution for (0.1) so is $\theta(t)+2\pi k\ \forall k\in\mathbb{Z}^N$. Therefore we shall call *distinct* the solutions of (0.1) whose difference does not belong to $2\pi\mathbb{Z}^N:=\{2\pi k\ \forall k\in\mathbb{Z}^N\}$. As pointed out by many authors (e.g. [5, 3, 1]), if the forcing terms f_k have *zero* mean value (i.e. $\int_0^T f_k=0$, $k=1,\dots,N$) then this symmetry is preserved by the variational principle associated to (0.1). Namely, T-periodic solutions of

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(0.1) correspond to critical points of a suitable functional (bounded from below) well defined in the Hilbert manifold $M = T^N \times E$, where T^N is the N-dimensional torus and

$$E = \left\{ \theta = (\theta_1, \ldots, \theta_N) : \theta_k \in H^1([0, T]),\ \theta_k(0) = \theta_k(T) \text{ and } \int_0^T \theta_k = 0,\ k = 1, \ldots, N \right\}.$$

Hence by means of the Ljusternik–Schnirelman theory, the rich topology of M guarantees that problem (0.1) with $\int_0^T f_k = 0$ admits at least $N + 1$ $(= 1 + \text{cup length } T^N)$ T-periodic solutions (see [4] and [5]).

In addition, if all the T-periodic solutions of (0.1) are "nondegenerate," then by Morse theory one obtains the existence of at least 2^N (= sum of Betti numbers of T^N) of them (see e.g. [2]).

However, Fournier–Willem [3] for the double pendulum and subsequently Chang–Long–Zehnder [1] for the N-pendulum have pointed out that if reasonable conditions are satisfied by the data then it is always possible to guarantee 2^N distinct T-periodic solutions for (0.1).

The purpose of this note is to give an appropriate extension of this result to the case of forcing terms whose mean value is not necessarily equal to zero. This new situation appears much more delicate since Eqs. (0.1) impose strong restrictions on these mean values. In fact, summing up Eqs. (0.1) and integrating in $[0, T]$ one easily sees that a necessary condition for (0.1) to have a T-periodic solution is that there exists $\tau_k \in [0, 2\pi), k = 1, \ldots, N$, such that

$$\sum_{k=1}^N \frac{1}{T} \int_0^T f_k(t)dt = g \sum_{k=1}^N M_k l_k \sin \tau_k. \tag{0.2}$$

In particular, the sum of the mean values of the f_k's cannot be arbitrarily large.

Here we shall treat the case where only *one* of the forcing terms, say f_N, is not restricted to have mean value zero. Thus given $f_j = f_j(t)$ T-periodic functions with $\int_0^T f_j = 0, j = 1, \ldots, N$ and $c \in \mathbb{R}$; after performing explicit calculations in (0.1), we are led to the following problem:

$$(1)_c \quad \begin{cases} \dfrac{d}{dt}\left(M_j l_j^2 \dot\theta_j + \sum\limits_{k=1}^{j-1} M_j l_j l_k \cos(\theta_j - \theta_k)\dot\theta_k + \sum\limits_{k=j+1}^{N} M_k l_k l_j \cos(\theta_k - \theta_j)\dot\theta_k \right) \\ \quad + \sum\limits_{k=1}^{j-1} M_j l_j l_k \sin(\theta_i - \theta_k)\dot\theta_k\dot\theta_j - \sum\limits_{k=j+1}^{N} M_k l_k l_j \sin(\theta_k - \theta_j)\dot\theta_k\dot\theta_j \\ \quad + g M_j l_j \sin\theta_j = f_j(t), \quad j = 1, \ldots, N-1 \\ \text{and} \\ \dfrac{d}{dt}\left(M_N l_N^2 \dot\theta_N + \sum\limits_{k=1}^{N-1} M_N l_N l_k \cos(\theta_N - \theta_N)\dot\theta_k \right) + \sum\limits_{k=1}^{N-1} M_N l_N l_k \sin(\theta_N - \theta_k)\dot\theta_k\dot\theta_j \\ \quad + g M_N l_N \sin\theta_N = f_N(t) + c; \quad \theta(0) = \theta(T); \quad \dot\theta(0) = \dot\theta(T). \end{cases}$$

Notice that the variational principle associated to $(1)_c$ with $c \neq 0$ gives rise to an unbounded functional multivalued in $M = T^N \times E$ (unlike the case $c = 0$). So Ljusternick–Schnirelman type arguments as employed in [5, 1 and 3] cannot be applied any longer.

Our approach instead consists in reducing $(1)_c$ to a finite-dimensional variational problem. More precisely, we shall see how to obtain solutions for $(1)_c$ once we obtain critical points of a smooth function of the type:

$$G_c(\theta_1, \dots, \theta_N) = g(\theta_1, \dots, \theta_N) + c\theta_N \tag{0.3}$$

with $g(\theta + 2\pi k) = g(\theta) \quad \forall k \in \mathbb{Z}^N$.

Thus by exploiting the special structure of g, roughly speaking, we shall prove that if $\|f_k\|_\infty$ is not too large (see condition $(f)_1$ below) $(k = 1, \dots, N)$, then there exists $T^* > 0$ such that for every $0 < T < T^*$ there exist constants $d_1 \leqq \dots \leqq d_{2^{N-1}} < 0 < D_{2^{N-1}} \leqq \dots \leqq D_1$ with the following property:

i) if $c \in (d_j, D_j)$ for some $j = 1, \dots, 2^{N-1} \mapsto (1)_c$ admits at least $2j$ distinct solutions;
ii) if $c = d_j = \dots = d_{j+r-1}$, $1 \leqq j + r \leqq 2N^{-1}$ and for $j > 1$, $c > d_{j-1} \mapsto (1)_c$ admits at least $2(j-1) + r$ distinct solutions;
iii) if $c = D_j = \dots = D_{j+r-1}$ and for $j > 1$, $c < D_{j-1} \mapsto (1)_c$ admits at least $2(j-1) + r$ distinct solutions. ■

Such restrictions on c were expected by (0.2).

In conclusion, we point out that the problem of finding critical points for functions of the type (0.3) is interesting by itself. Various interesting phenomena were observed in [6]. For example, despite the fact that g always admits at least $N + 1$ critical points, it may still happen that $g(\theta_1, \dots, \theta_N) + c\theta_N$ has no critical points for every $c \neq 0$. We refer to [6] for concrete examples and further discussions.

1. Statement of the Result

Given $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$, set $a_{i,j}(\theta) = l_i l_j M_{\max(i,j)} \cos(\theta_i - \theta_j)$, $i, j = 1, \dots, N$. Thus $A = A(\theta) = (a_{i,j}(\theta))_{i,j=1,\dots,N}$ defines a symmetric positive definite $N \times N$ matrix, and for $\theta, \xi \in \mathbb{R}^N$, $t \in \mathbb{R}$ we have:

$$\mathscr{L}(\theta, \xi, t) = \tfrac{1}{2} A(\theta)\xi \cdot \xi + g \sum_{j=1}^{N} M_j l_j \cos\theta_j + f(t) \cdot \theta,$$

where $f(t) = (f_1(t), \dots, f_n(t))$ and $\cdot$ denotes the usual scalar product in $\mathbb{R}^N$.

Let $\lambda_0 > 0$ be the ellipticity constant for $A(\theta)$, i.e

$$A(\theta)\xi \cdot \xi \geqq \lambda_0 |\xi|^2 \quad \forall \xi, \theta \in \mathbb{R}^N.$$

Introduce

$$v_{j,1} = \begin{cases} 0 & j = 1 \\ M_j \max\limits_{1 \leqq k \leqq j-1} l_k & 2 \leqq j \leqq N \end{cases}; \qquad v_k = \max\{v_{k,1}, v_{k,2}\},$$

$$v_{j,2} = \begin{cases} 0 & j = N \\ \max\limits_{j+1 \leqq k \leqq N} M_k l_k & 1 \leqq j \leqq N-1 \end{cases};$$

and denote with $\| \quad \|_\infty$ the L^∞ norm. We obtain

Theorem 1. *Let $f(t)=(f_1(t),\dots,f_N(t)$ be a continuous T-periodic function with*

$$(\mathrm{f})_1 \quad \int_0^T f=0; \quad \|f\|_\infty < g\frac{\lambda_0 M_k}{N\nu_k} \quad \forall k=1,\dots,N\left(\|f\|_\infty := \sum_{j=1}^N \|f_j\|_\infty\right).$$

There exists $T^*>0$ (depending on m_i, l_i, f_i, $i=1,\dots,N$) such that if $0<T<T^*$, then $\exists d_1 \leqq \dots \leqq d_{2^{N-1}} < 0 < D_{2^{N-1}} \leqq \dots \leqq D_1$ with the following property:

i) *if $c\in(d_j, D_j) \mapsto (1)_c$ admits at least $2j$ distinct solutions;*
ii) *if $c=d_j=\dots=d_{j+r-1}$ and for $j>1$, $c>d_{j-1} \mapsto (1)_c$ admits at least $2(j-1)+r$ distinct solutions;*
iii) *if $c=D_j=\dots=D_{j+r-1}$ and for $j>1$, $c<D_{j-1} \Rightarrow (1)_c$ admits at least $2(j-1)+r$ distinct solutions.* ■

To be precise we shall obtain a slightly more general version of Theorem 1 (see Theorem 2) where the assumption $0<T<T^*$ is replaced by more general condition involving m_k, l_k, f_k and T (see $(T)_1, (T)_2$ and $(T)_3$). From those conditions T^* can be explicitly estimated. Similar conditions were introduced in [1 and 3].

In fact, when $c=0$, our result reduces to those obtained in [1 and 3], with the difference that while we require $(\mathrm{f})_1$, our m_i and l_i are basically unrestricted unlike in [1 and 3].

