Commun. Math. Phys. 130, 111–121 (1990)

Communications in
Mathematical
Physics

Large Deviation Estimates in the Stochastic Quantization of φ_2^4

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Abstract. We study in the small noise limit the behaviour of field trajectories for the process constructed by the authors in connection with the stochastic quantization of φ_2^4. Due to the presence of infinite renormalization the usual large deviation techniques do not apply immediately and a new strategy has to be developed. We prove some estimates analogous to the Freidlin–Ventzel inequalities. From these it follows that the field trajectories suitably smeared in space over a scale r_0 behave, when the noise is small, as the projection on the same scale of a field obeying a regularized stochastic equation with a large cut-off. However the estimates are not uniform in the cut-off and an interesting feature of the problem is that the scale over which the field is smeared determines whether the noise is sufficiently small for the estimates to apply.

0. Preliminaries

There exists a well developed theory of small random perturbations of dynamical systems evolving in R^n or on some finite dimensional manifold. This goes under the name of Freidlin–Ventzel theory[1] as these authors developed several basic ideas in this domain. Their fundamental estimates turned out to be equivalent to large fluctuation results of Varadhan[7]. The Freidlin–Ventzel approach was then extended to stochastic nonlinear partial differential equations of parabolic type in one space dimension besides time[2]. This extension of the F-V estimates follows from a careful but otherwise straightforward adaptation of the arguments developed for the finite dimensional case. The situation is entirely different if the number of space dimensions D is greater than 1. The prototype of equations we want to consider is

$$\frac{\partial \varphi}{\partial \tau} = \Delta\varphi - \varphi - V'(\varphi) + \varepsilon\frac{\partial W}{\partial \tau}, \tag{0.1}$$

where $\partial W/\partial \tau$ is a white noise in all variables. $V(\varphi)$ is an even polynomial in φ. These equations may be called stochastic Landau–Ginzburg equations. For $D \geqq 2$

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an equation like (0.1) is meaningless because the white noise makes the non-linear term $V'(\varphi)$ meaningless. In view of the connection of such equations with Quantum Field Theory known as stochastic quantization, it is natural to try a procedure of renormalization which consists in modifying the non-linear part in such a way that it becomes a good stochastic variable. However this introduces infinite subtractions so that even the modified equation cannot be taken literally. In ref. [3] the following equation closely related to (0.1) was studied (for $\varepsilon = 1$. The ε of this paper is not to be confused with the parameter ε of [3]. The ε of [3] is called ρ in this paper.)

$$d\varphi = -\frac{1}{2}((-\Delta+1)^{\rho}\varphi + \lambda(-\Delta+1)^{-1+\rho}:\varphi^3:)dt + \varepsilon dW,$$
$$E(W(t,x)W(t',x)) = \min(t,t')(-\Delta+1)^{-1+\rho}(x,x'). \tag{0.2}$$

Δ is the Laplacian in $D=2$ and ρ is a sufficiently small positive number, in [I] $\rho<(1/10)$ [4]. The space variable is restricted to a compact $\Lambda \subset R^2$. $:\varphi^3:$ is the Wick product defined by

$$\begin{aligned}:\varphi^3: &= \varphi^3 - 3\varepsilon^2(-\Delta+1)^{-1}(x,x)\varphi \\ &= \varphi^3 - 3\varepsilon^2 C(x,x)\varphi \\ C(x,y) &= (-\Delta+1)^{-1}(x,y).\end{aligned} \tag{0.3}$$

Since $C(x,x)$ is infinite (0.2) in not a stochastic differential equation in a strong sense. It can be given a meaning in the following way: we first write it as a formal integral equation

$$\varphi_t = Z_t - \frac{\lambda}{2}\int_0^t ds e^{-1/2(t-s)C^{-\rho}} C^{1-\rho}:\varphi_s^3:, \tag{0.4}$$

where Z_t is the distribution valued Gaussian process solution of the linear equation

$$dZ_t = -\tfrac{1}{2}C^{-\rho}Z_t dt + \varepsilon dW_t. \tag{0.5}$$

Then it can be shown that a process φ_t can be constructed such that

$$\hat{Z}_t = \varphi_t + \frac{\lambda}{2}\int_0^t ds e^{-1/2(t-s)C^{-\rho}} C^{1-\rho}:\varphi_s^3: \tag{0.6}$$

