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A Unified Approach to String Scattering Amplitudes

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Abstract. 1) *Physics*. In the calculation of g-loop string tachyon amplitudes with n scattering points the distinguished Polyakov measure $d\pi_{g,n}$ on the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n punctures arises. We give an interpretatiton of this measure as the modulus squared of a holomorphic section $\mu_{g,n}$ (the Mumford form) of a certain holomorphic line bundle, i.e., we prove an analog of the Belavin-Knizhnik theorem $d\pi_{g,n} = |\mu_{g,n}|^2$ in the amplitudic case. We give an expression for this measure through the determinants of the Laplace operators over ghosts and over multivalued fields with monodromy prescribed by momenta at the scattering points. We show also that the form $\mu_{g,n}$ $(n \geqq 0)$ for the partition function and n-point amplitudes can be obtained from a unified over all n, universal Mumford form.

2) *Mathematics*. The following new concepts from the theory of complex algebraic curves are investigated: divisors with complex coefficients, complex powers of holomorphic line bundles, determinants of Laplace operators over multivalued functions, etc. The corresponding generalizations of the determinant line bundles, the Weil-Deligne pairings, the Quillen and the Arakelov-Deligne metrics are constructed. A suggested by string amplitude considerations analog of the Mumford theorem on holomorphic triviality of the bundle $\lambda_2 \otimes \lambda_1^{-13}$ over the moduli space is given. This analog asserts the existence of a canonical flat metric on a certain line bundle $\lambda_2 \otimes \lambda_1^{-13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle^{-1} \right)$ (see the main body of the text). There exist two differences: the latter bundle is not holomorphically trivial but has a canonical flat metric, and, being defined on the Teichmüller space $T_{g,n}$, this bundle can be pulled down only on an infinite-sheeted covering of the moduli space $\mathcal{M}_{g,n}$. The universal isometries and the relative curvatures from the second part of the paper may be interesting, too.

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Introduction

1. The String Partition Function

Let X be a closed Riemann surface (more precisely, a compact complex algebraic curve) of genus g, γ be a metric on X, which is considered as a Hermitian structure on the tangent bundle of X. We suppose $g \geqq 2$ throughout the paper in order to keep a certain unity of exposition and to pay no attention to pieces of subtlety of the small genus cases, as well as we suppose the ground field to be $\mathbb{C}$, although in some moments we could take an arbitrary algebraically closed field. But, undoubtedly, small genus needs to be considered separately.

Let also $\mathcal{M}_g$ be the moduli space of conformal classes of metrics on X. There is a one-to-one correspondence between conformal classes of metrics and complex structures on X. Then such a metric γ lies in the conformal class determined by the complex structure, i.e. γ is Kähler.

In the critical dimension $d=26$ the functional integral which gives the partition function of the Polyakov bosonic string can be reduced to a finite-dimensional one via the Faddeev-Popov trick. The g-loop contribution Z_g to the *partition function* is therefore the following integral:

$$Z_g = \int_{\mathcal{M}_g} \mathrm{d}\pi_g .$$

The integrand $\mathrm{d}\pi_g$ is called the *Polyakov measure.* It can be expressed in terms of an arbitrary basis $\{\varphi_1, \ldots, \varphi_g\}$ in the space $\Gamma(X, \Omega)$ of holomorphic differentials on X and an arbitrary basis $\{W_1, \ldots, W_{3g-3}\}$ in the space $\Gamma(X, \Omega^{\otimes 2})$ of holomorphic quadratic differentials:

$$\mathrm{d}\pi_g = \frac{W_1 \wedge \bar{W}_1 \wedge \ldots \wedge W_{3g-3} \wedge \bar{W}_{3g-3}}{\left(\frac{2}{i}\right)^{3g-3} \det(\varphi_i, \varphi_j)^{13}} \cdot \left(\frac{\det' \Delta_1}{\det(\varphi_i, \varphi_j) \cdot \int_X \frac{i}{2} \gamma \mathrm{d}z \mathrm{d}\bar{z}} \right)^{-13} \cdot \frac{\det' \Delta_2}{\det(W_i, W_j)} .$$

Here the brackets (,) denote the L_2-scalar product of the sections of Hermitian line bundles Ω and $\Omega^{\otimes 2}$ (Ω is the complex cotangent bundle of X and $\Omega^{\otimes 2}$ is its tensor square) with the metrics γ^{-1} and γ^{-2}, Δ_1 and Δ_2 are the Laplace operators ($\Delta = \bar{\partial}^* \bar{\partial}$) acting on smooth sections of Ω and $\Omega^{\otimes 2}$, det′ means the regularized determinant. The two latter factors in our expression for $\mathrm{d}\pi_g$ can be interpreted as powers of the norms in the sense of the Quillen metrics of the sections $\varphi_1 \wedge \ldots \wedge \varphi_g$ and $W_1 \wedge \ldots \wedge W_{3g-3}$ of the *determinant line bundles*

$$\lambda_1 = \det \mathbb{R}\Gamma(X, \Omega)$$

and

$$\lambda_2 = \det \mathbb{R}\Gamma(X, \Omega^{\otimes 2})$$

over the moduli space. [The fibres of these bundles over the point of $\mathcal{M}_g$ corresponding to a complex curve X are the maximal exterior powers of the spaces $\Gamma(X, \Omega)$ and $\Gamma(X, \Omega^{\otimes 2})$, correspondingly.] By definition, the *Quillen norms* squared are

$$\|\varphi_1 \wedge \ldots \wedge \varphi_g\|_Q^2 = \frac{\det(\varphi_i, \varphi_j) \cdot \int_X \frac{i}{2} \gamma \mathrm{d}z \mathrm{d}\bar{z}}{\det' \Delta_1} , \tag{1}$$

$$\|W_1 \wedge \ldots \wedge W_{3g-3}\|_Q^2 = \frac{\det(W_i, W_j)}{\det' \Delta_2} . \tag{2}$$

The Belavin-Knizhnik theorem permits us to get rid of these, non-holomorphic factors in the expression for $\mathrm{d}\pi_g$. In fact, the Mumford theorem asserts that the bundle $\lambda_2 \otimes \lambda_1^{-13}$ [1] is holomorphically trivial, i.e. there exists an isomorphism f: $\lambda_2 \otimes \lambda_1^{-13} \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}_g}$, defined up to a constant factor, $\mathcal{O}_{\mathcal{M}_g}$ being the trivial line bundle over the moduli space (moreover, according to the Beilinson-Manin theorem [1] there exists a canonical trivialization f). The *Belavin-Knizhnik theorem* [2] says that one can choose f so that it will be an isometry, provided the left-hand side with the tensor product of the Quillen metrics and the right-hand side with the trivial metric. Then taking $\mu_g = W_1 \wedge \ldots \wedge W_{3g-3} \otimes (\varphi_1 \wedge \ldots \wedge \varphi_g)^{-13}$, so that $f(\mu_g) = 1$, one has $\|\mu_g\|_Q = 1$ and

$$\mathrm{d}\pi_g = \mu_g \wedge \bar{\mu}_g ,$$

where $\mu_g \wedge \bar{\mu}_g$ is understood as the first factor in the expression for $\mathrm{d}\pi_g$ above. The form μ_g is called the *Mumford form.*

2. The String Amplitudes

Suppose additionally that an ordered set $\mathfrak{m}$ of distinct punctures $Q_1, \ldots, Q_n$ and an isotopical class of a disk B in $X \backslash \mathfrak{m}$, such that $\mathfrak{m} \subset B \subset X$, are given. For each point Q_i there fixed a vector $\mathbf{p}_i$ in the space-time $\mathbb{C}^{13}$, where the standard Hermitian metric is fixed. Each vector $\mathbf{p}_i$ is interpreted as *momentum vector*. These vectors satisfy the conditions:

1. $\sum_{i=1}^{n} \mathbf{p}_i = 0$ (the momentum-conservation law),
2. the Hermitian square $(\mathbf{p}_i, \mathbf{p}_i)$ equals 1 for every i (the mass of tachyon equals $\sqrt{-1}$).

Then the *tachyon scattering amplitude* is the integral

$$A(g; \mathbf{p}_1, \ldots, \mathbf{p}_n) = \int_{\mathcal{M}_{g,n}} \mathrm{d}\pi_{g,n} ,$$

where $\mathcal{M}_{g,n}$ is the moduli space of punctured Riemann surfaces and

$$\mathrm{d}\pi_{g,n} = \mathrm{d}\pi_g \cdot \prod_{j=1}^{n} \frac{i}{2} \gamma \mathrm{d}z_j \mathrm{d}\bar{z}_j \cdot \prod_{i,j=1}^{n} G(Q_i, Q_j)^{(\mathbf{p}_i, \mathbf{p}_j)} , \tag{3}$$

$G = \exp(g)$, $-g/\pi$ is the Green function of the scalar Laplace operator $\Delta = \bar{\partial}^* \bar{\partial}$, $g(z,z) = \lim_{z' \to z} (g(z',z) - \log \|z' - z\|)$, $\| \ \|$ being the distance on X in the metric γ.

Theorem. $\mathrm{d}\pi_{g,n} = \mu_{g,n,B} \wedge \bar{\mu}_{g,n,B}$, *where* $\mu_{g,n,B}$ *is a local holomorphic section of a flat Hermitian line bundle* $\lambda_2 \otimes \lambda_1^{-13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle \right)^{-1}$ *over the moduli space* $\mathcal{M}_{g,n,B}$ *of the data* $(X, Q_1, \ldots, Q_n, B)$. *Here* $D^\nu = \sum_{i=1}^{n} p_i^\nu \cdot Q_i$ *is a complex divisor with the momentum components as coefficients.*

The proof of the theorem will be given in 2.3. The definition of a complex divisor and the construction of the line bundle $\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle$ are contained in Sect. 1. The section $\mu_{g,n,B}$ is defined to have the norm 1, so it is locally unique modulo a constant factor $\exp(i\varphi)$, $\varphi \in \mathbb{R}$.

[1] Powers of line bundles mean tensor powers throughout the text

The space $\mathcal{M}_{g,n,B}$ is a quotient-space of the Teichmüller space $T_{g,n}$ by a subgroup of the modular group (see 2.1), but $\mu_{g,n,B} \wedge \bar{\mu}_{g,n,B}$ is modular-invariant and so can be pulled down on $\mathcal{M}_{g,n}$.

3. What Is $\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle$*?*

Now we give some commentary on the notions appearing in the formulation of Theorem 2.

If a usual divisor is a formal sum $\sum_{P \in X} n_P \cdot P$ over a finite number of points P with integral multiplicities n_P, then a *complex divisor* on X may have complex coefficients at the points Q_i, but has integral degree. To every complex divisor D there corresponds a usual holomorphic line bundle $\mathcal{O}(D)$ over X, which is defined to have a multivalued section $\mathbb{1}_D$ with asymptotics z^{n_P} at P (i.e. $\mathrm{ord}_P \mathbb{1}_D = n_P$), single-valued outside the disk B. $\mathcal{O}(D)$ is a usual holomorphic line bundle because its glueing functions one can choose to be single-valued (see 1.5).

Consider a meromorphic multivalued function f, single-valued outside B. (Multivalued here means having constant multiplicators, see 1.2.) To such a function f one can attach the divisor $\Sigma\, \mathrm{ord}_P f \cdot P$ which is called *principal*. The corresponding group of classes of complex divisors modulo principal is isomorphic to the group of classes of ordinary (integral) divisors (see Sect. 1).

Beginning from two holomorphic line bundles $\mathcal{O}(D)$ and $\mathcal{O}(D')$ over X one can construct a canonical one-dimensional complex vector space $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$, which generates a holomorphic line bundle over the moduli space under variation of parameters. If D' were equal to ΣP (i.e. all the non-zero coefficients were equal to 1), then $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle := \det \mathcal{O}(D)|_{D'}$ would be merely the maximal exterior power (determinant) of the vector space $\mathcal{O}(D)|_{D'}$. The definition in the case of an arbitrary D' is more complicated, but does not differ from the case of integral divisors, see 1.7. If $\deg D = \deg D' = 0$ and the line bundles $\mathcal{O}(D)$, $\mathcal{O}(D')$ are provided with Hermitian metrics, then there is a canonical Hermitian metric on $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$. If the former metrics are flat then the latter is flat, too, and it does not depend on metrics on $\mathcal{O}(D)$, $\mathcal{O}(D')$. In this case the metrics on $\mathcal{O}(D)$, $\mathcal{O}(D')$ can be expressed via the Green function of the scalar Laplace operator, see 1.9.

4. The Determinants of Laplace Operators over Multivalued Functions

One can define Quillen metric on an arbitrary determinant line bundle $\det \mathbb{R}\Gamma(X, \mathcal{O}(D))$. If D is a complex divisor, then the one-dimensional complex vector space $\det \mathbb{R}\Gamma(X, \mathcal{O}(D))$ is defined as the alternated tensor product $\det \mathrm{H}^0(X, \mathcal{O}(D)) \otimes (\det \mathrm{H}^1(X, \mathcal{O}(D)))^{-1}$ of the determinants of the cohomology groups of the bundle $\mathcal{O}(D)$. If, additionally, $\mathcal{O}(D)$ is metrized then the Quillen metric on $\det \mathbb{R}\Gamma(X, \mathcal{O}(D))$ is defined in 1.12 similar to (1) and (2), using the determinant of the Laplace operator acting on the smooth sections of $\mathcal{O}(D)$. For $D = \Sigma n_P \cdot P$ complex, these sections can be considered as smooth multivalued functions f on X admitting prescribed singularities and having a prescribed branching at the points of $\mathrm{supp} D$: $f(z, \bar{z}) \cdot z^{n_P}$ must be smooth and single-valued at P, z being a holomorphic coordinate at P.

Proposition. *There exists a canonical isometry*

$$\lambda_2\otimes\lambda_1^{-13}\otimes\left(\bigotimes_{\nu=1}^{13}\langle\mathcal{O}(D^\nu),\mathcal{O}(D^\nu)\rangle\right)^{-1}=\lambda_2\otimes\lambda_1^{13}$$
$$\otimes\left(\bigotimes_{\nu=1}^{13}\det\mathbb{R}\Gamma(X,\mathcal{O}(D^\nu))\otimes\det\mathbb{R}\Gamma(X,\mathcal{O}(-D^\nu))\right)^{-1}, \qquad (4)$$

all the terms provided with the metrics introduced above.

In the case of integral divisors the proposition is a sequence of one result of Deligne [3], connecting $\langle\mathcal{O}(D),\mathcal{O}(D')\rangle$ with $\det\mathbb{R}\Gamma(X,\mathcal{O}(D))$ and $\det\mathbb{R}\Gamma(X,\mathcal{O}(D'))$. This result has a straightforward generalization to the case of complex divisors (10).

Thus, the *generalized Mumford form* $\mu_{g,n,B}$ is a holomorphic section of norm 1 of the flat Hermitian bundle (4) over $\mathcal{M}_{g,n,B}$.

5. *The Universal Mumford Form*

The measures $d\pi_g$ and $d\pi_{g,n}$ for the partition function and the amplitudes are obtained from a single measure via the following construction, see 4.6. Let $J(X)$ be the Jacobian of the curve X, J the total space of the bundle with fibre $J(X)$ over a point $X\in\mathcal{M}_g$, J^m the same with fibre $J^m(X)$, and J^t with fibre $J^t(X)$, the dual Abelian variety to $J(X)$. Consider the holomorphic line bundle $\mathcal{B}:=(\mathrm{id}\times\Phi)^*(\mathcal{P})$ over $J\times J$, where $\mathcal{P}$ is the Poincaré bundle (see 3.1) over $J\times J^t$, $\Phi:J\to J^t$ is the natural principal polarization (see 3.1). Take the restriction $\mathcal{C}$ of $\mathcal{B}$ to J relative to the diagonal embedding

$$\Delta: J\hookrightarrow J\times J\,.$$

Finally, consider the line bundle $\mathcal{C}^{\boxtimes 13}$ over J^{13}. In 4.6 a canonical flat metric on this bundle is defined. We assert that as a *universal Mumford form* for the partition function and the amplitudes one can take a covariantly constant holomorphic section μ_U of norm 1 of the bundle $\lambda_2\otimes\lambda_1^{-13}\otimes\mathcal{C}^{\boxtimes 13}$. That means that the measures $d\pi_g$ and $d\pi_{g,n}$ are the modulus squared of the pull-back of the universal Mumford form relative to the morphism

$$\varphi=\varphi(g;\mathbf{p}_1,\ldots,\mathbf{p}_n):\mathcal{M}_{g,n,B}\to J^{13},$$

which is identical on the base $\mathcal{M}_g$ and acts in the fibre $X^n\backslash\Delta$ (see 2.1 concerning the diagonal Δ) over a curve $X\in\mathcal{M}_g$ due to the formula

$$\varphi:(X^n\backslash\Delta,\ \text{class of a disk } B)\to J^{13}(X),$$
$$(Q_1,\ldots,Q_n)\mapsto\boxtimes_{\nu=1}^{13}\mathcal{O}(D^\nu),$$

where $D^\nu=\sum_{i=1}^n p_i^\nu\cdot Q_i$. One can show (4.6) that there exists a canonical isometry

$$\lambda_2\otimes\lambda_1^{-13}\otimes\left(\bigotimes_{\nu=1}^{13}\langle\mathcal{O}(D^\nu),\mathcal{O}(D^\nu)\rangle\right)^{-1}=\varphi^*(\lambda_2\otimes\lambda_1^{-13}\otimes\mathcal{C}^{\boxtimes 13}),$$

and that

$$\mu_{g,n,B} = \varphi^*(\mu_U).$$

Therefore,

$$d\pi_{g,n} = \varphi^*(\mu_U) \wedge \overline{\varphi^*(\mu_U)}.$$

At the same time,

$$d\pi_g = d\pi_{g,0},$$

if $\varphi(g;\phi): X^0 \to J^{13}(X)$ is the constant map into $(0, \ldots, 0) \in J^{13}(X)$.

