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Endomorphism Valued Cohomology and Gauge-Neutral Matter

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Abstract. For several physically interesting Calabi-Yau manifolds, we count and parametrize gauge-neutral matter particles occurring in corresponding superstring compactifications. To this end, we use the technique of exact and spectral sequences and then describe and discuss our results in the more familiar tensor notation.

0. Preliminaries

An appreciable subset of consistent and possibly realistic superstring models is constructed on an "internal," complex 3-dimensional Calabi-Yau manifold [1], denoted $\mathcal{M}$. The particles of the low energy effective model correspond to elements of various cohomology groups on $\mathcal{M}$. For a very large family of (three dimensional) Calabi-Yau manifolds all relevant such groups have been determined in the literature [2], except for $H^1(\mathcal{M}, \operatorname{End}\mathcal{T}_{\mathcal{M}})$, where $\operatorname{End}\mathcal{T}_{\mathcal{M}}$ denotes the bundle of *traceless* endomorphisms of $\mathcal{T}_{\mathcal{M}}$, the holomorphic tangent bundle of $\mathcal{M}$. Elements of $H^1(\mathcal{M}, \operatorname{End}\mathcal{T}_{\mathcal{M}})$ correspond to matter particles which are neutral with respect to any Yang-Mills gauge interaction but interact directly with the particles of the standard model. Even though these particles tend to receive large masses, they may have a desirable phenomenological impact [3].

In this paper we determine $H^*(\mathcal{M}, \operatorname{End}\mathcal{T}_{\mathcal{M}})$ for a number of Calabi-Yau manifolds that lead to phenomenologically interesting models. We relate the elements of $H^1(\mathcal{M}, \operatorname{End}\mathcal{T}_{\mathcal{M}})$ to cohomological data entirely on an ambient space $\mathcal{W}$, in which $\mathcal{M}$ is embedded. To this end we use the technique of exact and spectral sequences (TESS) as in [2], except that we shall now employ it to its full extent instead of using only the vanishing part.

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For the reader unacquainted with TESS, Sect. 1 will present its barest essentials [1] following [2, 5]. This will establish our notation and conventions and we also describe the standard plan of attack for such a computation. To develop familiarity with TESS, we first unleash it on a pet example – a quintic hypersurface in $\mathbb{P}^4$ (Sect. 2).

In Sects. 3–5, we study the physically most interesting Calabi-Yau manifolds [2]:

$$\begin{bmatrix} 2 \| 3 \\ 2 \| 3 \end{bmatrix}^{2;83}_{-162}, \quad \begin{bmatrix} 3 \| 3 & 1 \\ 2 \| 0 & 3 \end{bmatrix}^{8;35}_{-54}, \quad \begin{bmatrix} 3 \| 3 & 0 & 1 \\ 3 \| 0 & 3 & 1 \end{bmatrix}^{14;23}_{-18}. \tag{0.1}$$

The first and the third were utilized by G. Tian and S.-T. Yau while R. Schimmrigk used the second one to construct a multiply connected Calabi-Yau manifold of the phenomenologically favourable $|\chi_E| = 6$ [7]. Section 6 contains a summary of our results and, understanding that exact and spectral sequences still mystify most of the physics audience, also a description using conventional tensor notation. Comparison with existing literature [8, 9, 10] is done *en route*.

1. Notation and Technique

In this article we focus on the three Calabi-Yau manifolds (0.1) but nothing in principle obstructs repeating our computations for any complete intersection Calabi-Yau (CICY) manifold [11, 6] $\mathcal{M}$. Indeed, the method is not restricted to Calabi-Yau manifolds – in principle one can use it to compute the cohomology of any complete intersection inside any product of compact complex homogeneous spaces with coefficients in any bundle arising by restriction of a homogeneous bundle combined with any bundle induced from the tangent bundle on $\mathcal{M}$. In particular, all Hodge numbers can be determined.

Whilst the method is "elementary" (since, as we shall see, it boils down to "mere linear algebra"), that is not to say that it is simple to apply. The implementation of the method can be *extremely* tedious. Perhaps it should be mechanized. In the case of our three examples, each is embedded in a $\mathscr{W} \stackrel{\text{def}}{=} \mathbb{P}_1^{n_1} \times \dots \mathbb{P}_m^{n_m}$ by means of a system of homogeneous holomorphic polynomial constraints $\{f(z)=0\}$. In this section we describe how the cohomology on $\mathcal{M}$ is related to that on $\mathscr{W}$.

1.1. Submanifold Cohomology from the Ambient One

The technique of exact and spectral sequences (TESS) is just tailored to relate the cohomology on $\mathcal{M}$ with cohomological data entirely on $\mathscr{W}$. It hinges on two relations. The first one is captured by the short exact sequence

$$0 \longrightarrow \mathscr{T}_{\mathcal{M}} \stackrel{\iota}{\longrightarrow} \mathscr{T}_{\mathscr{W}}|_{\mathcal{M}} \stackrel{f'}{\longrightarrow} \mathscr{E}|_{\mathcal{M}} \longrightarrow 0 \tag{1.1}$$

[1] For brevity we do assume at least nodding acquaintance with the application of exact sequences and accompanying long exact cohomology sequences. Should need arise, we refer the reader to consult [2] as an introduction and [4] for complete details

[2] Following [6], the [|("bra") column displays the dimensions of the $\mathbb{P}^n$ factors in the embedding space $\mathscr{W}$ as row-entries. Each column in the |] ("ket") part represents a constraint polynomial $f(z)$ with the r^{th} row-entry in the column being the degree of homogeneity of $f(z)$ with respect to the $\mathbb{P}_r^{n_r}$ factor in $\mathscr{W}$. We display the Hodge numbers b_{11} and b_{21} in the superscript and the Euler characteristic χ_E in the subscript

of vector bundles over $\mathcal{M}$, where $\mathcal{E}$ is the bundle over $\mathcal{W}$ a section of which defines $\mathcal{M}$ as its zero locus. Restricting this bundle to $\mathcal{M}$ gives the normal bundle of the embedding. This is the content of the sequence (1.1).

Reminder. In a sequence

$$\ldots \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \longrightarrow \ldots .$$

The image of α, denoted $\mathrm{im}\,\alpha$, consists of all elements $b_\alpha \in B$ for each of which there is some $a \in A$ such that $\alpha(a) = b_\alpha$; the kernel of β, denoted $\ker\beta$, consists of all $b^\beta \in B$ which are annihilated by β. The sequence is exact precisely if $\ker\beta = \mathrm{im}\,\alpha$, $\ker\gamma = \mathrm{im}\,\beta$, etc. Since $\{b^\beta\} = \{b_\alpha\} = \{\alpha(a)\}$, $\ker\beta$ can be identified with a quotient of A with whatever was mapped into A by the map immediately preceding it. Similarly, the cokernel of β, denoted $\mathrm{cok}\,\beta$, is the quotient $\{C/\mathrm{im}\,\beta\}$ and can be identified with $\mathrm{im}\,\gamma \subset D$.

Our complete intersection Calabi-Yau manifolds are defined by a number of polynomial equations. Thus, the bundle $\mathcal{E}$ is simply the direct sum of line bundles. Sequence (1.1) is equivalent to equating the quotient $\{\mathcal{T}_{\mathcal{W}}|_{\mathcal{M}}/\mathcal{T}_{\mathcal{M}}\}$ with $\mathcal{E} \stackrel{\mathrm{def}}{=} \oplus \mathcal{E}_f$, where each defining polynomial $f(z)$ is a section of a corresponding line bundle $\mathcal{E}_f$ over $\mathcal{W}$. Sequence (1.1) induces a long exact cohomology sequence

$$\ldots \longrightarrow H^q(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \xrightarrow{\iota} H^q(\mathcal{M}, \mathcal{T}_{\mathcal{W}}) \xrightarrow{f'} H^q(\mathcal{M}, \mathcal{E}) \longrightarrow H^{q+1}(\mathcal{M}, \mathcal{T}_{\mathcal{M}}) \longrightarrow \ldots$$

of which the q^{th} cohomology group we stack, by increasing q, below the corresponding bundle; the sequence threads from left to right, row by row downwards. We specify the map f' more precisely in Appendix A but suffice it here to remark that it may be viewed as an element of $(\mathcal{T}^*_{\mathcal{W}} \otimes \mathcal{E})|_{\mathcal{M}}$.

The other relation is encoded in the sheaf exact Koszul sequence

$$0 \longrightarrow \wedge^K \mathcal{E}^* \xrightarrow{f} \ldots \xrightarrow{f} \mathcal{E}^* \xrightarrow{f} \mathcal{O}_{\mathcal{W}} \xrightarrow{\varrho} \mathcal{O}_{\mathcal{M}} \longrightarrow 0, \qquad (1.2)$$

providing a *resolution of* $\mathcal{O}_{\mathcal{M}}$, the structure sheaf of $\mathcal{M}$. All the maps preceding ϱ are induced by contracting an element of $\wedge^k\mathcal{E}^*$ with f, the section of a vector bundle $\mathcal{E}$. The sequence (1.2) of sheaves of germs of holomorphic functions valued in the indicated bundles (which we denote by the same symbols hoping to cause no undue confusion) is exact over all of $\mathcal{W}$ and remains exact if tensored by (the sheaf of holomorphic functions valued in) any vector bundle $\mathcal{V}$ over $\mathcal{W}$.

From the sheaf exact sequence (1.2) tensored by any $\mathcal{V}$, $H^*(\mathcal{M}, \mathcal{V})$ is determined using the accompanying spectral sequence $\{E_i^{*,*}(\mathcal{V}), d_i\}$. Suffice it here to give only the algorithm for this computation; see [2, 4] for further information.

For the exact sequence (1.2) tensored by $\mathcal{V}$:

1. Compute the chart of cohomology groups $E_1^{q,k}(\mathcal{V}) \stackrel{\mathrm{def}}{=} H^q(\wedge^k\mathcal{E}^* \otimes \mathcal{V})$, for $q = 0, \ldots, \dim\mathcal{W}$ and $k = 0, \ldots, K$. We shall display the q^{th} row $E_1^{q,k}(\mathcal{V})$ (for $k = K, \ldots, 0$) below the sequence (1.2), stacking the rows by increasing q.
2. Set $i = 1$.
3. Find all non-vanishing differentials $d_i : E_i^{q,k}(\mathcal{V}) \to E_i^{q-i+1,k-i}(\mathcal{V})$.

4. Set

$$E_{i+1}^{q,k}(\mathscr{V})=\left\{\ker E_i^{q,k}(\mathscr{V})\xrightarrow{d_i}E_i^{q-i+1,k-i}(\mathscr{V})/d_i(E_i^{q+i-1,k+i}(\mathscr{V}))\right\},\qquad \forall q,k$$

and shift $i\mapsto i+1$.
5. Go to 3, unless $i>K$, in which case exit.

The result of the algorithm is the chart $E_{K+1}^{*,*}(\mathscr{V})$ of groups often denoted $E_\infty^{*,*}(\mathscr{V})$. These groups abut to the $\mathscr{V}$-valued cohomology on $\mathscr{M}$, inasmuch as $H^q(\mathscr{M},\mathscr{V})$ may be thought of as "composed" of the $E_\infty^{q+k,k}(\mathscr{V})$ $(k=0,\ldots,K)$. We note however that there is no way to identify this with the *direct* sum $\bigoplus_{k=0}^{K} E_\infty^{q+k,k}(\mathscr{V})$. Nevertheless, it does follow that

$$\operatorname{rank} H^q(\mathscr{M},\mathscr{V})=\sum_{k=0}^{K}\operatorname{rank} E_\infty^{q+k,k}(\mathscr{V}). \tag{1.3}$$

Clearly, the determination of d_i and their action (so that one knows how to form the quotients in step 4) is the backbone of this technique. The crucial point is that all differentials d_i are determined by the defining polynomials $\{f\}$ and, in principle, all the steps of the algorithm can always be completed. The examples soon to come will, we hope, clarify how this is accomplished. Whilst similar computations can be carried out for all CICY manifolds, they can easily grow out of hand. To use this method effectively requires considerable practise. Even so, the computations for the Schimmrigk and Tian-Yau manifolds took several days. It would probably be better to mechanize the procedure with a computer program.

Note that when $K=1$, the sequence (1.2) becomes a short exact sequence and induces a long exact cohomology sequence. The latter can now be understood as a collapsed spectral sequence in which there are only level-1 differentials $d_1=f$ and $E_\infty^{*,*}=E_2^{*,*}$.

