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Universal Schwinger Cocycles of Current Algebras in $(D+1)$-Dimensions: Geometry and Physics

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Abstract. We discuss the universal version of the Schwinger terms of current algebra (we call it the universal Schwinger cocycle) for $p=3$ (here p denotes the class of the Schatten ideal I_p, which is related to the $(D+1)$ space-time dimensions by $p=(D+1)/2$) in detail, and give a conjecture of the general form of the cocycle for any p. We also discuss the infinite charge renormalizations, the highest weight vector and state vectors for $p=3$. Last, we give brief comments on the problems caused by the difficulties to construct the measure of infinite-dimensional Grassmann manifolds.

1. Introduction

In particle physics, current algebra has been introduced in the study of strong interactions. It was assumed that the time-component of a current generates a closed algebra in the classical level. More explicitly, we consider a Dirac field in $D+1$-dimension coupled to an external Yang–Mills field A. Let G be a compact semi-simple Lie group and $\underline{g}$ its algebra. The current is

$$J^i(x) = \Psi^\dagger(x)\lambda^i\Psi(x). \tag{1}$$

We define

$$J(f) \overset{\text{def}}{=} \int dx f^i(x) J^i(x), \tag{2}$$

where $f(x) = f^i(x)\lambda^i : X \to \underline{g}$, is a mapping valued in the Lie algebra.

This operator satisfies

$$[J(f), J(g)] = J([f,g]). \tag{3}$$

But, in the quantum level, this relation is modified as follows:

$$[J(f), J(g)] = J([f,g]) + c(f,g;A). \tag{4}$$

This $v(f,g;A)$ is called the Schwinger term [F]. This requires the representations of the Abelian extension $\widehat{\mathrm{Map}(X;\underline{g})}$ of $\mathrm{Map}(X;\underline{g})$,

$$(0 \to \mathrm{Map}(A;C) \to \widehat{\mathrm{Map}(X;\underline{g})} \to \mathrm{Map}(X;\underline{g}) \to 0).$$

In the present paper, we construct universal objects from the point of K-theory; Gr_p, Det_p, Det_p^*, etc., for making the geometrical meaning of Schwinger terms and abelian extension. We discuss the two-cycle (universal Schwinger cocycle) for $p = 3$ (the suffix p denotes a suitable class of the Schatten ideal, which is related to the $(D + 1)$ space-time dimensions by $p = (D + 1)/2$), the two-cocyle for general p, the infinite charge renormalizations, and the highest weight vector. We also give brief comments for the measure of infinite-dimensional Grassmann manifolds.

Our results are the generalization of the work by Mickelsson and Rajeev [MR].

2. Embedding of Map$(X; G)$ into the Infinite-Dimensional Group

Let X be the D-dimensional compact spin manifold, (for example, the D-dimensional torus), and define

$$\mathrm{Map}(X: G) = \{g: X \to G, \text{ smooth maps}\}, \tag{5}$$

then this space becomes a group by pointwise multiplication.

In the previous section, we considered the current algebra in the level of a Lie algebra. But, for our purpose, it is better to consider the same things in the level of a Lie group. Then, instead of Map$(X; G)$, we treat a larger group which acts on a Hilbert space.

Now we consider the Hilbert space H consisting of free fermion fields Ψ carrying a unitary representation ρ of G.

Since a Dirac operator D on X has discrete eigenvalues, let H_+ be the space of the eigenstates with positive eigenvalues of the operator D, and H_- the space of the eigenstates with its non-positive eigenvalues. Let the basis of each eigenspace of D be

$$\{e_1, e_2, \dots\}: \text{orthogonal basis of } H_+, \tag{6}$$

$$\{e_0, e_{-1}, \dots\}: \text{orthogonal basis of } H_-. \tag{7}$$

Then $H = H_+ \oplus H_-$. We define the sign operator ε,

$$\varepsilon \stackrel{\text{def}}{=} \frac{D}{|D|}. \tag{8}$$

(If D has a zero eigenvalue, we set $\varepsilon = -1$.)

We define the operator $M(f): H \to H$ such that

$$[M(f)\Psi](x) \stackrel{\text{def.}}{=} \rho(f(x))\Psi(x). \tag{9}$$

$M(f)$ is decomposed as follows,

$$M(f) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{10}$$

where

$$\begin{cases} a: H_+ \to H_+, & d: H_- \to H_- \\ b: H_- \to H_+, & c: H_+ \to H_- \end{cases} \tag{11}$$

We define the Schatten Ideal

$$I_{2p}=\{A\in B(H)|\,\|A\|_{2p}=[\operatorname{tr}(A^{\dagger}A)^{p}]^{1/2p}<\infty\},\tag{12}$$

where $B(H)$ is the space of all bounded operators on H [S], [C].

We also define

$$GL_p\overset{\text{def}}{=}\{A\in GL(H)|\operatorname{tr}[\varepsilon,A]^{2p}<\infty]\quad(p=1,2,\ldots),\tag{13}$$

where $GL(H)$ is the set of all invertible operators on H.