has the same probability distributions as Z_t. This is what probabilists call a weak solution. To construct φ_t one regularizes (0.4) by substituting Z_t with Z_{tN}, its projection on the first N vectors of an appropriate orthogonal basis and correspondingly C with C_N everywhere including the Wick product (0.3). The regularized equation has a strong solution φ_{tN} and defines a Markovian semigroup

$$E_{\varphi_{0N}}(f(\varphi_{tN})) = E_{\varphi_{0N}}(f(Z_{tN})e^{\xi_{tN}}), \tag{0.7}$$

where ξ_{tN} is the Girsanov exponent

$$\xi_{tN} = -\frac{\lambda}{2\varepsilon}\int_0^t(:Z_{sN}^3:, dW_{sN}) - \frac{\lambda^2}{8\varepsilon^2}\int_0^t ds(:Z_{sN}^3:, C_N^{1-\rho}:Z_{sN}^3:) \tag{0.8}$$

where the scalar products are in $L^2(\Lambda)$.

Using the methods of constructive field theory it was shown in [I] that there exists a stochastic variable ξ_{t_∞} such that

$$E_{\varphi 0}(e^{\xi_{t\infty}}) = 1 \quad \mu_{C_\varepsilon} - \text{a.e.} \quad \text{in } \varphi_0, \tag{0.9}$$

where μ_{C_r} in the gaussian measure of covariance $C_\varepsilon = \varepsilon^2(-\Delta + 1)^{-1} = \varepsilon^2 C$ on the dual Sobolev space H_{-1}. Therefore φ_t is defined by the equation

$$E_{\varphi 0}(f(\varphi_t)) = E_{\varphi 0}(f(Z_t)e^{\xi_{t\infty}}). \tag{0.10}$$

f is in $L_2(d\mu)$ where $d\mu$ is the equilibrium measure

$$d\mu = d\mu_{C_\varepsilon} \exp\left(- \lambda/4\varepsilon^2 \int_\Lambda d^2x :\phi^4: \right). \tag{0.11}$$

:: is Wick product with respect to the covariance C_ε as in (0.3).

In [I] for Λ we chose a square and Dirichlet boundary conditions were imposed. In this paper we want to make a start in the study of the large deviation problem for the distribution valued field φ_t. This consists in studying the behaviour of typical trajectories for φ_t when $\varepsilon \to 0$. As a first step in this direction we establish some analogs of the well known F–V estimates for the field φ_t. This is a non-trivial problem. One must first realize that due to the circumstance that φ is a distribution one can ask meaningful questions only on some regularized version of it, e.g. $\varphi_g = \varphi_t * g$, where φ_g is the convolution with an appropriate test function in the space variables. But the really new problem is given by the fact that φ_t is not a strong solution of a stochastic differential equation which is the only case for which a theory of small stochastic perturbations exists. Therefore one has to devise new methods to cope with the problem. The approach we shall follow here consists in showing that the probabilities we are interested in can be approximated by corresponding probabilities for a truncated process φ_{tN} provided N is "sufficiently large." However N must be finite because we cannot expect uniformity in N in the estimates. Therefore N must be chosen in some way that is naturally suggested by the problem. A relevant thing is that the main object in this game, the action functional for large fluctuations, for sufficiently small ε does not depend on the renormalization counter term appearing in the Wick product (0.3). To simplify the discussion instead of studying the trajectories of φ_{tg} for arbitrary g we shall consider $(\varphi_t)_M$ the projection of φ_t on the first M vectors of the orthonormal basis. In the following to simplify the notation we shall denote this projection φ_{tM}. No confusion should arise with the truncated process φ_{tN} as N will always denote the cut-off.

We take this opportunity to correct in an appendix to this paper some errors in [I] which however do not affect in the least its conclusions.

1. The First F–V Estimate

We want an answer to the following question: what is the probability that the projection φ_{tM} of our field on the first M eigenvectors of the operator $(-\Delta + 1)$ with Dirichlet boundary conditions on $\partial\Lambda$ be close in the time interval $[0, T]$ to a preassigned function $f_t(x)$ which also lies in the same subspace and is continuous in t?