2. Variational Formulation in $\mathbb{R}^N$

Define the Hilbert space:

$$H=\{\theta=(\theta_1,\dots,\theta_N): \theta_j\in H^1([0,T]);\ \theta_j(0)=\theta_j(T), j=1,\dots,N\}$$

equipped with the scalar product

$$(\theta,\phi)=\int_0^T \dot\theta\cdot\dot\phi + \int_0^T \theta\cdot\phi, \quad \theta,\phi\in H.$$

If $\theta\in H$ and $1\leqq p\leqq\infty$ denote by

$$\|\theta\|_p = \sum_{k=1}^N \|\theta_k\|_p \quad \text{with} \quad \|\theta_k\|_p = \left(\int_0^T |\theta_k|^p\right)^{1/p}.$$

Given $f(t)=(f_1(t),\dots,f_N(t))$ with $\int_0^T f=0$ and $c\in\mathbb{R}$ define the functional $I_c: H\to\mathbb{R}$ as follows

$$I_c(\theta)=\frac{1}{2}\int_0^T A(\theta)\theta\cdot\theta + g\int_0^T \sum_{j=1}^N M_j l_j \cos\theta_j + \int_0^T f\cdot\theta + c\int_0^T \theta_N.$$

Straightforward calculations show that for every $s\in\mathbb{N}$, I_c is s-time Frechét differentiable. Furthermore, critical points of I_c are solutions for $(1)_c$ and vice versa. Notice that if $c\neq 0$, I_c is unbounded in H and

$$I_c(\theta+2\pi k)=I_c(\theta)+2\pi k_N cT$$

$\forall\theta\in H$ and $k=(k_1,\dots,k_N)\in\mathbb{Z}^N$. Hence I_c is multivalued in $M=T^N\times E$.

For every $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{N}$ define:

$$\Lambda_\alpha=\left\{\theta=(\theta_1,\ldots,\theta_N)\in H:\frac{1}{T}\int_0^T\theta_j=\alpha_j\right\}.$$

Notice that I_c is bounded from below in Λ_α. We have

Lemma 2.1. *For every $\alpha\in\mathbb{R}^N$ there exists $\theta_\alpha\in\Lambda_\alpha$ such that*

$$I_0(\theta_\alpha)=\inf_{\theta\in\Lambda_\alpha}I_0(\theta). \tag{2.1}$$

Furthermore $I_c(\theta_\alpha)=I_0(\theta_\alpha)+cT\alpha_N=\inf_{\Lambda_\alpha}I_c(\theta)$.

Proof. Obvious modifications of the arguments given in [1 and 3] show that I_0 satisfies the Palais–Smale (PS) condition in Λ_α. The conclusion then follows by standard arguments. (See Lemata 1.1 and 1.2 of [6].) ■

Set

$$\rho=\sum_{j=1}^N l_j v_j;\quad \lambda=\max_{1\leqq j\leqq N} l_j v_j;\quad M=\max_{1\leqq j\leqq N} l_j M_j$$

and

$$v=\max_{1\leqq j\leqq N}(gM_jl_j+\|f_j\|_\infty). \tag{2.2}$$

The purpose of the next lemma is to derive estimates for $\dot\theta_\alpha$ uniformly in α.

Lemma 2.2. *For every $\alpha\in\mathbb{R}^N$ and θ_α satisfying $(2.1)_\alpha$ we have:*

i) $\|\dot\theta_\alpha\|_2<AT^{3/2}$ *with* $A=(vN/\lambda_0\pi)$;
ii) $\|\dot\theta_\alpha\|_\infty\leqq Td(T)$ *with* $d(T)=(N/\lambda_0)(\|f\|_\infty+TA(\rho AT+gM))$;
iii) $\|\ddot\theta_\alpha\|_\infty\leqq(N/\lambda_0)[\|f\|_\infty+T^2(\rho d^2(T)+gAM+\lambda A^2)]$.

Proof. Set $\theta_\alpha=(\theta_1^\alpha,\ldots,\theta_N^\alpha)$ and $\theta_k^\alpha=\hat\theta_k^\alpha+\alpha_k$ with $\int_0^T\hat\theta_k^\alpha=0$, $k=1,\ldots,N$. We have:

$$gT\sum_{j=1}^N M_jl_j\cos\alpha_j+cT\alpha_N=I_c(\alpha)\geqq I_c(\theta_\alpha). \tag{2.3}$$

If $\hat\theta_j^\alpha=0$ $\forall j=1,\ldots,N$ then (i) is certainly valid. Assume $\hat\theta_j^\alpha\neq0$ for some j, from (2.3) we have:

$$\begin{aligned}0&\geqq\frac{1}{2}\int_0^T A(\theta)\dot\theta_\alpha\cdot\dot\theta_\alpha+g\sum_{j=1}^N l_jM_j\int_0^T(\cos(\hat\theta_j^\alpha+\alpha_j)-\cos\alpha_j)+\sum_{j=1}^N\int_0^T f_j\hat\theta_j^\alpha\\&>\frac{1}{2}\lambda_0\sum_{j=1}^N\|\dot\theta_j^\alpha\|^2-g\sum_{j=1}^N l_jM_j\|\hat\theta_j^\alpha\|_1-\sum_{j=1}^N\|f_j\|_\infty\|\hat\theta_j^\alpha\|_1\\&\geqq\|\dot\theta_\alpha\|_2\left(\frac{\lambda_0}{2N}\|\dot\theta_\alpha\|_2-\frac{T^{3/2}}{2\pi}v\right).\end{aligned}$$

Thus $\|\dot\theta_\alpha\|_2<(Nv/\lambda_0\pi)T^{3/2}$.

In order to obtain (i) and (ii) notice that θ_α satisfies:

$$\frac{d}{dt}\left(M_j l_j^2 \dot\theta_j^\alpha + \sum_{k=1}^{j-1} M_j l_k l_j \cos(\theta_j^\alpha - \theta_k^\alpha)\dot\theta_k^\alpha + \sum_{k=j+1}^{N} M_k l_k l_j \cos(\theta_k^\alpha - \theta_j^\alpha)\dot\theta_k^\alpha\right)$$

$$+ \sum_{k=1}^{j-1} M_j l_k l_j \sin(\theta_j^\alpha - \theta_k^\alpha)\dot\theta_k^\alpha\dot\theta_j^\alpha - \sum_{k=j+1}^{N} M_k l_k l_j \sin(\theta_k^\alpha - \theta_j^\alpha)\dot\theta_k^\alpha\dot\theta_j^\alpha + gM_j l_j \sin\theta_j^\alpha$$

$$= f_j(t) + \frac{1}{T}\int_0^T\left[\sum_{k=1}^{j-1} M_j l_j l_k \sin(\theta_j^\alpha - \theta_k^\alpha)\dot\theta_k^\alpha\dot\theta_j^\alpha - \sum_{k=j+1}^{N} M_k l_k l_j \sin(\theta_k^\alpha - \theta_j^\alpha)\dot\theta_k^\alpha\dot\theta_j^\alpha\right]$$

$$+ \frac{1}{T} g \int_0^T M_j l_j \sin\theta_j^\alpha, \quad j = 1,\ldots,N.$$

Thus dropping the superscript α and denoting by $(A(\theta)\ddot\theta)_j$ the j^{th} component of the vector $A(\theta)\ddot\theta$ we obtain:

$$(A(\theta_\alpha)\ddot\theta_\alpha)_j = -\sum_{k=1}^{j-1} M_j l_j l_k \sin(\theta_j - \theta_k)\dot\theta_k^2 + \sum_{k=j+1}^{N} M_k l_k l_j \sin(\theta_k - \theta_j)\dot\theta_k^2 + f_j(t)$$

$$+ gM_j l_j \sin\theta_j - \frac{1}{T} g \int_0^T M_j l_j \sin\theta_j + - \int_0^T \sum_{k=1}^{j-1} M_j l_k l_j \sin(\theta_j - \theta_k)\dot\theta_k\dot\theta_j$$

$$- \frac{1}{T}\int_0^T \sum_{k=j+1}^{N} M_k l_k l_j \sin(\theta_k - \theta_j)\dot\theta_k\dot\theta_j := w_j(t).$$

Set $w(t) = w_1(t),\ldots,w_N(t)$; so $A(\theta_\alpha)\ddot\theta_\alpha = w$. Hence

$$\ddot\theta_\alpha(t) = A^{-1}(\theta_\alpha)w(t),$$

and therefore

$$\|\ddot\theta_\alpha\|_1 \leqq \frac{N}{\lambda_0}\|w\|_1.$$

So

$$\|w_j\|_1 \leqq \sum_{k=1}^{j-1} M_j l_k l_j \int_0^T |\dot\theta_k|^2 + \sum_{k=j+1}^{N} M_k l_k l_j \int_0^T |\dot\theta_k|^2 + g\int_0^T M_j l_j \left|\sin\theta_j - \frac{1}{T}\int_0^T \sin\theta_j\right|$$

$$+ \|f\|_1 + \sum_{k=1}^{j-1} M_j l_k l_j \int_0^T |\dot\theta_k||\dot\theta_j| + \sum_{k=j+1}^{N} M_k l_k l_j \int_0^T |\dot\theta_k||\dot\theta_j|.$$