1.2. The Plan of Attack

By tensoring Sequence (1.1) with $\mathscr{T}_\mathscr{M}^*$ and the dual of Sequence (1.1) with $\mathscr{E}$ and with $\mathscr{T}_\mathscr{W}$ respectively, we obtain three short exact sequences that fit together into the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathscr{T}_\mathscr{W}\otimes\mathscr{E}^*)|_\mathscr{M} & & (\mathscr{E}\otimes\mathscr{E}^*)|_\mathscr{M} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathscr{T}_\mathscr{W}\otimes\mathscr{T}_\mathscr{W}^*)|_\mathscr{M} & & (\mathscr{E}\otimes\mathscr{T}_\mathscr{W}^*)|_\mathscr{M} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{T}_\mathscr{M}\otimes\mathscr{T}_\mathscr{M}^* & \longrightarrow & \mathscr{T}_\mathscr{W}|_\mathscr{M}\otimes\mathscr{T}_\mathscr{M}^* & \longrightarrow & \mathscr{E}|_\mathscr{M}\otimes\mathscr{T}_\mathscr{M}^* & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{1.4}$$

with exact rows and columns. Thus the vector bundle $\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$ over $\mathcal{M}$, intrinsic to $\mathcal{M}$, is related to the restriction to $\mathcal{M}$ of various vector bundles over $\mathcal{W}$.

The short exact sequences appearing in the diagram (1.4) induce long exact cohomology sequences which are interwoven accordingly. From here, we relate $H^*(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$ to the $(\mathcal{E}\otimes\mathcal{E}^*)|_{\mathcal{M}}$, $(\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}})|_{\mathcal{M}}$ and $(\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{W}})|_{\mathcal{M}}$-valued cohomology groups on $\mathcal{M}$. The latter cohomology groups are determined from appropriate spectral sequences in terms of cohomological data entirely on $\mathcal{W}$. To complete the plan, we shall need to evaluate various cohomology groups on $\mathcal{W}$. Using the Künneth formula, these are given in terms of cohomology groups on the $\mathbb{P}^{n_r}_r$ factors which we now discuss.

1.3. Homogeneous Bundles and Cohomology on $\mathbb{P}^n$

The complex projective space $\mathbb{P}^n=\mathbb{P}(\mathbb{C}^{n+1})$ is the space of complex lines in $\mathbb{C}^{n+1}$, i.e., a point in $\mathbb{P}^n$ is a 1-dimensional linear subspace $L\subset\mathbb{C}^{n+1}$. Roughly speaking then, a holomorphic vector bundle over $\mathbb{P}^n$ is a holomorphically varying family of vector spaces parametrized by $L\in\mathbb{P}^n$. One can therefore specify such a vector bundle by assigning, in a holomorphic manner, a vector space to each $L\in\mathbb{P}^n$.

Consider the assignment:

$$L\mapsto L^a\otimes(b_1\ldots b_n)[\mathbb{C}^{n+1}/L],$$

where L^a denotes the $|a|$-fold tensor product of L (L^*, the dual of L, if $a<0$) and is still 1-dimensional. By $(b_1\ldots b_n)$, where $b_i\leqq b_{i+1}$, for $i=1,\ldots,(n-1)$, we denote the $U(n)$ Young tableau [3] with b_i boxes in the i^{th} row (from its bottom), sticking out to the right (left if $b_i<0$) of the vertical "spine" of the tableau. Alternatively, noting that $(1\ldots1)=\det$, we also have that $(b_1\ldots b_n)=\det^{b_1}(0(b_2-b_1)\ldots(b_n-b_1))$ which the unsettled reader may utilize to avoid Young tableaux with boxes to the left. The vector bundle corresponding to the assignment above shall be denoted by $(a|b_1\ldots b_n)$ and we write the same for its sheaf of germs of sections.

In fact, all irreducible homogeneous vector bundles over $\mathcal{P}^n$ can be represented in this way. We note that $(a|b_1\ldots b_n)^*=(-a|-b_n\ldots-b_1)$ and that tensor products of vector bundles can now be manipulated as Kronecker products of Young tableaux. The decomposition $\mathrm{Y}\otimes\mathrm{Y}'=\bigoplus_i \mathrm{Y}_i$ then corresponds to a (holomorphic) decomposition of a tensor product of two homogeneous bundles into a direct sum of irreducible homogeneous bundles. E.g.:

$$(1|-10\ldots0)\otimes(-1|0\ldots01)=(0|0\ldots0)\oplus(0|-10\ldots01),$$

corresponds to

$$\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{T}^*_{\mathbb{P}^n}=[\operatorname{tr}(\mathcal{T}_{\mathbb{P}^n}\otimes\mathcal{T}^*_{\mathbb{P}^n})\approx\mathcal{O}]\oplus\operatorname{End}\mathcal{T}_{\mathbb{P}^n}. \tag{1.5}$$

For any homogeneous bundle $\mathcal{V}$ over $\mathbb{P}^n$, there exists a simple algorithm for determining the $\mathcal{V}$-valued cohomology over $\mathbb{P}^n$ [14] usually known as the Bott-

[3] Recall that $\mathbb{P}^n$ also equals the quotient $\{U(n+1)/(U(n)\times U(1))\}$. For a quick reference on Young Tableaux, see [13]

Borel-Weil theorem (BBW, for short). In the notation just reviewed, for $(a|b_1\dots b_n)$, the algorithm consists of [5]:

1. Add the sequence $0,1,\dots,n$ to the respective entries in $(a|b_1\dots b_n)$.
2. If any two entries in the result of Step 1 are equal, all cohomology vanishes; otherwise proceed.
3. Swap the minimum number (q) of neighbouring entries required to produce a strictly increasing sequence.
4. Subtract the sequence $0,1,\dots,n$ from the result of 3, to obtain $(\beta_0\beta_1\dots\beta_n)$, where $\beta_\alpha \leqq \beta_{\alpha+1}$, for $\alpha=0,\dots,(n-1)$.

Then $H^q(a|b_1\dots b_n)=(\beta_0\beta_1\dots\beta_n)[\mathbb{C}^{n+1}]$ and all other $(a|b_1\dots b_n)$-valued cohomology vanishes.

This tic-tac-toe algorithm not only determines the non-vanishing cohomology groups and their dimension, but also assigns a Young tableau to each. Indeed, if $\mathscr{V}$ is any homogeneous vector bundle on $\mathbb{P}^n$, then $H^q(\mathbb{P}^n,\mathscr{V})$ provides a finite dimensional representation of $GL(n+1,\mathbb{C})$ and the Young tableau which results from the BBW algorithm identifies this representation (in case $\mathscr{V}$ is irreducible). For example,

$$H^0(-k|0\dots 0)=(-k0\dots 0)[\mathbb{C}^{n+1}]\sim f_{(a_1\dots a_k)}\,.$$

As usual, parentheses around indices denote symmetrization. These represent sections of the k^{th} power of the hyperplane bundle, corresponding to k^{th} order polynomials $f_{a_1\dots a_k}z^{a_1}\dots z^{a_k}$. Similarly, $H^0(-1|0\dots 1)=(-10\dots 1)\ [\mathbb{C}^{n+1}]\sim\lambda_a^b$, with $\operatorname{tr}[\lambda_a^b]=0$. These represent sections of the tangent bundle corresponding to linear reparametrizations generated by $z^a\lambda_a^b\partial_b$, where $\partial_b=\partial/\partial z^b$.

In view of the Künneth formula,

$$H^q(X\times Y,\mathscr{V})=\bigoplus_{\gamma=0}^{q} H^\gamma(X,\mathscr{V}|_X)\otimes H^{q-\gamma}(Y,\mathscr{V}|_Y)\,,$$

we now have all the cohomology valued in homogeneous bundles over $\mathscr{W}$ at our fingertips. Through Sequence (1.2) and the sequences in the diagram (1.4), so is $H^1(\mathscr{M},\operatorname{End}\mathscr{T}_{\mathscr{M}})$.

1.4. Some Auxiliary Information

It will be very handy to make use of Serre duality:

$$H^q(X,\mathscr{V})^*=H^{\dim X-q}(X,\mathscr{V}^*\otimes\mathscr{K}_X)\,, \tag{1.6}$$

where $\mathscr{K}_X$ is the canonical bundle of X. For a Calabi-Yau manifold $\mathscr{M}$, the first Chern class vanishes and it follows that $\mathscr{K}_{\mathscr{M}}$ is holomorphically trivial. Serre duality is thereby simplified. For a CICY we can be more precise. Consider, for example, the case of a hypersurface of degree $n+1$ in $\mathbb{P}^n$. In the short exact sequence (1.1), $\mathscr{E}$ is the line bundle $(-(n+1)|0\dots 00)$ whilst $\mathscr{T}_{\mathscr{W}}=(-1|0\dots 01)$. It follows that $\wedge^{n-1}\mathscr{T}_{\mathscr{M}}=(1|1\dots 11)|_{\mathscr{M}}$. Thus, we can see directly that $\mathscr{K}_{\mathscr{M}}\equiv\wedge^{n-1}\mathscr{T}^*_{\mathscr{M}}$ is holomorphically trivial, being the restriction to $\mathscr{M}$ of $(-1|-1\dots -1-1)$. Note that although this is trivial as a holomorphic line bundle, it is not trivial as a *homogeneous* bundle. The above relation (1.6) becomes:

$$H^q(\mathcal{M},\mathcal{V})=H^{n-q}(\mathcal{M},\mathcal{V}^*)^*\otimes(11\ldots11) \tag{1.7}$$

and often provides a useful check of direct computations. The general case of a CICY in $\mathbb{P}_1^{n_1}\times\ldots\mathbb{P}_m^{n_m}$ may be treated similarly. The triviality of the canonical bundle comes down to the condition (in the notation of [6]) that each row of the "ket" sum to one more than the corresponding entry of the "bra."

We also quote [15]

Theorem 1. *For a stable holomorphic vector bundle $\mathcal{V}$ over a compact projective manifold X*

$$H^0(X,\mathrm{End}\,\mathcal{V})=0\,.$$

Since the tangent bundle of a Calabi-Yau manifold is stable, $H^0(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=0$ and

$$\begin{aligned}H^0(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}_{\mathcal{M}}^*)&=H^0(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}}\oplus\mathcal{O})=H^0(\mathcal{M})=\mathbb{C}\,,\\ H^1(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}_{\mathcal{M}}^*)&=H^1(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}}\oplus\mathcal{O})=H^1(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})\,.\end{aligned} \tag{1.8}$$

$H^2(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})$ is now determined by Serre duality and we shall use this to check our direct computations.

2. A Pet Manifold

As a warm-up pet example and an opportunity to explain the application of TESS in full force, consider a smooth quintic hypersurface in $\mathbb{P}^4$, $\mathcal{M}\in[4,5]$:

$$\mathcal{M}\hookrightarrow\mathbb{P}^4: f(z)\overset{\text{def}}{=}f_{abcde}z^az^bz^cz^dz^e=0\,.$$

Being a quintic polynomial, $f(z)$ is a section of the line bundle $\mathcal{E}=(-5|0000)$.

2.1. Submanifold Cohomology from the Ambient Cohomology

2.1.1. $\mathcal{E}\otimes\mathcal{E}^*=(0|0000)$

Since $\mathcal{E}$ is a line bundle over $\mathbb{P}^4$, $\mathcal{E}\otimes\mathcal{E}^*=\mathcal{O}$ is the trivial line bundle on $\mathbb{P}^4$. Since there is only one defining function, sequence (1.2) is a short exact sequence

$$0\to(5|0000)\to\mathcal{O}_{\mathcal{W}}\to\mathcal{O}_{\mathcal{M}}\to0\,,$$

the corresponding spectral sequence is therefore the long exact sequence on cohomology and, by BBW,

$$H^0(\mathcal{M})=(00000)\,,\qquad H^3(\mathcal{M})=(11111)$$

and all others vanish. Note that this is consistent with Serre duality (1.7).