The initial condition for f_t is that $\varphi_{0M} = f_0(x)$, i.e. it coincides with the projection of the initial condition φ_0. For the distance between φ_{tM} and f_t we can take the sup norm $\rho(\varphi_M, f) = \sup_{\substack{t\in[0,T]\\ x\in\Lambda}} |\varphi_{tM}(x) - f_t(x)|$. We try to estimate from below, when $\varepsilon \to 0$,

$$P_{\varphi 0}(\rho(\varphi_M, f) < \delta)$$

which exists μ_{C_ε} – a.e. in φ_0. δ is an arbitrary positive number. Using (0.10) the above probability can be written

$$\begin{aligned} P_{\varphi 0}(\rho(\varphi_M, f) < \delta) &= E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta) e^{\xi_{T\infty}}) \\ &= E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta)\chi(|\xi_{T\infty} - \xi_{TN}| < a) e^{\xi_{T\infty}}) \\ &\quad + E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta)\chi(|\xi_{T\infty} - \xi_{TN}| > a) e^{\xi_{T\infty}}), \end{aligned} \tag{1.1}$$

where N and a are for the moment arbitrary but $N > M$ and the latter is fixed. ξ_{tN} is given by (0.8). χ is the indicator function of the event in the argument.

From (1.1) we have

$$\begin{aligned} P_{\varphi 0}(\rho(\varphi_M, f) < \delta) &\geqq E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta)\chi(|\xi_{T\infty} - \xi_{TN}| < a) e^{\xi_{T\infty}}) \\ &\geqq E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta)\chi(|\xi_{T\infty} - \xi_{TN}| < a) e^{\xi_{TN}}) e^{-a}. \\ &= E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta) e^{\xi_{TN}}) e^{-a} \\ &\quad - E_{\varphi 0}(\chi(\rho(Z_M, f) < \delta)\chi(|\xi_{T\infty} - \xi_{TN}| > a) e^{\xi_{TN}}) e^{-a}. \\ &\geqq E_{\varphi 0N}(\chi(\rho(Z_M, f) < \delta) e^{\xi_{TN}}) e^{-a} \\ &\quad - E_{\varphi 0}(\chi(|\xi_{T\infty} - \xi_{TN}| > a) e^{\xi_{TN}}) e^{-a}. \end{aligned} \tag{1.2}$$

In the first term we substituted φ_0 with φ_{0N} as the other components do not evolve any more. The first term now can be estimated with the usual F–V techniques (since it involves the truncated process φ_{tN} (see (0.7)) which is a strong solution of the cut-off stochastic differential equation) and the question is whether *we can choose a and N in such a way that the second term can be made smaller than the first.* It is clear that to make the second term small it is convenient to take both N and a large. However we have to be careful that the factor e^{-a} does not spoil the estimate of the first term. Furthermore, since the F–V estimate of the first term will not be uniform in N we cannot take N too large.

Let us choose $a = h_1/\varepsilon^2$, where h_1 is a small positive number. Given then $h_2 > 0$ the F–V theory gives for the first term for fixed N and ε sufficiently small,

$$E_{\varphi 0N}(\chi(\rho(Z_M, f) < \delta) e^{\xi_{TN}}) \geqq \exp\left[-\frac{I^{*N}(f) + h_2}{\varepsilon^2}\right], \tag{1.3}$$

where $I^{*N}(f)$ is defined in the following way. Let

$$I^N(\bar{f}) = \frac{1}{2}\int_0^T dt \left\| \left(\frac{\partial \bar{f}}{\partial t} C_N^{-1+\rho} - \frac{1}{2}\Delta \bar{f} + \frac{1}{2}\bar{f} + \frac{1}{2}\lambda \bar{f}^3 \right) C_N^{(1-\rho)/2} \right\|^2. \tag{1.4}$$

Then

$$I^{*N}(f) = \inf I^N(\bar{f}), \quad \bar{f}_t : \bar{f}_{tM} = f_t, \quad \bar{f}_0 = \varphi_{0N}. \tag{1.4'}$$

$\bar{f}_t$ has N modes in its representation. We remark that the Wick product for ε

sufficiently small does not affect $I^N(\bar{f})$ since $:\bar{f}^3: = \bar{f}^3 - 3\varepsilon^2 C_N(x,x)\bar{f}$ is finite and $:\bar{f}^3: \underset{\varepsilon \to 0}{\longrightarrow} \bar{f}^3$. In conclusion the first contribution on the right-hand side of (1.2) is larger than

$$\exp\left[-\frac{I^{*N}(f)+h}{\varepsilon^2} \right] \tag{1.5}$$

with $h = h_1 + h_2$ for $\varepsilon < \varepsilon_0(N, \delta, h)$.