We note each $g\in GL_p$ can be written as

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix};\quad a,d\in\text{Fredholm and } b,c\in I_{2p}.\tag{14}$$

If $2p>d$, there exists a continuous injective homomorphism [MR]

$$M\colon \operatorname{Max}(X;G)\hookrightarrow GL_p.\tag{15}$$

3. Properties of Generalized Determinant [S]

We define the generalized determinant. For $A\in 1+I_p$,

$$R_p(A)=-1+(1+A)\exp\left\{\sum_{j=1}^{p-1}(-1)^j\frac{A^j}{j}\right\}\in I_1,\tag{16}$$

and therefore $\det_p A$ is defined as

$$\det_p A\overset{\text{def}}{=}\det(1+R_p(A)).\tag{17}$$

This form is a bit abstract. However, if A satisfies the condition $\|A-1\|<1$, then we find [MR],

$$\det_p A=\exp\operatorname{tr}\left\{(-1)^{p-1}\frac{(A-1)^p}{p}+(-1)^p\frac{(A-1)^{p+1}}{p+1}+\cdots\right\}.\tag{18}$$

This determinant satisfies the following properties ($A,B\in 1+I_p$):

(i) A is invertible off $\det_p A\neq 0$.

(ii) If $A\in 1+I_{p-1}$, then

$$\det_p A=\det_{p-1}A\cdot\exp\left[(-1)^{p-1}\operatorname{tr}\frac{(A-1)^{p-1}}{p-1}\right].\tag{19}$$

(iii) There exists a symmetric polynomial $\gamma_p(A,B)$ such that

$$\det_p AB=e^{\gamma_p(A,B)}\det_p A\cdot\det_p B.\tag{20}$$

We list first a few examples:

$$\gamma_1(A,B)\equiv 0\quad(A,B\in 1+I_1),\tag{21}$$

$$\gamma_2(A,B)=-\operatorname{tr}(A-1)(B-1)\quad(A,B\in 1+I_2),\tag{22}$$

$$\gamma_3(A,B)=\operatorname{tr}\{\tfrac{1}{2}(A-1)(B-1)(A-1)(B-1)+(A-1)(B-1)(A-1)+(B-1)(A-1)(B-1)\}\quad (A,B\in 1+I_3). \tag{23}$$

Now, if we define

$$\omega_p(A,B)\overset{\text{def.}}{=}\det{}_p B\cdot e^{\gamma_p(A,B)}, \tag{24}$$

then [MR],

$$\omega_p(A,BC)=\omega_p(AB,C)\cdot\omega_p(A,B), \tag{25}$$

where $A,B,C\in 1+I_p$.

4. Construction of Abelian Extension of GL_p [MR]

We define the subgroup B_p of GL_p,

$$B_p\overset{\text{def.}}{=}\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in GL_p \,|\, a,d\in\text{Fredholm},\ b\in I_{2p}\right\}, \tag{26}$$

and define the homogeneous space GL_p/B_p. This space is identified with the Grassmann manifold Gr_p,

$$\mathrm{Gr}_p=\{W\subset H \,|\, W=g\cdot H_+,\ g\in GL_p\}. \tag{27}$$

We also define the orthogonal projections $\mathrm{pr}_\pm$,

$$\mathrm{pr}_\pm : W\to H_\pm . \tag{28}$$

Since the diagonal block of $g\in GL_p$ is Fredholm, pr_+ is Fredholm. The off-diagonal block of g is in the class of I_{2p}, so pr_- is in the class of I_{2p}.

We set GL^p as

$$GL^p\overset{\text{def.}}{=}GL(H_+)\cap(1+I_p), \tag{29}$$

where $GL(H_+)$ is the set of all invertible operators on H_+.

We define a group

$$\varepsilon_p=\{(g,q)\in GL_p\times GL(H_+)\,|\,aq^{-1}-1\in I_p\}, \tag{30}$$

whose group multiplication is

$$(g_1,q_1)\cdot(g_2,q_2)=(g_1g_2,q_1q_2). \tag{31}$$

The group GL^p acts from the right on ε_p by $(g,q)\cdot t=(g,qt)$, so we have $GL_p=\varepsilon_p/GL^p$.

Since B_p acts on ε_p by $(g,q)\cdot k=(gk,q\alpha)$, where $k=\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix}\in B_p$, we can define the Stiefel manifold

$$\mathrm{St}_p\overset{\text{def.}}{=}\varepsilon_p/B_p, \tag{32}$$

which has the canonical projection $\pi:\mathrm{St}_p\to\mathrm{Gr}_p$.

Let $w=\{w_1,w_2,\dots\}$ be the basis of $W\in\mathrm{Gr}_p$. Then

$$\mathrm{pr}_+(w_i)=\sum_{j=1}^{\infty}(w_+)_{ji}e_j. \tag{33}$$

We call the basis w admissible if $w_+ \in 1 + I_p$. Then we can show that every $W \in \mathrm{Gr}_p$ has an admissible basis.

From now on, we set

$$w \overset{\text{def.}}{=} \begin{pmatrix} w_+ \\ w_- \end{pmatrix}, \quad w_\pm = \mathrm{pr}_\pm(w), \tag{34}$$

where $w_+ \in 1 + I_p$ and $w_- \in I_{2p}$.