Using (4) we show in 4.6 that the universal Mumford form can be constructed as a holomorphic section of the Hermitian holomorphic line bundle

$$\lambda_2 \otimes \lambda_1^{-13} \otimes T_E^* \mathcal{O}(W_{g-1})^{13} \otimes i^* T_E^* \mathcal{O}(W_{g-1})^{13} \otimes (\det \mathbb{R}\Gamma(E))^{26}$$

over the space J^{13}, and besides, of the bundle

$$\lambda_2 \otimes \lambda_1^{-13} \otimes \mathcal{O}(\Theta)^{26} \otimes \det \mathbb{R}\Gamma(\kappa)$$

over a finite-sheeted covering of J^{13}, where $W_{g-1}(X)$ is the image of X^{g-1} in $J_{g-1}(X) = \{$the variety of classes of divisors on X of degree $g-1\}$ under the natural map $X^{g-1} \to J_{g-1}(X)$ (W_{g-1} being a divisor on J_{g-1}), E is a fixed holomorphic line bundle of degree $g-1$ on X, $T_E: J \to J_{g-1}$ means the translation by E, $i: J \to J$ is the antipod (inversion morphism), $\Theta \subset J$ is the zero-set of the Riemann theta-function, $\kappa \in J_{g-1}$ is the Riemann constant (determined by the equality $\Theta = T_\kappa^* W_{g-1}$) as well as the line bundle of degree $g-1$ on X whose class equals the Riemann constant, together with an isomorphism $\kappa \otimes \kappa = \Omega$. All the named line bundles are Hermitian: the metrics on $\mathcal{O}(W_{g-1})$, $\mathcal{O}(\Theta)$, E and κ are constructed using the notion of admissible metric (in other words, with the help of the Green functions of Laplacians) due to Arakelov and Faltings – see 3.4 and 4.4.

In conclusion, we need to emphasize that all the results of Sect. 3 are simple consequences of classical results which were known by Arakelov and Faltings. We also use in essence Moret-Bailly's paper [4].

6. *Perspectives*

In order to have a more physical sense, this paper should be completed with its super version. Having the consistent theory of SUSY-curves (= superconformal manifolds = super Riemann surfaces), see Baranov-Schwarz [5], Friedan [6], Beilinson-Manin [1] and also [7], Deligne [8] and Rosly et al. [9], this work seems to be not so hard. Divisors on SUSY-curves are sums of points in some sense, as for ordinary curves (this was proposed by Manin and stated in [9, 8]). The technique of super Quillen metrics is an item of [9], the Weil-Deligne pairings $\langle\ ,\ \rangle$ are trivial in the super case [7–9], but one should use a purely super pairing $[\ ,\ ]$ (see [9]) to prove more fine facts.

The problem of the generalization to the case of not only tachyon amplitudes remains open. This may be useful in the proof of the factorization property on the level of Mumford forms: that is, for example, an equality of type $\mu_g = \mu_{g',1} \cdot \mu_{g'',1}$, when our curve, approaching the boundary of the moduli space $\mathcal{M}_g$, splits into two curves of genus g' and g'' (I am indebted to G. Moore for this remark).

It seems interesting to rewrite our results from the moduli space to the universal Grassmannian manifold, especially the results concerning the universal Mumford form. This must be compatible with the factorization property.

7. Notes

1. Another way of mathematical understanding of the amplitudic integrand is presented in the work of Baranov and Schwarz [5]. The term $d\pi_{g,n}/d\pi_g$ is interpreted by them as a measure on the product of 13 spaces of differentials of the third kind on X: the ν^{th} space consists of differentials with poles only at $Q_1, \ldots, Q_n$ and with residues p_i^ν at Q_i. Our picture seems to be the $\exp \int$ of their one in a sense.

2. The complex powers of holomorphic line bundles in two-dimensional conformal field theory appear also in Beilinson and Schechtman's paper [10]. In that approach a line bundle $\mathscr{L}$ is replaced by its *Atiyah algebra* $A_{\mathscr{L}}$, that is the sheaf of first order differential operators on $\mathscr{L}$ with highest symbol $\mathrm{id}_{\mathbb{L}} \otimes \partial/\partial z$. There takes place the exact sequence $0 \to \mathscr{E}nd\, \mathscr{L} \to A_{\mathscr{L}} \to T_X \to 0$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}nd\,\mathscr{L} \arrow[r] \arrow[d, "c"'] & A_{\mathscr{L}} \arrow[r] \arrow[d] & T_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{E}nd\,\mathscr{L} \arrow[r] & ? \arrow[r] & T_X \arrow[r] & 0, \end{tikzcd}$$

where c is the operator of multiplication by $c \in \mathbb{C}$ and T_X is the tangent sheaf of X, as usual, can be completed to commutative with a sheaf $A_{\mathscr{L}^c}$ interpreted as the Atiyah algebra of the (non-existing) bundle $\mathscr{L}^c$, $\mathscr{L}$ to the power c. The Atiyah algebra keeps incomplete information about its bundle, because an isomorphism $A_{\mathscr{L}} \xrightarrow{\sim} A_{\mathcal{O}}$ implies the existence of a canonical flat Hermitian connection on $\mathscr{L}$, but it does not yield the triviality of $\mathscr{L}$. It is remarkable that using Atiyah algebras one can apply local arguments in the Riemann-Roch type theorems.

1. Calculus of Complex Divisors on Riemann Surface

1. Complex Divisors

Let X be a complex compact curve of genus g, with a fixed ordered set $\mathfrak{m} = \{Q_1, \ldots, Q_n\}$ of n distinct points on X and a closed disk B such that $\mathfrak{m} \subset B$. A *complex divisor* is a formal sum

$$D = \sum_{P \in X} n_P \cdot P,$$

where

$$n_P \in \begin{cases} \mathbb{Z} \text{ for } P \in X \backslash \mathfrak{m}, \\ \mathbb{C} \text{ for } P \in \mathfrak{m}, \end{cases}$$

$$\deg D := \sum_{P \in X} n_P \in \mathbb{Z},$$

and only a finite number of $n_P \neq 0$. The corresponding *group of complex divisors* denote by $\mathrm{Div}(X, \mathfrak{m}, B)$.

2. Multivalued Meromorphic Functions

Let $p:\widetilde{X\backslash\mathfrak{m}}\to X\backslash\mathfrak{m}$ be the universal covering with the complex structure, pulled up from the base. Denote by H the kernel of the natural epimorphism $\pi_1(X\backslash\mathfrak{m})\to\pi_1(X)$ determined by the embedding $X\backslash\mathfrak{m}\hookrightarrow X$:

$$1\to H\to\pi_1(X\backslash\mathfrak{m})\to\pi_1(X)\to 1\,.$$

We shall call a holomorphic function φ on $\widetilde{X\backslash\mathfrak{m}}$ (more correctly, a section of the sheaf $p_*\mathcal{O}_{\widetilde{X\backslash\mathfrak{m}}}$) a *multivalued holomorphic function* on X, if

1. φ is $\pi_1(X\backslash B)$-invariant.
2. For every $\sigma\in H$, $\varphi^\sigma=f_\sigma\cdot\varphi$, where $\varphi^\sigma(x):=\varphi(\sigma x)$ and f_σ is a constant (f_σ is called *multiplicator*).
3. The branches of φ, as branches of a multivalued analytic function on X, have only removable singularities in $\mathfrak{m}$, that is, for any $Q_i\in\mathfrak{m}$ and any sequence $\{a_m\}$ in $\widetilde{X\backslash\mathfrak{m}}$, such that $p(a_m)\to Q_i$ when $m\to\infty$, there exists a limit $\lim\limits_{m\to\infty}\varphi(a_m)$ depending only on Q_i:

$$\varphi(Q_i):=\lim_{m\to\infty}\varphi(a_m)\,.$$

Denote by $\mathcal{O}'$ the sheaf of holomorphic multivalued functions on X. The corresponding sheaf of fields of fractions is denoted by $\mathcal{M}'$. Sections of the sheaf $\mathcal{M}'$ we shall call *multivalued meromorphic functions* on X. The following simple lemma describes the local behaviour of such functions.

Lemma. *Let z be a holomorphic coordinate on X near $Q_i\in\mathfrak{m}\subset X$. Then*

1. *if $\varphi\in\mathcal{O}'$, then either*

$$\varphi(z)=z^A\cdot\sum_{j=0}^{\infty}\alpha_j z^j\,,\quad where\quad 0<\mathrm{Re}A<1\,,$$

or

$$\varphi(z)=\sum_{j=0}^{\infty}\alpha_j z^j\,,$$

2. *if $\varphi\in\mathcal{M}'$, then*

$$\varphi(z)=z^A\cdot\sum_{j=n_0}^{\infty}\alpha_j z^j\,,\quad where\quad 0\leqq\mathrm{Re}A<1\,.\quad\square$$

Note. One must remember that these expansions have also a monodromy at the other points Q_i.

Definition. The number $A+n_0$ is called the order $\mathrm{ord}_{Q_i}\varphi$ of the multivalued meromorphic function φ at the singular point Q_i.

Let $\varphi\in\Gamma(X,\mathcal{M}')$ be a globally defined multivalued meromorphic function on X. Then $\sum\limits_{P\in X}\mathrm{ord}_P\varphi=0$, because $d\log\varphi$ is a differential of the third kind on X and the sum of its residues vanishes.

Definition. We call principal a divisor of the type

$$\mathrm{div}\,\varphi:=\sum_{P\in X}\mathrm{ord}_P\varphi\cdot P\,.$$

Then define the *group* $\mathrm{Cl}(X, \mathfrak{m}, B)$ *of classes of complex divisors* as the quotient-group of the group $\mathrm{Div}(X, \mathfrak{m}, B)$ by the sub-group of principal divisors.

3. Complex Cartier Divisors

Complex Cartier divisors are the elements of the group $\mathrm{CaDiv}(X, \mathfrak{m}, B) := \mathrm{H}^0(X, (\mathcal{M}')^*/(\mathcal{O}')^*)$. A complex Cartier divisor is called *principal*, if it belongs to the image of $\mathrm{H}^0(X, (\mathcal{M}')^*)$ under the natural homomorphism

$$\mathrm{H}^0(X, (\mathcal{M}')^*) \to \mathrm{H}^0(X, (\mathcal{M}')^*/(\mathcal{O}')^*).$$

Proposition. *The group* $\mathrm{Div}(X, \mathfrak{m}, B)$ *is naturally isomorphic to the group* $\mathrm{CaDiv}(X, \mathfrak{m}, B)$, *the principal divisors corresponding to the principal ones.*

Proof is analogic to the classical one. □

4. The Jacobian and the Abel Map

Proposition. *The group* $\mathrm{Cl}(X, \mathfrak{m}, B)$ *is isomorphic to the group* $\mathrm{Cl}(X)$ *of classes of ordinary (integral) divisors on* X.

Proof. It is sufficient to prove that every complex divisor D is equivalent to an integral divisor.

Consider the *Abel map* to the *Jacobian* $J(X)$ of X,

$$\varphi : \mathrm{Div}(X, \mathfrak{m}, B) \to J(X) := \mathbb{C}^g/\Lambda,$$

$$\sum_i n_i \cdot P_i \mapsto \left(\sum_i n_i \cdot \int_P^{P_i} \omega_1, \ldots, \sum_i n_i \cdot \int_P^{P_i} \omega_g \right) \bmod \Lambda,$$

where $\{\omega_1, \ldots, \omega_g\}$ is a basis of the space of holomorphic differentials (i.e., differentials of the first kind), $P \in \partial B$ is a base point, coinciding with no P_i, and the integration path from P to P_i must be contained in B if $P_i \in \mathfrak{m}$. Λ is the *periods lattice*: it consists of the vectors $(\int \omega_1, \ldots, \int \omega_g)$, the integration in each one being over a loop in X.

According to the Jacobi inversion theorem, the point $\varphi(D) \in J(X)$ is the image $\varphi(D')$ of an integral divisor D' of degree d. Then $\varphi(D - D') \in \Lambda$ and it remains to show that $D - D'$ is principal, that is, there exists a multivalued meromorphic function f on X such that $\operatorname{div} f = D - D'$. Further the construction follows a distinguished proof of the Abel theorem (see, e.g., Lang [11]): take a differential ω of the third kind having only first order poles in $\operatorname{supp}(D - D')$ with residues $\operatorname{res}_P \omega = \operatorname{ord}_P(D - D')$, $P \in X$. Then, with the help of the bilinear relations between differentials of the first and the third kind (the fundamental polygon of X must be chosen so that it contains B), one can change the periods of ω along the cycles in $X \backslash B$ to be $2\pi\sqrt{-1} \cdot$ (integer) by adding a linear combination of differentials of the first kind. Now set $f := \exp \int \omega$. It is a multivalued meromorphic function on X with required singularities. □

5. Complex Divisors and Invertible Sheaves

Recall that the group $\mathrm{Div}(X)$ of divisors with integral coefficients is isomorphic to the group of invertible $\mathcal{O}$-sub-modules of the $\mathcal{O}$-module $\mathcal{M}$ over X (the latter group is nothing but the group of holomorphic line bundles provided with a meromorphic section). Similarly, the group $\mathrm{Div}(X, \mathfrak{m}, B)$ is isomorphic to the group of invertible $\mathcal{O}'$-sub-modules of $\mathcal{M}'$ (= the group of holomorphic line bundles with multivalued glueing functions and provided with a multivalued meromorphic section): to every complex Cartier divisor determined by a set $f_i \in \Gamma(U_i, (\mathcal{M}')^*)$, such that $f_i/f_j \in \Gamma(U_i \cap U_j, (\mathcal{O}')^*)$, in a covering $\bigcup_i U_i = X$, there corresponds the $\mathcal{O}'$-sub-module in $\mathcal{M}'$ generated by the elements f_i^{-1} over each U_i. The emphasized section of this sub-module is $1 \in \mathcal{M}'$.

Really, the group of invertible $\mathcal{O}'$-sub-modules in $\mathcal{M}'$ is isomorphic to the group of invertible $\mathcal{O}$-sub-modules in $\mathcal{M}'$ (= the group of ordinary holomorphic line bundles provided with a multivalued meromorphic section). Indeed, take the complex divisor $D = \sum n_P \cdot P$ corresponding to a given $\mathcal{O}'$-module and let us construct canonically an $\mathcal{O}$-module $\mathcal{O}(D)$ corresponding to D. Choose a covering of X, for example, a covering consisting of two maps $U_1 := \{$a δ-neighbourhood of B for a small $\delta > 0\}$, $U_2 := X \backslash B$, and take a multivalued meromorphic function f_1 on U_1 such that $\mathrm{ord}_P f_1 = n_P$ for $P \in U_1$ and a multivalued meromorphic function f_2 on U_2 such that $\mathrm{ord}_P f_2 = n_P$ for $P \in U_2$ and $f_2^{\gamma_i} = \exp(2\pi\sqrt{-1} \cdot n_{Q_i}) \cdot f_2$, $i = 1, \ldots, n$, where γ_i is a loop in B containing the single point Q_i. Then f_1/f_2 is a single-valued non-zero holomorphic function on $U_1 \cap U_2$, i.e. $f_1/f_2 \in \Gamma(U_1 \cap U_2, \mathcal{O}^*)$, and it determines an $\mathcal{O}$-sub-module $\mathcal{O}(D)$ in $\mathcal{M}'$ having f_2/f_1 as glueing function. Thus, $\mathcal{O}(D)$ is the sheaf of sections of an ordinary holomorphic line bundle which we denote by $\mathcal{O}(D)$, too.

The meromorphic section of $\mathcal{O}(D)$ determined by f_1 and f_2 is multivalued and it is denoted by $\mathbb{1}_D$. The $\mathcal{O}$-sub-module $\mathcal{O}(D)$ in $\mathcal{M}'$ has also single-valued meromorphic sections: these are $\mathbb{1}_D \cdot \{$the multivalued meromorphic functions $f \in \Gamma(X, \mathcal{M}')$ such that $f^{\gamma_i} = \exp(-2\pi\sqrt{-1} \cdot n_{Q_i}) \cdot f$, $i = 1, \ldots, n\}$. Such a section has divisor $D_1 = \mathrm{div} f + D$ which is integral and equivalent to D. Hence, $\mathcal{O}(D)$ is isomorphic to the $\mathcal{O}$-sub-module $\mathcal{O}(D_1)$ in $\mathcal{M}$. Let us summarize these conclusions in the

Proposition. 1. $\mathrm{Div}(X, \mathfrak{m}, B) \simeq \{$*the group of invertible* $\mathcal{O}'$-*sub-modules in* $\mathcal{M}'\} \simeq \{$*the group of invertible* $\mathcal{O}$-*sub-modules in* $\mathcal{M}'\}$.

2. *For* $D \in \mathrm{Div}(X, \mathfrak{m}, B)$ *the corresponding invertible* $\mathcal{O}$-*sub-module* $\mathcal{O}(D)$ *in* $\mathcal{M}'$ *admits a single-valued meromorphic section and, equivalently, is isomorphic to an* $\mathcal{O}$-*sub-module* $\mathcal{O}(D_1)$ *in* $\mathcal{M}$, D_1 *being integral*, $D_1 \sim D$.