2.1.2. $\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*}=(-4|-1000)$

Tensoring the exact sequence (1.2) with $\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*}$ (recall: $K=1$) yields[4]:

q	$(1\|-1000)\xrightarrow{f}$	$(-4\|-1000)\xrightarrow{\varrho}$	$(\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})\|_{\mathscr{M}}$
0	0	$(-4-1000)_{224}$	$H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})$
1	$(00000)_1$	0	$\Rightarrow 0$
2	0	0	$\Rightarrow 0$
3	0	0	$\Rightarrow 0$
4	0	0	$\equiv 0$

From the first two rows,

$$
\begin{array}{ccccccccc}
0\longrightarrow & H^0(-4|-1000)_{224} & \longrightarrow & H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*}) & \longrightarrow & H^1(1|-1000)_1 & \longrightarrow 0\,, \\
 & \| & & & & \| & \\
 & (-4-1000)_{224} & & & & (00000)_1 &
\end{array}
$$

where the subscripts denote the dimensions of the respective cohomology groups (represented by the Young tableaux in the chart) so that $\dim H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})=224+1=225$. Note that this does not determine $H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})$ uniquely in terms of $H^0(-4|-1000)$ and $H^0(1|-1000)$ (see p. 168 of Bott and Tu in [4]) but we write

$$H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})=(00000)_1+(-4-1000)_{224}$$

instead of the Sequence (2.1) and reverve "$\oplus$" for the direct sum.

We note that $H^q(\mathscr{M},\mathscr{T}_{\mathscr{W}}\otimes\mathscr{E}^*)=H^{3-q}(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})^*\otimes(11111)$ by Serre duality (1.7) and, from the previous result on $H^q(\mathscr{M},\mathscr{E}\otimes\mathscr{T}_{\mathscr{W}}^{*})$,

$$H^3(\mathscr{M},\mathscr{T}_{\mathscr{W}}\otimes\mathscr{E}^*)=(11125)_{224}+(11111)_1\,,$$

and all other $\mathscr{T}_{\mathscr{W}}\otimes\mathscr{E}^*$-valued cohomology vanishes.

2.1.3. $\mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}_{\mathscr{W}}^{*}=(0|-1001)\oplus(0|000)$

The trace part $(0|0000)$ is trivial and we can borrow the results for $\mathscr{E}\otimes\mathscr{E}^*$ from above. For the traceless, $\mathrm{End}\,\mathscr{T}_{\mathscr{W}}$ part, we tensor Sequence (1.2) with $(0|-1001)$. Since both $H^*(5|-1001)$ and $H^*(0|-1001)$ vanish, so does $H^*(\mathscr{M},\mathrm{End}\,\mathscr{T}_{\mathscr{W}})$ and so

$$H^*(\mathscr{M},\mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}_{\mathscr{W}}^{*})=H^*(\mathscr{M})\,.$$

2.2. *Towards* $H^*(\mathscr{M},\mathrm{End}\,\mathscr{T}_{\mathscr{M}})$

With the information we have collected, we return to the diagram (1.4).

4 To save some space, we shall abbreviate $H^q(\mathbb{P}^n,(a|b_1\ldots b_n))$ to $H^q(a|b_1\ldots b_n)$ and likewise omit explicitly written products of $\mathbb{P}^n$'s. Our notation for the cohomology on the submanifold $\mathscr{M}$ shall, however, always contain the symbol $\mathscr{M}$

2.2.1. $H^*(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})$

From the rightmost column of the diagram (1.4), we now have that

q	$(\mathcal{E}\otimes\mathcal{E}^*)\vert_{\mathcal{M}}$	$\xrightarrow{f'}$	$(\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}})\vert_{\mathcal{M}}$	$\xrightarrow{\iota^\dagger}$	$\mathcal{E}\vert_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$
0	(00000)	$\xrightarrow{f'}$	[(00000)+(−4−1000)]	$\longrightarrow$	$H^0(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})$
1	0		0		$\Rightarrow 0$
2	0		0		$H^2(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})$
3	(11111)		0		$\Rightarrow 0$

The rows $q=1, 2$, and 3 tell us that $H^2(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})=(11111)_1$ and $H^q(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})$ vanishes for $q=1,3$. By exactness merely, f' must be $1-1$ for $q=0$, so that $H^0(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})$ is the quotient of the sum in the middle and the image f' (00000), whence $\dim H^0(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{M}})=225-1=224$. In fact, as computed in Appendix A, f' maps this $(00000)=H^0(0|0000)$ onto the $(00000)=H^1(1|-1000)$ appearing as a quotient in the sequence (2.1) above. Thus, we may conclude that

$$H^0(\mathcal{M}, \mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}})=(-4-1000)_{224}\,.$$

2.2.2. $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$

From the central column of the diagram (1.4), we have that

q	$(\mathcal{T}_{\mathcal{W}}\otimes\mathcal{E}^*)\vert_{\mathcal{M}}$	$\xrightarrow{f'}$	$(\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{W}})\vert_{\mathcal{M}}$	$\xrightarrow{\iota^\dagger}$	$\mathcal{T}_{\mathcal{W}}\vert_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$
0	0		(00000)	$\longrightarrow$	$H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$
1	0		0		$\Rightarrow 0$
2	0		0		$H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$
3	[(11111)+(11125)]	$\xrightarrow{f'}$	(11111)	$\longrightarrow$	$H^3(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$

Here, the isomorphism $H^0(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})=(00000)$ is readily obtained from row $q=0$ while the next row immediately tells us that $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})=0$. However, to determine $H^q(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$, for $q=2$, and 3 we must again find out what precisely the action of f' is. The argument of Appendix A may be adapted and we conclude that $H^3(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$ vanishes and that $H^2(\mathcal{M}, \mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})=(11125)_{224}$.

2.2.3. Finally, $H^*(\mathcal{M}, \mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$

From the base row of the diagram (1.4), we now have that

q	$\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$	$\longrightarrow$	$\mathcal{T}_{\mathcal{W}}\vert_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$	$\xrightarrow{f'}$	$\mathcal{E}\vert_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}}$
0	$H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$	$\longrightarrow$	(00000)	$\xrightarrow{f'_0}$	(−4−1000)
1	$H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$		0		0
2	$H^2(\mathcal{M}, \mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$	$\longrightarrow$	(11125)	$\xrightarrow{f'_2}$	(11111)
3	$H^3(\mathcal{M}, \mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})$		0		0

The only invariant element at our disposal is $f\in(-50000)$. In particular, there is no invariant element in $(-4-1000)$ and so f_0' is obliged to vanish. We may conclude that $H^0(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})=(00000)$ and $H^1(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})=(-4-1000)$. Similarly, f_2' must vanish and finally we obtain that

$$\begin{aligned}
&H^0(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=0\,,\\
&H^1(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=(-4-1000)_{224}\,,\\
&H^2(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=(11125)_{224}\,,\\
&H^3(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=0\,.
\end{aligned}\tag{2.2}$$

Notice agreement with Eq. (1.8) and with Serre duality (1.7). Also, we have just computed $\dim H^1(\mathcal{M},\mathrm{End}\,\mathcal{T}_{\mathcal{M}})=224$, in agreement with [8, 9].

3. The Bi-cubic Hypersurface in $\mathbb{P}^2\times\mathbb{P}^2$

Among the physically more interesting manifolds, we start with $\mathcal{M}\in\begin{bmatrix}2\,\|\,3\\2\,\|\,3\end{bmatrix}^{2;83}_{-162}$, i.e., the space of solutions of a bi-cubic constraint

$$f(x,y)\stackrel{\mathrm{def}}{=}f_{abc\alpha\beta\gamma}x^ax^bx^cy^\alpha y^\beta y^\gamma=0\,.$$

The polynomial f is a section of $\mathcal{E}=\begin{pmatrix}-3 & 0 & 0\\ -3 & 0 & 0\end{pmatrix}$, in the obvious notation for the tensor product $(-3|00)_x\otimes(-3|00)_y$. The tangent bundle is $\mathcal{T}_{\mathcal{W}}=\mathcal{T}_x\oplus\mathcal{T}_y$, where $\mathcal{T}_x=\begin{pmatrix}-1 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and $\mathcal{T}_y=\begin{pmatrix}0 & 0 & 0\\ -1 & 0 & 1\end{pmatrix}$. Therefore:

$$\begin{aligned}
\mathcal{E}\otimes\mathcal{E}^*&=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\\
\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}}&=\begin{pmatrix}-2 & -1 & 0\\ -3 & 0 & 0\end{pmatrix}\oplus\begin{pmatrix}-3 & 0 & 0\\ -2 & -1 & 0\end{pmatrix},\\
\mathcal{T}_{\mathcal{W}}\otimes\mathcal{E}^*&=\begin{pmatrix}2 & 0 & 1\\ 3 & 0 & 0\end{pmatrix}\oplus\begin{pmatrix}3 & 0 & 0\\ 2 & 0 & 1\end{pmatrix},\\
\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{W}}&=2\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\oplus\begin{pmatrix}-1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\oplus\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & 1\end{pmatrix}\oplus\begin{pmatrix}0 & -1 & 1\\ 0 & 0 & 0\end{pmatrix}\\
&\quad\oplus\begin{pmatrix}0 & 0 & 0\\ 0 & -1 & 1\end{pmatrix}.
\end{aligned}$$

3.1. Restricting the Ambient Cohomology to the Submanifold

We employ the spectral sequence separately for each summand. Again, since there is only one defining function, the Koszul resolution (1.2) is simply a short exact sequence

$$0\longrightarrow\begin{pmatrix}3 & 0 & 0\\ 3 & 0 & 0\end{pmatrix}\stackrel{f}{\longrightarrow}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\longrightarrow\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{\mathcal{M}}\longrightarrow 0$$

and the spectral sequence is equivalent to the corresponding long exact sequence on cohomology. As the BBW algorithm and the Künneth formula easily shows, the action of f in these computations is always zero ($E_1^{*,*}(\mathscr{V})=E_\infty^{*,*}(\mathscr{V})$ and the spectral sequence collapses). The following results are easily obtained for cohomology on $\mathscr{M}$:

Bundle	Non-zero cohomology	Bundle	Non-zero cohomology
$\left(\begin{array}{c\|cc}0&0&0\\0&0&0\end{array}\right)$	$H^0=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^1_1$ $H^3=\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^1_1$		
$\left(\begin{array}{c\|cc}-2&-1&0\\-3&0&0\end{array}\right)$	$H^0=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^1_1+\begin{pmatrix}-2&-1&0\\-3&0&0\end{pmatrix}^8_{10}$	$\left(\begin{array}{c\|cc}-3&0&0\\-2&-1&0\end{array}\right)$	$H^0=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^1_1+\begin{pmatrix}-3&0&0\\-2&-1&0\end{pmatrix}^{10}_8$
$\left(\begin{array}{c\|cc}2&0&1\\3&0&0\end{array}\right)$	$H^3=\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}^8_{10}+\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^1_1$	$\left(\begin{array}{c\|cc}3&0&0\\2&0&1\end{array}\right)$	$H^3=\begin{pmatrix}1&1&4\\1&2&3\end{pmatrix}^{10}_8+\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^1_1$
$\left(\begin{array}{c\|cc}-1&0&1\\1&-1&0\end{array}\right)$	$H^1=\begin{pmatrix}-1&0&1\\0&0&0\end{pmatrix}^8_1$ $H^2=\begin{pmatrix}1&1&1\\0&1&2\end{pmatrix}^1_8$	$\left(\begin{array}{c\|cc}1&-1&0\\-1&0&1\end{array}\right)$	$H^1=\begin{pmatrix}0&0&0\\-1&0&1\end{pmatrix}^1_8$ $H^2=\begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}^8_1$
$\left(\begin{array}{c\|cc}0&0&0\\0&-1&1\end{array}\right)$	All cohomology vanishes	$\left(\begin{array}{c\|cc}0&-1&1\\0&0&0\end{array}\right)$	All cohomology vanishes

The superscripts and subscripts are the dimensions of the $\mathbb{P}^2_x$- and $\mathbb{P}^2_y$-factor cohomology groups; the total dimension is therefore the product. Notice symmetry obtained by interchanging the factors $\mathbb{P}^2_x \leftrightarrow \mathbb{P}^2_y$ corresponding to swopping the rows above. Also there is Serre duality:

$$H^q(\mathscr{M},\mathscr{V})=H^{3-q}(\mathscr{M},\mathscr{V}^*)^*\otimes\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix} \tag{3.1}$$

[cf. Eq. (1.7)].

