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Ising Model on the Generalized Bruhat-Tits Tree

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Abstract. The partition function and the correlation functions of the Ising model on the generalized Bruhat-Tits tree are calculated. We computed also the averages of these correlation functions when the corresponding vertices are attached to the boundary of the generalized Bruhat-Tits tree.

1. Introduction

The Ising model on the Cayley tree turns out to be very interesting [1,2]. The Cayley tree T is manifestly determined to be a connected infinite graph with no loops, each vertex of T being connected with exactly $p+1$ nearest neighbour vertices by links. If p is a prime number, the Cayley tree is called the Bruhat-Tits tree. The branch B_z is defined to be a connected subtree with the only boundary vertex z of the graph $T\backslash B_z$ in the interior of T. By definition the branch contains no cycles. Let us introduce the generalized Bruhat-Tits tree F_g. It consists of a finite connected graph F_g^R with g independent loops, which is called a *reduced* graph, the branches B_x, $x \in F_g^R$, and each vertex is connected by links with exactly $p+1$ nearest neighbours (for every link, two endpoints of which are identified with a vertex, we include the vertex itself twice into the number of its nearest neighbours). If the vertex $x \in F_g^R$ has only one nearest neighbour $y \in F_g^R$, $x \neq y$, then p branches B_x and the link $[x, y]$ form the branch B_y. Hence instead of the reduced graph F_g^R we may consider the reduced graph $F_g^R \backslash [x, y]$. From now on F_0^R is merely a single vertex and $p+1$ branches should be added to this vertex in order to construct the Bruhat-Tits tree $F_0 = T$, for $g > 0$ each vertex $x \in F_g^R$ has $2 \leqq n(x) \leqq p+1$ nearest neighbours in F_g^R and $b(x) = p+1-n(x)$ branches should be added to x in order to construct the generalized Bruhat-Tits tree F_g. Due to [3–5] the Bruhat-Tits tree $T \equiv F_0$ may be interpreted as the coset space $PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$, where $PGL(2, \mathbb{K})$ is the group of fractional linear transformations of the projective line $P^1(\mathbb{K})$ over a ring $\mathbb{K}$ (we deal with the field of p-adic numbers $\mathbb{Q}_p$ and with the ring of the p-adic integers $\mathbb{Z}_p$). The element of $GL(2, \mathbb{Q}_p)$ is called hyperbolic if it has eigenvalues which p-adic norms are different. A Schottky group Γ_g is a free subgroup of $PGL(2, \mathbb{Q}_p)$ with g generators, all non-unit elements of which are hyperbolic. Usually the generalized Bruhat-Tits tree F_g may be interpreted as a coset space T/Γ_g, where Γ_g is some Schottky group [4–6].

Since the configuration σ takes the values ± 1 the Ising model action may be rewritten in the form

$$\beta \sum_{|x-y|=1} \sigma(x)\sigma(y) = \beta N_1 - \beta/2 \sum_{|x-y|=1} (\sigma(x)-\sigma(y))^2 ,$$

where N_1 is the total number of the links of the lattice. If σ is allowed to take any real values we obtain the action for $\mathbb{R}$ Ising model

$$S(\phi) = -\beta/2 \sum_{|x-y|=1} (\phi(x) - \phi(y))^2$$

by omitting the unessential term βN_1 and by changing $\sigma \to \phi$. This model is a lattice version of the free massless field. On the broad class of the lattices including the generalized Bruhat-Tits tree the correlation functions of the $\mathbb{R}$ Ising model with the free boundary conditions may be computed exactly [7]. By using the special average of these correlation functions the multiloop p-adic string amplitudes were calculated for the scattering of N identical tachyons attached to the boundary of the generalized Bruhat-Tits tree [8, 9].

In this paper we calculate the partition function and the correlation functions of the Ising model on the generalized Bruhat-Tits tree with the free boundary conditions. We also compute the averages of these correlation functions which are analogous to the p-adic string amplitudes.