3. *The group of isomorphism classes of line bundles* $\mathcal{O}(D)$, $D \in \mathrm{Div}(X, \mathfrak{m}, B)$, *is isomorphic to the group* $\mathrm{Pic}\, X$ *of isomorphism classes of ordinary holomorphic line bundles and, consequently, to the group* $\mathrm{Cl}(X)$. □

6. Green Functions of Complex Divisors

Let $\omega_1, \ldots, \omega_g$ be an orthonormal basis of the space of differentials of the first kind with the scalar product

$$\langle \omega, \omega' \rangle := \frac{\sqrt{-1}}{2} \int_X \omega \wedge \overline{\omega'}.$$

Fix the (1,1)-form

$$\eta=\frac{\sqrt{-1}}{2g}\sum_{j=1}^{g}\omega_j\wedge\overline{\omega_j}$$

on X. It is called the *canonical Kähler form* of X. A *Green function* of a complex divisor

$$D=\sum_{P\in X} n_P\cdot P$$

is a complex C^∞-differentiable function $g_D(z)$ on $X\backslash \operatorname{supp} D$, satisfying the condition

$$\partial\bar\partial g_D(z)=2\pi\sqrt{-1}(\deg D\cdot\eta-\delta_D(z))$$

on the level of currents, where $\delta_D(z)$ is the delta-function of the divisor D, defined by the equality $\int f(z)\delta_D(z)=\sum n_P f(P)$. In other words, $g_D(z)$ satisfies:

1. $\partial\bar\partial g_D(z)=2\pi\sqrt{-1}\deg D\cdot\eta$ on $X\backslash\operatorname{supp} D$,
2. $g_D(z)=n_P\log|z|+\alpha(z)$ in a neighbourhood U of each point P, z being a holomorphic coordinate in U, α being smooth.

Note. If $D=z'$ is a point $z'\in X$ and one normalizes $g_D(z)$ by the requirement $\int_X g_D(z)\eta=0$, then $-g(z,z')/\pi:=-g_D(z)/\pi$ is the distinguished Green function of the scalar Laplace operator Δ: $\Delta(f)\cdot\eta=\frac{1}{2\sqrt{-1}}\cdot\partial\bar\partial f$ (or $\Delta=\bar\partial^*\partial$). In the relative case (see Sect. 4) the normalization $g_D(a)=0$ for a fixed $a\in X$ is more useful.

Theorem. *For any complex divisor on X there exists a Green function. It is unique up to a constant.*

Proof. If $D=\Sigma n_i\cdot P_i$ is given then set $g_D(z):=\Sigma n_i\cdot g(z,P_i)$. □

7. *Admissible Metrics on Holomorphic Line Bundles*

Let D be a complex divisor, $\mathcal{O}(D)$ the corresponding holomorphic line bundle over X. Then we call *admissible* every Hermitian metric $\|\ \|$ on $\mathcal{O}(D)$ such that its Chern form $\mathbf{c}_1(\mathcal{O}(D))$ is proportional to η (really, it implies $\mathbf{c}_1(\mathcal{O}(D))=\deg D\cdot\eta$). Recall that the *Chern form* is, by definition, the (1,1)-form

$$\mathbf{c}_1(\mathcal{O}(D))=\frac{1}{2\pi\sqrt{-1}}\partial\bar\partial\log\|s\|,$$

where s is a regular local single-valued holomorphic section of $\mathcal{O}(D)$.

If $\mathcal{O}(D)$ is Hermitian it is desirable to know the norm of a multivalued meromorphic section m of $\mathcal{O}(D)$. The norm may be constructed in a usual way: if $m=f\cdot s$, f being a multivalued meromorphic function and s being a single-valued meromorphic section, then

$$\|m\|:=|f|\cdot\|s\|.$$

Thus, $\|m\|$ is a real multivaled function on X.

But one can define also single-valued "norms" of multivalued sections. Let us normalize $G_D(z)$ using one of normalizations mentioned in Point 6. Then define the norm of the canonical multivalued meromorphic section $\mathbb{1}_D$ at a point $z \in X$ by the formula:

$$\|\mathbb{1}_D\|(z) = \sqrt{G_D(z) \cdot G_{\bar{D}}(z)} = \prod_i G_{P_i}(z)^{n_i/2} \cdot G_{P_i}(z)^{\bar{n}_i/2} = \prod_i G_{P_i}(z)^{\mathrm{Re}\, n_i},$$

where $D = \Sigma n_i P_i$, $\bar{D} = \Sigma \bar{n}_i P_i$. The norm $\|\mathbb{1}_D\|(z)$ one can regard as the norm of the Weil-Deligne pairing $\langle \mathbb{1}_D, \mathbb{1}_z \rangle$, see Point 9.

This construction will not be in use in this paper.

8. *The Weil-Deligne Pairing*

Let $\mathscr{L}_1$, $\mathscr{L}_2$ be two holomorphic line bundles. Define a complex vector space $\langle \mathscr{L}_1, \mathscr{L}_2 \rangle$ as the space of linear combinations of expressions of the type

$$\langle l_1, l_2 \rangle,$$

where l_1 and l_2 are single-valued (i.e., having integral divisors) meromorphic sections of $\mathscr{L}_1$ and $\mathscr{L}_2$ with non-intersecting divisors, factorized modulo the relations

$$\langle f \cdot l_1, l_2 \rangle = f(\operatorname{div} l_2) \cdot \langle l_1, l_2 \rangle,$$
$$\langle l_1, g \cdot l_2 \rangle = g(\operatorname{div} l_1) \cdot \langle l_1, l_2 \rangle,$$

where f and g are single-valued meromorphic functions such that $f(\operatorname{div} l_2) := \prod_{P \in X} f(P)^{\operatorname{ord}_P l_2} \neq 0, \infty$ in the first formula and $g(\operatorname{div} l_1) := \prod_{P \in X} g(P)^{\operatorname{ord}_P l_1} \neq 0, \infty$ in the second one. Correctness of the definition is provided by the Weil reciprocity law (cf. [12]):

$$f(\operatorname{div} g) = g(\operatorname{div} f).$$

One can easily see that the space $\langle \mathscr{L}_1, \mathscr{L}_2 \rangle$ is a one-dimensional complex vector space. We will call it the *Weil-Deligne pairing* of $\mathscr{L}_1$ and $\mathscr{L}_2$.

9. *The Arakelov-Deligne Metric*

The metric mentioned in the head is a metric on the one-dimensional vector space $\langle \mathscr{L}_1, \mathscr{L}_2 \rangle$ and it is determined canonically by Hermitian metrics on $\mathscr{L}_1$, $\mathscr{L}_2$. We define this metric by setting the norm of a non-zero element $\langle l_1, l_2 \rangle$ (l_1, l_2 being meromorphic single-valued sections of $\mathscr{L}_1$, $\mathscr{L}_2$ with non-intersecting divisors) to be equal to

$$\|\langle l_1, l_2 \rangle\| := \exp\left(\int_X \mathrm{c}_1(\mathscr{L}_1) \cdot \log \|l_2\| + \log(\|l_1\|(\operatorname{div} l_2))\right), \tag{5}$$

where $\log(\|l_1\|(\operatorname{div} l_2)) := \sum_{P \in X} \operatorname{ord}_P l_2 \cdot \log \|l_1\|(P)$, as in Point 8.

Note. In the particular case, when the metrics on $\mathscr{L}_1$ and $\mathscr{L}_2$ are admissible and are normed by the condition of vanishing the integral in (5) (and the same, substituting l_1 for l_2), one obtains the ∞-component of the *Arakelov intersection number* $\langle\ ,\ \rangle$ of two integral divisors $\operatorname{div} l_1$ and $\operatorname{div} l_2$:

$$\langle \operatorname{div} l_1, \operatorname{div} l_2 \rangle := \|\langle l_1, l_2 \rangle\| = \|l_1\|(\operatorname{div} l_2).$$

10. The Arakelov Intersection Number for Complex Divisors

Let $\mathscr{L}_1$, $\mathscr{L}_2$ be two Hermitian holomorphic line bundles of degree 0. For any two their multivalued sections l_1, l_2 with non-intersecting divisors we shall define here a real number

$$\|\langle l_1, l_2\rangle\| \in \mathbb{R}$$

which coincides for single-valued l_1 and l_2 and for flat metrics on $\mathscr{L}_1$ and $\mathscr{L}_2$ with the Arakelov intersection number $\langle \operatorname{div} l_1, \operatorname{div} l_2\rangle$ defined in the previous point.

The definition is

$$\|\langle l_1, l_2\rangle\| := \sqrt{\prod_i G^{\bar{n}_i}_{\operatorname{div} l_2}(P_i)\, G^{n_i}_{\overline{\operatorname{div} l_2}}(P_i)}\,, \tag{6}$$

where $\overline{\operatorname{div} l_2}$ means the divisor with coefficients complex conjugate to $\operatorname{div} l_2$, $\operatorname{div} l_1 = \sum_i n_i P_i$ and $G_D(z) := \exp g_D(z)$, $g_D(z)$ being a Green function of the divisor D (see Point 6). The result does not depend on the choice of Green function, because $\deg \mathscr{L}_1 = \Sigma n_i = 0$. The symbol $\|\langle\ ,\ \rangle\|$ is symmetric:

$$\|\langle l_1, l_2\rangle\| = \|\langle l_2, l_1\rangle\|;$$

this can be viewed from the formula

$$\|\langle l_1, l_2\rangle\| = \sqrt{\prod_i G^{\bar{n}_i}_{\operatorname{div} l_2}(P_i) \prod_j G^{\bar{n}'_j}_{\operatorname{div} l_1}(P'_j)}\,, \tag{7}$$

where $\operatorname{div} l_2 = \sum_j n'_j P'_j$. In fact, $G_{\operatorname{div} l_1}(z) = \prod_i G^{n_i}_{P_i}(z)$ and $G_P(Q) = G_Q(P)$, so (7) is equivalent to (6). These arguments imply also the formula

$$\|\langle l_1, l_2\rangle\| = \prod_{i,j} G_{P_i}(P'_j)^{\operatorname{Re}(n_i \bar{n}'_j)}\,. \tag{8}$$

If l_1, l_2, m are such sections of holomorphic line bundles $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{M}$ that the expression below is well-defined then

$$\|\langle l_1 \otimes l_2, m\rangle\| = \|\langle l_1, m\rangle\| \cdot \|\langle l_2, m\rangle\|\,.$$

There are some properties being especial for complex divisors: if $\operatorname{supp} D_1$, $\operatorname{supp} D_2 \subset \mathfrak{m}$, then

$$\|\langle \mathbb{1}_{\alpha D_1}, \mathbb{1}_{D_2}\rangle\| = \|\langle \mathbb{1}_{D_1}, \mathbb{1}_{D_2}\rangle\|^{\alpha} \quad \text{for} \quad \alpha \in \mathbb{R},$$

and

$$\|\langle \mathbb{1}_{\alpha D_1}, \mathbb{1}_{D_2}\rangle\| = \|\langle \mathbb{1}_{D_1}, \mathbb{1}_{\bar{\alpha} D_2}\rangle\| \quad \text{for} \quad \alpha \in \mathbb{C}.$$

Thereby, the symbol $\|\langle\ ,\ \rangle\|$ is Hermitian. More precisely, it is the modulus of the exponent of a Hermitian form on the vector space of complex divisors of degree 0 with support in $\mathfrak{m}$. This Hermitian form is easy to write down (cf. (8)):

$$\sum_{i,j} n_i \bar{n}'_j g(Q_i, Q_j)\,.$$

11. Cohomology and Its Determinant

Since $\mathscr{L} = \mathcal{O}(D)$ for any complex divisor is merely an invertible sheaf (or holomorphic line bundle), one can define its *cohomology* $\mathrm{H}^0(X, \mathscr{L})$, $\mathrm{H}^1(X, \mathscr{L})$ in one of standard ways. We recall the definition of the Dolbeault cohomology.

Consider the operator $\bar{\partial}: \Omega^{0,0}(\mathscr{L}) \to \Omega^{0,1}(\mathscr{L})$, where $\Omega^{0,q}(\mathscr{L})$ is the space of smooth $(0, q)$-forms on X with coefficients in $\mathscr{L}(q=0,1)$. Then, by definition, $\mathrm{H}^0(X, \mathscr{L}) := \operatorname{Ker}\bar{\partial}$, $\mathrm{H}^1(X, \mathscr{L}) := \operatorname{Coker}\bar{\partial}$. These are finite-dimensional vector spaces, because X is compact. $\mathrm{H}^0(X, \mathscr{L})$ is evidently the space of holomorphic single-valued sections of $\mathscr{L}$. There are two fundamental formulas:

$$\mathrm{H}^1(X, \mathscr{L}) = \mathrm{H}^0(X, \Omega \otimes \mathscr{L}^*)^*$$

(the Serre duality) and

$$\chi(\mathscr{L}) := \dim \mathrm{H}^0(X, \mathscr{L}) - \dim \mathrm{H}^1(X, \mathscr{L}) = \deg \mathscr{L} + 1 - g \tag{9}$$

(the Riemann-Roch theorem).

The *determinant of cohomology* is the one-dimensional vector space $\det \mathbb{R}\Gamma(\mathscr{L}) := \det \mathrm{H}^0(X, \mathscr{L}) \otimes \det \mathrm{H}^1(X, \mathscr{L})^{-1}$, where $\det V := \Lambda^{\dim V}(V)$ is the maximal exterior power and $L^{-1} := L^*$. The determinant of cohomology is closely connected with the Weil-Deligne pairing: there take place the canonical isomorphisms

$$\langle \mathscr{L}_1, \mathscr{L}_2 \rangle = \det \mathbb{R}\Gamma(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes \det \mathbb{R}\Gamma(\mathscr{L}_1)^{-1} \otimes \det \mathbb{R}\Gamma(\mathscr{L}_2)^{-1} \otimes \det \mathbb{R}\Gamma(\mathcal{O}) \tag{10}$$

and

$$\det \mathbb{R}\Gamma(\mathscr{L})^2 \otimes \det \mathbb{R}\Gamma(\mathcal{O})^{-2} = \langle \mathscr{L} \otimes \Omega^*, \mathscr{L} \rangle \tag{11}$$

(see Deligne [3]), where all powers are tensor ones. The latter formula has a formal (but not random) resemblance with the Riemann-Roch formula for complex surfaces and at the same time, contains the classical Riemann-Roch theorem for complex curves (9): if one fixes a complex number $c \neq 0$ and considers the isomorphism $c\cdot : \mathscr{L} \to \mathscr{L}$ (multiplication by c) then via the functoriality property of $\det \mathbb{R}\Gamma(\ )$ and $\langle\ ,\ \rangle$ the left-hand side of (11) will be multiplied by $c^{2\chi(\mathscr{L})}$ and the right-hand side will be multiplied by $c^{2\deg \mathscr{L} - \deg \Omega}$. The isomorphism (11) is functorial, too, hence one has

$$\chi(\mathscr{L}) = \deg \mathscr{L} + 1 - g,$$

which is nothing but (9).

12. The Quillen Metrics

Similar to the Weil-Deligne pairing, the determinant of the cohomology $\det \mathbb{R}\Gamma(\mathscr{L})$ is endowed with a canonical metric provided that $\mathscr{L}$ and Ω are Hermitian. By definition, the element of $\det \mathbb{R}\Gamma(\mathscr{L})$ has the form

$$\mathbf{l} = (l_1 \wedge \ldots \wedge l_m) \otimes (l'_1 \wedge \ldots \wedge l'_n)^{-1},$$

where $\{l_1, \ldots, l_m\}$ is a basis of $\mathrm{H}^0(X, \mathscr{L})$ (hence, $l_1 \wedge \ldots \wedge l_m$ is a basis of $\det \mathrm{H}^0(X, \mathscr{L})$) and $\{l'_1, \ldots, l'_n\}$ is a basis of $\mathrm{H}^1(X, \mathscr{L})$ (hence, $l'_1 \wedge \ldots \wedge l'_n$ is a basis of $\det \mathrm{H}^1(X, \mathscr{L})$ and $(l'_1 \wedge \ldots \wedge l'_n)^{-1}$ is a basis of $(\det \mathrm{H}^1(X, \mathscr{L}))^{-1}$). We define the *Quillen metric* $\|\mathbf{l}\|_Q$ of the element $\mathbf{l}$ so that

$$\|\mathbf{l}\|^2 := \det(l_i, l_j) / (\det(l'_i, l'_j) \cdot \det{}' \Delta_{\mathscr{L}}),$$

where the scalar product (l_i, l_j) is the L_2-scalar product in the space of harmonic sections of $\mathscr{L}$ and (l'_i, l'_j) is the L_2-scalar product in the space of harmonic (0,1)-forms with coefficients in $\mathscr{L}$. Besides that, $\det' \Delta_{\mathscr{L}}$ is the regularized determinant of the Laplace operator acting on smooth (single-valued) sections of $\mathscr{L}$:

$$\log \det' \Delta_{\mathscr{L}} := -\zeta'(0),$$

where $\zeta(s)$ is the analytical continuation of $\Sigma \lambda_i^{-s}$ (the sum over all non-zero eigenvalues of $\Delta_{\mathscr{L}}$ for $s \gg 0$). This metric behaves smoothly under a variation of parameters. That would not be true if one did not include the term $\det' \Delta_{\mathscr{L}}$.

The remarkable fact noticed by Deligne [3] is that the canonical isomorphisms (10) and (11) permit to connect the Quillen and the Arakelov-Deligne metrics.

Theorem (Deligne [3]). *Suppose Hermitian metrics on holomorphic line bundles $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}$ and on the complex cotangent bundle Ω are given. Then the canonical isomorphisms*

$$\det \mathbb{R}\Gamma(\mathscr{L}_1 \otimes \mathscr{L}_2) \otimes \det \mathbb{R}\Gamma(\mathscr{L}_1)^{-1} \otimes \det \mathbb{R}\Gamma(\mathscr{L}_2)^{-1} \otimes \det \mathbb{R}\Gamma(\mathscr{O}) = \langle \mathscr{L}_1, \mathscr{L}_2 \rangle$$

and

$$\det \mathbb{R}\Gamma(\mathscr{L})^2 \otimes \det \mathbb{R}\Gamma(\mathscr{O})^{-2} = \langle \mathscr{L} \otimes \Omega^*, \mathscr{L} \rangle$$

are isometries, the left-hand side provided with the tensor product of the Quillen metrics and the right-hand side with the Arakelov-Deligne metrics. □

We will call the first isometry the *Deligne formula* and the second one the *Deligne-Riemann-Roch theorem.*

Note. The machinery of the determinants of cohomology and the Weil-Deligne pairings is especially powerful in the relative case, when one deals with families of complex curves and of holomorphic line bundles over them. This will be an item of the next sections. Here we need to emphasize that for any smooth projective morphism $\pi: X \to S$ of smooth complex algebraic varieties with a fibre being a connected complex curve one can define the fibrewise generalizations of $\det \mathbb{R}\Gamma(\det \mathbb{R}\pi_*)$ and of the Deligne isomorphisms (10), (11) and Theorem 12, where one must only substitute $\det \mathbb{R}\Gamma$ for $\det \mathbb{R}\pi_*$.