Before discussing the results derived from the diagram (1.4), note that the map $j:\mathscr{T}_\mathscr{W}\to\mathscr{E}$, corresponding to the two components $\mathscr{T}_\mathscr{W}=\mathscr{T}_x\oplus\mathscr{T}_y$, is furnished by the two differentials $dx\cdot(\partial f/\partial x)\sim\left(\begin{array}{c|cc}-2&-1&0\\-3&0&0\end{array}\right)$ and $dy\cdot(\partial f/\partial y)\sim\left(\begin{array}{c|cc}-3&0&0\\-2&-1&0\end{array}\right)$ (Appendix A can easily be adapted to give rigorous meaning and computational utility to these differentials). We shall denote these two components of j by f' and $\dot{f}$, respectively. From the rightmost column of the diagram (1.4) we obtain that $H^1(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_\mathscr{M})$ and $H^3(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_\mathscr{M})$ vanish, that $H^2(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_\mathscr{M})=H^4\left(\begin{array}{c|cc}3&0&0\\3&0&0\end{array}\right)^1_1$ and that $H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_\mathscr{M})$ is

$$\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^1_1+\left[\begin{pmatrix}-2&-1&0\\-3&0&0\end{pmatrix}^8_{10}\oplus\begin{pmatrix}-3&0&0\\-2&-1&0\end{pmatrix}^{10}_8\right]. \tag{3.2}$$

This is a vector space of dimension 161.

From the central column of the diagram (1.4), we obtain:

$$H^0(\mathscr{M},\mathscr{T}_\mathscr{W}\otimes\mathscr{T}^*_\mathscr{M})=2\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^1_1$$

$$H^1(\mathscr{M},\mathscr{T}_\mathscr{W}\otimes\mathscr{T}^*_\mathscr{M})=\begin{pmatrix}-1&0&1\\0&0&0\end{pmatrix}^8_1\oplus\begin{pmatrix}0&0&0\\-1&0&1\end{pmatrix}^1_8,$$

and

$$0\longrightarrow \begin{matrix}\begin{pmatrix}1&1&1\\0&1&2\end{pmatrix}^{1}_{8}\\ \begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}^{8}_{1}\end{matrix}\longrightarrow H^2(\mathcal{M},\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})\longrightarrow\left[\begin{matrix}\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}^{8}_{10}\\ \begin{pmatrix}1&1&4\\1&2&3\end{pmatrix}^{10}_{8}\end{matrix}+2\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^{1}_{1}\right]$$

$$\xrightarrow{j}2\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^{1}_{1}\longrightarrow H^3(\mathcal{M},\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})\longrightarrow 0.$$

From here on, the stacks represent direct sums. Along the argument in Appendix A, the two components of j, f', and $\dot{f}$, map $H^3\begin{pmatrix}2&\vline&0&1\\3&\vline&0&0\end{pmatrix}^{1}_{1}$ and $H^3\begin{pmatrix}3&\vline&0&0\\2&\vline&0&1\end{pmatrix}^{1}_{1}\left(=2\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^{1}_{1}\right)$ onto the two copies of $H^4\begin{pmatrix}3&\vline&0&0\\3&\vline&0&0\end{pmatrix}^{1}_{1}\left(\text{also }=2\begin{pmatrix}1&1&1\\1&1&1\end{pmatrix}^{1}_{1}\right)$. Computing as in Appendix A, this mapping turns out to be surjective. We thus find that $H^3(\mathcal{M},\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})$ vanishes and

$$H^2(\mathcal{M},\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{M}})=\begin{matrix}\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}^{8}_{10}\\ \begin{pmatrix}1&1&4\\1&2&3\end{pmatrix}^{10}_{8}\end{matrix}+\begin{matrix}\begin{pmatrix}1&1&1\\0&1&2\end{pmatrix}^{1}_{8}\\ \begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}^{8}_{1}\end{matrix},$$

a vector space of dimension 176.

3.2. The Endomorphism Valued Cohomology on the Submanifold

Finally, from the base row of the diagram (1.4), we obtain the accompanying long exact cohomology sequence, which falls into two sequences (dual to each other), the first of which is

$$0\longrightarrow H^0(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})\xrightarrow{\iota}2\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}\xrightarrow{j}\left[\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}+\begin{matrix}\begin{pmatrix}-2&-1&0\\-3&0&0\end{pmatrix}\\ \begin{pmatrix}-3&0&0\\-2&-1&0\end{pmatrix}\end{matrix}\right]$$

$$\xrightarrow{\delta}H^1(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})\xrightarrow{\iota}\begin{matrix}\begin{pmatrix}0&0&0\\-1&0&1\end{pmatrix}\\ \begin{pmatrix}-1&0&1\\0&0&0\end{pmatrix}\end{matrix}\longrightarrow 0.$$

The map j is again represented by the two components, f' and $\dot{f}$. Their action, however, cannot be $1-1$, since there are two copies of $\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$ to be mapped into only one copy of the same $\left(\text{it is impossible on the grounds of invariance to map into } \begin{pmatrix}-2&-1&0\\-3&0&0\end{pmatrix} \text{ or } \begin{pmatrix}-3&0&0\\-2&-1&0\end{pmatrix}\right)$. Nevertheless, j has maximal rank $(=1)$ and so

$$H^0(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})=\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}^{1}_{1},$$

in agreement with Eq. (1.8). This leaves

$$H^1(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})=\begin{array}{c}\begin{pmatrix}-1&0&1\\0&0&0\end{pmatrix}^8_1\\ \begin{pmatrix}0&0&0\\-1&0&1\end{pmatrix}^1_8\end{array}+\begin{array}{c}\begin{pmatrix}-2&-1&0\\-3&0&0\end{pmatrix}^8_{10}\\ \begin{pmatrix}-3&0&0\\-2&-1&0\end{pmatrix}^{10}_8\end{array}, \tag{3.3}$$

a vector space of dimension 176, as found in [9]. The remaining part of the exact sequence yields

$$H^2(\mathcal{M},\mathcal{T}_{\mathcal{M}}\otimes\mathcal{T}^*_{\mathcal{M}})=\begin{array}{c}\begin{pmatrix}1&2&3\\1&1&4\end{pmatrix}^8_{10}\\ \begin{pmatrix}1&1&4\\1&2&3\end{pmatrix}^{10}_8\end{array}+\begin{array}{c}\begin{pmatrix}1&1&1\\0&1&2\end{pmatrix}^1_8\\ \begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}^8_1\end{array},$$

in agreement with (3.1).

4. The Schimmrigk Manifold

Now we turn to $\mathcal{M}\in\left[\begin{array}{c||cc}3&3&1\\2&0&3\end{array}\right]^{8;35}_{-54}$. Let us denote the defining polynomials by

$$f(x)\stackrel{\text{def}}{=}f_{abc}x^ax^bx^c\,,\qquad g(x,y)\stackrel{\text{def}}{=}g_{a\alpha\beta\gamma}x^ay^\alpha y^\beta y^\gamma\,,$$

where $x\in\mathbb{P}^3$ and $y\in\mathbb{P}^2$. Here f and g are sections of $\left(\begin{array}{c|ccc}-3&0&0&0\\0&0&0&\end{array}\right)$ and $\left(\begin{array}{c|ccc}-1&0&0&0\\-3&0&0&\end{array}\right)$, respectively and $\mathcal{E}$ is now the rank-two direct sum of these two line bundles. The tangent bundle on $\mathcal{W}$ is given similarly as in the previous case,

$$\mathcal{T}_{\mathcal{W}}=\left(\begin{array}{c|ccc}-1&0&0&1\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}0&0&0&0\\-1&0&1&\end{array}\right).$$

We therefore have

$$\begin{aligned}
\mathcal{E}\otimes\mathcal{E}^*&=2\left(\begin{array}{c|ccc}0&0&0&0\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}-2&0&0&0\\3&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}2&0&0&0\\-3&0&0&\end{array}\right),\\
\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}}&=\left(\begin{array}{c|ccc}-2&-1&0&0\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}-3&0&0&0\\1&-1&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}0&-1&0&0\\-3&0&0&\end{array}\right)\\
&\quad\oplus\left(\begin{array}{c|ccc}-1&0&0&0\\-2&-1&0&\end{array}\right),\\
\mathcal{T}_{\mathcal{W}}\otimes\mathcal{E}^*&=\left(\begin{array}{c|ccc}2&0&0&1\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}3&0&0&0\\-1&0&1&\end{array}\right)\oplus\left(\begin{array}{c|ccc}0&0&0&1\\3&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}1&0&0&0\\2&0&1&\end{array}\right),\\
\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{W}}&=2\left(\begin{array}{c|ccc}0&0&0&0\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}-1&0&0&1\\1&-1&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}1&-1&0&0\\-1&0&1&\end{array}\right)\\
&\quad\oplus\left(\begin{array}{c|ccc}0&-1&0&1\\0&0&0&\end{array}\right)\oplus\left(\begin{array}{c|ccc}0&0&0&0\\0&-1&1&\end{array}\right).
\end{aligned}$$

4.1. Restricting the Ambient Cohomology to the Submanifold

Again, we compute the cohomology groups on $\mathscr{M}$ valued in each of these irreducible bundles by using the corresponding spectral sequence. To this end, note that the resolution (1.2) is now

$$0 \to \left(\begin{array}{c|ccc} 4 & 0 & 0 & 0 \\ 3 & 0 & 0 & \end{array}\right) \begin{array}{c} \xrightarrow{f} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & \end{array}\right) \xrightarrow{g} \\ \xrightarrow{g} \left(\begin{array}{c|ccc} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right) \xrightarrow{-f} \end{array} \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right) \to \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right)_{\mathscr{M}} \to 0. \tag{4.1}$$

The arrows labeled by f and g denote maps induced by contraction with the respective polynomials.

This time not all sequences converge at the first level and one has to determine the action of the spectral sequence differentials, all induced by f and g. For example, consider the spectral sequence for $\mathscr{E}_1 \otimes \mathscr{T}_y^* = \left(\begin{array}{c|ccc} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)$:

q	$\left(\begin{array}{c\|ccc} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & \end{array}\right)$	$\left(\begin{array}{c\|ccc} -2 & 0 & 0 & 0 \\ 4 & -1 & 0 & \end{array}\right)$ $\left(\begin{array}{c\|ccc} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)$	$\left(\begin{array}{c\|ccc} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)$	$\xrightarrow{\varrho}$ $(\mathscr{E}_1 \otimes \mathscr{T}_y^*)\|_{\mathscr{M}}$
0	0	0	0	$\Rightarrow 0$
1	0	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1 \xrightarrow{f}$	$\begin{pmatrix} -3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}^{20}_1$	see below
2	0	$\begin{pmatrix} -2 & 0 & 0 & 0 \\ & 0 & 1 & 2 \end{pmatrix}^{10}_8$	0	$\Rightarrow 0$
3	0	0	0	$\Rightarrow 0$
4	0	0	0	$\equiv 0$
5	0	0	0	$\equiv 0$

The map f is of maximum rank and yields the only non-vanishing differential in this spectral sequence. The corresponding chart of $E_2^{q,k}(\mathscr{E}_1 \otimes \mathscr{T}_y^*)$, the next and final level, is obtained by replacing $E_1^{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$ with 0 and $E_1^{1,0} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}$ by its quotient with $f\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$. Therefore, the only non-zero $\mathscr{E}_1 \otimes \mathscr{T}_y^*$-valued cohomology on $\mathscr{M}$ is the first cohomology:

$$H^1\left(\mathscr{M}, \left(\begin{array}{c|ccc} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)\right) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ & 0 & 1 & 2 \end{pmatrix}^{10}_8 + \left[\operatorname{cok}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1 \xrightarrow{f} \begin{pmatrix} -3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}^{20}_1\right]_{19}.$$

In this way we obtain all the cohomology on $\mathcal{M}$, valued in the various bundles listed above. The results are:

Bundle	Non-zero cohomology
$\left(\begin{array}{c\|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right)$	$H^0=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1 \quad H^3=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \end{pmatrix}^1_1$
$\left(\begin{array}{c\|ccc} -2 & 0 & 0 & 0 \\ 3 & 0 & 0 & \end{array}\right)$	$H^1=\ker\begin{pmatrix} -1 & 0 & 0 & 0 \\ & 1 & 1 & 4 \end{pmatrix}^4_{10} \xrightarrow{g} \begin{pmatrix} -2 & 0 & 0 & 0 \\ & 1 & 1 & 1 \end{pmatrix}^{10}_1$
$\left(\begin{array}{c\|ccc} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right)$	$H^0=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1+\begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^{20}_1 \quad H^1=\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & \end{pmatrix}^6_1$
$\left(\begin{array}{c\|ccc} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)$	$H^1=\begin{pmatrix} -2 & 0 & 0 & 0 \\ & 0 & 1 & 2 \end{pmatrix}^{10}_8+\left[\operatorname{cok}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1 \xrightarrow{f} \begin{pmatrix} -3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}^{20}_1\right]$
$\left(\begin{array}{c\|ccc} 0 & -1 & 0 & 0 \\ -3 & 0 & 0 & \end{array}\right)$	$H^0=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1 \quad H^1=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \end{pmatrix}^4_1$
$\left(\begin{array}{c\|ccc} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & \end{array}\right)$	$H^0=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}^1_1+\begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & \end{pmatrix}^4_8$
$\left(\begin{array}{c\|ccc} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & \end{array}\right)$	$H^1=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \end{pmatrix}^4_1 \quad H^2=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & \end{pmatrix}^4_1$
$\left(\begin{array}{c\|ccc} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & \end{array}\right)$	All cohomology vanishes
$\left(\begin{array}{c\|ccc} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & \end{array}\right)$	$H^1=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & \end{pmatrix}^4_8+\begin{pmatrix} -1 & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{pmatrix}^{15}_1 \quad H^2=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \end{pmatrix}^1_8$

where we have omitted entries that can be obtained using Serre duality. In the present case, Eq. (1.6) yields

$$H^q(\mathcal{M},\mathcal{V})=H^{3-q}(\mathcal{M},\mathcal{V}^*)^*\otimes\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \end{pmatrix}.$$

The next step requires using Sequence (1.1), which for this example becomes

$$0\longrightarrow\mathcal{T}_{\mathcal{M}}\longrightarrow\begin{bmatrix}\left(\begin{array}{c|ccc} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \end{array}\right) \\ \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & \end{array}\right)\end{bmatrix}\xrightarrow{j}\begin{bmatrix}\left(\begin{array}{c|ccc} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right) \\ \left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ -3 & 0 & 0 & \end{array}\right)\end{bmatrix}\longrightarrow 0. \tag{4.2}$$

Merely from the way that j occurs in Sequence (4.2), it follows that

$$j\sim\left(\begin{array}{c|ccc} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{array}\right)\oplus\left(\begin{array}{c|ccc} -3 & 0 & 0 & 0 \\ 1 & -1 & 0 & \end{array}\right)\oplus\left(\begin{array}{c|ccc} 0 & -1 & 0 & 0 \\ -3 & 0 & 0 & \end{array}\right)$$
$$\oplus\left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & \end{array}\right). \tag{4.3}$$

These four components can be represented by f', $\dot{f}$, g', and $\dot{g}$ in a notation analogous to the one of the previous section and can be investigated as in Appendix A.

We tensor now the dual of the Sequence (4.2) with each of $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{T}_x$, and $\mathscr{T}_y$. This yields four separate short exact sequences. The accompanying long exact cohomology sequences determine the $\mathscr{E}_i \otimes \mathscr{T}^*_{\mathscr{M}}$-valued and the $\mathscr{T}_i \otimes \mathscr{T}^*_{\mathscr{M}}$-valued cohomology on $\mathscr{M}$. The decomposition (4.3) and arguments as in Appendix A are sufficient to locate all non-vanishing maps between the cohomology groups, as induced by j. This information is then plugged into the long exact cohomology sequence accompanying the short exact row sequence in the diagram (1.4).

Throughout, we use our knowledge about the map j to determine its action in these various sequences and therefore the unknown cohomology groups. To spare the reader of the boring details of this computation and save some space, we only present the last few steps, these being the most complicated.

4.2. Endomorphism Valued Cohomology on the Submanifold

In the short exact row sequence in the diagram (1.4) we study the action of j in

$$\ldots \longrightarrow H^q(\mathscr{M}, \mathscr{T}_{\mathscr{M}} \otimes \mathscr{T}^*_{\mathscr{M}}) \longrightarrow H^q(\mathscr{M}, \mathscr{T}_{\mathscr{W}} \otimes \mathscr{T}^*_{\mathscr{M}}) \xrightarrow{j} H^q(\mathscr{M}, \mathscr{E} \otimes \mathscr{T}^*_{\mathscr{M}}) \longrightarrow \ldots$$

for each q separately and thereby determine $H^q(\mathscr{M}, \mathscr{T}_{\mathscr{M}} \otimes \mathscr{T}^*_{\mathscr{M}})$.

For $q=0$, we have

$$2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \xrightarrow{j} \begin{matrix} \begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \end{matrix} . \tag{4.4}$$

As usual, j is here obliged to map to $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$ and is of maximal rank $(=1)$.

This leaves us with $H^0(\mathscr{M}, \mathscr{T}_{\mathscr{M}} \otimes \mathscr{T}^*_{\mathscr{M}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix}$ as in (1.8) and

$$\operatorname{cok} j = \begin{matrix} \begin{pmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & \end{pmatrix} \end{matrix}$$

contributes via the connecting homomorphism to $H^1(\mathscr{M}, \mathscr{T}_{\mathscr{M}} \otimes \mathscr{T}^*_{\mathscr{M}})$. For $q=1$, we have a rather longish stack of cohomology groups on both sides of the map j (see Fig. 1) and the gory details are as follows. This is less complicated than the wiring diagram that Fig. 1 first resembles – the mappings are essentially determined by invariance. In Fig. 1 we have written any mapping derived from f or g as just f or g. For example,

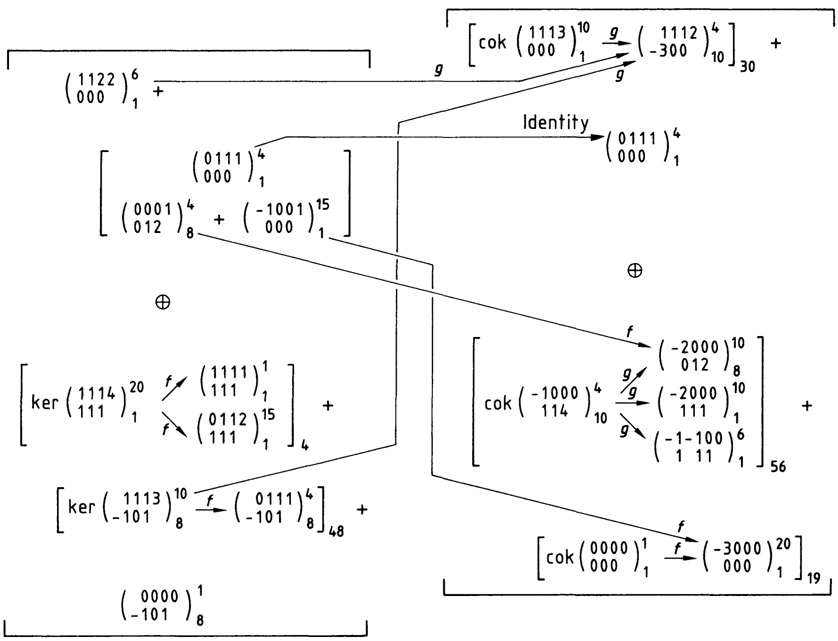

Fig. 1. The mapping $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{W}} \otimes \mathcal{T}^*_{\mathcal{M}}) \to H^1(\mathcal{M}, \mathcal{E} \otimes \mathcal{T}^*_{\mathcal{M}})$

$$\left[\operatorname{cok}\begin{pmatrix}-1 & 0 & 0 & 0\\ & 1 & 1 & 4\end{pmatrix}^4_{10} \begin{matrix} \overset{g}{\nearrow} \begin{pmatrix}-1 & -1 & 0 & 0\\ & 1 & 1 & 1\end{pmatrix}^6_1 \\ \overset{g}{\longrightarrow} \begin{pmatrix}-2 & 0 & 0 & 0\\ & 1 & 1 & 1\end{pmatrix}^{10}_1 \\ \overset{g}{\searrow} \begin{pmatrix}-2 & 0 & 0 & 0\\ & 0 & 1 & 2\end{pmatrix}^{10}_8 \end{matrix}\right]_{56}$$

comes from

$$\left[\operatorname{cok} H^2\begin{pmatrix}-1 & 0 & 0 & 0\\ 6 & 0 & 0 & \end{pmatrix}^4_{10} \begin{matrix} \overset{g}{\nearrow} H^2\begin{pmatrix}-1 & -1 & 0 & 0\\ 3 & 0 & 0 & \end{pmatrix}^6_1 \\ \overset{g}{\longrightarrow} H^2\begin{pmatrix}-2 & 0 & 0 & 0\\ 3 & 0 & 0 & \end{pmatrix}^{10}_1 \\ \overset{g}{\searrow} H^2\begin{pmatrix}-2 & 0 & 0 & 0\\ 4 & -1 & 0 & \end{pmatrix}^{10}_8 \end{matrix}\right]_{56}$$

In obtaining this cokernel in the first place, it is important to know that the linear transformation

$$g:\begin{pmatrix}-1 & 0 & 0 & 0\\ & 1 & 1 & 4\end{pmatrix}^4_{10} \begin{matrix} \nearrow \begin{pmatrix}-1 & -1 & 0 & 0\\ & 1 & 1 & 1\end{pmatrix}^6_1 \\ \longrightarrow \begin{pmatrix}-2 & 0 & 0 & 0\\ & 1 & 1 & 1\end{pmatrix}^{10}_1 \\ \searrow \begin{pmatrix}-2 & 0 & 0 & 0\\ & 0 & 1 & 2\end{pmatrix}^{10}_8 \end{matrix} \tag{4.5}$$

is injective (has no kernel) for generic choice of the second defining polynomial, i.e., for generic $g_{a\alpha\beta\gamma}=g_{a(\alpha\beta\gamma)}$[5]. We verify this here explicitly and defer further similar discussions to Sect. 6.

The $\begin{pmatrix} -1 & 0 & 0 & 0 \\ & 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ & 0 & 0 & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & \end{pmatrix}$ Young tableau corresponds to a tensor $T_a^{\alpha\beta\gamma}=T_a^{(\alpha\beta\gamma)}$. The combined kernel of the three maps in (4.5) consists of tensors $T_a^{(\alpha\beta\gamma)}$ which are annihilated by the maps, i.e., the solutions to the system of three tensorial equations

$$\begin{aligned} g_{\alpha\beta\gamma[a}T_{b]}^{\alpha\beta\gamma}&=0 && 6 \text{ equations},\\ g_{\alpha\beta\gamma(a}T_{b)}^{\alpha\beta\gamma}&=0 && 10 \text{ equations},\\ (\delta_{\mu\delta}^{\gamma\nu}-\tfrac{1}{3}\delta_{\delta\mu}^{\gamma\nu})g_{\alpha\beta\gamma(a}T_{b)}^{\alpha\beta\delta}&=0 && 80 \text{ equations}, \end{aligned}$$

corresponding to the three maps, respectively. [The $(\delta_{\mu\delta}^{\gamma\nu}-\frac{1}{3}\delta_{\delta\mu}^{\gamma\nu})$ prefactor projects the trace free part.]

Now, as 96 equations in only 40 unknowns, it is reasonable to expect that these equations force $T_a^{\alpha\beta\gamma}=0$ for a generic choice of $g_{a\alpha\beta\gamma}$. By choosing a quite specific $g_{a\alpha\beta\gamma}$, for example

$$g_{1112}=g_{2223}=g_{3133}=g_{4123}=1,$$

and all other $g_{a\alpha\beta\gamma}=0$, it is possible explicitly to show that this is the case. Further verifications of this form are needed in drawing conclusions from our identification of

$$H^1(\mathcal{M},\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}_{\mathcal{M}}^*)\to H^1(\mathcal{M},\mathcal{E}\otimes\mathcal{T}_{\mathcal{M}}^*)$$

given above. We feel that there should be some general theory which can be brought to bear here.