The right action of $t \in GL^p$ is the basis transformation $w_i' = \sum_j w_j t_{ji}$.

This action of GL^p on St_p is written shortely as

$$\begin{pmatrix} w_+ \\ w_- \end{pmatrix} \mapsto \begin{pmatrix} w_+ t \\ w_- t \end{pmatrix}, \tag{35}$$

and induces the right action on $\mathrm{St}_p \times C$ as follows,

$$(w, \lambda) \cdot t \overset{\text{def.}}{=} (wt, \lambda \omega_p(w_+, t)^{-1}). \tag{36}$$

Then we have the homogeneous space,

$$\mathrm{Det}_p \overset{\text{def}}{=} (\mathrm{St}_p \times C)/GL^p, \tag{37}$$

which is the line bundle over the Grassmannian Gr_p, whose projection is

$$[(w, \lambda)] \mapsto \text{the space spanned by the basis } \{w_1, w_2, \ldots\}. \tag{38}$$

Let $F = F(w) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ be the linear operator in $H = H_+ \oplus H_-$ such that $F|_W = +1$, $F|_{W^\perp} = -1$, and $F^2 = 1$ on H, where W is the plane determined by the basis $w = \{w_i\}$(we can set $F = 2w(w^\dagger w)^{-1} w^\dagger - 1$, especially $F = 2ww^\dagger - 1$ for $w^\dagger w = 1$).

We consider smooth functions $\alpha(g, q; w)$ on $\varepsilon_p \times \mathrm{St}_p$ such that

$$\frac{\alpha(g, q; wt)}{\alpha(g, q; w)} = \frac{\omega_p(w_+, t)}{\omega_p((gwq^{-1})_+, qtq^{-1})} \tag{39}$$

for $t \in GL^p$. A general solution of this equation is given by

$$\alpha(g, q; w) = f(g, q; W) \frac{\det_p w_+}{\det_p (gwq^{-1})_+} \cdot \frac{\det_p \frac{1}{2}(q^{-1}a(F_{11} + 1) + q^{-1}bF_{21})}{\det_p \frac{1}{2}(F_{11} + 1)}, \tag{40}$$

where $f: \varepsilon_p \times \mathrm{Gr}_p \to C^\times$ is an arbitrary smooth function [MR].

We define a group $\varepsilon_p \times \mathrm{Map}(\mathrm{Gr}_p, C^\times)$, whose group structure is defined by

$$(g_1, q_1, \mu_1)(g_2, q_2, \mu_2) = (g_1 g_2, q_1 q_2, \mu_1(g_2 \cdot F)\mu_2(F) \cdot \alpha(g_1, q_1; q_2 w q_2^{-1}) \alpha(g_2, q_2; w) \alpha(g_1 g_2, q_1 q_2; w)^{-1}). \tag{41}$$

$\varepsilon_p \times \mathrm{Map}(\mathrm{Gr}_p, C^\times)$ acts on Det_p by the formula

$$(g, q, \mu) \cdot (w, \lambda) = (gwq^{-1}, \mu(\pi(w)) \cdot \lambda \cdot \alpha(g, q; w)). \tag{42}$$

There is an Abelian extension $\widehat{GL_p}$ of GL_p by $\mathrm{Map}(\mathrm{Gr}_p, C^\times)$,

$$\widehat{GL_p} = (\varepsilon_p \times \mathrm{Map}(\mathrm{Gr}_p, C^\times))/N_p, \tag{43}$$

where N_p is the kernel of this action (the normal subgroup) consisting of elements $(1, q, \mu_q)$, where

$$\mu_q(w) = \alpha(1, q, w)^{-1} \cdot \omega_p(w_+, q^{-1})^{-1}, \quad q \in GL^p, \tag{44}$$

$$(1 \to \mathrm{Map}(\mathrm{Gr}_p, C^\times) \to \widehat{GL_p} \to GL_p \to 1).$$

We note that if $p = 1$, the above sequence is

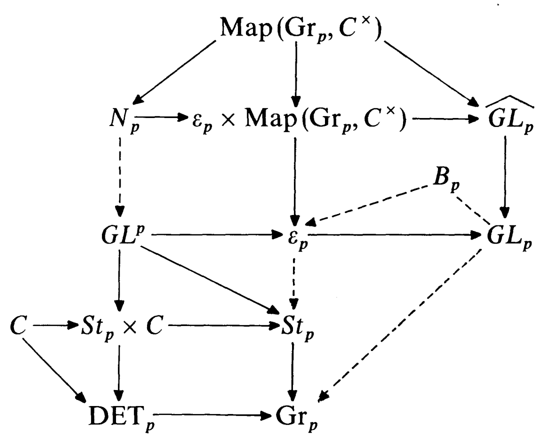
$$1 \to \mathrm{Map}(\mathrm{Gr}_1, C^\times) \to \widehat{GL_1} \to GL_1 \to 1,$$

which seems an Abelian extension but is reduced to a central extension [PS],

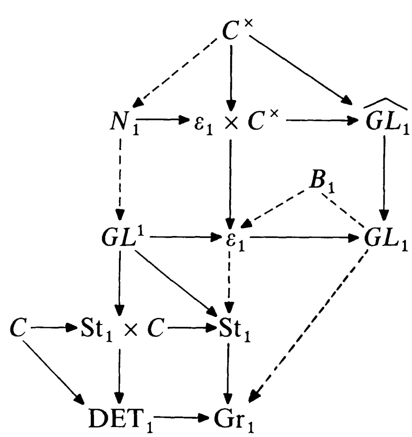
$$1 \to C^\times \to \widehat{GL_1} \to GL_1 \to 1.$$

Here we list various bundles constructed by [MR].