2. The Generalized Mumford Form on the Moduli Space

1. The Teichmüller and the Moduli Spaces of Punctured Surfaces

Let $T_{g,n}$ be the Teichmüller space of type (g,n), where $g \geqq 2$ is the genus of a Riemann surface X and n is the number of punctures. Denote by $\mathrm{Diff}^+(g,n)$ the group of orientation-preserving diffeomorphisms $X \to X$ mapping each puncture to itself, and by $\mathrm{Diff}^0(g,n)$ its normal sub-group consisting of diffeomorphisms isotopic to id. The *mapping class group* $\mathrm{Map}_{g,n} = \mathrm{Diff}^+(g,n)/\mathrm{Diff}^0(g,n)$ acts on $T_{g,n}$ by change of marking and the orbit space

$$\mathscr{M}_{g,n} = T_{g,n}/\mathrm{Map}_{g,n}$$

is called the *moduli space*.

Consider a Riemann surface X of genus g with an ordered set $\mathfrak{m}$ of n punctures and an isotopical class of a disk $B \subset X$ containing all the punctures. Let us show that $T_{g,n}$ parametrizes such objects $(X, \mathfrak{m}, B)$. In fact, $T_{g,n}$ parametrizes surfaces *marked* by generators $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$, $\gamma_1, \ldots, \gamma_n$ of $\pi_1(X \backslash \mathfrak{m})$ satisfying the conditions:

$$(\alpha_i, \beta_j) = \delta_{ij},$$

$$(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = (\alpha_i, \gamma_j) = (\beta_i, \gamma_j) = (\gamma_i, \gamma_j) = 0,$$

$$\left(\prod_{i=1}^{g} [\alpha_i, \beta_i]\right) \gamma_1, \ldots, \gamma_n = 1.$$

Here (,) is the geometric intersection number, [,] is the commutator.

Let $X \to T_{g,n}$ be the universal family of Riemann surfaces and let X_t, $t \in T_{g,n}$, be a single surface from this family. A curve on X_t representing the product $\gamma_1, \ldots, \gamma_n$ is the boundary of a disk B on X_t containing all the punctures, and the isotopical class of B in $X_t \backslash \mathfrak{m}$ is uniquely determined by marking. So $T_{g,n}$ is a parameter space for $(X, \mathfrak{m}, B)$.

We define the *moduli space of triples* $(X, \mathfrak{m}, B)$ as the quotient space

$$\mathcal{M}_{g,n,B} = T_{g,n}/G,$$

where G is the sub-group of $\mathrm{Map}_{g,n}$ preserving the isotopical class of the disk B in X.

In order to describe G we represent $\mathrm{Map}_{g,n}$ as the extension

$$1 \to \pi_1(X^n \backslash \Delta) \to \mathrm{Map}_{g,n} \to \mathrm{Map}_{g,0} \to 1,$$

where Δ is the union of $n-1$ diagonals $\{(x_1, x_1, x_3, x_4, \ldots, x_n)\}$, $\{(x_1, x_2, x_2, x_4, \ldots, x_n)\}, \ldots, \{(x_1, x_2, x_3, \ldots, x_{n-1}, x_{n-1})\}$ in X^n, and the homomorphism f: $\mathrm{Map}_{g,n} \to \mathrm{Map}_{g,0}$ is induced by considering a diffeomorphism of a surface with punctures as a diffeomorphism of the same one without them. The kernel $\mathrm{Ker} f$ of f consists of classes of those elements of $\mathrm{Diff}^+(g, n)$ which are isotopic to id in $\mathrm{Diff}^+(g, 0)$. Every isotopy from diffeomorphism in $\mathrm{Ker} f$ to id acting on punctures gives a loop in $X^n \backslash \Delta$, the space of ordered sets of n distinct points in X. Thereby, this correspondence gives an isomorphism $\mathrm{Ker} f \xrightarrow{\sim} \pi_1(X^n \backslash \Delta)$.

The sub-group G in $\mathrm{Map}_{g,n}$ is mapped on $\mathrm{Map}_{g,0}$ surjectively. Indeed, one can choose in a given class in $\mathrm{Map}_{g,0}$ a diffeomorphism acting identically in the disk $B \subset X$. The class in $\mathrm{Map}_{g,n}$ of this diffeomorphism lies evidently in G.

The sequence of immersions

$$B^n \backslash \Delta \to X^n \backslash \Delta \to X^n$$

yields the exact sequence of group homomorphisms

$$\pi_1(B^n \backslash \Delta) \to \pi_1(X^n \backslash \Delta) \to \pi_1(X)^n \to 1.$$

The group $G \cap \pi_1(X^n \backslash \Delta)$ contains the image of $\pi_1(B^n \backslash \Delta)$, because the latter is formed by the diffeomorphisms of X whose isotopy to id never takes any puncture out of the disk B. $\pi_1(X)^n$ acts on the loops $\gamma_i \in \pi_1(X \backslash \mathfrak{m})$ by adjunction:

$$(\gamma_1, \ldots, \gamma_n) \mapsto (g_1 \gamma_1 g_1^{-1}, \ldots, g_n \gamma_n g_n^{-1}) \quad \text{for} \quad (g_1, \ldots, g_n) \in \pi_1(X)^n,$$

and so the product $\gamma_1, \ldots, \gamma_n$ is preserved as a cycle iff $g_1 = g_2 = \ldots = g_n$.

Now we can summarize our discussion in the

Proposition. *The exact sequences of natural group homomorphisms*

$$1\to\pi_1(X^n\backslash\Delta)\to\mathrm{Map}_{g,n}\to\mathrm{Map}_{g,0}\to 1,\ \pi_1(B^n\backslash\Delta)\to\pi_1(X^n\backslash\Delta)\to\pi_1(X)^n\to 1$$

may be rewritten for the sub-group G in $\mathrm{Map}_{g,n}$ *preserving the isotopical class of a disk B ($\mathfrak{m}\subset B\subset X$) as follows:*

$$1\to G\cap\pi_1(X^n\backslash\Delta)\to G\to\mathrm{Map}_{g,0}\to 1,\ \pi_1(B^n\backslash\Delta)\to G\cap\pi_1(X^n\backslash\Delta)\to\pi_1(X)\to 1\,.\quad\square$$

Corollary. *There exists a natural projection*

$$\mathcal{M}_{g,n,B}\to\mathcal{M}_g$$

with a fibre isomorphic to $(H^n\backslash\Delta)/\pi_1(X)$, where H is the upper half-plane, $X=H/\pi_1(X)$, Δ is the diagonal in H^n defined analogically to the diagonal in X^n and $\pi_1(X)$ acts on $H^n\backslash\Delta$ diagonally. $\square$

The spaces $T_{g,n}$ and $\mathcal{M}_{g,n,B}$ have natural complex structures and $\dim T_{g,n}=\dim\mathcal{M}_{g,n,B}=3g-3+n$.

2. *The Mumford Form: Amplitudic Case*

Now assume $\pi:X\to S$ to be an algebraic family of Riemann surfaces with an ordered set of n punctures $Q_1,\dots,Q_n$ and an isotopical class of a disk B containing the punctures. An interesting physical situation arises when S coincides with $\mathcal{M}_{g,n,B}$ – the moduli space of such objects (see Point 1). Further, assume that at every puncture Q_i there is given a (constant) momentum vector $\mathbf{p}_i\in\mathbb{C}^{13}$. There hold the following equations:

1. $\sum_{i=1}^{n}\mathbf{p}_i=0$ ("the momentum-conservation law"),
2. $(\mathbf{p}_i,\mathbf{p}_i)=1$ for every i, where $(\ ,\ )$ is the standard Hermitian metric in $\mathbb{C}^{13}$ ("the mass of tachyon equals $\sqrt{-1}$"). Consider the following list of Hermitian holomorphic line bundles on X:

the bundle $\Omega:=\Omega^1_{X/S}$ of holomorphic Abelian differentials along the fibre of π with an arbitrary Hermitian metric (i.e., a Kähler metric on Riemann surface),

the bundle $\Omega^{\otimes 2}$ of *holomorphic quadratic differentials* along the fibres of π with tensor square metric,

thirteen bundles $\mathcal{O}(D^\nu)$, $\nu=1,\dots,13$, over X corresponding to the complex divisors $D^\nu=\sum_i p_i^\nu\cdot Q_i$, with arbitrary flat Hermitian metrics (note that the divisors Q_i are homotopic to each other because one can transform Q_i into Q_j inside B, and so every $\mathcal{O}(D^\nu)$ is topologically trivial).

Consider the following Hermitian holomorphic line bundles over the base S of the family:

the determinant bundles $\det\mathbb{R}\pi_*(\Omega)$, $\det\mathbb{R}\pi_*(\Omega^{\otimes 2})$ (their fibres over a point $s\in S$ are $\det\mathbb{R}\Gamma(X_s,\Omega)$ and $\det\mathbb{R}\Gamma(X_s,\Omega^{\otimes 2})$, correspondingly), with the Quillen metrics,

the Weil-Deligne bundles $\langle\mathcal{O}(D^\nu),\mathcal{O}(D^\nu)\rangle$, $\nu=1,\dots,13$, with the Arakelov-Deligne metrics.

Proposition. *The tensor product metric on the holomorphic line bundle*

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2})}{\det \mathbb{R}\pi_*(\Omega)^{\otimes 13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle \right)} \tag{12}$$

over $\mathcal{M}_{g,n,B}$ *is flat. Hence this bundle locally admits a holomorphic covariantly constant section* $\mu_{g,n,B}$ *of norm* 1 *which is unique up to a factor* $\exp(i\varphi)$, $\varphi \in \mathbb{R}$.

Definition. Such a section $\mu_{g,n,B}$ we call the Mumford form in the case of amplitudes.

Note. It is sufficient to require $\mathcal{O}(D^\nu)$ to be relatively flat (see 4.1). Indeed, the metric on $\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle$ does not depend on a pulled up from S factor of metric on $\mathcal{O}(D^\nu)$. But flat and relatively flat metrics on $\mathcal{O}(D^\nu)$ differ by such a factor, only (cf. Lemma 4.3).

Proof. Each of the Hermitian bundles

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2})}{\det \mathbb{R}\pi_*(\Omega)^{\otimes 13}} \tag{13}$$

and

$$\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle \tag{14}$$

is flat; the first, additionally, is trivial according to the Mumford theorem, i.e., there exists an isomorphism

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2})}{\det \mathbb{R}\pi_*(\Omega)^{\otimes 13}} \xrightarrow{\sim} \mathcal{O}_S. \tag{15}$$

The machinery of the Quillen and the Arakelov-Deligne metrics (see Deligne [3]) can be roughly reformulated as the rule:

if two line bundles arising (as $\det \mathbb{R}\pi_* \mathcal{L}$ or $\langle \mathcal{L}, \mathcal{M} \rangle$) over the base S from Hermitian line bundles over a family $\pi: X \to S$ are canonically isomorphic via the Grothendieck-Riemann-Roch type arguments, then such arguments are applicable to the curvatures of these bundles, provided that the metrics are constructed in a certain canonical way (such as the Arakelov-Deligne and the Quillen metrics).

In the case of the bundle (13) the trivialization (15) can be chosen so that it will be an isometry provided that the metric on $\mathcal{O}_S$ is trivial: $\|1\| = 1$, and flat, consequently.

The curvature form of the Arakelov-Deligne metric on $\langle \mathcal{L}, \mathcal{M} \rangle$ is given by the formula [3]:

$$\mathbf{c}_1(\langle \mathcal{L}, \mathcal{M} \rangle) = \int_{X/S} \mathbf{c}_1(\mathcal{L}) \wedge \mathbf{c}_1(\mathcal{M}) \tag{16}$$

(integration over fibres of $\pi: X \to S$). Since we choose a flat metric on $\mathcal{O}(D^\nu)$, the metric on (14) is also flat. $\square$

Corollary. *The holomorphic line bundle*

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes \det \mathbb{R}\pi_*(\Omega)^{\otimes 13}}{\bigotimes_{\nu=1}^{13} (\det \mathbb{R}\pi_*(\mathcal{O}(D^\nu)) \otimes \det \mathbb{R}\pi_*(\mathcal{O}(-D^\nu)))}$$

is canonically isometric to the bundle

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2})}{\det \mathbb{R}\pi_*(\Omega)^{\otimes 13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle \right)}. \quad \square$$

3. From the Mumford Form to the Polyakov Measure

The modulus squared of the Mumford form $\mu_{g,n,B}$ gives a measure on the moduli space $\mathcal{M}_{g,n,B}$ after a procedure similar to the Belavin-Knizhnik one. This measure is modular invariant and so it can be pulled down to the moduli space $\mathcal{M}_{g,n}$, giving the Polyakov measure (3). We now describe this procedure.

Proof of Theorem 2 from Introduction. Assume that a section α of the bundle (12) is constructed (locally on moduli) beginning from local bases $\{f\}$, $\{\omega_1, \ldots, \omega_g\}$, $\{W_1, \ldots, W_{3g-3}\}$ and $\{\langle s^\nu, t^\nu\rangle\}$ in the spaces of holomorphic sections of the bundles $\mathcal{O}_X$, Ω, $\Omega^{\otimes 2}$ and $\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle$, correspondingly, where s^ν and t^ν are (single-valued) meromorphic sections of $\mathcal{O}(D^\nu)$ with non-intersecting divisors. That means that

$$\alpha = \frac{W_1 \wedge \ldots \wedge W_{3g-3}}{f^{13} \cdot (\omega_1 \wedge \ldots \wedge \omega_g)^{\otimes 13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle s^\nu, t^\nu \rangle \right)}.$$

Then we set

$$\alpha \wedge \bar{\alpha} = \frac{W_1 \wedge \bar{W}_1 \wedge W_2 \wedge \bar{W}_2 \wedge \ldots \wedge W_{3g-3} \wedge \bar{W}_{3g-3}}{\left(\frac{2}{\sqrt{-1}}\right)^{3g-3} |f|^{26} \cdot \det \left(\frac{\sqrt{-1}}{2} \int_{X/S} \omega_i \wedge \bar{\omega}_j \right)^{13}}$$

$$\times \frac{\prod_{i=1}^{n} \frac{\sqrt{-1}}{2} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i}{\prod_{\nu=1}^{13} (\|\langle u^\nu, t^\nu\rangle\|^2 \|\langle v^\nu, \mathbb{1}_{D^\nu}\rangle\|^2) \prod_{i=1}^{n} |z_i|^2}. \tag{17}$$

Here $s^\nu = u^\nu \cdot \mathbb{1}_{D^\nu}$, $t^\nu = v^\nu \cdot \mathbb{1}_{D^\nu}$, u^ν and v^ν being multivalued meromorphic functions, $\mathbb{1}_{D^\nu}$ is the canonical multivalued meromorphic section of the bundle $\mathcal{O}(D^\nu)$, $\mathbb{1}_{Q_i}$ is the canonical single-valued holomorphic section of $\mathcal{O}(Q_i)$, z_i is a holomorphic coordinate near Q_i, vanishing at Q_i. The expression

$$\prod_{\nu=1}^{13} (\|\langle u^\nu, t^\nu\rangle\|^2 \|\langle v^\nu, \mathbb{1}_{D^\nu}\rangle\|^2) \prod_{i=1}^{n} |z_i|^2 = \prod_{\nu=1}^{13} (|u^\nu(\operatorname{div} t^\nu)|^2 |v^\nu(D^\nu)|^2) \prod_{i=1}^{n} |z_i|^2, \tag{18}$$

each factor being itself equal to 0 or ∞, is entirely well-defined as it has removable singularities as a function of the points $Q_1, \ldots, Q_n$ and the points of the support of $\operatorname{div} t^\nu$. The term (18) is expressed in terms of the Green functions of *principal* divisors, because u^ν and v^ν are not sections of line bundles, but merely functions.