2. Correlation Functions

In order to find the correlation functions and partition function for the infinite generalized Bruhat-Tits tree F_g we calculate them first on a finite subgraph $K \subset F_g$. A finite subgraph K of the generalized Bruhat-Tits tree F_g may be considered as a finite cell complex. It consists of zero- and one-dimensional cells: vertices and links. Every cell $\pm s_i^q$ is labelled by the integer $q = 0, 1$ (dimension) and by the number ± 1 (orientation). The cells with the opposite orientation s^q and $-s^q$ both belong to the lattice K. An integer-valued odd ($c^q(-s_i^q) = -c^q(s_i^q)$) function c^q on the q-dimensional cells is called a q-chain of the complex K. c^q can be regarded as a formal sum $\sum m_i s_i^q$, where the integers $m_i = c^q(s_i^q)$. A set of q-chains is an Abelian group: $c^q + c'^q = \sum (m_i + m_i') s_i^q$. It is denoted by $C^q(K, \mathbb{Z})$. It is possible to introduce the inner product on $C^q(K, \mathbb{Z})$: $\langle c^q, c'^q \rangle = \sum m_i m_i'$. We define the boundary operator ∂ on $C^q(K, \mathbb{Z})$ by $\partial s^0 = 0$ and $\partial [x_i, x_j] = x_j - x_i$. By linearity it is easy to extend the boundary operator on $C^q(K, \mathbb{Z})$. We define the coboundary operator ∂^* by the following relations $\partial^* c^1 = 0$ and $\langle \partial^* c^0, c'^1 \rangle = \langle c^0, \partial(c'^1) \rangle$. A kernel $Z_1(K, \mathbb{Z})$ of a homomorphism ∂: $C^1(K, \mathbb{Z}) \to C^0(K, \mathbb{Z})$ is called a group of cycles of the complex K. The image $B_0(K, \mathbb{Z})$ of a homomorphism ∂: $C^1(K, \mathbb{Z}) \to C^0(K, \mathbb{Z})$ is called a group of boundaries of the complex K. The image $B^1(K, \mathbb{Z})$ of a homomorphism ∂^*: $C^0(K, \mathbb{Z}) \to C^1(K, \mathbb{Z})$ is called a group of coboundaries of the complex K.

A homomorphism of $C^q(K, \mathbb{Z})$ into the Abelian group $\mathbb{Z}_2 = \{\pm 1\}$ (Don't confuse with 2-adic integers) is a q-chain of the complex K with coefficients in $\mathbb{Z}_2$. A set of all these homomorphisms is an Abelian group which is denoted by $C^q(K, \mathbb{Z}_2)$. Each homomorphism $h^q \in C^q(K, \mathbb{Z}_2)$ is defined by its values on the q-chains $1 \cdot s_i^q \in C^q(K, \mathbb{Z})$ i.e. on the cells s_i^q. Thus h^q is an $\mathbb{Z}_2$-valued function on the q-dimensional cells of K. On $C^q(K, \mathbb{Z}_2)$ we introduce the boundary and coboundary operators: $\partial h^1(c^0) = h^1(\partial^* c^0)$ and $\partial^* h^0(c^1) = h^0(\partial c^1)$. For example $\partial^* h^0([x, y]) = h^0(x) h^0(y)$, because of $\sigma^{-1} = \sigma$ for $\sigma = \pm 1$. The group of cycles $Z_1(K, \mathbb{Z}_2)$, the groups of boundaries $B_0(K, \mathbb{Z}_2)$ and coboundaries $B^1(K, \mathbb{Z}_2)$ are defined in an obvious way. The group $\mathbb{Z}_2$ is selfdual: if $\sigma_1, \sigma_2 = \pm 1$, then

$$\langle \sigma_1 | \sigma_2 \rangle = \sigma_1^{J(\sigma_2)} = \sigma_2^{J(\sigma_1)}, \tag{2.1}$$

where $J(\sigma) = 1/2(1-\sigma)$. Analogously, if $c_1^q, c_2^q \in C^q(K, \mathbb{Z}_2)$ then

$$\langle c_1^q | c_2^q \rangle = \prod_{s_i^q \in K} \langle c_1^q(s_i^q) | c_2^q(s_i^q) \rangle, \tag{2.2}$$

where multiplication runs over all positively oriented links of the lattice K.