We now estimate the second term. By Schwartz inequality

$$E_{\varphi 0}(\chi(|\xi_{T_\infty} - \xi_{T_N}| > a)e^{\xi_{T_N}}) \leqq (E_{\varphi 0}(e^{2\xi_{T_N}}))^{1/2} \cdot (E_{\varphi 0}(\chi(|\xi_{T_\infty} - \xi_{T_N}| > a))^{1/2}.$$

By adapting (3.11) and (A.11) of [I] the first factor is bounded for large N by

$$\exp\left\{ \frac{\lambda}{8\varepsilon^2} \left| \int_\Lambda d^2x :\varphi_{0N}^4: \right| + \text{const}\, \varepsilon^2 O(\lambda_N^{2\rho} \ln^2 \lambda_N) \right\},$$

where λ_N is the eigenvalue corresponding to the highest mode in the development of Z_{tN}. The second factor is the interesting one. By Chebysheff inequality

$$E_{\varphi 0}(\chi(|\xi_{T_\infty} - \xi_{TN}| > a)) \leqq \frac{E(|\xi_{T_\infty} - \xi_{TN}|^{2j})}{a^{2j}}. \tag{1.6}$$

The difference $\xi_{T_\infty} - \xi_{TN}$ after performing the stochastic integral is of the form (see for example (0.18) of [I])

$$-\frac{\lambda}{2\varepsilon^2}(A_{T_\infty} - A_{TN}) - \frac{\lambda^2}{8\varepsilon^2}(B_{T_\infty} - B_{TN}), \tag{1.7}$$

where

$$A_{T\infty} - A_{TN} = \frac{1}{4}\int_\Lambda d^2x(:Z_T^4(x): - :Z_{TN}^4(x):) - \frac{1}{4}\int_\Lambda d^2x(:\varphi_0^4(x): - :\varphi_{0N}^4(x):) \tag{1.8a}$$

$$+ \frac{1}{2}\int_0^T ds[:(Z_s^3, C^{-\rho}Z_s): - :(Z_{sN}^3, C_N^{-\rho}Z_{sN}):],$$

$$B_{T_\infty} - B_{TN} = \int_0^T ds[(:Z_s^3:, C^{1-\rho}:Z_s^3:) - (:Z_{Ns}^3:, C_N^{1-\rho}:Z_{Ns}^3:)]. \tag{1.8b}$$

$C_N^\alpha(x,y)$ means as usual

$$C_N^\alpha(x,y) = \sum_{k=1}^N \lambda_k^{-\alpha} \Phi_k(x)\Phi_k(y),$$

where the λ_k are the eigenvalues of $-\Delta + 1$ in Λ. Φ_k are the corresponding eigenfunctions.

$$C_N^{-1} C_N = P_N,$$

where P_N is the projection on the subspace of the first N vectors of the basis.

Let us introduce the notation

$$I_1=\int_\Lambda d^2y(:\varphi_0^4:-:\varphi_{0N}^4:),\quad I_2=\int_\Lambda d^2y(:Z_T^4:-:Z_{TN}^4:),$$

$$I_3=\int_0^T ds[:(Z_s^3,C^{-\rho}Z_s):-:(Z_{sN}^3,C_N^{-\rho}Z_{sN}):),$$

$$I_4=\int_0^T ds[(:Z_s^3:,C^{1-\rho}:Z_s^3:)-(:Z_{Ns}^3:C_N^{1-\rho}:Z_{Ns}^3:)]. \tag{1.9}$$

We have then

$$E_{\varphi 0}|\xi_{T\infty}-\xi_{TN}|^{2j}\leqq 2^{2j-1}\left\{\left(\frac{\lambda}{2\varepsilon^2}\right)^{2j}[\tfrac{1}{2}I_1^{2j}+\tfrac{1}{2}E_{\varphi 0}I_2^{2j}+E_{\varphi 0}I_3^{2j}]\right.$$
$$\left.+\left(\frac{\lambda^2}{8\varepsilon^2}\right)^{2j}E_{\varphi 0}I_4^{2j}\right\}. \tag{1.10}$$

We have now the following lemma.