Since

$$\left|\sin\theta_j(t) - \frac{1}{T}\int_0^T \sin\theta_j(s)ds\right| \leqq \int_0^T |\cos\theta_j||\dot\theta_j| \leqq \sqrt{T}\|\dot\theta_j\|_2 \quad \forall t\in[0,T],$$

we conclude

$$\|w_j\| \leqq l_j v_{j,1}\sum_{k=1}^{j-1}\|\dot\theta_k\|_2^2 + l_j v_{j,2}\sum_{k=j+1}^{N}\|\dot\theta_k\|_2^2 + l_j v_{j,1}\|\dot\theta_j\|_2\left(\sum_{k=1}^{j-1}\|\dot\theta_k\|_2\right)$$

$$+ l_j v_{j,2}\|\dot\theta_j\|_2\sum_{k=j+1}^{N}\|\dot\theta_k\|_2 + gM_j l_j\sqrt{T}\|\dot\theta_j\|_2 + \|f_j\|_1$$

$$\leqq l_j v_j\left(\sum_{\substack{k=1\\k\neq j}}^{N}\|\dot\theta_k\|_2^2+\|\dot\theta_j\|_2\sum_{\substack{k=1\\j\neq k}}^{N}\|\dot\theta_k\|_2\right)+gM_jl_j\sqrt{T}\|\dot\theta_j\|_2+T\|f_j\|_\infty$$

$$\leqq l_j v_j\left(\sum_{k=1}^{N}\|\dot\theta_k\|_2\right)^2+gM_jl_j\sqrt{T}\|\dot\theta_j\|_2+T\|f_j\|_\infty.$$

So by (i) we obtain

$$\|\ddot\theta_\alpha\|_1\leqq\frac{N}{\lambda_0}\sum_{j=1}^{N}\|w_j\|_1\leqq\frac{NT}{\lambda_0}(\|f_\infty+AT(\rho AT+gM))=Td(T),$$

and therefore

$$\|\dot\theta_\alpha\|_\infty\leqq\|\ddot\theta_\alpha\|_1\leqq Td(T).$$

To obtain (iii) we argue similarly, since

$$\|\ddot\theta\|_\infty\leqq\frac{N}{\lambda_0}\|w\|_\infty=\frac{N}{\lambda_0}\sum_{j=1}\|w_j\|_\infty$$

and

$$\|w_j\|_\infty\leqq l_jv_j\sum_{\substack{k=1\\k\neq j}}^{N}\|\dot\theta_k\|_\infty^2+gM_jl_j\sqrt{T}\|\dot\theta_j\|_2+\|f_j\|_\infty+l_jv_j\frac{1}{T}\int_0^T\sum_{k\neq j}|\dot\theta_k||\dot\theta_j|$$

$$\leqq l_jv_j\sum_{\substack{k=1\\k\neq j}}^{N}\|\dot\theta_k\|_\infty^2+gM_jl_j\sqrt{T}\|\dot\theta_j\|_2+\|f_j\|_\infty+l_jv_j\|\dot\theta_j\|_2\left(\frac{1}{T}\sum_{\substack{k=1\\k\neq j}}^{N}\|\dot\theta_k\|_2\right)$$

$$\leqq l_jv_jT^2d^2(T)+\sqrt{T}\|\dot\theta_j\|_2(M_jl_jg+Al_jv_j)+\|f_j\|_\infty\quad\text{(by (i) and (ii))}.$$

Thus:

$$\|\ddot\theta_\alpha\|_\infty\leqq\frac{N}{\lambda_0}[\|f\|_\infty+T^2(\rho d^2(T)+gAM+\lambda A^2)].\quad\blacksquare$$

Set

$$c(T)=\frac{\lambda_0}{N}-T^2\left[\frac{gM}{4\pi^2}+\lambda_1A(N-1)\left(\frac{1}{\sqrt{3}}+\frac{AT^2}{12}\right)\right],$$

where $\lambda_1=\max\limits_{1\leqq j\leqq N} l_jv_{j,1}$.

A crucial result in our approach is given by

Proposition 2.1. *Assume $c(T)>0$. For every $\alpha\in\mathbb{R}^N$ there exists a unique $\theta_\alpha\in\Lambda_\alpha$ satisfying $(2.1)_\alpha$. Furthermore the map $\alpha\to\theta_\alpha$ is analytic and*

$$\theta_{\alpha+2\pi k}=\theta_\alpha+2\pi k\quad\forall k\in\mathbb{Z}^N.$$

We shall start with the following

Lemma 2.3. *For every $\theta\in H$ with $\|\dot\theta\|^2\leqq AT^{3/2}$ and $v\in H$ with $\int_0^T v=0$ we have*

$$(I_c''(\theta)v,v)\geqq c(T)\|\dot v\|_2^2.\tag{2.4}$$

Proof. Given $\theta=(\theta_1,\ldots,\theta_N)$, $v=(v_1,\ldots, v_N)$ and $w=(w_1,\ldots,w_N)\in H$, straightforward calculations give:

$$\begin{aligned} I_c'(\theta)v &= \int_0^T A(\theta)\dot\theta\cdot\dot v - \int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \sin(\theta_i-\theta_j)(v_i-v_j)\dot\theta_i\dot\theta_j \\ &\quad + \int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \cos(\theta_i-\theta_j)(\dot\theta_i\dot v_j+\dot v_i\dot\theta_j) - g\int_0^T \sum_{j=1}^N M_j l_j \sin\theta_j v_j \\ &\quad + \int_0^T f\cdot v + c\int_0^T v_N, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} (I_c''(\theta)v,w) &= \int_0^T A(\theta)\dot v\cdot\dot w - \int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \sin(\theta_i-\theta_j) \\ &\quad \cdot[(w_i-w_j)(\dot v_i\dot\theta_j+\dot v_j\dot\theta_i)+(v_i-v_j)(\dot w_i\dot\theta_j+w_j\dot\theta_i)] \\ &\quad - \int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \cos(\theta_i-\theta_j)\dot\theta_i\dot\theta_j(w_i-w_j)(v_i-v_j) \\ &\quad - g\int_0^T \sum_{j=1}^N M_j l_j \cos\theta_j v_j w_j = (I_0''(\theta)v,w). \end{aligned} \tag{2.6}$$

Therefore:

$$\begin{aligned} (I_c''(\theta)v,v) &= \int_0^T A(\theta)\dot v\cdot\dot v - 2\int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \sin(\theta_i-\theta_j)(v_i-v_j)(\dot v_i\dot\theta_j+\dot v_j\dot\theta_i) \\ &\quad - \int_0^T \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i \cos(\theta_i-\theta_j)\dot\theta_i\dot\theta_j(v_i-v_j)^2 - g\int_0^T \sum_{j=1}^N M_j l_j \cos\theta_j v_i^2 \\ &\geqq \lambda_0 \sum_{j=1}^N \|\dot v_j\|_2^2 - \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i [\|v_i-v_j\|_\infty(\|\dot v_i\|_2\|\dot\theta_j\|_2+\|\dot v_j\|_2\|\dot\theta_i\|_2) \\ &\quad + \|v_i-v_j\|_\infty^2\|\dot\theta_i\|_2\|\dot\theta_j\|_2] - gM\sum_{j=1}^M \|v_j\|_2^2. \end{aligned}$$

Since $\|\dot\theta\|_2 \leqq AT^{3/2}$ and $\int_0^T v=0$ we obtain:

$$\begin{aligned} (I_c''(\theta)v,v) &\geqq \left(\frac{\lambda_0}{N} - g\frac{MT^2}{4\pi^2}\right)\|\dot v\|_2^2 - \sum_{j=2}^N \sum_{i=1}^{j-1} M_j l_j l_i [A^2T^3(\|v_i\|_\infty+\|v_j\|_\infty)^2 \\ &\quad + 2AT^{3/2}(\|v_i\|_\infty+\|v_j\|_\infty)(\|\dot v_i\|_2+\|\dot v_j\|_2)] \\ &\geqq \left(\frac{\lambda_0}{N} - g\frac{MT^2}{4\pi^2}\right)\|\dot v\|_2^2 - \lambda_1\left(\frac{A^2T^4}{12}+\frac{AT^2}{\sqrt3}\right)\sum_{j=2}^N \sum_{i=1}^{j-1}(\|\dot v_i\|_2+\|\dot v_j\|_2)^2 \\ &\quad \cdot\left(\frac{\lambda_0}{N} - T^2\left[\frac{gM}{4\pi^2}+\lambda_1 A(N-1)\left(\frac{1}{\sqrt3}+\frac{AT^2}{12}\right)\right]\right)\|\dot v\|_2^2. \quad \blacksquare \end{aligned}$$

Lemma 2.4. *Assume $c(T)>0$. Let $\theta^{(1)}+\alpha$, $\theta^{(2)}+\alpha\in\Lambda_\alpha$ satisfy:*

$$I'(\theta^{(i)}+\alpha)(\theta_c^{(1)}-\theta^{(2)})=0, \quad i=1,2,$$

and

$$\|\dot{\theta}^{(i)}\|_2 \leqq AT^{3/2}, \qquad i = 1, 2.$$

Then $\theta^{(1)} = \theta^{(2)}$.

Proof. For $\tau \in [0,1]$ set $\theta_\tau = \tau\theta^{(1)} + (1-\tau)\theta^{(2)} + \alpha \in \Lambda_\alpha$ and $v = \theta^{(1)} - \theta^{(2)}$. Thus $\int_0^T v = 0$ and $\|\dot{\theta}_\tau\|_2 \leqq AT^{3/2}$ $\forall \tau \in [0,1]$. Define $f(\tau) = I_c(\theta_\tau)$, $\tau \in [0,1]$. Since $f'(0) = f'(1) = 0$ there exists $\tau_0 \in (0,1)$ such that $0 = f''(\tau_0) = (I''(\theta_{\tau_0})v, v) \geqq c(T)\|\dot{v}\|_2^2$ by Lemma 2.3. By assumption $c(T) > 0$; consequently $v = 0$. ■

Proof of Proposition 2.1. The uniqueness of θ_α is an immediate consequence of Lemmata 2.2 and 2.4 since $\forall v \in H$ with $\int_0^T v = 0$ we have $I'(\theta_\alpha)v = 0$. Consider the map

$$F: \mathbb{R}^n \times E \to E$$

with

$$F(\alpha, \theta) = I_0'(\theta + \alpha) - \frac{1}{T}\sum_0^T I_0'(\theta + \alpha),$$

where we recall $E = \left\{\theta \in H : \int_0^T \theta = 0\right\}$.