(i) $p > 1$:



(ii) $p = 1$:



(Avoiding confusions, two kinds of lines are used, and they stand for the bundle maps.)

Near the unit element $g = 1$, we can define the local section $\Gamma: \widehat{GL_p} \to GL_p$ by

$$\Gamma(g) = (g, a, 1) \bmod N_p. \tag{45}$$

The extension is in general determined by the two-cocycle. In this case, the two-cocycle is computed by

$$\Gamma(g_1) \cdot \Gamma(g_2) = \Gamma(g_1 g_2) \cdot (1, 1, \xi(g_1, g_2)), \tag{46}$$

where $\xi(g_1, g_2) \in \mathrm{Map}(GL_p, C^\times)$.

By the associativity, ξ satisfies the following condition:

$$\xi(g_1, g_2 g_3; \pi(\omega)) \cdot \xi(g_2, g_3; \pi(\omega)) = \xi(g_1 g_2, g_3; \pi(\omega)) \cdot \xi(g_1, g_2; \pi(g_3 \omega)). \tag{47}$$

It is, in general, complicated to treat the group extension, but the corresponding Lie algebra is rather simpler. So we consider the infinitesimal version of the group extension,

$$(0 \to \mathrm{Map}(\mathrm{Gr}_p, C) \to \widehat{gl_p} \to gl_p \to 0).$$

The Lie algebra $\widehat{gl_p}$ of $\widehat{GL_p}$ is equivalent to $gl_p \oplus \mathrm{Map}(\mathrm{Gr}_p, C)$ as a vector space, where gl_p is the Lie algebra of GL_p.

The commutator in gl_p is

$$[(X, \mu), (Y, \nu)] = ([X, Y], X \cdot \nu - Y \cdot \mu + \eta(X, Y; F)), \tag{48}$$

where η is an antisymmetric bilinear form on gl_p taking values in $\mathrm{Map}(\mathrm{Gr}_p, C)$ and $X \cdot \nu$ is a Lie derivative of a function ν on Gr_p to the direction of the vector field X defined by the GL_p action on Gr_p.

From the Jacobi identity, we have

$$\begin{aligned} &\eta([X, Y], Z; F) + \eta([Y, Z], X; F) + \eta([Z, X], Y; F) \\ &\quad - Z \cdot \eta(X, Y; F) - X \cdot \eta(Y, Z; F) - Y \cdot \eta(Z, X; F) = 0. \end{aligned} \tag{49}$$

Now let's compute the two-cocycle. By Eq. (41), we have

$$\begin{aligned} &\{\Gamma(g_1)\Gamma(g_2)\Gamma(g_1^{-1})\Gamma(g_2^{-1})\}(w, \lambda) \\ &\quad = (g_1, g_2 g_1^{-1} g_2^{-1}, a(g_1)a(g_2)a(g_1^{-1})a(g_2^{-1}), \mu)(w, \lambda), \end{aligned} \tag{50}$$

where μ is

$$\begin{aligned} \mu(\pi(w)) = {} & \alpha(g_2^{-1}, a(g_2^{-1}); w) \\ & \cdot \alpha(g_1^{-1}, a(g_1^{-1}); g_2^{-1} w a(g_2^{-1})^{-1}) \\ & \cdot \alpha(g_2, a(g_2); g_1^{-1} g_2^{-1} w a(g_2^{-1})^{-1} a(g_1^{-1})^{-1}) \\ & \cdot \alpha(g_1, a(g_1); g_2 g_1^{-1} g_2^{-1} w a(g_2^{-1})^{-1} a(g_1^{-1})^{-1} a(g_2)^{-1}) \\ & \cdot \alpha(g_1 g_2 g_1^{-1} g_2^{-1}, a(g_1)a(g_2)a(g_1^{-1})a(g_2^{-1}); w)^{-1}. \end{aligned} \tag{51}$$

To obtain the two-cocycle $\eta_p(X, Y; F)$, we set $g_1 = e^{tX}$ and $g_2 = e^{sY}$, and compute

$$\frac{\partial^2}{\partial s \partial t} \mu(\pi(w))|_{s=t=0} = \eta_p(X, Y; F). \tag{52}$$

We can get the two-cocycle $\eta_p(X, Y; F)$ with the suitable choice of $f(g, q; W)$ in Eq. (40). (In is not easy to choose $f(g, q; W)$ for each p.)