Let us give a heuristic motivation of the definition (17). This motivation will have an exact sense in the case of integral momenta: $p_i^\nu \in \mathbb{Z}$ for all i, ν. For ν fixed (and omitted for simplicity) consider the following sequence of canonical isomorphisms:

$$\begin{aligned}\langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle &= \langle \mathcal{O}(D), \mathcal{O}(D)\rangle = \langle \mathcal{O}(\Sigma p_i Q_i), \mathcal{O}(\Sigma p_i Q_i)\rangle \\ &= \bigotimes_{i \neq j} \langle \mathcal{O}(p_i Q_i), \mathcal{O}(p_j Q_j)\rangle \bigotimes_i \langle \mathcal{O}(p_i Q_i), \mathcal{O}(p_i Q_i)\rangle \\ &= \bigotimes_{i \neq j} \langle \mathcal{O}(Q_i), \mathcal{O}(Q_j)\rangle^{\otimes p_i \bar{p}_j} \bigotimes_i \langle \mathcal{O}(Q_i), \mathcal{O}(Q_i)\rangle^{p_i \bar{p}_i}.\end{aligned}$$

Restoring the index ν anew and multiplying over $\nu = 1, \ldots, 13$, we get $\bigotimes_\nu \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle = \bigotimes_{i \neq j} \langle \mathcal{O}(Q_i), \mathcal{O}(Q_j)\rangle^{\otimes (\mathbf{p}_i, \mathbf{p}_j)} \bigotimes_i \langle \mathcal{O}(Q_i), \mathcal{O}(Q_i)\rangle$. According to the *adjunction formula*, the bundle $\langle \Omega(Q), \mathcal{O}(Q)\rangle = \langle \Omega, \mathcal{O}(Q)\rangle \otimes \langle \mathcal{O}(Q), \mathcal{O}(Q)\rangle$ is canonically trivialized by the residue map and so $\langle \mathcal{O}(Q), \mathcal{O}(Q)\rangle = \langle \Omega, \mathcal{O}(Q)\rangle^{-1}$. Thereby,

$$\bigotimes_\nu \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle = \bigotimes_{i \neq j} \langle \mathcal{O}(Q_i), \mathcal{O}(Q_j)\rangle^{\otimes (\mathbf{p}_i, \mathbf{p}_j)} \bigotimes_i \langle \Omega, \mathcal{O}(Q_i)\rangle^{-1}. \tag{19}$$

From this point of view, in order to get the formula (17) we have factorized the section $\bigotimes_\nu \langle s^\nu, t^\nu\rangle$ of the bundle $\bigotimes_\nu \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu)\rangle$ via the isomorphism (19): $\bigotimes_\nu \langle s^\nu, t^\nu\rangle = x \otimes y$, and then put $\bigotimes_\nu \langle s^\nu, t^\nu\rangle \wedge \bigotimes_\nu \overline{\langle s^\nu, t^\nu\rangle} := \|x\|^2 \cdot y \wedge \bar{y}$, where $\|x\|$ is the norm of x in the sense of "non-constant" part of the metric, the part depending on the choice of a section. Precisely, that means that we take the part

$$\prod_{\nu=1}^{13} (\|\langle u^\nu, t^\nu\rangle\| \cdot \|\langle v^\nu, \mathbb{1}_{D^\nu}\rangle\|) \prod_{i=1}^{n} |z_i|$$

of the Arakelov-Deligne norm

$$\prod_{\nu=1}^{13} (\|\langle u^\nu, t^\nu\rangle\| \cdot \|\langle v^\nu, \mathbb{1}_{D^\nu}\rangle\|) \prod_{i=1}^{n} |z_i| \cdot \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \|\langle \mathbb{1}_{Q_i}, \mathbb{1}_{Q_j}\rangle\|^{\mathrm{Re}(\mathbf{p}_i, \mathbf{p}_j)}$$

of the section $\bigotimes_\nu \langle s^\nu, t^\nu\rangle$. (Note, that in the Belavin-Knizhnik procedure we analogically take the "non-constant" part

$$\frac{1}{|f|^2 \cdot \det\left(\int_{X/S} \frac{\sqrt{-1}}{2} \omega_i \wedge \bar{\omega}_j\right)} = \frac{1}{|f|^2 \cdot \det(\omega_i, \omega_j)}$$

of the Quillen norm (here: squared)

$$\frac{\det' \Delta_1}{\det(\omega_i, \omega_j) \cdot |f|^2 \cdot \int_{X/S} \frac{\sqrt{-1}}{2} \gamma dz d\bar{z}} = \frac{\det' \Delta_1}{\det(\omega_i, \omega_j) \cdot (f, f)}$$

of the section $f^{-1} \cdot (\omega_1 \wedge \ldots \wedge \omega_g)^{-1}$.)

Returning to the Mumford form, note that $\alpha/\|\alpha\| = \mu_{g,n,B} \times \exp(i\varphi)$, $\varphi \in \mathbb{R}$, because $\|\mu_{g,n,B}\| = 1$ by definition. Hence $\alpha \wedge \bar{\alpha}/\|\alpha\|^2 = \mu_{g,n,B} \wedge \bar{\mu}_{g,n,B}$.

In order to clarify the connection between the Mumford form and the Polyakov measure we now calculate $\|\alpha\|$. One can easily state that

$$\|\alpha\|^2=\frac{\|W_1\wedge\ldots\wedge W_{3g-3}\|_Q^2}{\|f\cdot\omega_1\wedge\ldots\wedge\omega_g\|^{26}\cdot\prod_\nu\|\langle s^\nu,t^\nu\rangle\|^2}=\frac{\det(W_i,W_j)}{\det'\Delta_2}$$

$$\times\left(\frac{\det'\Delta_1}{\det(\omega_i,\omega_j)\cdot(f,f)}\right)^{13}\cdot\frac{1}{\prod_{\nu=1}^{13}(\|\langle u^\nu,t^\nu\rangle\|^2\,\|\langle v^\nu,\mathbb{1}_{D^\nu}\rangle\|^2)}$$

$$\times\frac{1}{\prod_{i,j=1}^{n}\|\langle\mathbb{1}_{Q_i},\mathbb{1}_{Q_j}\rangle\|^{2\operatorname{Re}(\mathbf{p}_i,\mathbf{p}_j)}}.$$

The scalar products are taken in the sense of L_2-metrics over the spaces of harmonic functions, 1-differentials or quadratic differentials. The scalar products (ω_i,ω_j) can be rewritten as $\int(\sqrt{-1}/2)\omega_i\wedge\bar\omega_j$ and (f,f) as $|f|^2\cdot\int(\sqrt{-1}/2)\gamma\mathrm{d}z\mathrm{d}\bar z$. Recall that here as in (18) some terms may equal 0 or ∞, but all the expression is well-defined.

Therefore

$$\mu_{g,n,B}\wedge\bar\mu_{g,n,B}=\alpha\wedge\bar\alpha/\|\alpha\|^2=\left(\frac{\sqrt{-1}}{2}\right)^{3g-3}W_1\wedge\bar W_1\wedge\ldots\wedge\bar W_{3g-3}$$

$$\times\frac{\det'\Delta_2}{\det(W_i,W_j)}\cdot\left(\frac{\int(\sqrt{-1}/2)\gamma\mathrm{d}z\mathrm{d}\bar z}{\det'\Delta_1}\right)^{13}\cdot\left(\prod_{i=1}^{n}\frac{\sqrt{-1}}{2}\mathrm{d}z_i\wedge\mathrm{d}\bar z_i/|z_i|^2\right)$$

$$\times\prod_{i,j=1}^{n}\|\langle\mathbb{1}_{Q_i},\mathbb{1}_{Q_j}\rangle\|^{2\operatorname{Re}(\mathbf{p}_i,\mathbf{p}_j)}.$$

The sole moment which remains to clarify is the equality

$$\prod_i\gamma\frac{\sqrt{-1}}{2}\mathrm{d}z_i\wedge\mathrm{d}\bar z_i\prod_{i,j}G(Q_i,Q_j)^{2\operatorname{Re}(\mathbf{p}_i,\mathbf{p}_j)}$$

$$=\left(\prod_i\frac{\sqrt{-1}}{2}\mathrm{d}z_i\wedge\mathrm{d}\bar z_i/|z_i|^2\right)\prod_{i,j}\|\langle\mathbb{1}_{Q_i},\mathbb{1}_{Q_j}\rangle\|^{2\operatorname{Re}(\mathbf{p}_i,\mathbf{p}_j)}.$$

In fact,

$$G(Q_i,Q_j)=\|\langle\mathbb{1}_{Q_i},\mathbb{1}_{Q_j}\rangle\|\quad\text{for}\quad i\neq j$$

and

$$G(Q_i,Q_i):=\lim_{Q_i'\to Q_i}(G(Q_i',Q_i)/\|Q_i'-Q_i\|)=\|\langle\mathbb{1}_{Q_i},\mathbb{1}_{Q_i}\rangle\|/(\sqrt{\gamma}|z_i|),$$

where $\|Q_i'-Q_i\|$ is the distance between Q_i and Q_i' in the metric γ on X.

Thus, comparing the expression got for $\mu\wedge\bar\mu$ with (3), we see that

$$\mathrm{d}\pi_{g,n}=\mu_{g,n,B}\wedge\bar\mu_{g,n,B}\tag{20}$$

over $\mathcal{M}_{g,n,B}$. It is well-known that $\mathrm{d}\pi_{g,n}$ is modular-invariant so $\mu_{g,n,B}\wedge\bar\mu_{g,n,B}$ can be pulled down to $\mathcal{M}_{g,n}$. And finally, the dependence on the choice of $\mu_{g,n,B}$ vanishes, because the factor $\exp(i\varphi)$ vanishes in (20). $\square$

3. The Universal Mumford Form

1. The Poincaré Line Bundle

Let J be the Jacobian of X, $J^t = \mathrm{Pic}^0 J =$ (the group of isomorphism classes of holomorphic line bundles on J which are topologically trivial) be the dual Abelian variety of J, and $\Phi: J \xrightarrow{\sim} J^t$ be the canonical principal polarization. (If $J = V/\Lambda$, where V is a complex vector space ($V = \mathrm{H}^0(X, \Omega)^*$) and Λ is a complete lattice in V ($\Lambda = \mathrm{H}^1(X, \mathbb{Z})$), then $J^t = \bar{V}^*/\Lambda^*$, where $\bar{V}^*$ is the space of antilinear functionals on V and $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. The isomorphism $\Phi: J \xrightarrow{\sim} J^t$ is induced by the intersection form $\Lambda \otimes \Lambda \to \mathbb{Z}$.) The product $J \times J^t$ is endowed with the *Poincaré line bundle* which we denote by $\mathscr{P}$. This bundle is uniquely determined by two conditions:

1. for $x \in J^t$ the line bundle induced by $\mathscr{P}$ over $J \times \{x\}$ is in the class x,
2. for $e \in J$, the unit in the group J, the line bundle induced by $\mathscr{P}$ over $\{e\} \times J^t$ is holomorphically trivialized.

The line bundle $\mathscr{B}$ over $J \times J$ is defined as the pull-back of $\mathscr{P}$ under the morphism $\mathrm{id} \times \Phi: J \times J \to J \times J^t$:

$$\mathscr{B} = (\mathrm{id} \times \Phi)^* \mathscr{P}.$$

The bundles $\mathscr{P}$ and $\mathscr{B}$ have unique Hermitian metrics satisfying the requirements:

1. their curvatures equal zero,
2. they are compatible with the corresponding trivializations – $\mathscr{P}$ at $\{e\} \times \{e^t\}$ and $\mathscr{B}$ at $\{e\} \times \{e\}$, e^t being the class of the trivial line bundle over J.

Such metrics exist, because $\mathscr{B}$ and $\mathscr{P}$ are topologically trivial.

The reason to use these bundles in our context is the existence of a connection between $\langle \mathscr{L}, \mathscr{M} \rangle$ and $\mathscr{B}$.

Let $m \geqq 0$ be an integer,

$$\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m, \ \sum_{i=1}^{m} p_i = 0, \ (x_1, \ldots, x_m) \in X^m, \ D = \sum p_i x_i,$$

$\mathrm{cl}\,\mathcal{O}(D)$ be the class in J of the line bundle $\mathcal{O}(D)$ on X and

$$\varphi(\mathbf{p}): X^m \to J$$

be the *Abel map*

$$(x_1, \ldots, x_m) \mapsto \mathrm{cl}\,\mathcal{O}(D).$$

Similarly, for such another integer $n \geqq 0$ and $\mathbf{p}' = (p'_1, \ldots, p'_n) \in \mathbb{Z}^n$, $\sum_{i=1}^{n} p'_i = 0$, we have the line bundle $\mathcal{O}(D')$ on X, $D' = \sum p'_i x'_i$, $(x'_1, \ldots, x'_n) \in X^n$, and the Abel map

$$\varphi(\mathbf{p}'): X^n \to J.$$

Taking $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$ according to Sect. 1 and letting $(x_1, \ldots, x_m, x'_1, \ldots, x'_n) \in X^m \times X^n$ running we get a line bundle on $X^m \times X^n$. Considering $\mathcal{O}(D)$ as a line bundle over $X \times X^m$ endow it with a Hermitian metric which becomes flat for every $(x_1, \ldots, x_m)$ being fixed. Then we do the same for $\mathcal{O}(D')$ and consequently $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$ has a Hermitian metric – the Arakelov-Deligne one. It does not depend on the choice of metrics on $\mathcal{O}(D)$ and $\mathcal{O}(D')$. This results from the following proposition which is in essence due to Moret-Bailly [4].

Proposition. *There exists a canonical isometry*

$$\langle \mathcal{O}(D), \mathcal{O}(D')\rangle = (\varphi(\mathbf{p}) \times \varphi(\mathbf{p}'))^*(\mathcal{B}^{-1}) \tag{21}$$

for $\varphi(\mathbf{p}) \times \varphi(\mathbf{p}'): X^m \times X^n \to J \times J$.

Note. In order to generalize this fact to the case of a *complex* divisor D we must consider a family Y^m of m distinct points $x_1, \dots, x_m \in X$, lying in a disk $B \subset X$ which may be transformed isotopically when $x_1, \dots, x_m$ vary. Then the Abel map $\varphi(\mathbf{p}): Y^m \to J$ for $\mathbf{p} \in \mathbb{C}^m$, $\sum p_i = 0$, is well-defined (as well as $\varphi(\mathbf{p}'): Y^n \to J$, $\mathbf{p}' \in \mathbb{C}^n$, $\sum p_i' = 0$) and (21) is an isometry. The proof below remains valid without principal corrections.

Proof. Consider a *universal line bundle* U_0 on $X \times J$. This is a Hermitian bundle with the property:

for $\delta \in J$ the line bundle induced by U_0 over $X \times \{\delta\}$ is flat and lies in the class δ.

This bundle is defined modulo tensorization by a Hermitian holomorphic line bundle pulled back from J.

Using the isomorphicity of $\mathcal{O}(D)$ and $(\mathrm{id} \times \varphi(\mathbf{p}))^* U_0$ over each fiber of the projection $\pi: X \times X^m \to X^m$ we have an isomorphism

$$\mathcal{O}(D) \xrightarrow{\sim} (\mathrm{id} \times \varphi(\mathbf{p}))^*(U_0) \otimes \pi^*(\mathcal{M}), \tag{22}$$

where $\mathcal{M}$ is a line bundle over X^m. The curvature of $\mathcal{O}(D)$ being restricted to a fibre of π equals zero and so we can choose a Hermitian metric on $\mathcal{M}$ such that (22) is an isometry. Having produced these constructions for the case of $\mathbf{p}'$, D', $\varphi(\mathbf{p}'): X^n \to J$, etc. we come to an isometry

$$\langle \mathcal{O}(D), \mathcal{O}(D')\rangle \xrightarrow{\sim} (\varphi(\mathbf{p}) \times \varphi(\mathbf{p}'))^* \langle U_0, U_0\rangle .$$

It is a canonical isometry (i.e. it does not depend on the isometry (22), on the bundle $\mathcal{M}$ and on the metrics on U_0 and $\mathcal{M}$) because $\deg U_0 = 0$ over X by construction.

We are through if we use the following

Lemma. *There exists a canonical isometry*

$$\langle U_0, U_0\rangle = \mathcal{B}^{-1} .$$

In particular, for the classes $\mathrm{cl}\,\mathcal{L}$, $\mathrm{cl}\,\mathcal{M} \in J$ of line bundles $\mathcal{L}, \mathcal{M}$ *on* X *we have a canonical isometry of the fibres*

$$\langle \mathcal{L}, \mathcal{M}\rangle = \mathcal{B}^{-1}_{(\mathrm{cl}\,\mathcal{L}, \mathrm{cl}\,\mathcal{M})} .$$

The *proof* of the Lemma would require introducing some new notions, so we refer the reader to the paper of Moret-Bailly [4, 2.9.4, 4.14.1]. □

2. The Divisor W_{g-1}

Denote by W_{g-1} the image of the Abel map

$$\varphi: X^{g-1} \to J_{g-1},$$
$$(x_1, \dots, x_{g-1}) \mapsto \mathrm{cl}\, \mathcal{O}(x_1 + \dots + x_{g-1}) .$$

(Under the notation of the previous point it would be better to write $\varphi=\varphi(1,\ldots,1)$ $-g-1$ units in brackets.) W_{g-1} is the zero-set of a section of the line bundle $(\det \mathbb{R}\pi_* U_{g-1})^{-1}$, where U_{g-1} is a universal line bundle over $X\times J_{g-1}$ defined similarly to U_0 in Point 1 modulo tensorization by a line bundle over J_{g-1}, $\pi: X\times J_{g-1}\to J_{g-1}$ being the second projection. The matter is that $\pi_* U_{g-1}=\mathbb{R}^1\pi_* U_{g-1}=0$ over $N=J_{g-1}\setminus W_{g-1}$ and so over N the bundle $(\det \mathbb{R}\pi_* U_{g-1})^{-1}=\mathcal{O}_N$ has the unit section which can be continued to a unique section of $(\det \mathbb{R}\pi_* U_{g-1})^{-1}$ with support W_{g-1}.

In other words, one has a canonical isomorphism

$$\mathcal{O}(W_{g-1})=(\det \mathbb{R}\pi_* U_{g-1})^{-1}, \tag{23}$$

where $\mathcal{O}(W_{g-1})$ is the line bundle assigned to the divisor W_{g-1} in J_{g-1}. (For a more detailed exposition of this construction see, for example, the paper of Moret-Bailly [4].)

3. *Theta-Function and Theta-Divisor*

There exists another natural divisor Θ on J – it is the zero-set of the Riemann theta-function. This divisor Θ is called the theta-divisor. Let us recall some basic facts on theta.

Choose a *marking* $(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g)$ of our Riemann surface X. It is a set of real closed cycles on X. A marking is supposed to be a symplectic basis in $\mathrm{H}_1(X,\mathbb{Z})$, i.e.,

$$(\alpha_i,\beta_j)=\delta_{ij},$$
$$(\alpha_i,\alpha_j)=(\beta_i,\beta_j)=0.$$

Consider also a basis $\varphi_1,\ldots,\varphi_g$ in $\mathrm{H}^0(X,\Omega)$, normalized by the conditions $\int_{\alpha_i}\varphi_j=\delta_{ij}$. After that we can identify J with $\mathbb{C}^g/\Lambda$, where Λ is the lattice generated by the columns of the *periods matrix*

$$(I|\tau):=\left(\int_{\alpha_j}\varphi_i\Big|\int_{\beta_j}\varphi_i\right).$$

According to the *Riemann relations*, ${}^t\tau=\tau$ ("t" means transposition) and $\operatorname{Im}\tau>0$.