Obviously F defines an analytic map between the given spaces. Now for $\alpha \in \mathbb{R}^N$, let $\theta_\alpha = \theta_\alpha^0 + \alpha$ be the unique element of Λ_α satisfying $(2.1)_\alpha$. We have:

$$F(\alpha, \theta_\alpha^0) = 0,$$

and for every $v \in E$,

$$\frac{\partial F}{\partial \theta}(\alpha, \rho)v = I''(\theta + \alpha)v.$$

Hence by Lemma 2.3 and a standard application of the Fredholm alternative (from (2.6) $I''(\theta)$ is self-adjoint) we conclude that $(\partial F/\partial\theta)(\alpha, \theta_\alpha^0)$ defines an isomorphism from E into E. So if we denote

$$B_\rho(\alpha) = \left\{\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N : \sum_{j=1} |\alpha_i - \beta_j|^2 < \rho^2\right\},$$

by the implicit function theorem we have that for every $\alpha_0 \in \mathbb{R}^n$ there exists $\varepsilon > 0$ and an analytic map $\sigma: B_\varepsilon(\alpha_0) \to E$ satisfying:

$$\sigma(\alpha_0) = \theta_{\alpha 0}$$

and

$$0 = F(\alpha, \sigma(\alpha)) = I'(\alpha + \sigma(\alpha)) - \frac{1}{T}\int_0^T I'(\alpha + \sigma(\alpha)) \quad \forall \alpha \in B_\varepsilon(\alpha_0).$$

In addition, since $\|\dot{\theta}_{\alpha_0}\|_2 < AT^{3/2}$ for $\varepsilon > 0$ small we may assume $\|\dot{\sigma}(\alpha)\|_2 \leqq AT^{3/2}$ for all $\alpha \in B_\varepsilon(\alpha_0)$. Hence by Lemma 2.4 we conclude $\sigma(\alpha) = \theta_\alpha^0$. ■

Set

$$g(\alpha) = I_0(\theta_\alpha), \quad \alpha \in \mathbb{R}^N.$$

So g defines an analytic map and $g(\alpha + 2\pi k) = g(\alpha)\ \forall k \in \mathbb{Z}^N$, $\forall \alpha \in \mathbb{R}^N$. Since

$$I'_c(\theta_\alpha) = (I'_c(\theta_\alpha)e_1, \ldots, I'_c(\theta_\alpha)e_N)$$

with $e_j = (0, 0, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{ component}}}{1}, 0, \ldots, 0) \in \mathbb{R}^N$; it is readily verified that θ_α is a critical point for I_c if and only if α is a critical point for the function:

$$G_c(\alpha) = I_c(\theta_\alpha) = g(\alpha) + cT\alpha_N.$$

Thus θ_α is a solution for $(1)_c$ if and only if $(\partial G_c/\partial \alpha_j)(\alpha) = 0\ \forall j = 1, \ldots, N$.

As already observed in general, the fact that g always admits at least $N+1$ critical points may have no influence on the number of critical points for G_c even for small c (see [6]).

What comes to help in our situation is the particular structure of $g(\alpha) = I_0(\theta_\alpha)$, as we shall see in the next section.

3. A Generalized Version of Theorem 1

First of all, let us introduce the following notations:

$$L = \max_{j=1,\ldots,N} l_j, \quad c_1 = \min_{1 \leqq k \leqq N-1} A^2\left(\frac{1}{\sqrt{2}} v_k l_k + (N-2)ML\right),$$

$$c_2 = \min_{1 \leqq k \leqq N-1} g\frac{l_k M_k}{\sqrt{2}A^2}\left(\frac{1}{\sqrt{2}} l_k v_k + (N-2)ML\right)^{-1} \tag{3.1}$$

and

$$e(T) = d^2(T) + \rho d^2(T) + gAM + \lambda A^2.$$

Set $\mathscr{S}_j(\alpha) = I'_c(\theta_\alpha)e_j$, $j = 1, \ldots, N$. Notice: $\mathscr{S}_j(\alpha) = I'_0(\theta)e_j\ \forall j = 1, \ldots, N-1$. The main ingredient in the proof of Theorem 1 is the following:

Proposition 3.1. *Let $f = f(t) \in \mathbb{R}^N$ be a continuous T-periodic function such that*

$$\text{(f)}_1 \qquad \int_0^T f = 0, \quad \|f\|_\infty < g\frac{\lambda_0 M_k}{N v_k}, \quad k = 1, \ldots, N.$$

If T satisfies

$$\text{(T)}_1 \qquad c(T) > 0 \quad \textit{and} \quad T^2 < \frac{\sqrt{3}\pi}{2A^2};$$

$$\text{(T)}_2 \qquad T^2 e(T) < \frac{gM_k}{v_k} - \frac{N}{\lambda_0}\|f\|_\infty \quad \forall k = 1, \ldots, N;$$

$$\text{(T)}_3 \qquad T^2\left(1 + \frac{A}{2\sqrt{3}}T^2\right)^2 < \frac{c(T)c_1(c_2 - T^2)}{\lambda^2 A^2 (N-1)}.$$

Then for every $j = 1, \dots, N-1$ *there exists* 2^j *analytic maps* $\gamma_k: \mathbb{R}^{N-j} \to \mathbb{R}^j$ *with the following properties*:

(i) $\gamma_k(\alpha_{j+1}, \dots, \alpha_N) = \gamma_k(\alpha_{j+1} + 2\pi k_{j+1}, \dots, \alpha_N + 2\pi k_N) \quad \forall k_s \in \mathbb{Z},\ s = j+1, \dots, N.$
(ii) $\mathscr{S}_r(\gamma_k(\alpha_{j+1}, \dots, \alpha_N), \alpha_{j+1}, \dots, \alpha_N) = 0 \quad \forall (\alpha_{j+1}, \dots, \alpha_N) \in \mathbb{R}^{N-j},\ \ k = 1, \dots, 2^j$ *and* $r = 1, \dots, j$.
(iii) $\text{Range}\,\gamma_k \cap \text{Range}\,\gamma_k + 2\pi\mathbb{Z}^j = \varnothing$ *if* $k \neq h$, *where* $\text{Range}\,\gamma_h + 2\pi\mathbb{Z}^j = \{\gamma + 2\pi k\colon \gamma \in \text{Range}\,\gamma_h$ *and* $k \in \mathbb{Z}^j\}$.

Define

$$b_j(\alpha) = (I_c''(\theta_\alpha)e_j, e_j) = (I_0''(\theta_\alpha)e_j, e_j)$$

$\alpha \in \mathbb{R}^N$, $j = 1, \dots, N$.

To obtain Proposition 3.1 we need some preliminary results. We shall start with the following:

Lemma 3.1. *Under the assumptions of Proposition 3.1, if for some* $k = 1, \dots, N-1$, $\alpha = (\alpha_1, \dots, \alpha_N)$ *satisfies*

(a) $\alpha_k \in [-\pi, \pi)$, $\mathscr{S}_k(\alpha) = 0$ *and* $b_k(\alpha) \leqq 0$; *then* $\alpha_k \in (-\pi/4, \pi/4)$, *or*
(b) $\alpha_k \in [0, 2\pi)$, $\mathscr{S}_k(\alpha) = 0$ *and* $b_k(\alpha) \geqq 0$, *then* $\alpha_k \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$.

Proof. Using (2.5) we have:

$$\begin{aligned}
0 = \mathscr{S}_k(\alpha) &= \int_0^T \sum_{i=1}^{k-1} M_k l_k l_i \sin(\theta_i^\alpha - \theta_k^\alpha)\dot\theta_i^\alpha \dot\theta_k^\alpha - \int_0^T \sum_{j=k+1}^{N} M_j l_j l_k \sin(\theta_k^\alpha - \theta_j^\alpha)\dot\theta_k^\alpha \dot\theta_j^\alpha \\
&\quad - gM_k l_k \int_0^T \sin\theta_k^\alpha = -l_k\Bigg[\int_0^T gM_k \sin\theta_k^\alpha - \sum_{i=1}^{k-1} M_k l_i \int_0^T \dot\theta_i^\alpha \dot\theta_k^\alpha(\sin\theta_i^\alpha \cos\theta_k^\alpha - \cos\theta_i^\alpha \sin\theta_k^\alpha) \\
&\quad + \sum_{j=k+1}^{N} M_j l_j \int_0^T \dot\theta_k^\alpha \dot\theta_j^\alpha(\sin\theta_i^\alpha \cos\theta_k^\alpha - \sin\theta_i^\alpha \cos\theta_k^\alpha) \\
&\quad + \sum_{j=k+1}^{N} M_j l_j \int_0^T \dot\theta_k^\alpha \dot\theta_j^\alpha(\sin\theta_k^\alpha \cos\theta_j^\alpha - \sin\theta_j^\alpha \cos\theta_k^\alpha) \\
&= -l_k \int_0^T \sin\theta_k^\alpha \Bigg[gM_k + \sum_{j=1}^{k-1} M_k l_j\left(\dot\theta_j^\alpha \dot\theta_k^\alpha \cos\theta_j^\alpha + \frac{d}{dt}(\sin\theta_j^\alpha \dot\theta_j^\alpha)\right) \\
&\quad + \sum_{j=k+1}^{N} M_j l_j\left(\dot\theta_j^\alpha \dot\theta_k^\alpha \cos\theta_j^\alpha + \frac{d}{dt}(\dot\theta_j^\alpha \sin\theta_j^\alpha)\right)\Bigg] = -l_k \int_0^T \sin\theta_k^\alpha \psi_k^\alpha,
\end{aligned}$$

where

$$\begin{aligned}
\psi_k^\alpha(t) = gM_k &+ \sum_{j=1}^{k-1} M_k l_j\left[\dot\theta_j^\alpha \dot\theta_k^\alpha \cos\theta_j^\alpha + \frac{d}{dt}(\dot\theta_j^\alpha \sin\theta_j^\alpha)\right] \\
&+ \sum_{j=k+1}^{N} M_j l_j\left[\dot\theta_j^\alpha \dot\theta_k^\alpha \cos\theta_j^\alpha + \frac{d}{dt}(\dot\theta_j^\alpha \sin\theta_j^\alpha)\right].
\end{aligned}$$

Similarly by (2.6) and analogous computations we obtain:

$$b_k(\alpha) = (I_c''(\theta_\alpha)e_k, e_k) = -l_k \int_0^T \cos\theta_k^\alpha \psi_k^\alpha.$$

Claim. $\psi_k^\alpha(t) \geqq 0$ *for all* $t \in \mathbb{R}$.