We must note that the calculations below are based on $\hat{U}_p$ (the unitary group) rather than $\widehat{GL_p}$ and, moreover, the only connected component of the unit element is exploited. The situation is the same as [MR] essentially.

5. Results

The results are as follows,

(i) $p=1$: Kac–Peterson [KP] (or [PS])

$$\alpha(g, q; w) \equiv 0, \tag{53}$$

$$\begin{aligned}\eta_1(X, Y; F) &= -\tfrac{1}{8}\operatorname{tr}[[\varepsilon, X], [\varepsilon, Y]]\varepsilon \\ &= \operatorname{tr}(b(X)c(Y) - b(Y)c(X)).\end{aligned} \tag{54}$$

(ii) $p=2$: Mickelsson–Rajeev [MR]

$$\alpha(g, q; w) = \exp[-\operatorname{tr}\{(1-q^{-1}a)(w_+ - 1) + q^{-1}b(\tfrac{1}{2}F_{21} - w_-)\}], \tag{55}$$

$$\begin{aligned}\eta_2(X, Y; F) &= \tfrac{1}{8}\operatorname{tr}[[\varepsilon, X], [\varepsilon, Y]](F-\varepsilon) \\ &= -\tfrac{1}{2}\operatorname{tr}\{(b(X)c(Y) - b(Y)c(X))(F_{11} - 1) \\ &\quad - b(X)(F_{22}+1)c(Y) + b(Y)(F_{22}+1)c(X)\}.\end{aligned} \tag{56}$$

(iii) $p=3$:

$$\alpha(g, q; w) = \exp\left[\tfrac{1}{2}\operatorname{tr}\left\{\tilde{\gamma}(g, q; w) + 2(q^{-1}a - 1)\left(\frac{F_{11}-1}{2}\right)^2\right\}\right], \tag{57}$$

where

$$\begin{aligned}\tilde{\gamma}(g, q; w) = \{&((q^{-1}a - 1)w_+ + q^{-1}bw_- + w_+ - 1)(w_+^\dagger - 1) \\ &\cdot((q^{-1}a - 1)w_+ + q^{-1}bw_- + w_+ - 1)(w_+^\dagger - 1) \\ &+ 2((q^{-1}a - 1)w_+ + q^{-1}bw_- + w_+ - 1)^2(w_+^\dagger - 1) \\ &+ 2((q^{-1}a - 1)w_+ + q^{-1}bw_- + w_+ - 1)(w_+^\dagger - 1)^2 \\ &- (w_+ - 1)(w_+^\dagger - 1)(w_+ - 1)(w_+^\dagger - 1) \\ &- 2(w_+ - 1)^2(w_+^\dagger - 1) - 2(w_+ - 1)(w_+^\dagger - 1)^2\},\end{aligned} \tag{58}$$

$$\begin{aligned}\eta_3(X, Y; F) &= \tfrac{1}{64}\operatorname{tr}\{2[[\varepsilon, X], [\varepsilon, Y]](F-\varepsilon)^3 \\ &\quad - [\varepsilon, X](F-\varepsilon)[\varepsilon, Y](F-\varepsilon)^2 + [\varepsilon, Y](F-\varepsilon)[\varepsilon, X](F-\varepsilon)^2\} \\ &= \tfrac{1}{4}\operatorname{tr}\{(b(X)c(Y) - b(Y)c(X))(F_{11}-1)^2 \\ &\quad + b(X)(F_{22}+1)^2c(Y) - b(Y)(F_{22}+1)^2c(X) \\ &\quad - b(X)(F_{22}+1)c(Y)(F_{11}-1) \\ &\quad + b(Y)(F_{22}+1)c(X)(F_{11}-1)\}.\end{aligned} \tag{59}$$

(iv) $p \geqq 4$: It is not easy to calculate.

From the above local formula of $\eta_p(X, Y; F)$ $(p = 1, 2, 3)$, we can guess the general formula $\eta_p(X, Y; F)$ as follows:

(v) p: any natural number

$$\eta_p(X, Y; F) = c_p \operatorname{tr}\left\{\sum_{l=0}^{p-1} (-1)^{(p-1)-l}(b(X)(F_{22}+1)^l c(Y)(F_{11}-1)^{(p-1)-l} - b(Y)(F_{22}+1)^l c(X)(F_{11}-1)^{(p-1)-l})\right\}, \tag{60}$$

where c_p is a constant which only depends on p.

In fact, we can show that this satisfies the Jacobi identity (Eq. (49)) (see Appendix). Therefore, $\eta_p(X, Y; F)$ above becomes the two-cocycle. (Of course, $\eta_p(X, Y; F)$ coincides each case (i), (ii), and (iii) with suitable c_p).

We conjecture that η_p (Eq. (60)) gives all of the cocycles.