Lemma 1. *There exists a Borel set $B_N\subset H_{-1}(\Lambda)$ with $\mu_{C_\varepsilon}(B_N)\geqq 1-(4/\lambda_N)$ such that for $\varphi_0\in B_N$,*

1. $I_1^{2j}\leqq c^j(j!)^4(\ln\lambda_N)^{mj}\lambda_N^{-(j-1)}(\varepsilon^2)^{4j}$,
2. $E_{\varphi 0}I_2^{2j}\leqq c^j(j!)^4(\ln\lambda_N)^{mj}\lambda_N^{-(j-1)}(\varepsilon^2)^{4j}$, (1.11)
3. $E_{\varphi 0}I_3^{2j}\leqq c_T^j(j!)^4(\ln\lambda_N)^{mj}\lambda_N^{-(1-2\rho)j+1}(\varepsilon^2)^{4j}$,
4. $E_{\varphi 0}I_4^{2j}\leqq c_T^j(j!)^6(\ln\lambda_N)^{mj}\lambda_N^{-(2-2\rho)j+1}(\varepsilon^2)^{6j}$.

c and c_T are appropriate constants.

Proof. The proof is similar for the cases 1–4 and we illustrate it for the case 3). First note

$$E_{\varphi 0}I_3^{2j}\leqq T^{2j-1}\int_0^T dsE_{\varphi 0}(:(Z_s^3,C^{-\rho}Z_s):-:(Z_{sN}^3,C_N^{-\rho}Z_{sN}):)^{2j}. \tag{1.12}$$

Using the fact that μ_{c_ε} is the invariant measure for the OU process we have

$$\int d\mu_{c_\varepsilon}(\varphi)E_\varphi(I_3^{2j})\leqq T^{2j-1}\int d\mu_{c_\varepsilon}(\varphi)(:(\varphi^3,C^{-\rho}\varphi):-:(\varphi_N^3,C^{-\rho}\varphi_N):)^{2j}$$
$$\leqq C_T^j(j!)^4(\ln\lambda)^{mj}\lambda^{-(1-2\rho)j}(\varepsilon^2)^{4j}. \tag{1.13}$$

In obtaining the last inequality we have used straightforward Feynman graph estimates. Now define $C_{jN\varepsilon}$ the expression in the second line of (1.13). By the Chebysheff inequality,

$$\mu_{c_\varepsilon}\{E_\varphi I_3^{2j}>\lambda_N C_{jN_\varepsilon}\}\leqq\frac{1}{\lambda_N C_{jN_\varepsilon}}\int d\mu_{c_\varepsilon}(\varphi)E_\varphi(I_3)^{2j}\leqq\lambda_N^{-1}. \tag{1.14}$$

This proves 3. The other cases are proved similary. By optimizing with respect to j from (1.6) and the lemma just proved we obtain that for $\varphi_0 \in B_N$,

$$P_{\varphi 0}\left(|\xi_{T_\infty} - \xi_{TN}| > \frac{h_1}{\varepsilon^2}\right) \leqq \exp\left\{-\frac{C}{\varepsilon^2}\left(\frac{h_1^2}{g(N)}\right)^{1/4}\right\} \tag{1.15}$$

for an appropriate choice of C and of the function $g(N)$ with $g(N) \xrightarrow[N\to\infty]{} 0$.