Using the estimate of Lemma 2.2 we have:

$$\begin{aligned}
\psi_k^\alpha(t) &\geqq gM_k - \sum_{j=1}^{k-1} M_k l_j(\|\dot\theta_j^\alpha\|_\infty \|\dot\theta_k^\alpha\|_\infty + \|\dot\theta_j^\alpha\|_\infty^2 + \|\ddot\theta_j^\alpha\|_\infty) \\
&\quad - \sum_{j=k+1}^{N} M_j l_j(\|\dot\theta_j^\alpha\|_\infty \|\dot\theta_k^\alpha\|_\infty + \|\dot\theta_j^\alpha\|_\infty^2 + \|\ddot\theta_j^\alpha\|_\infty) \\
&\geqq gM_k - v_k \sum_{\substack{j=1 \\ j\neq k}}^{N} (\|\dot\theta_j^\alpha\|_\infty \|\dot\theta_k^\alpha\|_\infty + \|\dot\theta_j^\alpha\|_\infty^2 + \|\ddot\theta_j^\alpha\|_\infty) \\
&\geqq gM_k - v_k\left(\left(\sum_{j=1}^{N} \|\dot\theta_j^\alpha\|_\infty\right)^2 + \sum_{j=1}^{N} \|\ddot\theta_j^\alpha\|_\infty\right) = gM_k - v_k(\|\dot\theta_\alpha\|_\infty^2 + \|\ddot\theta_\alpha\|_\infty) \\
&\geqq gM_k - v_k\left[\frac{N}{\lambda_0}\|f\|_\infty + T^2(d^2(T) + \rho d^2(T) + gAM + \lambda A^2)\right] > 0
\end{aligned}$$

by assumption $(\mathrm{T})_2$. Now if $\mathscr{S}_k(\alpha) = 0 \geqq b_k(\alpha)$, then

$$\int_0^T \sin(\hat\theta_k^\alpha + \alpha_k)\psi_k^\alpha = 0 \geqq \int_0^T \cos(\hat\theta_k^\alpha + \alpha_k)\psi_k^\alpha, \tag{3.2}$$

where $\theta_k^\alpha = \hat\theta_k^\alpha + \alpha_k$ and $\int_0^T \hat\theta_k^\alpha = 0$. Furthermore, by Lemma 2.2 and $(\mathrm{T})_1$, we have $\|\hat\theta_k^\alpha\|_\infty < \pi/4$. Arguing by contradiction, assume for example that $\alpha_k \geqq \pi/4$. So by (3.2) necessarily $\alpha_k > \pi - \pi/4 = \frac{3}{4}\pi$, which in turn gives $\theta_k^\alpha(t) \geqq \pi/2$ for all t. But since $\psi_k^\alpha > 0$, the second inequality in (3.2) requires $\alpha_k > \frac{3}{2}\pi - \pi/4 > \pi$ contradicting $\alpha_k \in [-\pi, \pi)$. A similar argument shows that $\alpha_k > -\pi/4$. Analogously, if we assume $\mathscr{S}_k(\alpha) = 0 \leqq b_k(\alpha)$ and $\alpha_k \in [0, 2\pi)$, then from

$$\int_0^T \sin(\hat\theta_k^\alpha + \alpha_k)\psi_k^\alpha = 0 \leqq \int_0^T \cos(\hat\theta_k^\alpha + \alpha_k)\psi_k^\alpha \quad \text{and} \quad \|\hat\theta_k^\alpha\|_\infty < \frac{\pi}{4},$$

one easily gets $\alpha_k \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$. ■

Lemma 3.2. *Under the assumptions of Proposition 3.1, if for some* $1 \leqq r \leqq N$ *we have:* $\mathscr{S}_{j_s}(\alpha) = 0$, $b_{j_s}(\alpha) \geqq 0$ $(\leqq 0)$ *with* $1 \leqq j_s \leqq N-1$ *and* $s = 1, \ldots, r$, *then*

$$\left(I_c''(\theta_\alpha)\sum_{s=1}^{r} \tau_{j_s} e_{j_s}, \sum_{s=1}^{r} \tau_{j_s} e_{j_s}\right) \geqq Tc_1(c_2 - T^2)\sum_{s=1}^{r} \tau_{j_s}^2 \quad \left(\leqq -Tc_1(c_2 - T^2)\sum_{s=1}^{r} \tau_{j_s}^2\right).$$

Proof. First of all notice that if

$$\begin{aligned}
0 = \mathscr{S}_k(\alpha) &= -l_k \int_0^T \sin(\hat\theta_k^\alpha + \alpha_k)\psi_k^\alpha \\
&= -l_k\left(\cos\alpha_k \int_0^T \sin\hat\theta_k^\alpha \psi_k^\alpha + \sin\alpha_k \int_0^T \cos\hat\theta_k^\alpha \psi_k^\alpha\right), \quad 1 \leqq k \leqq N-1,
\end{aligned}$$

then for $\alpha_k \neq \pi/2 + h\pi$, $h \in \mathbb{Z}$, we have

$$b_k(\alpha) = -l_k \int_0^T \cos(\hat{\theta}_k^\alpha + \alpha_k)\psi_k^\alpha = -l_k\left(\cos\alpha_k \int_0^T \cos\hat{\theta}_k^\alpha \psi_k^\alpha - \sin\alpha_k \int_0^T \sin\hat{\theta}_k^\alpha \psi_k^\alpha\right)$$

$$= -\frac{l_k}{\cos\alpha_k}\int_0^T \cos\hat{\theta}_k^\alpha \psi_k^\alpha.$$

Thus if $b(\alpha) \geqq 0$, by Lemma 3.1 we have $\alpha_k + 2\pi h \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$ for some $h \in \mathbb{Z}$, which implies:

$$b_k(\alpha) \geqq \frac{l_k}{\sqrt{2}}\int_0^T \psi_k^\alpha(t)dt \geqq \frac{l_k T}{\sqrt{2}}(gM_k - \nu_k A^2 T^2). \tag{3.3}$$

Similarly, if $b_k(\alpha) \leqq 0$, then $\alpha_k + 2\pi h \in (-\pi/4, \pi/4)$ for some $h \in \mathbb{Z}$ and

$$b_k(\alpha) \leqq -\frac{l_k T}{\sqrt{2}}(gM_k - \nu_k A^2 T^2). \tag{3.4}$$

Next observe that if $p > k$ by (2.6) we get:

$$|(I_c''(\theta_\alpha)e_p, e_k)| = |M_p l_p l_k \int_0^T \cos(\theta_k^\alpha - \theta_p^\alpha)\dot{\theta}_p^\alpha \dot{\theta}_k^\alpha| \leqq M_p l_p l_k A^2 T^3.$$

Thus if $b_{j_s}(\alpha) \geqq 0 = \mathscr{S}_{j_s}(\alpha)$, $s = 1, \ldots, r$, using (3.3) we obtain:

$$\begin{aligned}
&\left(I_c''(\theta_\alpha)\sum_{s=1}^r \tau_{j_s} e_{j_s}, \sum_{s=1}^r \tau_{j_s} e_{j_s}\right) \\
&= \sum_{s=1}^r \tau_{j_s}^2 b_{j_s}(\alpha) + \sum_{s \neq p} \tau_{j_s}\tau_{j_p}(I_c''(\theta_\alpha)e_{j_s}, e_{j_p}) \\
&\geqq \frac{T}{\sqrt{2}}\sum_{s=1}^r \tau_{j_s}^2 l_{j_s}(gM_{j_s} - \nu_{j_s}A^2T^2) - 2A^2MLT^3 \sum_{\substack{p,s=1 \\ s<p}}^r |\tau_{j_s}||\tau_{j_p}| \\
&\geqq T\sum_{s=1}^r \left(\frac{1}{\sqrt{2}} g l_{j_s} M_{j_s} - T^2\left(\frac{A^2}{\sqrt{2}} l_{j_s}\nu_{j_s} + A^2ML(r-1)\right)\right)\tau_{j_s}^2 \\
&\geqq c_1 T(c_2 - T^2)\sum_{s=1}^r \tau_{j_s}^2
\end{aligned}$$

and $T^2 < c_2$ by $(\mathrm{T})_3$.