Let us consider the Ising model on the lattice K. A configuration is a chain $\sigma^0 \in C^0(K, \mathbb{Z}_2)$, i.e. a function on the vertices of K taking the values ± 1. The Ising model action may be rewritten as

$$S(\sigma^0) = \beta \sum_{s_i^1 \in K} \partial^* \sigma^0(s_i^1), \tag{2.3}$$

where the summing runs over all positively oriented links of the lattice K. The partition function is

$$Z_k = 2^{-N_0} \sum_{\sigma^0 \in C^0(K, \mathbb{Z}_2)} e^{S(\sigma^0)}, \tag{2.4}$$

where N_0 is the total number of the vertices of the lattice K. The correlation function has the form

$$W_K(\chi^0) = Z_K^{-1} 2^{-N_0} \sum_{\sigma^0 \in C^0(K, \mathbb{Z}_2)} \langle \chi^0 | \sigma^0 \rangle e^{S(\sigma^0)}, \tag{2.5}$$

where the chain $\chi^0 \in C^0(K, \mathbb{Z}_2)$ takes the value -1 at the vertices $x_1, \ldots, x_m$ and takes the value 1 otherwise. Thus the definition (2.2) implies $\langle \chi^0 | \sigma^0 \rangle = \sigma^0(x_1) \ldots \sigma^0(x_m)$.

By Lemma 1 of [10] the correlation function $W_K(\chi^0)$ isn't zero only for boundaries $\chi^0 = \partial \chi^1$ and

$$W_K(\partial \chi^1) = Z_K^{-1} |B^1(K, \mathbb{Z}_2)|^{-1} \sum_{\phi^1 \in B^1(K, \mathbb{Z}_2)} \langle \chi^1 | \phi^1 \rangle \exp\left\{\beta \sum_{s_i^1 \in K} \phi^1(s_i^1)\right\}, \tag{2.6}$$

where $|B^1(K, \mathbb{Z}_2)|$ is the order of the finite group of the coboundaries $B^1(K, \mathbb{Z}_2)$. By using the Fourier transformation on the group $B^1(K, \mathbb{Z}_2)$ we obtain due to Proposition 1 of [10]

$$W_K(\partial \chi^1) = Z_K^{-1} (\operatorname{ch} \beta)^{N_1} \sum_{\xi^1 \in Z_1(K, \mathbb{Z}_2)} \exp\left\{-2\beta^* \sum_{s_i^1 \in K} J(\chi^1(s_i^1) \xi^1(s_i^1))\right\}, \tag{2.7}$$

where $J(\sigma) = 1/2(1-\sigma)$ and the number β^* is given by the following equation: $e^{-2\beta^*} = \operatorname{th} \beta$.

Let K be a finite connected subgraph of the generalized Bruhat-Tits tree F_g. Let K contain also the reduced graph F_g^R. The cell complex K is torsion free. Hence

$$\begin{aligned} \partial \chi^1(s_k^0) &= \exp\{i\pi(\partial c^1)(s_k^0)\}, \\ \xi^1(s_k^1) &= \exp\{i\pi \zeta^1(s_k^1)\}, \end{aligned} \tag{2.8}$$