We now have to discuss whether (1.15) can be smaller than (1.5) for appropriate choices of all the parameters involved. We have to be careful in order to avoid circularity of the argument. Let us analyze first the conditions for the validity of (1.5). We assume N given once for all and φ_{0N} satisfying

$$\sup_{x\in\Lambda} |\varphi_{0N}| < D. \tag{1.16}$$

Furthermore we assume

$$I^{*N}(f) < K, \tag{1.17}$$

which amounts to a restriction on f. Then

$$E_{\varphi_{0N}}(\chi(\rho(Z_M, f) < \delta) e^{\xi_{TN}}) e^{-h_1/\varepsilon^2} \geqq e^{-(I^{*N}(f)+h)/\varepsilon^2} \tag{1.18}$$

for $0 < \varepsilon < \varepsilon_0\ (N, D, K)$. We omit the obvious dependence of ε_0 on δ and h. We choose ε in $(0, \varepsilon_0)$ and we keep it fixed in the following. We now have

$$E_{\varphi 0}\left(\chi\left(|\xi_{T_\infty} - \xi_{TN}| > \frac{h_1}{\varepsilon^2}\right) e^{\xi_{TN}}\right) e^{-h_1/\varepsilon^2}$$

$$\leqq \exp\left\{\frac{\lambda}{8\varepsilon^2} |\int d^2x :\varphi_{0N}^4:| + \text{const}\, \varepsilon^2 O(\lambda_N^{2\rho} (\ln \lambda_N)^2) - \frac{h_1}{\varepsilon^2} - \frac{C}{\varepsilon^2}\left(\frac{h_1^2}{g(N)}\right)^{1/4}\right\}. \tag{1.19}$$

We now note that

$$|\int :\varphi_{0N}^4: d^2x| \leqq |\Lambda| (D^4 + \varepsilon^2 D^2 O(\ln \lambda_N) + \varepsilon^4 O(\ln^2 \lambda_N)). \tag{1.20}$$

Then if

$$K < C\left(\frac{h_1^2}{g(N)}\right)^{1/4} - \frac{\lambda}{8} |\Lambda| D^4 - H(\varepsilon, N), \tag{1.21}$$

where $H(\varepsilon, N)$ is an appropriate constant (1.19) will be exponentially smaller (in $1/\varepsilon^2$) than (1.18). It is clear that if for some N and ε (1.21) is not satisfied we can always increase N and decrease ε in such a way that its validity is implemented. Therefore we have proved

Theorem 1.1. *Given h_1, h, δ, N, D, K satisfying (1.16), (1.17), (1.21) with $\varepsilon < \varepsilon_0(N, D, K)$,*

$$P_{\varphi 0}(\rho(\varphi_M, f) < \delta) \geqq e^{-(I^{*N}(f) + \bar{h})/\varepsilon^2} \tag{1.22}$$

for a set of initial conditions φ_0 of μ_{c_ε} measure greater than

$$1 - \frac{4}{\lambda_N} - \mu_{c_\varepsilon}\left(\sup_{x\in\Lambda} |\varphi_{0N}| > D\right) \tag{1.23}$$

and some $\bar{h} > h$.

It is clear that the third term in (1.23) will be exponentially small for ε small. Some remarks are in order. The content of this theorem is that once an approximating truncated process is chosen and ε is so small that large deviation estimates are applicable to it, there exists a set of initial conditions for which these estimates are good approximations for the projections of the full process. What is different from the usual F–V theory is that due to the singular character of the trajectories of the process φ, the admissible initial conditions depend on ε. In fact the measures μ_{c_ε} are not absolutely continuous one with respect to the other for different ε.

2. The Second F–V Estimate

In the previous section we have introduced the functional $I^{*N}(f)$, where f is a function continuous in t and having M modes as far as its x dependence is concerned. $I^{*N}(f)$ is not in general lower semicontinuous but, as discussed by Freidlin[5], one can always work with a lower semicontinuous version. We then define

$$\phi^{*N}(s) = \{f : I^{*N}(f) \leqq s\}, \tag{2.1}$$

where f has M modes and $I^{*N}(f)$ is given by (1.4′). We want to study now the probability of the event $\{\rho(\varphi_M, \phi^{*N}(s)) > \delta\}$. We have