In any case, it now follows that $H^1(\mathcal{M},\operatorname{End}\mathcal{T}_{\mathcal{M}})=A+B+C$, where

$$A=\left[\ker\begin{pmatrix}1&1&1&4\\1&1&1&\end{pmatrix}_1^{20}\begin{array}{c}\overset{f}{\nearrow}\begin{pmatrix}1&1&1&1\\1&1&1&\end{pmatrix}_1^{1}\\ \underset{f}{\searrow}\begin{pmatrix}0&1&1&2\\1&1&1&\end{pmatrix}_1^{15}\end{array}\right]_4, \tag{4.6}$$

$$B=\left[\begin{array}{ccc} \begin{pmatrix}1&1&2&2\\0&0&0&\end{pmatrix}_1^{6} & \overset{g}{\searrow} & \\ \ker\begin{pmatrix}1&1&1&3\\0&0&0&\end{pmatrix}_1^{10} & \overset{g}{\longrightarrow} & \begin{pmatrix}1&1&1&2\\-3&0&0&\end{pmatrix}_{10}^{4} \\ \begin{pmatrix}1&1&1&3\\-1&0&1&\end{pmatrix}_8^{10} & \overset{g}{\nearrow}\;\underset{f}{\longrightarrow} & \begin{pmatrix}0&1&1&1\\-1&0&1&\end{pmatrix}_8^{4}\end{array}\right]_{24}, \tag{4.7}$$

$$C=\begin{pmatrix}0&0&0&0\\-1&0&1&\end{pmatrix}_8^{1}+\left[\begin{pmatrix}-2&-1&0&0\\0&0&0&\end{pmatrix}_1^{20}\oplus\begin{pmatrix}-1&0&0&0\\-2&-1&0&\end{pmatrix}_8^{4}\right]. \tag{4.8}$$

$H^2(\mathcal{M},\operatorname{End}\mathcal{T}_{\mathcal{M}})$ is given dually.

[5] We use round brackets to denote symmetrization and square brackets to denote skewing

In particular, $\dim H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T}_{\mathcal{M}}) = 88$. This should be confronted with the situation in Ref. [10] where a particular superconformal $1+1$-dimensional model is identified with compactification on the Calabi-Yau manifold studied in this section. The authors there find an amazing degree of agreement between the two models. They, however, also report a total of 154 gauge-invariant scalars in that superconformal $1+1$-dimensional model allegedly corresponding to $H^1(\mathcal{M}, \mathrm{End}\,\mathcal{T}_{\mathcal{M}})$ and also to certain extra states. The latter states come accompanied by the same number of extra $U(1)$ gauge particles [16]. With our present result, this could imply an extra $U(1)^{66}$ gauge invariance which, however, was not found in the superconformal model, suggesting that the identification is possibly false (see however note added).

5. The Tian-Yau Manifold

Finally, we turn to $\mathcal{M} \in \left[\begin{array}{c\|ccc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array}\right]^{14;23}_{-18}$. Now we have three defining polynomials

$$f(x) \stackrel{\text{def}}{=} f_{abc} x^a x^b x^c\,, \qquad g(y) \stackrel{\text{def}}{=} g_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma\,, \qquad h(x,y) \stackrel{\text{def}}{=} h_{a\alpha} x^a y^\alpha\,,$$

where $x \in \mathbb{P}^3_x$ and $y \in \mathbb{P}^3_y$. These polynomials are sections of $\left(\begin{array}{c|ccc} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, $\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right)$ and $\left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)$, respectively and $\mathcal{E}$ is the direct sum of these.

The resolution (1.2) is now

$$0 \longrightarrow \left(\begin{array}{c|ccc} 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{c} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{c|ccc} 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{c|ccc} 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right) \end{array} \longrightarrow \begin{array}{c} \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right) \\ \left(\begin{array}{c|ccc} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \longrightarrow \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{5.1}$$

$$\longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow 0\,,$$

where we have left the arrows unlabeled but it should not be too hard for the reader to identify them with contractions with the polynomials f, g, and h. One can now execute TESS as outlined in Sect. 1. Whilst (5.1) is a little more involved than (4.1) used for the Schimmrigk manifold, there is some extra symmetry under $\mathbb{P}^3_x \leftrightarrow \mathbb{P}^3_y$ which is helpful as in the bi-cubic and the computations turn out to be only slightly more difficult than those of the previous section.

Rather than follow this track, however, we shall adopt a different approach as will be described shortly. The reason for this is twofold: On one hand, this alternative approach will serve as an example for a far more general computation – a complex intersection in a homogeneous space for a complex semi-simple Lie group. This generalizes in a way the CICYs which have been, in the physics literature so far, considered to involve only complex projective spaces. On the other hand, our computations prove to be quite simpler in this alternative approach. We also note that the Tian-Yau manifold has never before been analyzed from this point of view.

In effect, what we shall do is firstly restrict to the zero locus of $h(x, y)$. Since $h_{a\alpha}$ is non-degenerate [6], it may be used to identify the homogeneous coördinates y^α as dual to x^a. In other words, we may use $h_{a\alpha}$ to lower the α-indices and turn them into a-indices:

$$y_a \stackrel{\text{def}}{=} h_{a\alpha} y^\alpha .$$

The equation $h(x,y)=0$ now reads $x^a y_a = 0$ and defines the natural incidence submanifold in $\mathbb{P}^3 \times \mathbb{P}^{3*}$. Alternatively, a point $x \in \mathbb{P}^3$ corresponds to a line (1-dimensional linear subspace) $L \subset \mathbb{C}^4$ whilst $y \in \mathbb{P}^{3*}$ corresponds to a hyperplane (3-dimensional linear subspace) $H \subset \mathbb{C}^4$ and the incidence submanifold consists of the flag manifold [7]:

$$\left\{ L \subset H : \begin{matrix} L \text{ and } H \text{ are linear subspaces of } \mathbb{C}^4, \\ \text{of dimension 1 and 3 respectively} \end{matrix} \right\} \cong \left\{ \frac{U(4)}{U(1) \times U(2) \times U(1)} \right\} .$$

This is a homogeneous space for $GL(4, \mathbb{C})$ and its cohomology may be analyzed by the BBW algorithm explained for the case of a projective space in Sect. 1.3. Indeed, the algorithm is formally identical but the bundles that one can start with now have the form $(a|bc|d)$ for $b \leqq c$, meaning that the fibre over $L \subset H$ is

$$L^a \otimes (bc) [H/L] \otimes [\mathbb{C}^4/H]^d .$$

More details can be found in [5].

Thus, we shall start with this flag manifold as our ambient space $\mathscr{W}$ but otherwise pursue TESS as outlined in Sect. 1. The advantage is that now we only have two defining equations:

$$f(x) \stackrel{\text{def}}{=} f_{abc} x^a x^b x^c \quad \text{and} \quad g(y) \stackrel{\text{def}}{=} g^{abc} y_a y_b y_c .$$

These polynomials are sections of

$$(-3|00|0) \quad \text{and} \quad (0|00|3)$$

respectively. The Koszul complex (1.2) reads

$$0 \longrightarrow (3|00|-3) \begin{matrix} \xrightarrow{f} (0|00|-3) \xrightarrow{g} \\ \xrightarrow[g]{} (3|00|0) \xrightarrow[-f]{} \end{matrix} (0|00|0) \longrightarrow (0|00|0)_{\mathscr{M}} \longrightarrow 0 . \tag{5.2}$$

There is a slight disadvantage, however, when it comes to the tangent bundle since, although homogeneous, it is reducible. In the notation of [5], there is a composition series:

$$\mathscr{T}_{\mathscr{W}} = (-1|00|1) + \begin{matrix} (-1|01|0) \\ (0|-10|1) \end{matrix} ,$$

[6] This is the generic choice; some more special cases are also admissible [17] and those may need to be examined on their own

[7] This is manifestly a generalization of $\mathbb{P}^4 = \{U(4)/U(3) \times U(1)\}$

meaning that $\mathcal{T}_{\mathcal{W}}$ is an *extension* (see p. 168 of Bott and Tu in [4])

$$0 \to \begin{matrix}(-1|01|0)\\(0|-10|1)\end{matrix} \to \mathcal{T}_{\mathcal{W}} \to (-1|00|1) \to 0\,. \tag{5.3}$$

It is not the trivial extension – roughly speaking, the possible extensions are classified by first cohomology

$$H^1\left(\begin{bmatrix}(-1|01|0)\\(0|-10|0)\end{bmatrix} \otimes (-1|00|1)^*\right) = H^1\begin{pmatrix}(0|01|-1)\\(1|-10|0)\end{pmatrix} = \mathbb{C}^2$$

and for $\mathcal{T}_{\mathcal{W}}$ one must take $\begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{C}^2$. This is reflected through various non-trivial connecting homomorphisms when (5.3) is used to compute cohomology.

As an example of this artillery in action, we shall first compute the Hodge numbers b_{1_q} for the Tian-Yau manifold. In other words, we shall compute $H^q(\mathcal{M}, \mathcal{T}^*_{\mathcal{M}})$. We need the exact sequence

$$0 \to \mathcal{E}^*|_{\mathcal{M}} \to \mathcal{T}^*_{\mathcal{W}}|_{\mathcal{M}} \to \mathcal{T}^*_{\mathcal{M}} \to 0\,,$$

dual to (1.1). The Koszul complex (5.2) and the BBW algorithm yield

Bundle	Non-vanishing cohomology	
$(3\|00\|0)$	$H^2 = (1113)$	$H^3 = \ker(0003) \xrightarrow{f} (0000)$
$(0\|00\|-3)$	$H^2 = (-3-1-1-1)$	$H^3 = \ker(-3000) \xrightarrow{g} (0000)$
$(1\|-10\|0)$	$H^1 = (0000)$	$H^2 = (0111)$
$(0\|01\|-1)$	$H^1 = (0000)$	$H^2 = (-1-1-10)$
$(1\|00\|-1)$	$H^3 = (-1001)$	

the first two of which give $H^*(\mathcal{M}, \mathcal{E}^*)$. Using the dual of (5.3) to compute $H^*(\mathcal{M}, \mathcal{T}^*_{\mathcal{W}})$ gives

$$\begin{aligned} H^1(\mathcal{M}, \mathcal{T}^*_{\mathcal{W}}) &= 2(0000)\,,\\ H^2(\mathcal{M}, \mathcal{T}^*_{\mathcal{W}}) &= (0111) \oplus (-1-1-10)\,,\\ H^3(\mathcal{M}, \mathcal{T}^*_{\mathcal{W}}) &= (-1001)\,. \end{aligned}$$

Finally, we obtain

q	$\mathcal{E}^*\|_{\mathcal{M}}$		$\to \mathcal{T}^*_{\mathcal{W}}\|_{\mathcal{M}}$	$\to \mathcal{T}^*_{\mathcal{M}}$
0	0		0	$\Rightarrow 0$
1	0		2(0000)	See below
	(1113)	$\xrightarrow{g}$	(0111)	
2	$(-3-1-1-1)$	$\xrightarrow{f}$	$(-1-1-10)$	See below
3	$\ker(0003) \to (0000)$ $\ker(-3000) \to (0000)$	$\xrightarrow{f}$ / $\xrightarrow{g}$	(-1001)	See below

Suffice it here merely to state that all indicated maps are onto. $H^3(\mathcal{M},\mathcal{T}^*_{\mathcal{M}})$ therefore also vanishes and we have

$$H^1(\mathcal{M},\mathcal{T}^*_{\mathcal{M}})=\frac{[\ker(1113)_{10}\xrightarrow{f}(0111)_4]_6}{[\ker(-3-1-1-1)_{10}\xrightarrow{g}(-1-1-10)_4]_6}+2(0000)_1\,,$$

of dimension 14 and

$$H^2(\mathcal{M},\mathcal{T}^*_{\mathcal{M}})=\ker\begin{array}{ccc}(0003)_{20} & \longrightarrow & (0000)_1\\ & \overset{f}{\searrow} & (-1001)_{15}\\ & \underset{g}{\nearrow} & \\ (-3000)_{20} & \xrightarrow{g} & (0000)_1\end{array},$$

of dimension 23, as in [12, 6].