where $c^1 \in C^1(K, \mathbb{Z})$ and $\zeta^1 \in Z_1(K, \mathbb{Z})$. Since the graph K is connected any chain ∂c^1 may be presented as $\partial\left(\sum_1^N m_i \kappa_{c, x_i}\right)$, where κ_{c, x_i} is some path from the arbitrary fixed vertex $c \in F_g^R$ to the vertex x_i and the integers m_i satisfy the condition $\sum m_i = 0$. (By definition the path κ_{c,x_i} is the sum of the different links connecting two vertices c and x_i.) In view of the first relation (2.8) it is possible to add to the chain ∂c^1 any chain $c^0=\sum n_i s_i^0$ with even integers n_i. Hence we may consider all $m_i=1$ and $N=\sum m_i=0\ (\mathrm{mod}\,2)$, i.e. N is even. Since the graph K contains the reduced graph F_g^R and any branch has no loops each cycle $\zeta^1\in Z_1(K,\mathbb{Z})$ has the form $\zeta^1=\sum_{i=1}^{g} m_i z_i$ where the loops $z_1,\dots,z_g$ form the basis of the group of cycles $Z_1(F_g^R,\mathbb{Z})$. It is possible to choose the basis in this way thus $z_k(s_i^1)=0,\ \pm 1$ for any link s_i^1 and $k=1,\dots,g$. In view of the second relation (2.8) we may consider all $m_i=0,1$. Thus the relation (2.8) may be rewritten as

$$\begin{aligned}\partial\chi^1(s_k^0)&=\partial\left(e^{i\pi\sum_1^{2N}\kappa_{c,x_j}}\right)(s_k^0)=e^{i\pi\sum_1^{2N}\delta_{x_j,s_k^0}}\\ \xi^1(s_k^1)&=\left(e^{i\pi\sum_1^{g}\varepsilon_j z_j}\right)(s_k^1),\end{aligned}\tag{2.9}$$

where the numbers $\varepsilon_j=0,1$.

It is easy to verify the following formula:

$$1/2\left(1-e^{i\pi\sum_1^{n}\varepsilon_k}\right)=\sum_{q=1}^{n}(-2)^{q-1}\sum_{\substack{k_1<\dots<k_q\\ k_j=1}}^{n}\varepsilon_{k_1}\dots\varepsilon_{k_q},\tag{2.10}$$

where the numbers $\varepsilon_k=0,1$.

In view of the formula (2.10) the substitution of the relations (2.9) into Eq. (2.7) yields for the correlation function

$$\begin{aligned}W_K\left(e^{i\pi\sum_{j=1}^{2N}\delta_{x_j,\cdot}}\right)&=Z_K^{-1}(\mathrm{ch}\,\beta)^{N_1}\\ &\times\sum_{\varepsilon_1,\dots,\varepsilon_g=0}^{1}\exp\left\{-2\beta^*\left[\sum_{m=1}^{2N}(-2)^{m-1}\sum_{k_1<\dots<k_m\atop 1}^{2N}\langle\kappa_{c,x_{k_1}},\dots,\kappa_{c,x_{k_m}}\rangle\right.\right.\\ &+\sum_{n=1}^{g}(-2)^{n-1}\sum_{l_1<\dots<l_n\atop 1}^{g}\langle\varepsilon_{l_1}z_{l_1},\dots,\varepsilon_{l_n}z_{l_n}\rangle\\ &+\sum_{m=1}^{2N}\sum_{n=1}^{g}(-2)^{m+n-1}\sum_{k_1<\dots<k_m\atop 1}^{2N}\sum_{l_1<\dots<l_n\atop 1}^{g}\langle\kappa_{c,x_{k_1}},\dots,\kappa_{c,x_{k_m}},\\ &\left.\left.\varepsilon_{l_1}z_{l_1},\dots,\varepsilon_{l_n}z_{l_n}\rangle\right]\right\}\end{aligned}\tag{2.11}$$

and for the partition function

$$Z_K=(\mathrm{ch}\,\beta)^{N_1}\sum_{\varepsilon_1,\dots,\varepsilon_g=0}^{1}\exp\left\{-2\beta^*\sum_{n=1}^{g}(-2)^{n-1}\sum_{l_1<\dots<l_n\atop 1}^{g}\langle\varepsilon_{l_1}z_{l_1},\dots,\varepsilon_{l_n}z_{l_n}\rangle\right\}\tag{2.12}$$

Here for the chains $c_1^1, \ldots, c_k^1 \in C^1(K, \mathbb{Z})$, $k=1,2,\ldots$, we introduced the product

$$\langle c_1^1, \ldots, c_k^1 \rangle = \sum_{s_i^1 \in K} (c_1^1(s_i^1) \ldots c_k^1(s_i^1)). \tag{2.13}$$

For the Bruhat-Tits tree $T=F_0$ the formula (2.12) gives the partition function calculated in the paper [1].