6. Infinite Charge Renormalization

We establish the relation $\eta_p(X, Y; F)$ and $\eta_{p+1}(X, Y; F)$ in this section. Consider the $\eta_p(X, Y; F)$, where $X, Y \in gl_{p+1}$ and $F \in \mathrm{Gr}_{p+1}$. Then this is a two-cocycle but divergent. So we must subtract the divergence in order to get the well-defined two-cocycle: we must find a one-cocycle $\beta_p(X; F)$ such that

$$(\eta_p + \delta\beta_p)(X, Y; F) = \eta_{p+1}(X, Y; F), \tag{61}$$

where

$$\delta\beta_p(X, Y; F) = \delta_X \beta_p(Y; F) - \delta_Y \beta_p(X; F) + \beta_p([X, Y]; F). \tag{62}$$

In [MR] ($p = 1$), (61) is interpreted as the "infinite charge renormalization." Here we generalize their interpretation to arbitrary p. The results are stated as follows:

(i) $p = 1$ [MR]

$$\beta_1(X; F) = \tfrac{1}{16} \operatorname{tr}[X, \varepsilon][F, \varepsilon]. \tag{63}$$

(ii) $p = 2$

$$\beta_2(X, F) = -\tfrac{1}{64} \operatorname{tr}[X, \varepsilon][F, \varepsilon](F - \varepsilon)^2. \tag{64}$$

We conjecture the form of β_p for any natural number p,

(iii) p: any natural number

$$\beta_p(X, F) = d_p \operatorname{tr}[X, \varepsilon][F, \varepsilon](F - \varepsilon)^{2(p-1)}, \tag{65}$$

where d_p is constant only depending on p.

7. The Highest Weight Vector and State Vectors

We define holomorphic cross sections of Det_p^* (the dual bundle of Det_p). These are identified with functions

$$\Psi : \mathrm{St}_p \to C, \quad \Psi(wt) = \Psi(w)\omega_p(w_+, t). \tag{66}$$

For example, $\Psi_0(w) = \det_p w_+$, which is the "highest weight vector" [MR]. Let $\Gamma(\mathrm{Det}_p^*)$ be the set of all cross sections Ψ.

We define $(i)=\{i_1,i_2,\ldots,i_n,\ldots\}\in N$, which is any finite set [M], and put $S=\{(i)\}$.

Then we define the matrix $w(i)$ by exchanging the rows labeled by (i) of w_+ for the corresponding rows of w_-:

$$w(i)=\begin{pmatrix} w_+^{(1)} \\ \vdots \\ w_-^{(i_1)} \\ \vdots \\ w_-^{(i_n)} \\ \vdots \end{pmatrix}. \tag{67}$$

It is trivial that $w(i)-w_+\in I_1$ and $w(i)=w_+$ if (i) is the empty set.

Now we define

$$\Psi_{(i)}(w)\overset{\text{def.}}{=}\det_p w(i)\cdot e^{\alpha_p(w(i),w_+)}. \tag{68}$$

Since $\Psi_{(i)}(w)$ should satisfy $\Psi_{(i)}(wt)=\Psi_{(i)}(w)\cdot\omega_p(w_+,t)$, we have

$$\gamma_p(w(i),t)-\gamma_p(w_+,t)=-\{\alpha_p(w(i)t,w_+t)-\alpha_p(w(i),w_+)\}, \tag{69}$$

see Eqs. (21)–(23).

We state our result.

(i) $p=2$ [M],

$$\Psi_{(i)}(w)=\det_2 w(i)\cdot e^{\operatorname{tr}(w(i)-w_+)}. \tag{70}$$

(ii) $p=3$,

$$\Psi_{(i)}(w)=\det_3 w(i)\cdot e^{-\operatorname{tr}((1/2)w(i)^2-2w(i)-(1/2)w_+^2+2w_+)}. \tag{71}$$

Therefore, we have shown that $\Psi_{(i)}$ is a holomorphic section for each $(i)\in S$.

We conjecture that $\{\Psi_{(i)}|(i)\in S\}$ is a holomorphic basis of $\Gamma(\mathrm{Det}_p^*)$:

$$\forall\Psi\in\Gamma(\mathrm{Det}_p^*)\Rightarrow\Psi=\sum_{(i)}c_{(i)}\Psi_{(i)}. \tag{72}$$

8. Discussion

We note again that all the above discussions were based on the $\hat{U}_p$ (the unitary subgroup of $\widehat{GL_p}$) rather than $\widehat{GL_p}$. We want to construct the representation of the $\widehat{GL_p}$ (or $\hat{U}_p$) on a "Hilbert space," (for the general discussion, see [MR]).

We can define an inner product on $\Gamma(\mathrm{Det}_p^*)$ as follows:

$$\langle\Psi_1,\Psi_2\rangle=\int_{\mathrm{Gr}_p} dm\,\bar{\Psi}_1(w)\Psi_2(w)l(w)^{-2}, \tag{73}$$

if the quasi-invariant measure dm on Gr_p exists (in general, for $p\geqq 2$, the measure dm is unknown, but for $p=1$, the measure may be given by [P1]), where

$$l(w)=\exp\{-\tfrac{1}{2}\gamma_p(w_+,w_+^\dagger)\}. \tag{74}$$

Given the inner product on $\Gamma(\mathrm{Det}_p^*)$, we can construct the representation:

$$T{:}\widehat{GL_p}\ (\text{respectively } \hat{U}_p) \to \Gamma(\mathrm{Det}_p^*), \tag{75}$$

$$(T(g,q,\lambda)\Psi)(w) = \lambda(g^{-1}F)^{-1}\alpha(g,q;g^{-1}wq)^{-1}\Psi(g^{-1}wq). \tag{76}$$

Then we can shown this representation is unitary.