$$\begin{aligned} P_{\varphi 0}(\rho(\varphi_M, \phi^{*N}(s)) > \delta) &= E_{\varphi 0}(\chi(\rho(Z_M, \phi^{*N}(s)) > \delta) e^{\xi_{T\infty}}) \\ &= E_{\varphi 0}(\chi(\rho(Z_M, \phi^{*N}(s)) > \delta)\chi(|\xi_{T\infty} - \xi_{TN}) < a) e^{\xi_{T\infty}}) \\ &\quad + E_{\varphi 0}(\chi(\rho(Z_M, \phi^{*N}(s)) > \delta)\chi(|\xi_{T\infty} - \xi_{TN}) > a) e^{\xi_{T\infty}}) \\ &\leqq E_{\varphi 0}(\chi(\rho(Z_M, \phi^{*N}(s)) > \delta) e^{\xi_{TN}}) e^{a} \\ &\quad + E_{\varphi 0}(\chi(|\xi_{T\infty} - \xi_{TN}| > a) e^{\xi_{T\infty}}) \\ &\leqq E_{\varphi 0N}(\chi(\rho(Z_M, \phi^{*N}(s)) > \delta) e^{\xi_{TN}}) e^{a} \\ &\quad + (E_{\varphi 0}(e^{2\xi_{T\infty}}))^{1/2} (E_{\varphi 0}(\chi(|\xi_{T\infty} - \xi_{TN}| > a))^{1/2}. \end{aligned} \tag{2.2}$$

The estimates now follow the same line of reasoning as in the previous section. The only difference is the estimate of $E_{\varphi 0}(e^{2\xi_{T\infty}})$ since now the process without cut-off is involved. It can be seen without difficulty that

$$(E_{\varphi 0}(e^{2\xi_{T\infty}})) \leqq e^{\lambda/4\varepsilon^2 |\int :\varphi_0^4: d^2x| + B_T|\Lambda|} \tag{2.3}$$

with $B_T > 0$ independent of φ_0 for a set of initial conditions of μ_{c_ε} measure greater than $1 - 4\lambda_N^{-1}$. Suppose now $|\int :\varphi_0^4: d^2x| \leqq R$.

Theorem 2.1. *Given* h_1, h, δ, N, D *as in Theorem 1.1,* $s < K$,

$$K < C\left(\frac{h_1^2}{g(N)}\right)^{1/4} - \frac{\lambda}{8}R - \varepsilon^2 B_T|\Lambda| \tag{2.4}$$

$\varepsilon < \varepsilon_0(N, D, K)$. *Then*

$$P_{\varphi 0}(\rho(\varphi_M, \phi^{*N}(s)) > \delta) \leqq e^{-(s-\bar{h})/\varepsilon^2} \tag{2.5}$$

for a set of initial conditions of μ_{c_ε} measure greater than

$$1 - 4\lambda_N^{-1} - \mu_{c_\varepsilon}\left(\sup_{x \in \Lambda} |\varphi_{0N}| > D\right) - \mu_{c_\varepsilon}(|\textstyle\int :\varphi_0^4: d^2x| > R)$$

and $h < \bar{h}$.

3. Concluding Remarks

If we compare out Theorems (1.1) and (2.1) with the usual F–V estimates one notices two main differences: a) there is an intrinsic ε-dependence of the admissible initial conditions; b) the estimates depend on the cut-off N of the auxiliary finite dimensional process. It is then natural to ask the following questions. Is it possible on the basis of our approach to reconstruct the main results of the F–V theory, e.g. to make estimates of the invariant measure? These require a rather detailed study of exit times and trajectories from domains containing the attractors of the unperturbed deterministic equations. From our point of view one should work with the deterministic part of the truncated equation which depends on N and ε in a nontrivial way.

The second question which spontaneously arises is whether we can do better, that is whether we can eliminate the explicit dependence on the cut-off of the auxiliary process. After all it seems reasonable to expect that the ε_0 below which large deviation estimates apply depends in the end only on the scale r_0 (that is M) over which we smear the field.

We also remark that in our discussion the coupling constant λ was kept fixed (with respect to ε). As it is easily seen this means that we stay out of the phase transition region where the stochastic equation acquires two effective equilibrium states. It seems possible to treat also this case, which is physically most interesting, by introducing suitably rescaled variables and then applying our methods.

Progress in all these directions would be relevant not only in connection with stochastic quantization. In fact our Landau–Ginzburg equations appear in a great variety of problems which we may roughly describe as stochastic hydrodynamics. In particular the scale dependence of large fluctuations is an interesting new phenomenon relevant for physics as emphasized in [6] where also an extensive qualitative and numerical analysis of a stochastic Landau–Ginzburg equation was carried out.

Appendix

Erratum to the authors' paper "On the stochastic quantization of Field theory" Commun. Math. Phys. **101**, 409 (1985), to which we refer for notations.