If the vertices $x_1, \ldots, x_{2N}$ are fixed, the limit $K \to F_g$ for the correlation function (2.11) is obtained by omitting the multiplier $(\mathrm{ch}\beta)^{N_1}$ in the formulas (2.11) and (2.12). We denote this limit by

$$W\left(e^{i\pi \sum_{j=1}^{2N} \delta_{x_j, \cdot}}\right).$$

3. Averages

Our aim now will be to compute the correlation functions with the vertices attached to the boundary of the generalized Bruhat-Tits tree.

If a vertex $x_i \notin F_g^R$ then by the definition of the graph F_g there exists the unique vertex $x^R \in F_g^R$ such that the path

$$\kappa_{c,x} = \kappa_{c,x^R} + \kappa_{x^R,x}, \tag{3.1}$$

where the path κ_{c,x^R} belongs to the reduced graph F_g^R and the unique path $\kappa_{x^R,x}$ lies in the branch B_{x^R}. Any half-infinite path (without returns) in B_{x^R} starting at an vertex x^R we call a *ray* $x^R \to x$. The set of all rays will be called the boundary ∂F_g of F_g. On ∂F_g we introduce the basis of open sets ∂B_x, where $x \in F_g \backslash F_g^R$, and ∂B_x consists of all rays having infinite intersection with the branch B_x. The measure μ_0 on ∂F_g is defined by the following relation:

$$\mu_0(\partial B_x) = p^{-\langle \kappa_{x^R,x} \rangle}. \tag{3.2}$$

The relation

$$|x^R \to x,\ y^R \to y|_p = p^{-\langle x^R \to x,\, y^R \to y \rangle} \tag{3.3}$$

defines the distance on ∂F_g.

The reduced graph F_0^R is merely a single vertex c. The boundary ∂F_0 can be naturally identified with p-adic projective line $P^1(\mathbb{Q}_p)$ [4, 5] with the measure related to the Haar measure dx on $\mathbb{Q}_p$ by the following relations:

$$\begin{aligned} d\mu_0(c \to x) &= dx, \quad |x|_p \leqq 1, \\ d\mu_0(c \to x) &= dx/|x|_p^2, \quad |x|_p > 1, \end{aligned} \tag{3.4}$$

where $|\cdot|_p$ is the standard p-adic norm on $\mathbb{Q}_p$, and the distance (3.3) on $P^1(\mathbb{Q}_p)$ is defined by its restriction on $\mathbb{Q}_p$:

$$\begin{aligned} &|c \to x, c \to y|_p = |x-y|_p, \quad |x|_p \leqq 1, \quad |y|_p \leqq 1; \\ &|c \to x, c \to y|_p = |x^{-1} - y^{-1}|_p, \quad |x|_p > 1, \quad |y|_p > 1; \\ &|c \to x, c \to y|_p = 1, \quad \text{otherwise}. \end{aligned} \tag{3.5}$$

We call the vertex $x^R \in F_g^R$ external if x^R is the end of $b(x^R) > 0$ branches in F_g. For $g > 0$ by definition of the reduced graph an external vertex $x^R \in F_g^R$ defines $b(x^R)$