Similarly in case $b_{j_s}(\alpha) \leqq 0 = \mathscr{S}_{j_s}(\alpha)$, $s = 1, \ldots, r$, by (3.4) we conclude:

$$\begin{aligned}
&\left(I_c''(\theta_\alpha)\sum_{s=1}^r \tau_{j_s} e_{j_s}, \sum_{s=1}^r \tau_{j_s} e_{j_s}\right) \\
&\leqq \sum_{s=1}^r b_{j_s}(\alpha)\tau_{j_s}^2 + 2A^2MLT^3 \sum_{\substack{s,p=1 \\ b<p}}^r |\tau_{j_s}||\tau_{j_p}|
\end{aligned}$$

$$\leqq -\left(T\sum \frac{1}{\sqrt{2}} l_{j_s}(gM_{j_s} - v_{j_s}A^2T^2)\tau_{j_s}^2 - 2A^2MLT^3 \sum_{\substack{s,p=1\\ s<p}}^{r} |\tau_{j_s}||\tau_{j_p}|\right)$$

$$\leqq -Tc_1(c_2 - T^2)\sum_{s=1}^{r} \tau_{j_s}^2. \quad \blacksquare$$

Lemma 3.3. *Under the assumptions of Proposition 3.1, if* $v=(v_1,\ldots,v_N)\in H$ *and* $\alpha\in\mathbb{R}^N$ *satisfy:*

(i) $\int_0^T v_N = 0$;

(ii) $\mathscr{S}_j(\alpha)=0$ *and* $b_j(\alpha)\geqq 0$ *whenever* $\int_0^T v_j \neq 0$. *Then* $(I_c''(\theta_\alpha)v,v)\geqq 0$ *and equality occurs if and only if* $v=0$.

Proof. If $\int_0^T v_j = 0$ for all $j=1,\ldots,N$, the claim follows by Lemma 2.3. Hence assume that there exists $j_1,\ldots,j_r\in\{1,\ldots,N-1\}$ such that $\int_0^T v_k \neq 0 \Leftrightarrow k=j_s$ for some $s=1,\ldots,r$. Write $v=\hat{v}+e$, where $\int_0^T \hat{v}=0$ and $e=\sum_{s=1}^{r}\tau_{j_s}e_{j_s}$ for some $\tau_{j_s}\in\mathbb{R}$.

For every $k=1,\ldots,N$ and $\alpha\in\mathbb{R}^N$ we have:

$$|(I_c''(\theta_\alpha)\hat{v},e_k)| \leqq v_k l_k \sum_{\substack{j=1\\ j\neq k}}^{N} [\|\dot{v}_j\|_2\|\dot{\theta}_k\|_2 + \|\dot{\theta}_j\|_2\|\dot{v}_k\|_2 + \|\dot{\theta}_j\|\,\|\dot{\theta}_k\|(\|\hat{v}_j\|_\infty + \|\hat{v}_k\|_\infty)]$$

$$\leqq v_k l_k (AT^{3/2}\|\dot{v}\|_2 + A^2T^3(N-1)\|\hat{v}\|_\infty)$$

$$\leqq v_k l_k AT^{3/2}\left(1+\frac{A^2T^2}{2\sqrt{3}}\right)\|\dot{v}\|_2.$$

Thus using Lemmata 2.3 and 3.2 we conclude

$$\begin{aligned}(I_c''(\theta_\alpha)v,v) &= (I_c''(\theta_\alpha)\hat{v},\hat{v}) + 2\sum_{s=1}^{r}\tau_{j_s}(I_c''(\theta_\alpha)\hat{v},e_{j_s}) + (I_c''(\theta_\alpha)e,e)\\ &\geqq c(T)\|\dot{v}\|_2^2 - 2\lambda AT^{3/2}\left(1+\frac{A^2T^2}{2\sqrt{3}}\right)\left(\sum_{s=1}^{r}|\tau_{j_s}|\right)\|\dot{v}\|_2\\ &\quad + \frac{c_1T(c_2-T^2)}{(N-1)}\left(\sum_{s=1}^{r}|\tau_{j_s}|\right)^2. \end{aligned} \tag{3.5}$$

Since by $(\mathrm{T})_3$,

$$\frac{c(T)c_1(c_2-T^2)}{N-1} > \lambda^2A^2T^2\left(1+\frac{A^2T^2}{2\sqrt{3}}\right)^2,$$

we get that the right-hand side (3.5) is non-negative and vanishes if and only if $\hat{v}=0$ and $\tau_{j_s}=0$ for all $s=1,\ldots,r$. $\blacksquare$

Lemma 3.4. *Under the assumptions of Proposition 3.1, let* $v=(v_1,\ldots,v_N)\in H$, $\alpha\in\mathbb{R}^N$ *and* $k\in\{1,\ldots,N\}$ *satisfy:*

(i) $\mathscr{S}_j(\alpha) = 0$ *for all* $j = 1, \ldots, k-1$.

(ii) $\int_0^T v_k = 1$, $(I_c''(\theta_\alpha)v, e_j) = 0$ *for all* $j = 1, \ldots, k-1$ *and* $(I_c''(\theta_\alpha)v, e_k) = (I_c''(\theta_c)v, v)$.

We have:

$$\text{if } (I_c''(\theta_\alpha)v, v) \leqq 0\ (\geqq 0), \text{ then } b_k(\alpha) < 0\ (>0)$$
$$\text{and } (I_c''(\theta_\alpha)v, v) < 0\ (>0).$$

Proof. If $b_j(\alpha) \geqq 0$ whenever $\int_0^T v_j \neq 0$ then by Lemmata 3.1 and 3.3 we have $b_j(\alpha) > 0$ for all such j's and $(I_c''(\theta_\alpha)v, v) > 0$ since $v \neq 0$. Hence assume there exists $p \geqq 1$ and $i_1 < \cdots < i_p \in \{1, \ldots, k\}$ such that $b_j(\alpha) < 0$ if and only if $j = i_s$ for some $s = 1, \ldots, p$. Write $v = \hat{v} + e$, where $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$ satisfies $\int_0^T \hat{v}_{i_s} = 0$, $s = 1, \ldots, p$ and $e = \sum_{s=1}^{p} \tau_{i_s} e_{i_s}$ for some $\tau_{i_s} \in \mathbb{R}$.

It is enough to show that:

$$\begin{array}{lll} \text{if } \ i_p \leqq k-1 & \text{then } \ (I_c''(\theta_\alpha)v, v) > 0 & \text{and} \\ \text{if } \ i_p = k & \text{then } \ (I_c''(\theta_\alpha)v, v) < 0. & \end{array}$$

Assume $i_p \leqq k-1$, then $\hat{v} \neq 0$ since $\int_0^T \hat{v}_k = 1$ and

$$\begin{aligned}
(I_c''(\theta_\alpha)v, v) &= (I_c''(\theta_\alpha)\hat{v}, \hat{v}) + 2 \sum_{s=1}^{p} \tau_{i_s}(I_c''(\theta_\alpha)\hat{v}, e_{i_s}) + (I_c''(\theta_\alpha)e, e) \\
&= (I_c''(\theta_\alpha)\hat{v}, \hat{v}) + 2 \sum_{s=1}^{p} \tau_{i_s}(I_c''(\theta_\alpha)v, e_{i_s}) - (I_c''(\theta_\alpha)e, e) \\
&= (I_c''(\theta_\alpha)\hat{v}, \hat{v}) - (I_c''(\theta_\alpha)e, e) > 0,
\end{aligned}$$

since by (ii), $(I_c''(\theta_\alpha)v, e_{i_s}) = 0$ for all $s = 1, \ldots, p$ and $(I_c''(\theta_\alpha)\hat{v}, \hat{v}) > 0 \geqq (I_c''(\theta_\alpha)e, e)$ by Lemmata 3.3 and 3.2. If $i_p = k$, then $\tau_{i_p} = 1$ and

$$\begin{aligned}
(I_c''(\theta_\alpha)v, e_k) &= (I_c''(\theta_\alpha)\hat{v}, \hat{v}) + 2 \sum_{s=1}^{p} \tau_{i_s}(I_c''(\theta_\alpha)v, e_{i_s}) - (I_c''(\theta_\alpha)e, e) \\
&= (I_c''(\theta_\alpha)\hat{v}, \hat{v}) + 2(I_c''(\theta_\alpha)v, e_k) - (I_c''(\theta_\alpha)e, e)
\end{aligned}$$

which gives

$$-(I_c''(\theta_\alpha)v, v) = -(I_c''(\theta_\alpha)v, e_k) = (I_c''(\theta_\alpha)\hat{v}, \hat{v}) - (I_c''(\theta_\alpha)e, e) > 0. \quad \blacksquare$$

We are finally ready to give the

Proof of Proposition 3.1. We shall use an induction procedure.

Step 1 ($j = 1$). In this step we shall construct two analytic maps $\gamma_1, \gamma_2 : \mathbb{R}^{N-1} \to \mathbb{R}$ satisfying (i)–(iii). More precisely, we shall obtain the following

Claim. *For every* $(\alpha_2, \ldots, \alpha_N) \in \mathbb{R}^{N-1}$ *there exist unique* $\gamma^+ = \gamma^+(\alpha_2, \ldots, \alpha_N) \in [-\pi, \pi)$ *and* $\gamma^- = \gamma^-(\alpha_2, \ldots, \alpha_N) \in [0, 2\pi)$ *such that*

$$g(\gamma^+, \alpha_2, \ldots, \alpha_N) = \max_\gamma g(\gamma, \alpha_2, \ldots, \alpha_N)$$

and

$$g(\gamma^-, \alpha_2, \dots, \alpha_N) = \min_{\gamma} g(\gamma, \alpha_2, \dots, \alpha_N).$$

(Recall $g(\alpha) = I_0(\theta_\alpha)$). Furthermore, the map $(\alpha_2, \dots, \alpha_N) \to \gamma^\pm(\alpha_2, \dots, \alpha_N)$ is analytic, $\gamma^\pm(\alpha_2 + 2\pi k_2, \dots, \alpha_N + 2\pi k_N) = \gamma^\pm(\alpha_2, \dots, \alpha_N)$ for all $k_j \in \mathbb{Z}$, $j = 2, \dots, N$ and

$$\operatorname{Range} \gamma^+ \cap \operatorname{Range} \gamma^- + 2\pi\mathbb{Z} = \varnothing.$$

So the conclusion will follow by taking $\gamma_1(\alpha_2, \dots, \alpha_N) = \gamma^+(\alpha_2, \dots, \alpha_N)$ and $\gamma_2(\alpha_2, \dots, \alpha_N) = \gamma^-(\alpha_2, \dots, \alpha_N)$.