The computation of $H^1(\mathcal{M},\operatorname{End}\mathcal{T}_{\mathcal{M}})$ is fairly lengthy and we present only the barest outline. The bundles with which we need to start the computation are

$$\mathcal{E}\otimes\mathcal{E}^*=2(0|00|0)\oplus(-3|00|-3)\oplus(3|00|3) \tag{5.4}$$

$$\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}}=\begin{matrix}(-2|-10|0)\\(1|-10|3)\\(-3|01|-1)\\(0|01|2)\end{matrix}+\begin{matrix}(-2|00|-1)\\(1|00|2)\end{matrix} \tag{5.5}$$

$$\mathcal{T}_{\mathcal{W}}\otimes\mathcal{E}^*=\begin{matrix}(2|00|1)\\(-1|00|-2)\end{matrix}+\begin{matrix}(2|01|0)\\(-1|01|3)\\(3|-10|1)\\(0|-10|-2)\end{matrix} \tag{5.6}$$

$$\mathcal{T}_{\mathcal{W}}\otimes\mathcal{T}^*_{\mathcal{W}}=\begin{matrix}(0|-10|1)\\(-1|01|0)\end{matrix}+\begin{matrix}2(0|-11|0)\\3(0|00|0)\\(-1|02|-1)\\(-1|11|-1)\\(1|-20|-1)\\(1|-1-1|1)\end{matrix}+\begin{matrix}(0|01|-1)\\(1|-10|0)\end{matrix} \tag{5.7}$$

5.1. Restricting the Ambient Cohomology to the Submanifold

Using the Koszul complex (5.2) to compute cohomology on $\mathcal{M}$ gives the results in Table 1 which together with the $x^a\leftrightarrow y_a$ duality

$$H^q(\mathcal{M},\mathcal{V})=H^{3-q}(\mathcal{M},\mathcal{V}_{x\leftrightarrow y})^*$$

and Serre duality [cf. (1.7)]

$$H^q(\mathcal{M},\mathcal{V})=H^{3-q}(\mathcal{M},\mathcal{V}^*)^*$$

is a list of all necessary cohomology. For example,

$$\begin{array}{ccc} H^2(\mathcal{M},(1|-10|0))=(0111) & \overset{x\leftrightarrow y}{\Longleftrightarrow} & H^2(\mathcal{M},(0|01|-1))=(-1-1-10) \\ \text{Serre}\Big\Updownarrow & & \Big\Updownarrow\text{Serre} \\ H^1(\mathcal{M},(-1|01|0))=(-1-1-10) & \overset{x\leftrightarrow y}{\Longleftrightarrow} & H^1(\mathcal{M},(0|-10|1))=(0111). \end{array}$$

Table 1. The relevant cohomology groups on $\mathbb{P}^3\times\mathbb{P}^3$

Bundle	Non-vanishing Cohomology
(0|00|0)	$H^0=(0000)_1 \quad H^3=(0000)_1$
(3|00|3)	$H^1=(1113)_{10} \quad H^2=(1113)_{10}$
(0|01|2)	$H^0=(0000)_1+(0012)_{20}$
(1|−10|3)	$H^1=(0111)_4+(0123)_{64}+[\mathrm{cok}(0000)_1\overset{g}{\longrightarrow}(0003)_{20}]_{19}$
(1|00|2)	$H^1=(1122)_6$
(1|−10|0)	$H^1=(0000)_1 \quad H^2=(0111)_4$
(1|−20|1)	$H^1=(-1001)_{15} \quad H^1-(-1001)_{15}$
(1|−11|0)	All cohomology vanishes
(1|−1−1|1)	All cohomology vanishes

5.2. Endomorphism Valued Cohomology on the Submanifold

The cohomology of $\mathcal{E}\otimes\mathcal{E}^*$ can be read off immediately from the table:

$$\begin{aligned} H^0(\mathcal{M},\mathcal{E}\otimes\mathcal{E}^*)&=2(0000),\\ H^1(\mathcal{M},\mathcal{E}\otimes\mathcal{E}^*)&=\ (1113)\oplus(-3-1-1-1),\\ H^2(\mathcal{M},\mathcal{E}\otimes\mathcal{E}^*)&=\ (1113)\oplus(-3-1-1-1),\\ H^0(\mathcal{M},\mathcal{E}\otimes\mathcal{E}^*)&=2(0000). \end{aligned}$$

The cohomology of $\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}}$ is slightly more subtle since it is given by an extension, derived from Seq. (5.3), and so an exact sequence must be used. It turns out, however, that, by invariance, the connecting homomorphisms are all zero and the naïve conclusion is valid:

q	$H^q(\mathcal{M},\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}})$
0	$2(0000)+[(0012)\oplus(-2-100)]$
1	$(1122)+(0111)+(0123)+[\mathrm{cok}(0000)\overset{g}{\longrightarrow}(0003)]$ $(-2-2-1-1)+(-1-1-10)+(-3-2-10)+[\mathrm{cok}(0000)\overset{f}{\longrightarrow}(-3000)]$
2	0
3	0

Now one can use the right-hand column of (1.4) easily to compute

q	$H^q(\mathscr{M}, \mathscr{E}\otimes\mathscr{T}^*_{\mathscr{M}})$
0	$[\ker(1113)\xrightarrow{f}(0111)]+(0012)$ $[\ker(-3-1-1-1)\xrightarrow{g}(-1-1-10)]+(-2-100)$
1	$(1113)+(1122)+(0123)+[\mathrm{cok}(0000)\xrightarrow{g}(0003)]$ $(-3-1-1-1)+(-2-2-1-1)+(-3-2-10)+[\mathrm{cok}(0000)\xrightarrow{f}(-3000)]$
2	$2(0000)$
3	0

The cohomology of $\mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{W}}$ is a little more tricky since the fact that it is given by extensions affects (though simplifies) the outcome. We find:

$$\begin{aligned} H^0(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{W}}) &= \ (0000)_1\,, \\ H^1(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{W}}) &= 2(-1001)_{15}\,, \\ H^2(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{W}}) &= 2(-1001)_{15}\,, \\ H^3(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{W}}) &= \ (0000)_1\,. \end{aligned}$$

The middle column of (1.4) allows one to compute $H^*(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{M}})$

q	$H^q(\mathscr{M}, \mathscr{T}_{\mathscr{W}}\otimes\mathscr{T}^*_{\mathscr{M}})$
0	(0000)
1	$(1122)+(0111)+(0123)+\left[\ker(0003)\begin{smallmatrix}\xrightarrow{f}(0000)\\ \xrightarrow[f]{}(-1001)\end{smallmatrix}\right]$ $(-2-2-1-1)+(-1-1-10)+(-3-2-10)+\left[\ker(-3000)\begin{smallmatrix}\xrightarrow{g}(0000)\\ \xrightarrow[g]{}(-1001)\end{smallmatrix}\right]$ $+2(-1001)$
2	$(0000)+[(0012)\oplus(-2-100)]$
3	0

Finally, we may use the bottom row of (1.4) to complete the calculations giving that both $H^0(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})$ and $H^3(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})$ vanish in agreement with (1.8), that $H^1(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})=A+B+C$, where

$$A=\left[\ker(0003)_{20}\begin{smallmatrix}\xrightarrow{f}(0000)_1\\ \xrightarrow[f]{}(-1001)_{15}\end{smallmatrix}\right]_4\oplus\left[\ker(-3000)_{20}\begin{smallmatrix}\xrightarrow{g}(0000)_1\\ \xrightarrow[g]{}(-1001)_{15}\end{smallmatrix}\right]_4 \tag{5.8}$$

$$B=\left[\ker(1113)_{10}\xrightarrow{f}(0111)_4\right]_6\oplus\left[\ker(-3-1-1-1)_{10}\xrightarrow{g}(-1-1-10)_4\right]_6, \tag{5.9}$$

$$C=(0012)_{20}\oplus(-2-100)_{20}, \tag{5.10}$$

and $H^2(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})$ dually.

Note that $\dim H^1(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})=60$.

6. Results and Remarks

Translating the entire preceding analysis into tensor notation would no doubt be favored by most physicists but is, however, comparable to rewriting MAXIMA into ASSEMBLER [8]. Instead, we here discuss our results comparing with the literature. Also, far more than just the dimension can be discerned from the above analysis. In particular, an explicit parametrization for $H^1(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}})$ can be obtained which can then be used to determine the relative strengths of the Yukawa couplings [18]. To see how this happens, we will shortly return to our pet example and then the three CICY manifolds in turn.

The basic relations between the homogeneous vector bundle notation used and the tensor notation on a $\mathbb{P}^n$ are

$$\begin{aligned}(-1|0\ldots0)&\sim\lambda_a z^a,\\(0|-10\ldots0)&\sim\lambda_a dz^a,\\(0|0\ldots01)&\sim\lambda^a\partial_a.\end{aligned}$$

All other relations can be obtained from these by multiplication as dictated by the Young tableaux. Whenever it will simplify expressions, we shall factor out powers of $(1\ldots1)$ which is trivial holomorphically. For example,

$$H^2(4|004)\overset{\mathrm{BBW}}{=}(1124)=(1111)\otimes(0013),\qquad(0013)\sim f^{a(bcd)},$$

and we shall use $f^{a(bcd)}$ to represent $H^2(4|004)$. Note also that $(1|-10\ldots0)$ is the cotangent bundle of $\mathbb{P}^n$ and $H^1(1|-10\ldots0)=(0\ldots)$ by BBW. On the other hand, cotangent bundle-valued 1-forms are, on a $\mathbb{P}^n$, represented by its Kähler class which we denote by J [and bear in mind that $J\mapsto 1\in(0\ldots0)$].

6.1. The Quintic in $\mathbb{P}^4$

As derived in Eqs. (2.2) and in complete agreement with [8, 9], $H^1(\mathscr{M}, \mathrm{End}\,\mathscr{T}_{\mathscr{M}}) = H^1(-4|-1000)=(-4-1000)$ and is represented by the tensor coefficients

$$p_{a(bcde)}\sim(-4-1000)\,[\mathbb{C}^5],$$

which are totally symmetric in the $(bcde)$ indices but vanish upon total symmetrization with a. Such a tensor occurs as a non-trivial variation of

$$df(z)=dz^a f_a(z)\overset{\mathrm{def}}{=}dz^a f_{abcde}z^b z^c z^d z^e,$$

the differential of the defining polynomial $f(z)$ and has a polynomial deformation theoretic interpretation à la Kodaira and Spencer [8, 9]. On comparison of our analysis of the quintic hypersurface in $\mathbb{P}^4$ with deformation theory methods as used in [8], it may appear that TESS is merely a meticulous and overly technical reprise. The three physically more interesting cases however suggest otherwise as we shall shortly see.

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[8] Some of the computing procedures however are indeed better understood at the level of tensors, as pointed out in Sect. 4

6.2. The Bi-cubic in $\mathbb{P}^2 \times \mathbb{P}^2$

From Eq. (3.3)

$$H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}^*_{\mathcal{M}}) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^8_1 \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}^1_8 + \begin{pmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{pmatrix}^8_{10} \begin{pmatrix} -3 & 0 & 0 \\ -2 & -1 & 0 \end{pmatrix}^{10}_8 ,$$

we can again read off a basis for $H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}} \otimes \mathcal{T}^*_{\mathcal{M}})$ as in the case of our pet example.

To that end, we start with the larger contributions.

$$\begin{pmatrix} -2 & -1 & 0 \\ -3 & 0 & 0 \end{pmatrix}^8_{10} \sim p_{a(bc)(\alpha\beta\gamma)} .$$

Very much like in the pet example, this is symmetric in the (bc) pair of indices but vanishes upon total symmetrization with a. Also, it occurs as a non-trivial variation of

$$d_x f(x, y) = dx^a f_a(x, y) = dx^a f_{abc\alpha\beta\gamma} x^b x^c y^\alpha y^\beta y^\gamma ,$$

again admitting a deformation theoretic interpretation. Of course, the analogous is true of the corresponding $\mathbb{P}^2_x \leftrightarrow \mathbb{P}^2_y$ companion,

$$d_y f(x, y) = dy^\alpha f_\alpha(x, y) = dy^\alpha f_{abc\alpha\beta\gamma} x^a x^b x^c y^\beta y^\gamma .$$

The two smaller, 8-dimensional contributions are represented by

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^8_1 \sim p^a_b , \quad \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}^1_8 \sim p^\alpha_\beta ,$$

both of which are traceless matrices. Deformation theory, as discussed in [9], does not account for these contributions.

6.3. The Schimmrigk Manifold

This is perhaps the most complicated example and we shall devote some more time to discuss each of the three contributions in (4.6)–(4.8) in turn.