($0 < b(x^R) \leqq p-1$) branches. Then the ray $x^R \to x$ starting at the external vertex $x^R \in F_g^R$ may be identified with the p-adic integer number $x \in \mathbb{Z}_p$ in the form

$$x = a_0 + a_1 p + a_2 p^2 + \dots, \tag{3.6}$$

where $0 \leqq a_0 \leqq b(x^R) - 1$ and $0 \leqq a_i \leqq p-1$ for $i > 0$. We denote the set of these numbers as $\mathbb{Z}_p[a_0 < b(x^R)]$. Thus for $g > 0$,

$$\partial F_g \cong \bigcup_{\substack{x^R \in F_g^R \\ b(x^R) > 0}} \mathbb{Z}_p[a_0 < b(x^R)]. \tag{3.7}$$

It is easy to verify that under this correspondence

$$|x^R \to x, y^R \to y|_p = (|x-y|_p)^{\delta_{x^R, y^R}}, \tag{3.8}$$

$$d\mu_0(x^R \to x) = dx. \tag{3.9}$$

The distance between k rays $x_1^R \to x_1, \dots, x_k^R \to x_k$ is defined similarly to the definition (3.3)

$$|x_1^R \to x_1, \dots, x_k^R \to x_k|_p = p^{-\langle x_1^R \to x_1, \dots, x_k^R \to x_k \rangle}. \tag{3.10}$$

The definition of the generalized Bruhat-Tits tree F_g and the relations (3.3), (3.10) imply

$$|x_1^R \to x_1, \dots, x_k^R \to x_k|_p = \left(\max_{i,j=1,\dots,k} |x_i^R \to x_i, x_j^R \to x_j|_p \right)^{\delta_{x_1^R, x_2^R} \dots \delta_{x_1^R, x_k^R}}. \tag{3.11}$$

We call the boundary ∂K of the graph K the set of all vertices from $K \subset F_g$ which have among the nearest neighbours the vertices from $F_g \backslash K$. Let $f_1(x^R \to x), \dots, f_{2N}(x^R \to x)$ be the positive continuous functions summable with the measure $d\mu_0(x^R \to x)$ on ∂F_g defined by the relation (3.2). We introduce the average

$$A_{2N}(f_1, \dots, f_{2N}) = \lim_{\substack{K_j \to F_g \\ j=1,\dots,2N}} \sum_{\{x_j \in \partial K_j\}} W\left(e^{i\pi} \sum_{j=1}^{2N} \delta_{x_j}, \cdot \right) \prod_{j=1}^{2N} \bar{f}_j(x_j), \tag{3.12}$$

where

$$\bar{f}_j(x_j) = (\mu_0(\partial B_{x_j}))^{-1} \int_{\partial B_{x_j}} d\mu_0(x^R \to x) f_j(x^R \to x). \tag{3.13}$$

If all graphs $K_j = K$ and all functions $f_j(x^R \to x) = 1$ the average $A_{2N}(1, \dots, 1)$ is the straightforward analogue of the p-adic string amplitude [8, 9].

We find the limit (3.12) for the special sequence of graphs $\{K_1\}$ such that

$$\partial K_l = \{x \in F_g | \langle \kappa_{x^R, x} \rangle = l,\ x \in B_{x^R},\ x^R \in F_g^R,\ b(x^R) > 0\}.$$

We suppose also that the supports of the functions $f_i(x^R \to x)$ and $f_j(x^R \to x)$ don't intersect for $i \neq j$.

For $g = 0$ the relations (2.11), (2.12), (3.2) and (3.12), (3.13) imply

$$\begin{aligned} A_{2N}(f_1, \dots, f_{2N}) = &\lim_{l \to \infty} \exp\{2Nl(\ln p - 2\beta^*)\} \\ &\times \sum_{\{x_j \in \partial K_l\}} \left(\prod_{j=1}^{2N} \int_{\partial B_{x_j}} d\mu_0(c \to x) f_j(c \to x) \right) \\ &\times \exp\left\{ -2\beta^* \sum_{m=2}^{2N} (-2)^{m-1} \sum_{k_1 < \dots < k_m}^{2N} \langle \kappa_{c, x_{k_1}}, \dots, \kappa_{c, x_{k_m}} \rangle \right\}. \end{aligned} \tag{3.14}$$