Proof of Claim. We shall start by showing that if

$$\alpha_0 = (\alpha_1^0, \dots, \alpha_N^0) \in \mathbb{R}^N \quad \text{satisfies} \quad \alpha_1^0 \in [0, 2\pi), \quad \frac{\partial g}{\partial \alpha_1}(\alpha_0) = 0 \leqq \frac{\partial^2 g}{\partial \alpha_1^2}(\alpha_0),$$

then $\alpha_1^0 \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$ and necessarily $(\partial^2 g/\partial \alpha^2)(\alpha_0) > 0$. To see this, notice that

$$0 = \frac{\partial g}{\partial \alpha_1}(\alpha_0) = I_c'(\theta_{\alpha_0}) \left.\frac{\partial \theta_\alpha}{\partial \alpha_1}\right|_{\alpha = \alpha_0} = I_c'(\theta_{\alpha_0}) e_1 = \mathscr{S}_1(\alpha_0)$$

and

$$\left(I_c''(\theta_{\alpha_0}) \left.\frac{\partial \theta}{\partial \alpha_1}\right|_{\alpha = \alpha_0}, e_1 \right) = \frac{\partial^2 g}{\partial \alpha_1^2}(\alpha_0) = \left(I_c''(\theta_{\alpha_0}) \left.\frac{\partial \theta_\alpha}{\partial \alpha_1}\right|_{\alpha = \alpha_0}, \left.\frac{\partial \theta_\alpha}{\partial \alpha_1}\right|_{\alpha = \alpha_0} \right).$$

Letting $v = \partial \theta_\alpha / \partial \alpha_1 |_{\alpha = \alpha_0}$ we have that v satisfies the assumptions of Lemma 3.4 with $k = 1$. So the conclusion easily follows with the help of Lemma 3.1. A similar argument shows that if $\alpha_1^0 \in [-\pi, \pi)$, $(\partial g/\partial \alpha_1)(\alpha_0) = 0 \geqq (\partial^2 g/\partial \alpha_1^2)(\alpha_0)$ then $\alpha_1^0 \in (-\pi/4, \pi/4)$ and $(\partial^2 g/\partial \alpha_1^2)(\alpha) < 0$. These two facts readily imply that if $\gamma_0 \in [0, 2\pi)$ and $(\partial g/\partial \alpha_1)(\gamma_0, \alpha_2, \dots, \alpha_N) = 0 \leqq (\partial^2 g/\partial \alpha_1^2)(\gamma_0, \alpha_2, \dots, \alpha_N)$, then necessarily $\gamma_0 = \gamma^-(\alpha_2, \dots, \alpha_N)$. Similarly if $\gamma_0 \in [-\pi, \pi)$, $(\partial g/\partial \alpha_1)(\gamma_0, \alpha_2, \dots, \alpha_N) = 0 \geqq (\partial^2 g/\partial \alpha_1^2)(\gamma_0, \alpha_2, \dots, \alpha_N)$, then $\gamma_0 = \gamma^+(\alpha_2, \dots, \alpha_N)$. Thus $\gamma^\pm(\alpha_2, \dots, \alpha_N)$ is unique.

Let $(\alpha_2^0, \dots, \alpha_N^0) \in \mathbb{R}^{N-1}$ be fixed. Set $\gamma_0^\pm = \gamma^\pm(\alpha_2^0, \dots, \alpha_N^0)$ and $\alpha_0^\pm = (\gamma_0^\pm, \alpha_2^0, \dots, \alpha_N^0) \in \mathbb{R}^N$. By the previous observations we know that $\gamma_0^+ \in (-\pi/4, \pi/4)$, $\gamma_0^- \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$ and $(\partial^2 g/\partial \alpha_1^2)(\alpha_0^+) < 0 < (\partial^2 g/\partial \alpha_1^2)(\alpha_0^-)$. Furthermore, $(\partial g/\partial \alpha_1)(\alpha_0^\pm) = 0$, so by the implicit function theorem, there exist $\varepsilon > 0$ and analytic curves:

$$\psi^+ : B_\varepsilon(\alpha_2^0, \dots, \alpha_N^0) \to [-\pi, \pi)$$
$$\psi^- : B_\varepsilon(\alpha_2^0, \dots, \alpha_N^0) \to [0, 2\pi),$$

satisfying

$$\psi^\pm(\alpha_2^0, \dots, \alpha_N^0) = \gamma_0^\pm, \quad \frac{\partial g}{\partial \alpha_1}(\psi^\pm(\alpha_2, \dots, \alpha_N), \alpha_2, \dots, \alpha_N) = 0$$

and

$$\frac{\partial^2 g}{\partial \alpha_1^2}(\psi^+(\alpha_2, \dots, \alpha_N), \alpha_2, \dots, \alpha_N) < 0 < \frac{\partial^2 g}{\partial \alpha_1^2}(\psi^-(\alpha_2, \dots, \alpha_N), \alpha_2, \dots, \alpha_N)$$

for all $(\alpha_2, \dots, \alpha_N) \in B_\varepsilon(\alpha_2^0, \dots, \alpha_N^0)$. Thus necessarily, $\psi^\pm(\alpha_2, \dots, \alpha_N) = \gamma^\pm(\alpha_2, \dots, \alpha_N)$,

$\gamma^+(\alpha_2,\ldots,\alpha_N)\in(-\pi/4,\pi/4)$ and $\gamma^-(\alpha_2,\ldots,\alpha_N)\in(\frac{3}{4}\pi,\frac{5}{4}\pi)$. This concludes the proof of Step 1.

Step 2 ($j\mapsto j+1$). Assume that for $1\leqq j\leqq N-2$ we have 2^j analytic maps $\gamma_k:\mathbb{R}^{N-j}\to\mathbb{R}^j$ satisfying (i)–(iii). Set $g_k(\alpha_{j+1},\ldots,\alpha_N)=g(\gamma_k(\alpha_{j+1},\ldots,\alpha_N),\alpha_{j+1},\ldots,\alpha_N)$. Hence g_k is analytic and 2π-periodic in each variable for all $k=1,\ldots,2^j$. Analogous to the previous step we are done once we obtain the following:

Claim. *For every* $\alpha_{j+2},\ldots,\alpha_N\in\mathbb{R}^{N-(j+1)}$ *there exist unique* $\gamma_k^+(\alpha_{j+2},\ldots,\alpha_N)\in[-\pi,\pi)$ *and* $\gamma_k^-(\alpha_{j+2},\ldots,\alpha_N)\in[0,2\pi)$ *satisfying*

$$g_k(\gamma_k^+(\alpha_{j+2},\ldots,\alpha_N),\alpha_{j+2},\ldots,\alpha_N)=\max_\gamma g_k(\gamma,\alpha_{j+2},\ldots,\alpha_N)$$

and

$$g_k(\gamma_k^-(\alpha_{j+2},\ldots,\alpha_N),\alpha_{j+2},\ldots,\alpha_N)=\min_\gamma g_k(\gamma,\alpha_{j+2},\ldots,\alpha_N).$$

Furthermore the map $(\alpha_{j+2},\ldots,\alpha_N)\to\gamma^\pm(\alpha_{j+2},\ldots,\alpha_N)$ is analytic, $\gamma_k^\pm(\alpha_{j+2}+2\pi k_{j+2},\ldots,\alpha_N+2\pi k_N)=\gamma^\pm(\alpha_{j+2},\ldots,\alpha_N)$ and

$$\operatorname{Range}\gamma_k^+\cap\operatorname{Range}\gamma_k^-+2\pi\mathbb{Z}=\varnothing\quad\text{for all}\quad k=1,\ldots,2^j.$$

Indeed, to conclude will be enough to take $\hat\gamma_k:\mathbb{R}^{N-(j+1)}\to\mathbb{R}^{j+1}$ with

$$\hat\gamma_k(\alpha_{j+2},\ldots,\alpha_N)=\begin{cases}(\gamma_k(\gamma_k^+(\alpha_{j+2},\ldots,\alpha_N),\alpha_{j+2},\ldots,\alpha_N),\gamma_k^+(\alpha_{j+2},\ldots,\alpha_N))\\ \qquad\qquad\text{if}\quad 1\leqq k\leqq 2^j\\ (\gamma_{k-2^j}(\gamma_{k-2^j}^-(\alpha_{j+2},\ldots,\alpha_N),\alpha_{j+2},\ldots,\alpha_N),\gamma_{k-2^j}^-(\alpha_{j+2},\ldots,\alpha_N)).\\ \qquad\qquad\text{if}\quad 2^j<k\leqq 2^{j+1}\end{cases}$$

In fact, since γ_k satisfies (i)–(iii) for all $k=1,\ldots,2^j$, it is easily verified that $\hat\gamma_k$ defines an analytic map and satisfies (i)–(iii) for all $k=1,\ldots,2^{j+1}$.