In this way the *Riemann theta-function* is defined as the series

$$\theta(z,\tau)=\sum_{m\in\mathbb{Z}^g}\exp\pi i({}^tm\tau m+2{}^tmz),$$

$z\in\mathbb{C}^g$, $\tau\in H_g$. One can easily show that for m, $n\in\mathbb{Z}^g$,

$$\theta(z+n,\tau)=\theta(z,\tau),$$
$$\theta(z+\tau m,\tau)=\theta(z,\tau)\exp\pi i(-{}^tm\tau m-2{}^tmz).$$

Hence, for every τ the zero-set of $\theta(z,\tau)$ in $\mathbb{C}^g$ is Λ-invariant and can be pulled down to a divisor

$$\Theta:=\{z\in\mathbb{C}^g|\theta(z,\tau)=0\}/\Lambda\subset\mathbb{C}^g/\Lambda=J,$$

which is called the *theta-divisor*.

Two divisors $\Theta \subset J$ and $W_{g-1} \subset J_{g-1}$ according to the Riemann theorem coincide up to a translation:

$$\Theta = T_\kappa^* W_{g-1}, \tag{24}$$

where κ, the *Riemann constant*, is such a point in J_{g-1} that $2\kappa = \mathrm{cl}\,\Omega \in J_{2g-2}$, and $T_\kappa : J \to J_{g-1}$ is the translation by κ. Θ and κ depend on marking in the contrary to W_{g-1}.

It seems not to be surprising that the universal bundles U_j over $X \times J_j$ for other j are connected with W_{g-1} and Θ in a way similar to (23). But the priority of the case $j = g-1$ is that $\det \mathbb{R}\pi_* U_{g-1}$ does not depend on the choice of U_{g-1}, because the Euler characteristic of U_{g-1} along the fibres of π vanishes.

To deal with other U_j one needs to understand how to fix U_j in suitable terms. Here is a variant of the answer.

Definition. For a line bundle E over $X \times J_j$ of $\deg E = m$ along the fibres of the second projection $\pi : X \times J_j \to J_j$ denote by U_j^E the universal line bundle characterized by two properties:

1. for $\delta \in J_j$ the line bundle induced by U_j^E over $X \times \{\delta\}$ is in the class δ,
2. the line bundle $\langle U_j^E, E \rangle$ over J_j is holomorphically trivialized (cf. the definition of the Poincaré line bundle in Point 1).

If $E = \mathcal{O}(a)$ [2], $a \in X$ is a point, then we denote U_j^E by U_j^a and 2. is equivalent to the condition

2′. the restriction of the line bundle U_j^a to $\{a\} \times J_j \subset X \times J_j$ is holomorphically trivialized.

As concerns the uniqueness of U_j^E, one can say the following. Two bundles U_j, U_j' with the property 1. differ by a tensor factor $\pi^* M$, where M is a line bundle over J_j:

$$U_j' = \pi^* M \otimes U_j.$$

Then $\langle U_j', E \rangle = M^{\otimes \deg E} \otimes \langle U_j, E \rangle = M^{\otimes m} \otimes \langle U_j, E \rangle$, so 2. yields a trivialization of $M^{\otimes m}$. Particularly 2′. yields a trivialization of M itself, so U_j^a is unique.

Having introduced U_j^E, we are ready to formulate the following proposition connecting determinant line bundles over J_j with the divisor W_{g-1}.

Proposition (Moret-Bailly [4, 2.5]). *Let E be a holomorphic line bundle over X of $\deg E = g-1-j$ and U_j^E be a universal line bundle over $X \times J_j$. Then there exist the canonical isomorphisms of holomorphic line bundles over J:*

1. $\det \mathbb{R}\pi_* U_j^E = T_E^* \mathcal{O}(-W_{g-1}) \otimes \det \mathbb{R}\pi_* \mathcal{O} \otimes (\det \mathbb{R}\pi_* E)^{-1}$,
2. $(\det \mathbb{R}\pi_* U_j^E)^{\otimes 2} = (T_E^* \mathcal{O}(-W_{g-1}))^{\otimes 2} \otimes \langle E, E^{-1} \otimes \Omega \rangle$.

If $E = \mathcal{O}((g-1-j) \cdot a)$ then we obtain canonical Moret-Bailly's isomorphism:

3. $\det \mathbb{R}\pi_* U_j^a = T_E^* \mathcal{O}(-W_{g-1}) \otimes (\Omega|_a)^{(g-j-1)(g-j)/2}$.

Note. In all the formulas $T_E^* \mathcal{O}(-W_{g-1})$ can be replaced by $T_{E-\kappa}^* \mathcal{O}(-\Theta)$ according to (24).

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[2] To be more correct, we need to denote $E = p_1^* \mathcal{O}(a)$, where $p_1 : X \times J_j \to J_j$ is the first projection, but we omit p_1^* if this yields no confusion

Proof. All the isomorphisms mentioned above yield from the isomorphism (23) via the Deligne-Riemann-Roch theorem (see Theorem 1.12 and Note after it).

At first, we note that

$$U_{g-1} = T_{-E}^*(U_j^E) \otimes E, \tag{25}$$

where $T_{-E}: J_{g-1} \to J_j$ is the translation by $\mathrm{cl}(-E) \in J_{j-g+1}$ (in order to see it one has to check up the universality property). Then the Deligne formula (10) gives

$$\det \mathbb{R}\pi_* U_{g-1} = T_{-E}^* \langle U_j^E, E \rangle \otimes T_{-E}^* \det \mathbb{R}\pi_* U_j^E \otimes \det \mathbb{R}\pi_* E \otimes (\det \mathbb{R}\pi_* \mathcal{O})^{-1}.$$

Here $\langle U_j^E, E \rangle$ is trivial by definition, and $\det \mathbb{R}\pi_* U_{g-1} = \mathcal{O}(-W_{g-1})$, so we have 1.

2. one gets from 1., applying the Deligne-Riemann-Roch theorem to $\det \mathbb{R}\pi_* E \otimes (\det \mathbb{R}\pi_* \mathcal{O})^{-1}$.

To obtain 3. "squared" one uses the adjunction formula $\langle \mathcal{O}(a), \mathcal{O}(a) \rangle = \langle \Omega, \mathcal{O}(a) \rangle^{-1} = (\Omega|_a)^{-1}$ and the isomorphism 2. The isomorphism 3. itself can be constructed using (23) and (25) in a slightly different way (see Moret-Bailly [4, 2.5]). □

4. *Principal Polarization and Admissible Metrics*

By definition of the Jacobian J, the holomorphic 1-forms on X are the same as on J:

$$\mathrm{H}^0(X, \Omega) = \mathrm{H}^0(J, \Omega_J^1).$$

If $\{\omega_1, \ldots, \omega_g\}$ is an orthonormal basis in this space for the Hermitian scalar product

$$(\omega, \omega') := \frac{\sqrt{-1}}{2} \int_X \omega \wedge \bar{\omega}',$$

then define the *canonical form*

$$\eta := \frac{\sqrt{-1}}{2g} \sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j$$

on X and the *principal polarization*

$$\eta_J := \frac{\sqrt{-1}}{2} \sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j$$

on J. The form η (or η_J) is canonical in the sense that it does not depend on the choice of basis $\{\omega_1, \ldots, \omega_g\}$. The form η_J is translation-invariant and the form η obviously satisfies

$$\int_X \eta = 1.$$

Note. One can show that this definition of principal polarization is equivalent to the definition given in Point 1, as a morphism $\Phi: J \to J^t$.

Definition. Let $\mathcal{L}$ be a holomorphic line bundle over J_j, $\mathcal{M}$ over X and $\mathcal{N}$ over X^n. Then

1. a metric on $\mathcal{L}$ is called admissible if its Chern form $\mathbf{c}_1(\mathcal{L})$ is translation-invariant (under the translations from J),

2. $\mathscr{L}$ is called polarizing if its Chern class $c_1(\mathscr{L})=\lambda\cdot \mathrm{cl}\,\eta_J$ for some $\lambda\in\mathbb{R}$ (really, $\lambda\in\mathbb{Z}$), J_j being identified with J by an arbitrary translation,
3. a metric on $\mathscr{M}$ is called admissible if its Chern form $\mathbf{c}_1(\mathscr{M})=\lambda\cdot\eta$ for some $\lambda\in\mathbb{R}$ (really, $\lambda\in\mathbb{Z}$),
4. a metric on $\mathscr{N}$ is called n-admissible, if its Chern form $\mathbf{c}_1(\mathscr{N})=\sum_{i=1}^{n}\lambda_i\cdot p_i^*\eta$, where $p_i: X^n\to X$, $i=1,\dots,n$, are the natural projections and $\lambda_i\in\mathbb{R}$ (really, $\mathbb{Z}$).

Now, let us endow the bundles considered in the previous points with certain canonical metrics. The first bundle will be U_j^E for a line bundle E over X provided with an admissible metric: U_j^E has a unique Hermitian metric satisfying the requirements (provided $\deg E\neq 0$):

1. it is admissible over $X\times\{\delta\}$ for every $\delta\in J_j$,
2. it is compatible with the trivialization 2. or 2'. in Definition 3.

The second will be the bundle $\mathcal{O}(-W_{g-1})$ over J_{g-1}. There are two ways to define a metric on this bundle:
a) to induce a metric on it via the canonical isomorphism (23):

$$\mathcal{O}(-W_{g-1})=\det\mathbb{R}\pi_* U_{g-1},$$

b) to induce a metric on it via the translation (24):

$$\mathcal{O}(-W_{g-1})=T^*_{-\kappa}\mathcal{O}(-\Theta).$$

Of course, we suppose the Quillen metric on $\det\mathbb{R}\pi_* U_{g-1}$ is given in the first case, or the metric on $T^*_{-\kappa}\mathcal{O}(-\Theta)$ defined below is given in the second case.

It would be useful to endow $\mathcal{O}(\Theta)$ with such a metric, that its Chern form $\mathbf{c}_1(\mathcal{O}(\Theta))$ equals η_J. Therefore, $\mathcal{O}(\Theta)$ with this metric would be polarizing and admissible. To define a Hermitian metric on $\mathcal{O}(\Theta)$ one needs to define the square $\|\theta\|^2$ of the section $\theta(z)$:

$$\|\theta(z)\|^2=h(z)\cdot|\theta(z)|^2, \tag{26}$$

where $h(z)$ is a real positive C^∞-function with the transition law determined by the requirement $\|\theta(z+m+\tau n)\|=\|\theta(z)\|$. There exists a canonical choice of such a metric h: let $H=(\mathrm{Im}\,\tau)^{-1}$, then

$$h(z)=(\det H)^{-1/2}\cdot\exp((\pi/2)\cdot{}^t(z-\bar z)H(z-\bar z)) \tag{27}$$

is a Gaussian type density along $\mathrm{Im}\,z$. (The function $\log\|\theta(z)\|$ on J is usually called the *Néron function* or the *Green function* of the divisor Θ.) Calculate the Chern form:

$$\mathbf{c}_1(\mathcal{O}(\Theta))=(2\pi i)^{-1}\partial\bar\partial\log h(z)=-(i/2)^t(d\bar z)H dz=-(i/2)^t\bar\varphi H\varphi,$$

where $\varphi:=(\varphi_1,\dots,\varphi_g)$ is the vector of the differentials normalized via A-periods. Let us change a basis: $\varphi=B\omega$, B is the transition matrix, $\omega=(\omega_1,\dots,\omega_g)$ is our orthonormal basis. Then the Riemann relation gives $(i/2)\int\varphi_i\wedge\bar\varphi_j=\mathrm{Im}\,\tau_{ij}$. But $(i/2)\int\omega_i\wedge\bar\omega_j=\delta_{ij}$, hence $B^t\bar B=\mathrm{Im}\,\tau$ and one can continue:

$$\mathbf{c}_1(\mathcal{O}(\Theta))=-(i/2)^t\bar\varphi H\varphi=-(i/2)^t\bar\omega^t\bar B(\mathrm{Im}\,\tau)^{-1}B\omega=-(i/2)^t\bar\omega\wedge\omega=\eta_J.$$

The bundle $\mathcal{O}(-\Theta)$ we metrize by h^{-1}. We set in the case b) that the bundle $\mathcal{O}(-W_{g-1})$ inherits this metric via the translation (24) to $\mathcal{O}(-\Theta)$ (in particular, $\mathcal{O}(-W_{g-1})$ is polarizing and the metric b) on it is admissible).

Provide the bundle Ω with an arbitrary Hermitian metric. Recall that we consider this metric as a Riemannian metric on X, compatible with the complex structure.

Finally, provide the one-dimensional vector space $\Omega|_a = \langle \Omega, \mathcal{O}(a) \rangle$ with the Arakelov-Deligne metric.

Then there holds the

Proposition (Faltings [13], Moret-Bailly [4, 4.14]). a) *Suppose that we have endowed the bundles* $\det \mathbb{R}\pi_* U_{g-1}$ *and* $\det \mathbb{R}\pi_* U_j^E$ *with the Quillen metrics and the other bundles with the metrics introduced above, the metric on* $\mathcal{O}(-W_{g-1})$ *being defined as in the case* a) *above, via the isomorphism*

0. $\det \mathbb{R}\pi_* U_{g-1} = \mathcal{O}(-W_{g-1})$.
Then the isomorphisms
1. $\det \mathbb{R}\pi_* U_j^E = T_E^* \mathcal{O}(-W_{g-1}) \otimes \det \mathbb{R}\pi_* \mathcal{O} \otimes (\det \mathbb{R}\pi_* E)^{-1}$,
2. $(\det \mathbb{R}\pi_* U_j^E)^{\otimes 2} = (T_E^* \mathcal{O}(-W_{g-1}))^{\otimes 2} \otimes \langle E, E^{-1} \otimes \Omega \rangle$
from Proposition 3 are isometries. If the metric on Ω *satisfies the condition that the residue map* $\langle \Omega(a), \mathcal{O}(a) \rangle \to \mathcal{O}_{J_j}$ *is an isometry, then*
3. $\det \mathbb{R}\pi_* U_j^a = T_E^* \mathcal{O}(-W_{g-1}) \otimes (\Omega|_a)^{(g-j-1)(g-j)/2}$,
where $E = \mathcal{O}((g-j-1) \cdot a)$, *is also an isometry.*
b) *If the metric on* $\mathcal{O}(-W_{g-1})$ *is defined as in the case* b) *above, via the translation to* $\mathcal{O}(-\Theta)$, *then the isomorphisms* 0.–3. *are isometries up to a constant factor (with the same additional requirement for* 3. *as in* a)*).*

Note. As in Proposition 2 one can replace $\mathcal{O}(-W_{g-1})$ by $T_{-\kappa}^*(\mathcal{O}(-\Theta))$.

Proof. a) is a straightforward consequence of the isomorphism (25) and the Deligne-Riemann-Roch (cf. Proof of Proposition 2).
b) The plan of the proof is to verify that the metrics on the left-hand side and the right-hand side of the isomorphism 0. are admissible.