The combined kernel in (4.6) is actually rather simple. It consists of elements of

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & \end{pmatrix} \sim p^{(abc)} ,$$

that vanish upon mapping, by f, into $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \end{pmatrix}$ (upon factoring out $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \end{pmatrix}$). Noting that these Young tableaux correspond to scalars and trace free matrices, the restrictions defining the kernel are easily enforced by

$$f_{abc} p^{(abc)} = 0 ,$$
$$(\delta^{ai}_{jd} - \tfrac{1}{4} \delta^{ai}_{dj}) f_{abc} p^{(bcd)} = 0 .$$

Since these two conditions can be combined, we have that

$$\left[\ker\begin{pmatrix}1&1&1&4\\1&1&1&\end{pmatrix}^{20}_{1}\begin{array}{l}\nearrow\begin{pmatrix}1&1&1&1\\1&1&1&\end{pmatrix}^{1}_{1}\\[1ex] \searrow\begin{pmatrix}0&1&1&2\\1&1&1&\end{pmatrix}^{15}_{1}\end{array}\right]_{4}\sim\{p^{(abc)}:f_{abc}p^{(bcd)}=0\}\,. \quad (4.6')$$

Similarly, the combined kernel in (4.7) becomes, after factoring out (1111) from all $\mathbb{P}^3$ bundles:

$$\left[\ker\begin{array}{ccc}\begin{pmatrix}0&0&1&1\\0&0&0&\end{pmatrix}^{6}_{1} & \searrow^{g} & \\ \begin{pmatrix}0&0&0&2\\0&0&0&\end{pmatrix}^{10}_{1} & \xrightarrow{g} & \begin{pmatrix}0&0&0&1\\-3&0&0&\end{pmatrix}^{4}_{10}\\ \begin{pmatrix}0&0&0&2\\-1&0&1&\end{pmatrix}^{10}_{8} & \nearrow^{g}\;\xrightarrow{f} & \begin{pmatrix}-1&0&0&0\\-1&0&1&\end{pmatrix}^{4}_{8}\end{array}\right]_{24}.$$

To describe this, we introduce tensor variables $p^{[ab]}$, $p^{(ab)}$, and $p_{\beta}^{(ab)\alpha}$ for the three Young tableaux on the left-hand side, respectively. To describe the kernel, we need to constrain these variables to vanish upon mapping over into the right-hand side Young tableaux. The last one of these is trace free in α, β and must satisfy a separate condition

$$f_{abc}p_{\beta}^{(ab)\alpha}=0\,,$$

corresponding to the kernel of the base row mapping. For a generic choice of f_{abc}, these are 32 independent equations in 80 components of $p_{\beta}^{(ab)\alpha}$ and leave 48 elements free to describe the kernel.

The combined kernel of the mapping into $\begin{pmatrix}0&0&0&1\\-3&0&0&\end{pmatrix}$ is described by

$$\phi_1 g_{b(\alpha\beta\gamma)}p^{[ab]}+\phi_2 g_{b(\alpha\beta\gamma)}p^{(ab)}+\phi_3 g_{b(\alpha\delta\gamma}p_{[ab]\delta\beta)}=0\,.$$

The parameters ϕ_i describe the triple mapping more precisely and are non-vanishing but otherwise arbitrary. For a generic choice of $g_{a\alpha\beta\gamma}$, these are 40 independent conditions on 6+10+48 (up to now free) components of our tensor variables, leaving 24 to span the combined kernel.

Lastly, the three Young tableaux in (4.8) are represented by

$$p_{\beta}^{\alpha}\,,\qquad p_{a(bc)}\,,\qquad p_{a\alpha(\beta\gamma)}\,.$$

The last two tensors, occurring as non-trivial variations of

$$d_x f(x)=dx^a f_{abc}x^b x^c\,,\quad\text{and}\quad d_y g(x,y)=dy^\alpha g_{a\alpha\beta\gamma}x^a y^\beta y^\gamma\,,$$

correspond to the result of the polynomial deformation method [10]; the other 36 components have been missed.

6.4. The Tian-Yau Manifold

Without much ado, we simply list the tensor variables and the corresponding constraints.

$$\left[\ker(0003)_{20} \begin{smallmatrix} \xrightarrow{f} (0000)_1 \\ \xrightarrow[f]{} (-1001)_{15} \end{smallmatrix}\right]_4 \sim \{p^{(abc)} : f_{abc}p^{(bcd)} = 0\},$$

$$[\ker(1113)_{10} \xrightarrow{f} (0111)_4]_6 \sim \{p^{(ab)} : f_{abc}p^{(bc)} = 0\},$$

$$(0012)_{20} \sim \{p^{a(bc)}\}.$$

Together with the $x^a \leftrightarrow y_a$ flipped contributions, these are 60 unrestricted components spanning $H^1(\mathscr{M}, \mathscr{T}_{\mathscr{M}} \otimes \mathscr{T}^*_{\mathscr{M}})$ for the Tian-Yau CICY.

Note added. After the completion of this article, it was discovered [19] that the number of End$\mathscr{T}_{\mathscr{M}}$-valued 1-forms may jump for specially symmetric choices of the Calabi-Yau manifold $\mathscr{M}$. Indeed, the special choice of defining polynomials, with which the comparison in [10] is made, implies 108 rather then 88 End$\mathscr{T}_{\mathscr{M}}$-valued 1-forms [19]. Even so, the exactly soluble model has 46 gauge-neutral massless chiral superfields more, only three of which are accompanied by extra $U(1)$ gauge invariances [9]. This discrepancy requires a thorough explanation for the two models to be regarded as equivalent.

A. The $f' : \mathscr{T}_{\mathscr{W}} \to \mathscr{E}$ Map

For any manifold (not necessarily Calabi-Yau) $\mathscr{M} \hookrightarrow \mathscr{W} \stackrel{\text{def}}{=} \mathbb{P}_1^{n_1} \times \ldots \times \mathbb{P}_m^{n_m}$ defined without multiplicity as the space of simultaneous solutions to a system of K polynomial constraints $f^{(a)}(z) = 0$, the short sequence

$$0 \longrightarrow \mathscr{T}_{\mathscr{M}} \xrightarrow{\iota} \mathscr{T}_{\mathscr{W}}|_{\mathscr{M}} \xrightarrow{f'} \left(\mathscr{E} \stackrel{\text{def}}{=} \bigoplus_{a=1}^{K} \mathscr{E}_a\right)\Big|_{\mathscr{M}} \longrightarrow 0$$

is exact, where

$$\mathscr{E}_a \stackrel{\text{def}}{=} \bigotimes_{r=1}^{m} (-\deg_{\mathbb{P}_r^{n_r}}(f^{(a)})|0\ldots0)$$

are the line bundles over $\mathscr{W}$ of which $f^{(a)}(z)$ are sections.

For the following discussion, the number of constraints and also the number of projective spaces whose product is the ambient space is irrelevant. We shall therefore consider the case where there is one of each, leaving the general treatment to the reader. In fact, the degree of the defining function is also unimportant and so, for simplicity, we shall discuss our pet example $\mathscr{M} \in [4, 5]$, a smooth quintic hypersurface in $\mathbb{P}^4$. The defining polynomial, $f(z) = f_{abcde}z^a z^b z^c z^d z^e$, is a section of $(-5|0000)$. In more traditional notation (see, e.g., Griffiths and Harris in [4]), this bundle is denoted $\mathcal{O}(5)$. The derivative of $f(z)$

$$f_a(z) = 5f_{abcde}z^b z^c z^d z^e = \partial_a f(z) = \frac{\partial f}{\partial z^a}$$

[9] We thank A. N. Schellekens for discussions on this point

is invariantly defined as a section of $\mathcal{O}_a(4)$ [i.e. $(-4|0000)\otimes(-10000)$ in our general notation]. As a holomorphic vector bundle, this is the direct sum of five copies of $\mathcal{O}(4)$ but as a homogeneous bundle, the index a indicates that $GL(5,\mathbb{C})$ acts according to the dual of the standard representation [i.e. as (-10000)]. This derivative is responsible for the mapping

$$(-1|0001)|_{\mathcal{M}}=\mathcal{T}_{\mathcal{W}}|_{\mathcal{M}}\xrightarrow{f'}\mathcal{E}|_{\mathcal{M}}=\mathcal{O}_{\mathcal{M}}(5)$$

the kernel of which is $\mathcal{T}_{\mathcal{M}}$. In other words, $\partial_a f$ should define a natural section f' of

$$\mathcal{T}^*_{\mathcal{W}}|_{\mathcal{M}}\otimes\mathcal{O}_{\mathcal{M}}(5)=(-4|-1000)_{\mathcal{M}}.$$

In order to make this precise, recall the Euler exact sequence

$$0\longrightarrow\mathcal{T}^*_{\mathcal{W}}\longrightarrow\mathcal{O}_a(-1)\xrightarrow{z^a}\mathcal{O}\longrightarrow 0$$

(i.e. $(-10000)\otimes(1|0000)=(0|0000)+(1|-10000)$ as in [5]). Tensoring with $\mathcal{O}(5)$ gives

$$0\longrightarrow(-4|-1000)\longrightarrow\mathcal{O}_a(4)\xrightarrow{z^a}\mathcal{O}(5)\longrightarrow 0$$

as an exact sequence on $\mathbb{P}^4$ and $\partial_a f$ provides a natural section of the middle bundle. Under the mapping z^a, $\partial_a f$ maps to $5f$; indeed, $z^a\partial_a$ is the Euler homogeneity operator and f has homogeneity 5. Since this vanishes on $\mathcal{M}$, $\partial_a f$ has image in $(-4|-1000)$ when restricted to $\mathcal{M}$. This is precisely f' and we can use this description to compute the induced mapping

$$H^0(0|0000)\xrightarrow{f'}H^0(\mathcal{M},\mathcal{E}\otimes\mathcal{T}^*_{\mathcal{W}})=H^0(\mathcal{M},(-4|-1000))$$

encountered in Sect. 2. To this end, consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (1|-1000) & \xrightarrow{\times f} & (-4|-1000) & \longrightarrow & (-4|-1000)|_{\mathcal{M}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_a(-1) & \xrightarrow{\times f} & \mathcal{O}_a(4) & \longrightarrow & \mathcal{O}_a(4)|_{\mathcal{M}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle z^a} & & \downarrow{\scriptstyle z^a} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\times f} & \mathcal{O}(5) & \longrightarrow & \mathcal{O}_{\mathcal{M}}(5) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. The first row may be used to compute the cohomology of $(1|-1000)$. Since all cohomology with coefficients in $\mathcal{O}(-1)$ vanishes, the connecting homomorphism gives an isomorphism

$$(0|0000)=\mathbb{C}=H^0(\mathbb{P}^4,\mathcal{O})\cong H^1(1|-1000),$$

a result that agrees with the BBW computation. Now consider the elements

$$5\in\Gamma(\mathbb{P}^4,\mathcal{O})\qquad \partial_a f\in\Gamma(\mathbb{P}^4,\mathcal{O}_a(4))\qquad f'\in\Gamma(\mathscr{M},(-4|-1000))$$

in the above diagram. They are related in a diagram chase:

$$5f=z^a\partial_a f,\qquad \partial_a f|_{\mathscr{M}}=f',$$

and we may conclude that

$$\begin{array}{ccccccc}
\mathbb{C}=H^0(0|0000) & \to & H^0(\mathscr{M},(-4|-1000)) & \to & H^0(\mathbb{P}^4,\mathcal{O}_a(4)) & \to & H^0(\mathbb{P}^4,\mathcal{O})=H^1(1|-1000)\\
\cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \cup\!\!|\\
1 & \longrightarrow & f' & \longrightarrow & \partial_a f & \longrightarrow & 5
\end{array}$$

We have therefore computed the mapping $\hat{f}'$:

$$\begin{array}{ccccccccc}
 & & & & H^0(0|0000) & \searrow^{\hat{f}'} & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H^0(-4|-1000) & \longrightarrow & H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}}) & \longrightarrow & H^1(1|-1000) & \longrightarrow & 0\\
 & & \| & & \| & & \| & & \\
 & & (-4-1000) & & H^0(\mathscr{M},(-4|-1000)) & & (00000) & &
\end{array}$$

as multiplication by 5. This shows that $H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}})$ splits

$$H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}})=(00000)\oplus(-4-1000).$$

In retrospect, it is clear that $H^0(0|0000)$ cannot map into $(-4-1000)$ because this would yield an invariant element whereas the only invariant element is $f\in(-50000)$, a distinct representation.

Finally, we may deduce the following result needed for Sect. 2. The exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\mathscr{E}\otimes\mathscr{E}^*)|_{\mathscr{M}} & \longrightarrow & (\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}})|_{\mathscr{M}} & \longrightarrow & \mathscr{E}|_{\mathscr{M}}\otimes\mathscr{T}^*_{\mathscr{M}} \longrightarrow 0\\
 & & \| & & \| & & \\
 & & (0|0000) & & (-4|-1000) & &
\end{array}$$

on $\mathscr{M}$ gives the exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathscr{M},(0|0000)) & \longrightarrow & H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}}) & \longrightarrow & H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{M}}) & \longrightarrow & 0.\\
 & & \| & & & & & & \\
 & & H^0(0|0000) & & & & & &
\end{array}$$

Since we have just computed $H^0(0|0000)\xrightarrow{f'}H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{W}})$ we may conclude that

$$H^0(\mathscr{M},\mathscr{E}\otimes\mathscr{T}^*_{\mathscr{M}})=(-4-1000).$$

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