Proof of the Claim. As before we see that if $(\alpha_{j+1}^0,\ldots,\alpha_N^0)\in\mathbb{R}^{N-j}$ satisfies

$$\alpha_{j+1}^0\in[0,2\pi),\ \frac{\partial g_k}{\partial\alpha_{j+1}}(\alpha_{j+1}^0,\ldots,\alpha_N^0)=0\leqq\frac{\partial^2 g_k}{\partial\alpha_{j+1}^2}(\alpha_{j+1}^0,\ldots,\alpha_N^0),$$

then $\alpha_{j+1}^0\in(\frac{3}{4}\pi,\frac{5}{4}\pi)$ and $(\partial^2 g_k/\partial\alpha_{j+1}^2)(\alpha_{j+1}^0,\ldots,\alpha_N^0)>0$. Indeed, set $\alpha_0=(\gamma_k(\alpha_{j+1}^0,\ldots,\alpha_N^0),\alpha_{j+1}^0,\ldots,\alpha_N^0)\in\mathbb{R}^N$; by (ii) we have $\mathscr{S}_r(\alpha_0)=0$ for all $r=1,\ldots,j$ and for $\gamma_k=(\gamma_{k,1},\ldots,\gamma_{k,j})$,

$$\begin{aligned}0&=\frac{\partial g_k}{\partial\alpha_{j+1}}(\alpha_{j+1},\ldots,\alpha_N)\\&=I'(\theta_{\alpha_0})\left(\sum_{p=1}^j\frac{\partial\theta_\alpha}{\partial\alpha_p}\bigg|_{\alpha=\alpha_0}\frac{\partial\gamma_{k,p}}{\partial\alpha_{j+1}}(\alpha_{j+1}^0,\ldots,\alpha_N^0)+\cdot\frac{\partial\theta_\alpha}{\partial\alpha_{j+1}}\bigg|_{\alpha=\alpha_0}\right)\\&=I'(\theta_{\alpha_0})e_{j+1}=\mathscr{S}_{j+1}(\alpha_0).\end{aligned}$$

Set

$$v=\sum_{p=1}^j\frac{\partial\gamma_{k,p}}{\partial\alpha_{j+1}}(\alpha_{j+1}^0,\ldots,\alpha_N^0)\frac{\partial\theta_\alpha}{\partial\alpha_p}\bigg|_{\alpha=\alpha_0}+\frac{\partial\theta_\alpha}{\partial\alpha_{j+1}}\bigg|_{\alpha=\alpha_0}.$$

We have: $\int_0^T v_{j+1} = 1$, $\mathscr{S}_r(\alpha_0) = 0$ for all $r = 1, \ldots, j+1$ and

$$(I''(\theta_{\alpha_0})v, e_{j+1}) = \frac{\partial^2 g_k}{\partial \alpha_{j+1}^2}(\alpha_{j+1}^0, \ldots, \alpha_N^0) = (I''(\theta_{\alpha_0})v, v) \geqq 0.$$

So Lemma 3.4 applies to v and α_0 with $k = j+1$ and together with Lemma 3.1 gives $\alpha_{j+1}^0 \in (\frac{3}{4}\pi, \frac{5}{4}\pi)$ and

$$\frac{\partial^2 g_k}{\partial \alpha_{j+1}^2}(\alpha_{j+1}, \ldots, \alpha_N) > 0.$$

Similarly we see that if $\alpha_{j+1} \in [-\pi, \pi)$,

$$\frac{\partial g_k}{\partial \alpha_{j+1}}(\alpha_{j+1}, \ldots, \alpha_N) = 0 \geqq \frac{\partial^2 g_k}{\partial \alpha_{j+1}^2}(\alpha_{j+1}, \ldots, \alpha_N),$$

then $\alpha_{j+1} \in (-\pi/4, \pi/4)$ and $(\partial^2 g/\partial \alpha_{j+1}^2)(\alpha_{j+1}, \ldots, \alpha_N) < 0$. These two facts readily give uniqueness for $\gamma_k^{\pm}(\alpha_{j+1}, \ldots, \alpha_N)$ for all $(\alpha_{j+1}, \ldots, \alpha_N) \in \mathbb{R}^{N-j}$. More precisely if $\gamma \in [0, 2\pi)$,

$$\frac{\partial g_k}{\partial \alpha_{j+1}}(\gamma, \alpha_{j+1}, \ldots, \alpha_N) = 0 \leqq \frac{\partial^2 g_k}{\partial \alpha_{j+1}^2}(\gamma, \alpha_{j+1}, \ldots, \alpha_N),$$

then $\gamma = \gamma_k^-(\alpha_{j+2}, \ldots, \alpha_N)$; and if $\gamma \in [-\pi, \pi)$,

$$\frac{\partial g_k}{\partial \alpha_{j+1}}(\gamma, \alpha_{j+2}, \ldots, \alpha_N) = 0 \geqq \frac{\partial^2 g_k}{\partial \alpha_{j+1}^2}(\gamma, \alpha_{j+2}, \ldots, \alpha_N),$$

then $\gamma = \gamma_k^+(\alpha_{j+2}, \ldots, \alpha_N)$.

As for the previous step, these observations together with the implicit function theorem (applied to $\partial g_k/\partial \alpha_{j+1}$ near each $(\gamma^{\pm}(\alpha_{j+2}, \ldots, \alpha_N), \alpha_{j+2}, \ldots, \alpha_N) \in \mathbb{R}^{N-j}$) yields the conclusions. ■

By means of Proposition 3.1 we obtain the following stronger version of Theorem 1.

Theorem 2. *If f and T satisfy the assumptions of Proposition 3.1, then the conclusion of Theorem 1 holds.*

Proof. Proposition 3.1 with $j = N-1$ gives 2^{N-1} analytic curves $\gamma_k : \mathbb{R} \to \mathbb{R}^{N-1}$, $k = 1, \ldots, 2^{N-1}$ with the properties:

(1) $\mathscr{S}_r(\gamma_k(\tau), \tau) = 0$ for all $\tau \in \mathbb{R}$; $r = 1, \ldots, N-1$ and $k = 1, \ldots, 2^{N-1}$;
(2) $\gamma_k(\tau + 2\pi p) = \gamma_k(\tau)$ for all $\tau, p \in \mathbb{Z}$ and $k = 1, \ldots, 2^{N-1}$;
(3) if $k \neq h$ then $\operatorname{Range} \gamma_k \cap \operatorname{Range} \gamma_h + 2\pi \mathbb{Z}^{N-1} = \emptyset$.

Set $\sigma_k(\tau) = (\gamma_k(\tau), \tau) \in \mathbb{R}^N$ and define the 2π-periodic analytic function $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(\tau) = -\frac{1}{T} I_0'(\theta_{\sigma_k(\tau)}) e_N.$$

Since

$$c - h_k(\tau) = \frac{1}{T} \mathscr{S}_N(\sigma_k(\tau)),$$

we have that θ_α is a critical point for I_c (i.e. a solution for $(1)_c$) if and only if there exists $k \in \{1,\dots,2^{N-1}\}$ and $\tau \in \mathbb{R}$ such that $\alpha = \sigma_k(\tau)$ and $h_k(\tau) = c$. Set $\hat{d}_k = \min_\tau h_k$ and $\hat{D}_k = \max_\tau h_k$. It is easy to check that $h_k(\pi/4) < 0 < h_k(\frac{3}{4}\pi)$, so $\hat{d}_k < 0 < \hat{D}_k$ for all $k = 1,\dots,2^{N-1}$. Therefore if $c \in (\hat{d}_k, \hat{D}_k)$ by the 2π-periodicity of h_k we obtain $\tau_{k,1} \neq \tau_{k,2} \in [0, 2\pi)$ with $h_k(\tau_{k,i}) = c$, $i = 1, 2$.

More generally, if $c \in \bigcap_{j=1}^{p} (\hat{d}_{k_j}, \hat{D}_{k_j})$ for some $p \geqq 1$, then there exist $\tau_{k_{j,1}} \neq \tau_{k_{j,2}} \in [0, 2\pi)$ with the property $h_{k_j}(\tau_{k_{j,i}}) = c$, $i = 1, 2$ and $j = 1,\dots,p$. From property (3) above we know that for $j \neq r$, $\sigma_{k_j}(\tau_{k_{j,i}}) \neq \sigma_{k_r}(\tau_{k_{r,s}}) + 2\pi k$ for all $\mathbb{Z}^N$, $i, s = 1, 2$. So we obtain $2p$ distinct solutions for $(1)_c$ in this situation.

Let $i_1,\dots,i_{2^{N-1}}$ and $i'_1,\dots,i'_{2^{N-1}}$ permutations of $\{1,\dots,2^{N-1}\}$ such that

$$\hat{d}_{i_1} \leqq \dots \leqq \hat{d}_{i_{2^{N-1}}} < 0 < \hat{D}_{i'_{2^{N-1}}} \leqq \dots \leqq \hat{D}_{i'_1}.$$

Set $d_k = \hat{d}_{i_k}$ and $D_k = \hat{D}_{i'_k}$, $k = 1,\dots,2^{N-1}$. If $c \in (d_j, D_j)$ for $j = 1,\dots,2^{N-1}$, then $c \in (d_k, D_k)$ for every $k = 1,\dots,j$. Therefore by the previous observations it is easy to obtain $2j$ distinct solutions for $(1)_c$ in this case.

If $c = d_j = \dots = d_{j+r-1}$ with $1 \leqq r \leqq 2^{N-1} - j$ our argument readily gives r distinct solutions for $(1)_c$. Furthermore, if $j \geqq 2$ then $c \in (d_{j-1}, 0) \subset (d_{j-1}, D_{j-1})$ from which we obtain another $2(j-1)$ additional solutions. The case $c = D_j = \dots = D_{j+r-1}$, $1 \leqq r \leqq 2^{N-1} - j$ follows similarly. ∎

Now Theorem 1 is an easy consequence of Theorem 2, since the conditions $(T)_1,\dots,(T)_3$ are surely satisfied as $T \mapsto 0$.

References

1. Chang, K. C., Long, Y., Zehnder, E.: Forced oscillations for the triple pendulum. preprint
2. Conley, C. C., Zehnder, E.: The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold. Invent. Math. **73**, 33–49 (1983)
3. Fournier, G., Willem, M.: Multiple solutions of the forced double pendulum equation. preprint
4. Palais, R.: The Lusternik–Schnirelman theory on Banach manifolds. Topology **5**, 115–132 (1966)
5. Rabinowitz, P.: On a class of functionals invariant under a $\mathbb{Z}^n$ action. Trans. AMS **310**, 303–311 (1988)
6. Tarantello, G.: Remarks on forced equations of the double pendulum type. Trans. AMS (to appear)

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