But unfortunately, Pickrell [P2] has shown that the unitary subgroup of the group extension $\widehat{GL_p}$ $(p>1)$ does not have separable Hilbert space representations which are nontrivial on the extension part. So there may be no quasi-invariant measure on Gr_p $(p>1)$.

Last we note that the results of [MR] and ours are deeply related to "Universal Yang–Mills Theory" proposed by Rajeev [R] and developed by us [TF]. We will discuss this point in another paper.

Appendix

In this appendix, we show that Eq. (60) satisfies the Jacobi identity (Eq. (49)). First of all, we list some useful formulas,

$$b([X,Y]) = a(X)b(Y) - a(Y)b(X) + b(X)d(Y) - b(Y)d(X), \tag{a1}$$

$$c([X,Y]) = c(X)a(Y) - c(Y)a(X) + d(X)c(Y) - d(Y)c(X), \tag{a2}$$

$$([Z,F])_{11} = a(Z)(F_{11}-1) + b(Z)F_{21} - (F_{11}-1)a(Z) - F_{12}c(Z), \tag{a3}$$

$$([Z,F])_{22} = c(Z)F_{12} + d(Z)(F_{22}+1) - F_{21}b(Z) - (F_{22}+1)d(Z), \tag{a4}$$

$$F_{12}(F_{22}+1) = -(F_{11}-1)F_{12}, \quad F_{21}(F_{11}-1) = -(F_{22}+1)F_{21}, \tag{a5, 6}$$

$$Z\cdot(F_{11}-1)^n = \sum_{k=0}^{n-1}(F_{11}-1)^k(-[Z,F])_{11}(F_{11}-1)^{(n-1)-k}, \tag{a7}$$

$$Z\cdot(F_{22}+1)^n = \sum_{k=0}^{n-1}(F_{22}+1)^k(-[Z,F])_{22}(F_{22}+1)^{(n-1)-k}. \tag{a8}$$

Now we consider the Lie derivatives,

$$
\begin{aligned}
&Z\cdot\eta_p(X,Y;F)\\
&\quad = \frac{d}{dt}\eta_p(X,Y;e^{-tZ}Fe^{tZ})|_{t=0} = c_p\,\mathrm{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}\right.\\
&\cdot\left(b(X)\left(\sum_{k=0}^{l-1}(F_{22}+1)^k(-[Z,F])_{22}(F_{22}+1)^{(l-1)-k}\right)c(Y)(F_{11}-1)^{(p-1)-l}\right.\\
&+b(X)(F_{22}+1)^l c(Y)\left(\sum_{k=0}^{(p-1)-(l-1)}(F_{11}-1)^k(-[Z,F])_{11}(F_{11}-1)^{(p-1)-k-(l+1)}\right)\\
&-c(X)\left(\sum_{k=0}^{(p-1)/(l+1)}(F_{11}-1)^k(-[Z,F])_{11}(F_{11}-1)^{(p-1)-(l+1)-k}\right)b(Y)(F_{22}+1)^l\\
&\left.\left.-c(X)(F_{11}-1)^{(p-1)-l}b(Y)\left(\sum_{k=0}^{l-1}(F_{22}+1)^k(-[Z,F])_{22}(F_{22}+1)^{(l-1)-k}\right)\right)\right\}
\end{aligned}
$$

$$= c_p \operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}\Big((c(X)(F_{11}-1)^{(p-1)-l}b(Y)-c(Y)(F_{11}-1)^{(p-1)-l}b(X))\right.$$

$$\cdot\Big(\sum_{k=0}^{l-1}(F_{22}+1)^k(c(Z)F_{12}+d(Z)(F_{22}+1)-F_{21}b(Z)$$

$$-(F_{22}+1)d(Z))(F_{22}+1)^{(l-1)-k}\Big)-b(X)(F_{22}+1)^l c(Y)-b(Y)(F_{22}+1)^l c(X))$$

$$\cdot\Big(\sum_{k=0}^{(p-1)-(l-1)}(F_{11}-1)^k(a(Z)(F_{11}-1)+b(Z)F_{21}-(F_{11}-1)a(Z)$$

$$\left.-F_{12}c(Z))(F_{11}-1)^{(p-1)-(l+1)-k}\Big)\Big)\right\}. \tag{a9}$$

Using (a3)–(a8), we have following results after simple but tedious calculations:

$$Z\cdot\eta_p(X,Y;F)$$

$$= c_p \operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}((a(Z)b(X)-b(X)d(Z))(F_{22}+1)^l c(Y)(F_{11}-1)^{(p-1)-l}\right.$$