Since the supports of the functions $f_i(x^R \to x)$ and $f_j(x^R \to x)$ don't intersect for $i \neq j$ the last sum in (3.14) absolutely converges as $l \to \infty$ to

$$\int_{P^1(\mathbb{Q}_p)^{\times 2N}} f_1(x_1) d\mu_0(x_1) \dots f_{2N}(x_{2N}) d\mu_0(x_{2N}) \times \prod_{m=2}^{2N} \prod_{k_1 < \dots < k_m}^{2N} \left(\max_{i,j=1,\dots,m} |x_{k_i}, x_{k_j}|_p \right)^{(-2)^{m-1} 2\beta^* (\ln p)^{-1}}. \tag{3.15}$$

Here we use the relations (3.3), (3.10), (3.11) and we denote by $d\mu_0(x)$ the measure given by the right-hand sides of the relations (3.4). The distance $|x, y|_p$ is given by the right-hand sides of the relations (3.5). We use also the correspondence $\partial F_0 \cong P^1(\mathbb{Q}_p)$ [4, 5] and replace the functions $f_j(c \to x)$ simply by $f_j(x)$.

It follows now from (3.14) that

$$A_{2N}(f_1, \dots, f_{2N}) = \begin{cases} 0, & 2\beta^* > \ln p \\ \infty, & 2\beta^* < \ln p \end{cases}. \tag{3.16}$$

If

$$2\beta^* = \ln p \tag{3.17}$$

inserting (3.15) into the right-hand side of (3.14) we obtain the non-trivial limit

$$A_{2N}(f_1, \dots, f_{2N}) = \int_{P^1(\mathbb{Q}_p)^{\times 2N}} f_1(x_1) d\mu_0(x_1) \dots f_{2N}(x_{2N}) d\mu_0(x_{2N}) \times \prod_{m=2}^{2N} \prod_{k_1 < \dots < k_m}^{2N} \left(\max_{i,j=1,\dots,m} |x_{k_i}, x_{k_j}|_p \right)^{(-2)^{m-1}}. \tag{3.18}$$

Here we considered the simplest case when the supports of the different functions $f_i(x)$ on $P^1(\mathbb{Q}_p)$ don't intersect. In order to extend the formula (3.18) to the general case it is necessary to study the convergence of the integral (3.18).

Let us consider the generalized Bruhat-Tits tree F_g with $g > 0$. By using the relations (2.11), (2.12), the decomposition (3.1) and the formulas (3.7)–(3.11) we obtain the relation (3.16). If β^* satisfies the condition (3.17) we have the non-trivial limit

$$A_{2N}(f_1, \dots, f_{2N}) = \sum_{\substack{\{x_i^R\} \in F_g^R \\ b(x^R) > 0}} W\left(e^{i\pi \sum_{j=1}^{2N} \delta_{x_j^R, \cdot}} \right) \times \int_{\mathbb{Z}_p[a_0 < b(x_1^R)]} f_1(x_1^R \to x_1) dx_1 \dots \int_{\mathbb{Z}_p[a_0 < b(x_{2N}^R)]} f_{2N}(x_{2N}^R \to x_{2N}) dx_{2N} \times \prod_{m=2}^{2N} \prod_{k_1 < \dots < k_m}^{2N} \left(\max_{i,j=1,\dots,m} |x_{k_i} - x_{k_j}|_p \right)^{(-2)^{m-1} \delta_{x_{k_1}^R, x_{k_2}^R} \dots \delta_{x_{k_1}^R, x_{k_m}^R}}, \tag{3.19}$$

where the correlation function

$$W\left(e^{i\pi \sum_{j=1}^{2N} \delta_{x_j^R, \cdot}} \right)$$

is given by the equations (2.11), (2.12) with the omitted multiplier $(\mathrm{ch}\,\beta)^{N_1}$. The proof of the formula (3.19) is exactly analogous to the case $g = 0$. To extend the formula (3.19) to the case when the supports of the functions $f_1(x^R \to x)$ intersect one needs to study the convergence of the integral (3.19).

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