1. *page 416 "Remark on Stochastic integral in (2.7)".*
The statement that the integral in (2.8) is a "martingale" is clearly in error as the integrand has a t-dependence (pointed out to us by S. K. Mitter whom we thank). Fortunately the martingale property is unnecessary. Equations (2.9–2.12) which remain valid (but not (2.13)) together with the easily obtained estimate for

$0 < s < t < T$,

$$E_0^{(w)}(\| I_t - I_s \|_E^4) \leqq C_T |t-s|^2 \tag{2.13}$$

replacing (2.13), show that I_t defined in the Ito sense, is a continuous Gaussian process in E. We also remark that (2.11) written as

$$I_t^{(N)} = \sum_{n=1}^{N} e^{-1/2t\lambda_n^\varepsilon} \int_0^t \lambda_n^{-1/2(1-\varepsilon)} e^{1/2s\lambda_n^\varepsilon} d\beta_s^{(n)} e_n$$

is a sum of Ito stochastic integrals (martingales) with deterministic coefficients converging to I_t (2.12), but neither $I_t^{(N)}$ nor the limit I_t are martingales.

2. *page 417*. Replace the first sentence of paragraph containing (2.17), "The stochastic integral $\cdots(1/\lambda_n)(1-e^{-\lambda_n^\varepsilon t})$" as follows:

The stochastic integral on the right-hand side of (2.16) can be written as

$$e^{-1/2t\lambda_n^\varepsilon} I_{tn},$$

where I_{tn} can be identified with Brownian motion $\beta_{\sigma_n(t)}$ with time change $t \to \sigma_n(t) = \lambda_n^{-1}(e^{t\lambda_n^\varepsilon} - 1)$, [13]. It is now trivial to check that the above Gaussian process has the same transition probability (1-dimensional distribution) as Brownian motion with time change

$$t \to \tau_n(t) = \lambda_n^{-1}(1 - e^{-t\lambda_n^\varepsilon}).$$

3. *page 433*. There is a typographical error in A. 15 which should read:

$$\mu_c\{M \leqq -\text{const}\,(K^{4\varepsilon}(\ln K)^2) - 1\} \leqq e^{-\text{const}\,K^{(2-4\varepsilon)/4}(\ln K)^{-m/4}}.$$

4. *page 434*, *Lemma 4* as stated is incorrect. It can be replaced by a somewhat weaker statement similar to Lemma 1 of Sect. 1 of the present paper. However this would allow to prove Proposition 2 of the appendix of [I], for a set of initial conditions of μ_c measure $\geqq 1-\alpha, \alpha$ arbitrary but strictly larger than zero. In other words the bound asserted by the proposition would not be uniform in α. To obtain the bound μ_c a.e. we can follow a different and simpler approach which allows to reduce A2 to A1 of [I]. Using the notation of paper [I], we have to prove that

$$E_\varphi\left(\exp\left\{-\int_0^t ds M(\varphi_s)\right\}\right) < \infty, \quad \mu_c - \text{a.e. in } \varphi. \tag{A.1}$$

Using Riemann sum approximations for the s-integral above, Fatou's Lemma and that μ_c is the invariant measure of the OU process it follows by a standard calculation that

$$\int d\mu_c E_\varphi\left(\exp\left\{-\int_0^t ds M(\varphi_s)\right\}\right) \leqq \int d\mu_c e^{-\lambda t M(\varphi)} < \infty. \tag{A.2}$$

By applying now the Fubini theorem from the joint measurability of the integral on the left-hand side with respect to φ and the OU process, we obtain (A.1). In this way all the results of [I] on the existence of the weak stochastic dynamics are true μ_c a.e. in the initial condition. This approach was given in P. K. Mitter

in "New perspectives in quantum field theory," M. Asorey et al. (eds.), Singapore: World Scientific 1986, pp. 181–307.

Acknowledgements. We would like to express our gratitude to the Laboratoire de Physique Thèorique et Hautes Energies, Université de Paris VI, the Department of Physics of Università "La Sapienza"—Roma, and the INFN—Sezisne di Roma for the support received during our collaboration.

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Communicated by G. Parisi

Received July 15, 1989; in revised form December 15, 1989