The metric on $\mathcal{O}(-W_{g-1})$ is admissible by construction. We want to prove that the Quillen metric on $\det \mathbb{R}\pi_* U_{g-1}$ is admissible, too. That means that $\mathbf{c}_1(\det \mathbb{R}\pi_* U_{g-1})$ is translation-invariant: for every line bundle E over X of $\deg E = 0$

$$\mathbf{c}_1(\det \mathbb{R}\pi_* T_E^* U_{g-1}) = \mathbf{c}_1(\det \mathbb{R}\pi_* U_{g-1}).$$

By universality of U_{g-1},

$$T_E^*(U_{g-1}) = E \otimes U_{g-1} \otimes \pi^* \mathcal{M}$$

for a certain line bundle $\mathcal{M}$ over J. Choose an admissible metric on E. Then the latter equality is an isometry for a certain metric on $\mathcal{M}$. Hence

$$\mathbf{c}_1(T_E^* U_{g-1}) = \mathbf{c}_1(U_{g-1}) + \nu,$$

where $\nu = \pi^* \mathbf{c}_1(\mathcal{M})$ is a form pulled up from J. Apply the Deligne-Riemann-Roch theorem to U_{g-1} relative to the second projection $\pi: X \times J \to J$:

$$(\det \mathbb{R}\pi_* U_{g-1})^{\otimes 2} = (\det \mathbb{R}\pi_* \mathcal{O})^{\otimes 2} \otimes \langle U_{g-1}, U_{g-1} \otimes \Omega^{-1} \rangle$$

(a canonical isometry). Then according to the Deligne formula (15)

$$\begin{aligned}
&\mathbf{c}_1(\det \mathbb{R}\pi_* T_E^* U_{g-1}) - \mathbf{c}_1(\det \mathbb{R}\pi_* U_{g-1}) \\
&\quad = (1/2)\textstyle\int (\mathbf{c}_1(U_{g-1}) + v) \wedge (\mathbf{c}_1(U_{g-1}) + v - \mathbf{c}_1(\Omega)) \\
&\qquad - (1/2)\textstyle\int \mathbf{c}_1(U_{g-1}) \wedge (\mathbf{c}_1(U_{g-1}) - \mathbf{c}_1(\Omega)) \\
&\quad = (1/2)v\textstyle\int (\mathbf{c}_1(U_{g-1}) - \mathbf{c}_1(\Omega)) + (1/2)v\int \mathbf{c}_1(U_{g-1}) + (1/2)\int v^2 \\
&\quad = (1/2)v\textstyle\int (2\mathbf{c}_1(U_{g-1}) - \mathbf{c}_1(\Omega)) = (1/2)v(2\deg_X U_{g-1} - \deg_X \Omega) = 0
\end{aligned}$$

(integration is over X everywhere). Thus, 0. in the case b) is proved.
The isometries 1.–3. are constructed analogically to a). □

This Proposition and Lemma 1 suggest that there must be a connection between the line bundle $\mathscr{B}$ over $J \times J$ (see Point 1) and the line bundles $\mathcal{O}(-W_{g-1})$ and $\mathcal{O}(-\Theta)$ over J – something similar to the Deligne formula (10):

Corollary (Moret-Bailly [4]). a) *For a line bundle E over X there exist the canonical isometries*

1. $\mathscr{B} = m^* T_E^* \mathcal{O}(W_{g-1}) \otimes p_1^* T_E^* \mathcal{O}(-W_{g-1}) \otimes p_2^* T_E^* \mathcal{O}(-W_{g-1}) \otimes (\det \mathbb{R}\pi_* E)^{-1}$,
2. $\mathscr{B} = m^* \mathcal{O}(\Theta) \otimes p_1^* \mathcal{O}(-\Theta) \otimes p_2^* \mathcal{O}(-\Theta) \otimes (\det \mathbb{R}\pi_* \kappa)^{-1}$,

where $m: J \times J \to J$ is the multiplication morphism, $p_1^*, p_2^*: J \times J \to J$ are the first and the second projections, $\pi: X \times J \to J$ (or $X \times J \times J \to J \times J$) is the second projection. All the bundles are Hermitian: the metric on $\mathscr{B}$ is defined in Point 1, the metrics on $\mathcal{O}(W_{g-1})$ and $\mathcal{O}(\Theta)$ are defined in Proposition 4, a) *and the metrics on E and κ are admissible, κ being a holomorphic line bundle over X whose class in J_{g-1} coincides with the Riemann constant.*

b) *If one defines the metrics on $\mathcal{O}(W_{g-1})$ and $\mathcal{O}(\Theta)$ as in Proposition 4,* b), *then the canonical isomorphisms* 1., 2. *are isometries up to a constant factor.*

Proof. According to Lemma 1 there defined a canonical isometry

$$\mathscr{B} = \langle p_1^* U_0^E, p_2^* U_0^E \rangle^{-1}.$$

By the Deligne formula this yields

$$\mathscr{B} = (\det \mathbb{R}\pi_*(p_1^* U_0^E \otimes p_2^* U_0^E))^{-1} \otimes \det \mathbb{R}\pi_* p_1^* U_0^E \otimes \det \mathbb{R}\pi_* p_2^* U_0^E \otimes (\det \mathbb{R}\pi_* \mathcal{O})^{-1}.$$

But, obviously,

$$p_1^* U_0^E \otimes p_2^* U_0^E = m^* U_0^E,$$

and so

$$\begin{aligned}
\mathscr{B} &= m^*(\det \mathbb{R}\pi_* U_0^E)^{-1} \otimes p_1^* \det \mathbb{R}\pi_* U_0^E \otimes p_2^* \det \mathbb{R}\pi_* U_0^E \otimes (\det \mathbb{R}\pi_* \mathcal{O})^{-1} \\
&= m^* T_E^* \mathcal{O}(W_{g-1}) \otimes p_1^* T_E^* \mathcal{O}(-W_{g-1}) \otimes p_2^* T_E^* \mathcal{O}(-W_{g-1}) \otimes (\det \mathbb{R}\pi_* E)^{-1}
\end{aligned}$$

by Proposition a), 1. (All the equalities are canonical isometries.) The rest of the proof is evident. □

5.

Let us return to the situation of Point 1, where the line bundles $\mathcal{O}(D)$ and $\mathcal{O}(D')$ over $X \times X^m$ and $X \times X^n$ were considered. Recall that there was fixed a vector

$\mathbf{p}=(p_1,\dots,p_m)\in\mathbb{Z}^m$, $\sum p_i=0$, and there was defined the divisor $D=\sum p_i x_i$ for every $(x_1,\dots,x_m)\in X^m$, the Abel map $\varphi(\mathbf{p}):X^m\to J$ sending $(x_1,\dots,x_m)$ into $\mathrm{cl}\,\mathcal{O}(D)$ (the same for $\mathbf{p}'$, D', etc.). For $\mathbf{p}$ complex one must use a family Y instead of X^m, as it was mentioned in Note after Proposition 1. The symbol π will denote both of the second projections $X\times X^m\to X^m$, $X\times J\to J$, we hope that it gives rise to no confusion.

Now one can easily reformulate Proposition 1 in terms of $\mathcal{O}(W_{g-1})$ or $\mathcal{O}(\Theta)$ instead of $\mathscr{B}$. But we are interested in the special case $D'=-D$. A simple calculation leads us to the

Corollary. a) *There take place the canonical isometries*

1. $\langle\mathcal{O}(D),\mathcal{O}(D)\rangle=\varphi(\mathbf{p})^*(T_E^*\mathcal{O}(W_{g-1})\otimes i^*T_E^*\mathcal{O}(W_{g-1}))\otimes(\det\mathbb{R}\pi_*E)^{\otimes 2}$,
2. $\langle\mathcal{O}(D),\mathcal{O}(D)\rangle=\varphi(\mathbf{p})^*(\mathcal{O}(\Theta)^{\otimes 2})\otimes(\det\mathbb{R}\pi_*\kappa)^{\otimes 2}$,

where $i:J\to J$ *is the inversion* $j\mapsto -j$ *and all the metrics are those of Proposition* a).

b) *With the metrics of Proposition* b), *the isomorphisms* 1. *and* 2. *are isometries up to a constant factor.* □

4. Remarks on Relative Case

From this very moment we consider a family of Riemann surfaces depending algebraically on parameters taken from an arbitrary (algebraic) base S (the model example will be the moduli space). To be more precise, instead of a single complex algebraic curve we deal with a projective smooth morphism $X\to S$ of complex algebraic varieties with smooth connected fibers of dimension 1.

The families which we have used earlier have J or X^n as a base, i.e. we examined only a variation of a line bundle over a fixed X. Now we consider variation of parameters where neither a bundle, nor X is fixed.

1. The Relative $\partial\bar\partial$-Lemma

Let us outline some distinguishing features of the main constructions of the previous sections in the relative case. The crucial point of the procedure of endowing holomorphic line bundles over X with metrics is the $\partial\bar\partial$-lemma, which will appear below after some preliminaries. But first of all, due to Bismut, Gillet, and Soulé [14] define the *relative Dolbeault complexes* of an arbitrary holomorphic family $\pi:X\to S$ (more precisely, π is a smooth morphism of connected complex manifolds with n-dimensional connected complex manifold as a fiber). For $s\in S$, $p=0,1,\dots,n$, let

$$0\to E_s^{p,0}\xrightarrow{\bar\partial}E_s^{p,1}\xrightarrow{\bar\partial}\dots E_s^{p,n}\to 0$$

denote the Dolbeault complex of the fiber X_s: $E_s^{p,q}$ is the vector space of smooth (p,q)-forms on X_s. Then standard sheaf arguments permit to unite the complexes $E_s^{p,\cdot}$ for all $s\in S$ in a complex

$$0\to E^{p,0}\xrightarrow{\bar\partial}E^{p,1}\xrightarrow{\bar\partial}\dots E^{p,n}\to 0$$

of infinite-dimensional vector bundles over S. A smooth section of the bundle $E^{p,q}$ is called a *relative* (p,q)*-form* and the map $\bar\partial$ is called the *relative differential* $\bar\partial$. Analogically define the *relative differential* ∂, and put $\mathrm{d}:=\partial+\bar\partial$.

For a Hermitian holomorphic line bundle $\mathscr{L}$ over the family $\pi: X \to S$ (that is merely over X) define the *relative Chern form* $\mathbf{c}_{1,X/S}\mathscr{L} \in \Gamma(E^{1,1}) := \Gamma(S, E^{1,1})$ as $\frac{1}{2\pi\sqrt{-1}} \partial\bar{\partial} \log \|l\|$, where l is a regular local holomorphic section of $\mathscr{L}$ (equivalently way, $\mathbf{c}_{1,X/S}\mathscr{L}$ is the image of the Chern form of $\mathscr{L}$ under the natural projection from the space of (1, 1)-forms on X to the space $\Gamma(E^{1,1})$ of relative (1, 1)-forms). If π is locally Hermitian (i.e., there is an open covering $\{U\}$ of S such that for any U a Hermitian metric g_U on $\pi^{-1}(U)$ is given) then $E^{p,q}$ will be Hermitian. If π is proper, then we define the adjoint operators $\bar{\partial}^*$, ∂^*, and d^* and the Laplacians $\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, $\Delta_{\partial} := \partial\partial^* + \partial^*\partial$ and $\Delta_{\mathrm{d}} := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$. If π is, moreover, *locally Kähler* (i.e., it is locally Hermitian with g_U Kähler for every U), then

$$\Delta_{\mathrm{d}} = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}.$$

Proposition. *Let $\pi: X \to S$ be a proper smooth locally Kähler morphism of connected complex manifolds with connected fibers. Suppose there are given a smooth relative (1, 1)-form μ which is relatively closed* ($\mathrm{d}\mu = 0$) *and a holomorphic line bundle $\mathscr{L}$ over X with*

$$\mathbf{c}_{1,X/S}\mathscr{L} = \mathrm{cl}\,\mu$$

(it is an equality of relative Chern classes: it means that $\mathbf{c}_{1,X/S}\mathscr{L} - \mu$ is d*-exact). Then locally on S there exists a Hermitian metric on $\mathscr{L}$, such that*

$$\mathbf{c}_{1,X/S}\mathscr{L} = \mu$$

exactly.

Proof. Let $\| \;\|$ be an arbitrary Hermitian metric on $\mathscr{L}$. Then its first Chern form is $\mathbf{c}_{1,X/S}\mathscr{L} = \frac{1}{2\pi\sqrt{-1}} \partial\bar{\partial} \log \|s\|$ for a local regular holomorphic section s of $\mathscr{L}$. In order to find a metric on $\mathscr{L}$ such that $\mathbf{c}_{1,X/S}\mathscr{L} = \mu$ one must multiply $\| \;\|$ by $\exp(\sigma(x))$, where $\sigma(x)$ is a smooth real function on X. The first Chern class will be equal to $\mathbf{c}'_{1,X/S}\mathscr{L} = \frac{1}{2\pi\sqrt{-1}} \partial\bar{\partial}\sigma + \mathbf{c}_{1,X/S}\mathscr{L} = \mu$. Thus we need to find a real smooth function σ being a solution of the equation

$$\partial\bar{\partial}\sigma = 2\pi\sqrt{-1}(\mu - \mathbf{c}_{1,X/S}\mathscr{L}).$$

The sole part which remains to be proved is the following:

Lemma *(the relative $\partial\bar{\partial}$-lemma). Let η be a smooth relative (p, q)-form which is relatively* d*-exact and ∂- and $\bar{\partial}$-closed. Then locally on S there exists a smooth relative $(p-1, q-1)$-form σ on X such that*

$$\partial\bar{\partial}\sigma = \eta.$$

If η is purely imaginary then σ may be chosen real.

Proof of Lemma. As well as its absolute prototype this lemma is based on the Hodge decomposition. But the latter is a slightly more fine fact from the theory of elliptic families.

Let $\mathscr{H}_s^{p,q}$ be the space of complex harmonic (p,q)-forms on a fibre $\pi^{-1}(s)$:

$$\mathscr{H}_s^{p,q} := \operatorname{Ker} \Delta_{\bar{\partial},s} = \operatorname{Ker} \bar{\partial}_s \cap \operatorname{Ker} \bar{\partial}_s^*$$

(all operators are assumed acting on $E_s^{p,q}$). The space $\mathscr{H}_s^{p,q}$ is finite-dimensional and

$$\mathrm{h}^{p,q}(s) := \dim \mathscr{H}_s^{p,q}$$

is the Hodge number of the fibre $\pi^{-1}(s)$. According to Deligne's theorem [15] $\mathrm{h}^{p,q}(s)$ is constant as a function of s. Thereby, locally on S, 0 is a discrete point of the spectrum of $\Delta_{\bar{\partial}}$. In particular, one has (cf. Bismut-Gillet-Soulé [14, Proof of Theorem 3.14]):

1. there exists a finite-dimensional vector bundle $\mathscr{H}_s^{p,q} \subset E_s^{p,q}$ over S with fibre $\mathscr{H}_s^{p,q}$,
2. there exists a projection $\mathscr{H}: E^{p,q} \to \mathscr{H}^{p,q}$ and an operator (Green operator)

$$G_{\bar{\partial}}: E^{p,q} \to E^{p,q},$$

satisfying the conditions: $G_{\bar{\partial}}(\mathscr{H}^{p,q}) = 0$, $\bar{\partial} G_{\bar{\partial}} = G_{\bar{\partial}} \bar{\partial}$, $\bar{\partial}^* G_{\bar{\partial}} = G_{\bar{\partial}} \bar{\partial}^*$ and there takes place the orthogonal decomposition

$$\eta = \mathscr{H}\eta + \Delta_{\bar{\partial}} G_{\bar{\partial}} \eta \tag{28}$$

for every section η of the bundle $E^{p,q}$.

Similar decompositions are defined for the operators ∂ and d, their Green operators will be denoted by G_∂ and G_d, correspondingly. Our fibration is Kähler, so $2G_\mathrm{d} = G_\partial = G_{\bar{\partial}}$ and all the operators d, d^*, ∂, ∂^*, $\bar{\partial}$, $\bar{\partial}^*$ commute with our Green operators.

Return to the proof of Lemma. Decompose η by (28) using $\bar{\partial}\eta = 0$:

$$\eta = \bar{\partial}\bar{\partial}^* G_{\bar{\partial}} \eta .$$

For the operator ∂ one has the decomposition

$$\bar{\partial}^* G_{\bar{\partial}} \eta = \partial \partial^* G_\partial (\bar{\partial}^* G_{\bar{\partial}} \eta),$$

since $\partial(\bar{\partial}^* G_{\bar{\partial}} \eta) = -\bar{\partial}^* G_{\bar{\partial}} \partial \eta = 0$ via $\partial \eta = 0$. Finally,

$$\eta = \bar{\partial}\partial(\partial^* \bar{\partial}^* G_{\bar{\partial}}^2 \eta),$$

and letting

$$\sigma = -\partial^* \bar{\partial}^* G_{\bar{\partial}}^2 \eta,$$

one obtains $\partial\bar{\partial}\sigma = \eta$.

Concerning the realness of σ, one can easily see that the construction of σ can be rewritten only in terms of real forms and operators, if one starts from a real form $\sqrt{-1}\,\eta$. □

2. The Relative Poincaré Line Bundle

As usual, in the relative case $J = \operatorname{Pic}^0 X/S := \operatorname{Pic}^0 X / \operatorname{Pic}^0 S$ is the relative Jacobian, fibred canonically over $S: J \to S$, with an ordinary Jacobian of curve as a fibre. Analogically, $J^t := \operatorname{Pic}^0 J/S$. The set $J^t(S)$ of S-points of J^t is in one-to-one correspondence with the group of isomorphism classes of topologically trivial holomorphic line bundles over J modulo the classes of the bundles pulled up from S. Then the *relative Poincaré line bundle* $\mathscr{P}$ is the unique bundle over $J \times J^t$ (all the Cartesian products should be fibered products in relative case), satisfying the conditions:

1. for $x \in J^t(S)$ the line bundle induced by $\mathscr{P}$ over $J \times \{x\}$ belongs to the class x,
2. for $e \in J(S)$, the unit section, the line bundle induced by $\mathscr{P}$ over $\{e\} \times J^t$ is holomorphically trivialized.

Put

$$\mathscr{B} := (\mathrm{id} \times \Phi)^* \mathscr{P},$$

where $\Phi : J \to J^t$ is the principal polarization (Φ is a morphism *over* S).

Hermitian metrics on $\mathscr{B}$ and $\mathscr{P}$ are defined (as in 3.1) to satisfy the requirements:

1. their relative curvatures equal zero as relative forms over S,
2. they are compatible with the trivializations: $\mathscr{P}$ at $\{e\} \times \{e^t\}$ and $\mathscr{B}$ at $\{e\} \times \{e\}$ (the latter points are now S-points, i.e. the sections of the projections to S), e^t being the class of $\mathcal{O}_J$ in $\mathrm{Pic}^0 J/S$.

Such a metric on each bundle ($\mathscr{B}$ or $\mathscr{P}$) exists due to Proposition 1 and is unique by obvious reasons.

Consider the Abel map

$$\varphi(\mathbf{p}) : X^m \to J,$$

defined as in Sect. 3, but now X and J are the relative curve and the relative Jacobian. The divisor $D = \sum p_i x_i$ is now a relative complex divisor. Provide the line bundle $\mathcal{O}(D)$ with such a Hermitian metric that its relative curvature via the projection $X \times X^m \to X^m$ equals zero. Having done the same for $\mathbf{p}'$, D', ... (see 3.5), we may consider the Deligne line bundle $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$ over $X^m \times X^n$ with the Arakelov-Deligne metric. Then there takes place the relative variant of Proposition 3.1.

Proposition. *There exists a canonical isometry*

$$\langle \mathcal{O}(D), \mathcal{O}(D') \rangle = (\varphi(\mathbf{p}) \times \varphi(\mathbf{p}'))^* (\mathscr{B}^{-1})$$

for $\varphi(\mathbf{p}) \times \varphi(\mathbf{p}') : X^m \times X^n \to J \times J$. *Furthermore, the bundle* $\langle \mathcal{O}(D), \mathcal{O}(D') \rangle$ *is flat.*

Note. For D and D' being complex one must replace X^m and X^n by Y^m and Y^n, as in Note after Proposition 3.4.