$$-(a(Z)b(Y)-b(Y)d(Z))(F_{22}+1)^l c(X)(F_{11}-1)^{(p-1)-l}$$

$$+(c(X)a(Z)-d(Z)c(X))(F_{11}-1)^{(p-1)-l}b(Y)(F_{22}+1)^l$$

$$-(c(Y)a(Z)-d(Z)c(Y))(F_{11}-1)^{(p-1)-l}b(X)(F_{22}+1)^l$$

$$+\sum_{k=0}^{(p-1)-l-1}(P_k(Z,X,Y)-P_k(X,Y,Z)+P_k(Y,X,Z)-P_k(Z,Y,X)$$

$$\left.+Q_k(Z,X,Y)-Q_k(X,Y,Z)+Q_k(Y,X,Z)-Q_k(Z,Y,X)))\right\}, \tag{a10}$$

where

$$P_k(X,Y,Z)=(F_{11}-1)^{(p-1)-l-1-k}b(X)(F_{22}+1)^l c(Y)(F_{11}-1)^k b(Z)F_{21}, \tag{a11}$$

$$Q_k(X,Y,Z)=c(X)(F_{11}-1)^{(p-1)-l-1-k}b(Y)(F_{22}+1)^l c(Z)(F_{11}-1)^k F_{12}, \tag{a12}$$

The others are given by the cyclic permutations of X, Y, and Z. Using the fact that

$$P_k(Z,X,Y)+(\text{cyclic permutations})+Q_k(Z,X,Y)+(\text{cyclic permutations})=0, \tag{a13}$$

we have,

$$Z\cdot\eta_p(X,Y;F)+X\cdot\eta_p(Y,Z;F)+Y\cdot\eta_p(Z,X;F)$$

$$= c_p \operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}\right.$$

$$\cdot((a(X)b(Y)-a(Y)b(X)+b(X)d(Y)-b(Y)d(X))(F_{22}+1)^l c(Z)(F_{11}-1)^{(p-1)-l}$$

$$+(a(Y)b(Z)-a(Z)b(Y)+b(Y)d(Z)-b(Z)d(Y))(F_{22}+1)^l c(X)(F_{11}-1)^{(p-1)-l}$$

$$+(a(Z)b(X)-a(X)b(Z)+b(Z)d(X)-b(X)d(Z))(F_{22}+1)^l c(Y)(F_{11}-1)^{(p-1)-l}$$

$$-(c(X)a(Y)-c(Y)a(X)+d(X)c(Y)-d(Y)c(X))(F_{11}-1)^{(p-1)-l}b(Z)(F_{22}+1)^l$$
$$-(c(Y)a(Z)-c(Z)a(Y)+d(Y)c(Z)-d(Z)c(Y))(F_{11}-1)^{(p-1)-l}b(X)(F_{22}+1)^l$$
$$\left.-(c(Z)a(X)-c(X)a(Z)+d(Z)c(X)-d(X)c(Z))(F_{11}-1)^{(p-1)-l}b(Y)(F_{22}+1)^l)\right\}$$
$$=c_p\operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}(b([Z,Y])(F_{22}+1)^l c(Z)(F_{11}-1)^{(p-1)-l}\right.$$
$$+b([Y,Z])(F_{22}+1)^l c(X)(F_{11}-1)^{(p-1)-l}$$
$$+b([Z,X])(F_{22}+1)^l c(Y)(F_{11}-1)^{(p-1)-l}$$
$$-c([X,Y])(F_{11}-1)^{(p-1)-l}b(Z)(F_{22}+1)^l$$
$$-c([Y,Z])(F_{11}-1)^{(p-1)-l}b(X)(F_{22}+1)^l$$
$$\left.-c([Z,X])(F_{11}-1)^{(p-1)-l}b(Y)(F_{22}+1)^l)\right\}. \tag{a14}$$

Since $b(\ ), c(\ )\in I_{2p}$ and $F_{11}-1$, $F_{22}+1\in I_p$, we can easily see that each term of (a14) is in trace class, and we can modify (a14) as follows:

$$Z\cdot\eta_p(X,Y;F)+X\cdot\eta_p(Y,Z;F)+Y\cdot\eta_p(Z,X;F)$$
$$=c_p\operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}(b([X,Y])(F_{22}+1)^l c(Z)(F_{11}-1)^{(p-1)-l}\right.$$
$$\left.-c([X,Y])(F_{11}-1)^{(p-1)-l}b(Z)(F_{22}+1)^l\right\}$$
$$+c_p\operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}(b([Y,Z])(F_{22}+1)^l c(X)(F_{11}-1)^{(p-1)-l}\right.$$
$$\left.-c([Y,Z])(F_{11}-1)^{(p-1)-l}b(X)(F_{22}+1)^l)\right\}$$
$$+c_p\operatorname{tr}\left\{\sum_{l=0}^{p-1}(-1)^{(p-1)-l}(b([Z,X])(F_{22}+1)^l c(Y)(F_{11}-1)^{(p-1)-l}\right.$$
$$\left.-c([Z,X])(F_{11}-1)^{(p-1)-l}b(Y)(F_{22}+1)^l)\right\}$$
$$=\eta_p([X,Y],Z;F)+\eta_p([Y,Z],X;F)+\eta_p([Z,X],Y;F). \tag{a15}$$

Thus we have just the Jacobi identity (Eq. (49)).

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