Proof. The arguments of Proposition 3.4 reduce the problem to the

Lemma. 1) *There exists a canonical isometry*

$$\langle U_0, U_0 \rangle = \mathscr{B}^{-1},$$

where U_0 *is a universal line bundle on* $X \times J$ *with relatively flat Hermitian metric.*
2. $\mathscr{B}$ *is flat.*

Proof of Lemma. 1) The part of the lemma which relates to the existence of a canonical isomorphism is due to Moret-Bailly [4, 2.9.4]. The isometricity of this isomorphism yields from the following simple general observations:

if one has

- a canonical isomorphism $\langle U_0, U_0\rangle = \mathscr{B}^{-1}$ over $J \times J \to S$,
- Hermitian metrics on both sides of this isomorphism,
- for $s \in S$ fixed our isomorphism is an isometry,

then it is an isometry over S.

The first point is mentioned above, for the second one see before the proposition, and the third one is provided by Lemma 3.1.
2. Taking into account the first part of the lemma, it is sufficient to prove the lemma on flatness of $\langle \mathscr{L}, \mathscr{M}\rangle$ (see the following point). □

3. *Flatness of* $\langle \mathscr{L}, \mathscr{M}\rangle$

Lemma. *Let $\pi: X \to S$ be a smooth proper map of complex manifolds of relative dimension* 1 *with connected fibres. Let $\mathscr{L}$ and $\mathscr{M}$ be two relatively flat Hermitian holomorphic line bundles. Then the bundle $\langle \mathscr{L}, \mathscr{M}\rangle$ is flat.*

Proof. If $\mathscr{L}$ and $\mathscr{M}$ were flat the assertion would be trivial due to Deligne's formula

$$\mathbf{c}_1(\langle \mathscr{L}, \mathscr{M}\rangle) = \int_{X/S} \mathbf{c}_1(\mathscr{L}) \wedge \mathbf{c}_1(\mathscr{M}).$$

In our situation, when $\mathscr{L}$, $\mathscr{M}$ are only relatively flat, let us cover S by open sets, where $\mathscr{L}$ and $\mathscr{M}$ are topologically trivial. If we denote by $\mathscr{L}_1$ and $\mathscr{M}_1$ the restrictions of $\mathscr{L}$ and $\mathscr{M}$ to such a set U and endow $\mathscr{L}_1$ and $\mathscr{M}_1$ with flat metrics, then $\mathscr{L} \otimes \mathscr{L}_1^{-1}$ and $\mathscr{M} \otimes \mathscr{M}_1^{-1}$ are Hermitian holomorphic line bundles pulled up from U and we have a canonical isometry

$$\langle \mathscr{L}, \mathscr{M}\rangle = \langle \mathscr{L}_1, \mathscr{M}_1\rangle$$

because $\deg_{X/S}\mathscr{L} = \deg_{X/S}\mathscr{M} = 0$. Hence, $\langle \mathscr{L}, \mathscr{M}\rangle$ is flat, too. □

4. *Relative Principal Polarization and Relative Admissible Metrics*

First of all, one can notice that *all the definitions and constructions of* 3.2–3.5 have been chosen so that they *are* easily *generalized to the relative case. Proposition 3.3 in the relative case is due to Moret-Bailly* [4].

As an example, we generalize the notions of polarizing bundle and admissible metric (Definition 3.4). To do this, let us give an equivalent, more convenient for relative the case, definition of the canonical form and the principal polarization. If $\{\omega_1, \ldots, \omega_g\}$ is an *arbitrary local basis* of the locally free sheaf $\pi_* \Omega^1_{X/S}$ over S and

$$M = \left((\sqrt{-1}/2) \int_{X/S} \omega_i \wedge \bar{\omega}_j\right)^{-1}$$

is the inverse of the Gram matrix of the standard Hermitian product of the basis elements, then

$$\eta = \frac{\sqrt{-1}}{2g}\, {}^t\omega M \bar{\omega} = \frac{\sqrt{-1}}{2g} \sum_{i,j=1}^{n} \omega_i \wedge m_{ij} \bar{\omega}_j$$

is called the *canonical form* on X/S, being considered as a relative (1, 1)-form on X/S, and

$$\eta_J=(\sqrt{-1}/2)^t\omega M\bar{\omega}=(\sqrt{-1}/2)\sum_{i,j=1}^{n}\omega_i\wedge m_{ij}\bar{\omega}_j$$

is called the *principal polarization* on J/S, being considered as a relative (1, 1)-form on J/S. The equivalence with the former definition is stated immediately after changing the basis.

Using this definition one can easily see that η and η_J are *smooth* relative forms.

Now we call a metric on a holomorphic bundle $\mathscr{L}$ over $J_k \to S$ *relatively admissible* if its relative Chern form $\mathbf{c}_{1,J_k/S}\mathscr{L}$ is invariant under the translations from $J(S)$, and we call a bundle $\mathscr{L}$ *relatively polarizing* if its relative Chern class $\mathbf{c}_{1,J_k/S}\mathscr{L}$ is proportional to $\mathrm{cl}\,\eta_J$ with integral proportionality coefficient, where cl means the class modulo relatively d-exact forms. The corresponding notions for $X \to S$ and $X^n \to S$ are defined analogically.

In order to deal with the Riemann theta-function let us specialize the base S of the family: S will be the moduli space $\mathscr{M}_g$ in Siegel's realization which follows. The set of complex $(g \times g)$-matrices τ satisfying the Riemann relations: ${}^t\tau=\tau$, $\operatorname{Im}\tau>0$, generate the *Siegel upper half-space* H_g. The points of H_g which are the period matrices of Riemann surfaces form a closed analytic subvariety N_g in H_g. Each Riemann surface with its complex structure and marking is reconstructed from τ up to a unique isomorphism. The moduli space $\mathscr{M}_g$ is the quotient-space $N_g/\mathrm{Sp}(2g,\mathbb{Z})$, $\mathrm{Sp}(2g,\mathbb{Z})$ acting by changes of marking. The Riemann theta-function $\theta(z,\tau)$ depends really on two variables: $z \in \mathbb{C}^g$, $\tau \in H_g$.

After these preliminaries it seems more useful to *leave the generalization of Proposition 3.4,* a), *Corollary 3.4,* a) *and Corollary 3.5,* a) *to the reader as an exercise.*

5. A Problem

The metric (26), (27) is in fact a Hermitian metric $h(z,\tau)$ on the bundle $\mathscr{O}(\Theta)$ over $J \to \mathscr{M}'_g$, where $\mathscr{M}'_g=N_g/\Gamma_{1,2}$ is a finite covering of the moduli space $\mathscr{M}_g$, $\Gamma_{1,2} \subset \mathrm{Sp}(2g,\mathbb{Z})$ is a modular subgroup consisting of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ such that the diagonals of matrices ${}^tAC$ and ${}^tBD$ are even, $\mathscr{M}'_g$ parametrizes the pairs $(X,\mathscr{L})$, $\mathscr{L}$ being a *theta-characteristic*, a holomorphic line bundle over X such that $\mathscr{L}\otimes\mathscr{L}$ is isomorphic to Ω. To see that $h(z,\tau)$ is a metric, one must verify that $h(z,\tau)\cdot|\theta(z,\tau)|^2$ is Λ- and $\Gamma_{1,2}$-invariant. In other words, $h(z,\tau)$ must have the same transition law as $|\theta(z,\tau)|^{-2}$. (Recall that under the transition $\tau m+n \in \Lambda$,

$$\theta(z+\tau m+n,\tau)=\exp\pi\sqrt{-1}(-m\cdot{}^t\tau m-2m\cdot{}^tz)\cdot\theta(z,\tau),$$

and under the transition $\gamma:=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma_{1,2}$,

$$\begin{aligned}&\theta({}^t(C\tau+D)^{-1}\cdot z,(A\tau+B)(C\tau+D)^{-1})\\&=\zeta\cdot\det(C\tau+D)^{1/2}\cdot\exp\pi\sqrt{-1}({}^tz(C\tau+D)^{-1}\cdot Cz)\cdot\theta(z,\tau),\end{aligned}$$

where ζ is 8^{th} root of unity, depending on γ – see Mumford [16].) One can easily check the required transition law for h.

If, in order to prove the relative variant of Proposition 3.4, b), we start from a relatively admissible metric on U_{g-1} then the arguments analogous to the proof of Proposition 3.4, b) show that the Quillen metric on the bundle $\det \mathbb{R}\pi_* U_{g-1}$ over

$$p: J_{g-1} \to \mathcal{M}'_g$$

is relatively admissible, as well as the metric on $\mathcal{O}(-W_{g-1}) = T^*_{-\kappa}\mathcal{O}(-\Theta)$ got from $h(z,\tau)$ by translation. Thereby, the isomorphism

$$\det \mathbb{R}\pi_* U_{g-1} = \mathcal{O}(-W_{g-1})$$

is an isometry up to multiplication by a function $f(m)$ on $\mathcal{M}'_g$.

The problem is to make this function equal to 1. I do not know how to do this, but one must evidently do the following.

Consider the section $\mathcal{M}'_g \to J_{g-1}$ of p, which takes each pair $(X, \mathcal{L}) \in \mathcal{M}'_g$ to the class $\mathrm{cl}\,\mathcal{L} \in J_{g-1}$ of its theta-characteristic $\mathcal{L}$. If one could choose a metric on $\mathcal{L}$ such that

$$\det \mathbb{R}\pi_* \mathcal{L} = \mathcal{O}(-W_{g-1})|_{\mathrm{cl}\,\mathcal{L}} = \mathcal{O}(-\Theta)|_{\mathrm{cl}\,\mathcal{L}-\kappa}$$

was an isometry then the function $f(m)$ would be equal to 1.

6. *The Universal Mumford Form: Construction*

Suppose the data of 2.2 is given, that is, an algebraic family $\pi: X \to S$ of Riemann surfaces with an ordered set of n punctures $Q_1, \ldots, Q_n$ and an isotopy class of a disk B containing all the punctures are considered. Besides that, there are fixed n impulse vectors $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{C}^{13}$ with some physically motivated conditions (see 2.2). Now the base S of the family will be the moduli space $\mathcal{M}_{g,n,B}$ (see 2.1), unless the opposite is mentioned specially.

We want to observe all such families with all permissible values of n and $\mathbf{p}_i$ on an equal footing. According to Corollary 2.1, $\mathcal{M}_{g,n,B}$ is a fibration over the space $\mathcal{M}_g$ (which parametrizes complex structures on X) with a fibre $(H^n \backslash \Delta)/\pi_1(X)$ (which parametrizes ordered sets of n punctures $x_1, \ldots, x_n$ belonging to a disk B in X, B being fixed up to an isotopy of B in $X \backslash \{x_1, \ldots, x_n\}$), where H is the upper half-plane. Therefore, there is defined a morphism φ over $\mathcal{M}_g$:

$$\begin{array}{ccc} \mathcal{M}_{g,n,B} & \longrightarrow & J^{13} \\ & \searrow \quad \swarrow & \\ & \mathcal{M}_g & \end{array},$$

where J^{13} is the 13^{th} power of the universal Jacobian over the moduli space $\mathcal{M}_g$ (its fibre over the point of $\mathcal{M}_g$ corresponding to a complex structure on X is the 13^{th} power of the Jacobian $J(X)$ of X). The morphism φ is defined as the Cartesian product

$$\varphi := \varphi(\mathbf{p}^1) \times \ldots \times \varphi(\mathbf{p}^{13}),$$

$\varphi(\mathbf{p}^\nu)$ being the Abel map

$$\varphi(\mathbf{p}^\nu): (H^n \backslash \Delta)/\pi_1(X) \to J,$$

$$(Q_1, \ldots, Q_n) \mapsto \mathrm{cl}\,\mathcal{O}\left(\sum_i p_i^\nu Q_i\right).$$

Endow the bundle Ω with an arbitrary Hermitian metric, $\Omega^{\otimes 2}$ with the tensor square metric, the bundle E, which is an arbitrary holomorphic line bundle over the universal curve $X \to \mathcal{M}_g$, with an arbitrary relatively admissible Hermitian metric and the corresponding determinant line bundles $\det \mathbb{R}\pi_*(\cdot)$ with the Quillen metrics. Consider the line bundle $\mathcal{B}$ over $J \times J \to \mathcal{M}_g$ with the metric which is *relatively flat and compatible with the trivialization of* $\mathcal{B}$ *over the unit section* $\{e\} \times \{e\}: \mathcal{M}_g \to J \times J$. The Hermitian line bundle $\mathcal{C}$ over J define as the restriction of $\mathcal{B}$ to the diagonal:

$$\mathcal{B} = \mathcal{C}|_{\mathrm{diag}(J)},$$
$$\mathrm{diag}: J \to J \times J .$$

If $\mathcal{L}$ is a bundle over J^{13} denote by $\mathcal{L}^{\boxtimes 13}$ its *exterior tensor power*: $\mathcal{L}^{\boxtimes 13} := p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes \ldots p_{13}^* \mathcal{L}$, where $p_i: J^{13} \to J$ is the i^{th} natural projection.

The metric on $\mathcal{O}(W_{g-1}) = T_{-\kappa}^* \mathcal{O}(\Theta)$ we introduce letting the canonical isomorphism (23)

$$\mathcal{O}(-W_{g-1}) = \det \mathbb{R}\pi_* U_{g-1}$$

to be an isometry for the Quillen metric on the determinant line bundle of U_{g-1}, the latter being endowed with a relatively admissible metric.

Theorem-Definition. a) *There exists a canonical isometry of two Hermitian holomorphic line bundles over* $J^{13} \to \mathcal{M}_g$:

$$\mathrm{MU} := \frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes \mathcal{C}^{\boxtimes 13}}{(\det \mathbb{R}\pi_* \Omega)^{\otimes 13}}$$
$$= \frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes T_E^* \mathcal{O}(W_{g-1})^{\boxtimes 13} \otimes i^* T_E^* \mathcal{O}(W_{g-1})^{\boxtimes 13} \otimes (\det \mathbb{R}\pi_* E)^{\otimes 26}}{(\det \mathbb{R}\pi_* \Omega)^{\otimes 13}},$$

where $i: J \to J$ *is the inversion morphism. (By* MU *we have denoted one of these bundles after their identification.)*

b) *The bundle* MU *is flat, so it locally admits a horizontal holomorphic section* μ_U *of norm 1. This section is called the universal Mumford form. It is unique up to a factor* $\exp(i\alpha)$, $\alpha \in \mathbb{R}$.

c) *For every set of impulses* $\mathbf{p}_1, \ldots, \mathbf{p}_n$ *there exists a canonical isometry of Hermitian holomorphic line bundles (see 2.2) over* $\mathcal{M}_{g,n,B}$

$$\frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2})}{(\det \mathbb{R}\pi_* \Omega)^{\otimes 13} \otimes \left(\bigotimes_{\nu=1}^{13} \langle \mathcal{O}(D^\nu), \mathcal{O}(D^\nu) \rangle \right)} = \varphi^*(\mathrm{MU}).$$

Moreover,

$$\mu_{g,n,B} = \varphi^*(\mu_U)$$

under this isometry.

Note. 1) If one replaces $\mathcal{M}_g$ by its finite covering $\mathcal{M}_g'$ (see Point 5) then there is one more canonical isometry

$$\mathrm{MU} = \frac{\det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes \det \mathbb{R}\pi_*(\kappa)^{\otimes 26} \otimes (\mathcal{O}(\Theta)^{\otimes 2})^{\boxtimes 13}}{(\det \mathbb{R}\pi_* \Omega)^{\otimes 13}},$$

κ having the metric such that $\kappa \otimes \kappa = \Omega$ is an isometry.

2) If S is an arbitrary base, as at the beginning of this point, then there is a natural morphism $S \to \mathcal{M}_{g,n,B}$ and if φ_1 denote the composition $S \to \mathcal{M}_{g,n,B} \xrightarrow{\varphi} J^{13}$ then one can replace φ in c) by φ_1.

Proof. a) trivially follows from the relative variant of Corollary 3.4, 1) and from the isometry

$$\mathscr{C} = \mathscr{B}^{-1}|_{\mathrm{antidiag}(J)},$$

which yields from Lemma 3.1 in the relative case and from the isometry $\langle \mathscr{L}, \mathcal{M}^{-1} \rangle = \langle \mathscr{L}, \mathcal{M} \rangle^{-1}$, antidiag: $J \to J \times J$ mapping j into $(j, -j)$. To prove the note, 1) one needs to use Corollary 3.4, 2) and the Serre duality: $\det \mathbb{R}\pi_*(\mathscr{L}^{-1} \otimes \kappa) = \det \mathbb{R}\pi_*(\mathscr{L} \otimes \kappa)$.
b) MU is flat because $\det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes (\det \mathbb{R}\pi_* \Omega)^{\otimes -13}$ is flat by the Belavin-Knizhnik theorem (or by the Deligne-Riemann-Roch) and because $\mathscr{B}$ and, hence, $\mathscr{C}$ is flat.
c) is a sequence of Proposition 2. □

Note. For $n=0$ the vectors $\mathbf{p}^1, \ldots, \mathbf{p}^{13}$ are vectors from the zero-dimensional space and it is natural to postulate that $D^\nu = 0$, $\nu = 1, \ldots, 13$. Then $\varphi : \mathcal{M}_g \to J^{13}$ maps $\mathcal{M}_g$ into the unit section $\{e\} \times \ldots \times \{e\}$ (13 times) of $J^{13} \to \mathcal{M}_g$. (The image of this section one can identify with $\mathcal{M}_g$). But $\mathscr{C}$ is canonically trivial over $\{\dot{e}\} \times \ldots \times \{e\}$ by construction, so

$$\varphi = \mathrm{id} : \mathcal{M}_g \to \mathcal{M}_g$$

and

$$\mathrm{MU} = \det \mathbb{R}\pi_*(\Omega^{\otimes 2}) \otimes (\det \mathbb{R}\pi_* \Omega)^{\otimes -13}.$$

Therefore, the theorem in the case $n=0$ states nothing but the existence of the Polyakov measure in the partition function integral.

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