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Hyperbolicity and Invariant Measures for General C^2 Interval Maps Satisfying the Misiurewicz Condition

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Abstract. In this paper we will show that piecewise C^2 mappings f on $[0,1]$ or S^1 satisfying the so-called Misiurewicz conditions are globally expanding (in the sense defined below) and have absolute continuous invariant probability measures of positive entropy. We do not need assumptions on the Schwarzian derivative of these maps. Instead we need the conditions that f is piecewise C^2, that all critical points of f are "non-flat," and that f has no periodic attractors. Our proof gives an algorithm to verify this last condition. Our result implies the result of Misiurewicz in [Mi] (where only maps with negative Schwarzian derivatives are considered). Moreover, as a byproduct, the present paper implies (and simplifies the proof of) the results of Mañé in [Ma], who considers general C^2 maps (without conditions on the Schwarzian derivative), and restricts attention to points whose forward orbit stay away from the critical points. One of the main complications will be that in this paper we want to prove the existence of invariant measures and therefore have to consider points whose iterations come arbitrarily close to critical points. Misiurewicz deals with this problem using an assumption on the Schwarzian derivative of the map. This assumption implies very good control of the non-linearity of f^n, even for high n. In order to deal with this non-linearity, without an assumption on the Schwarzian derivative, we use the tools of [M.S.]. It will turn out that the estimates we obtain are so precise that the existence of invariant measures can be proved in a very simple way (in some sense much simpler than in [Mi]). The existence of these invariant measures under such general conditions was already conjectured a decade ago.

Introduction

There are a large number of papers on iterations of piecewise smooth one-dimensional mappings $f: M \to M$, where $M = [0,1]$ or S^1. Initially all metric results for these maps assumed that f is piecewise expanding, see for example [La, Y.]. Later the condition that f needs to be expanding was somewhat relaxed. This was done by considering expanding maps which are induced from special maps, see

[Ru, Ja, Bo, Pi1 and Pi2, Sz]. Only when D. Singer introduced the concept of Schwarzian derivative in the study of these maps it became possible to study more general maps. Misiurewicz [Mi], Collet–Eckmann [C.E.] and others proved hyperbolicity and measure properties for these maps assuming that the Schwarzian derivative of these maps is negative.

However the condition that the Schwarzian derivative of f is negative can be expressed as a convexity condition on $1/\sqrt{f'}$. So this condition is not preserved under smooth coordinate changes, is not very natural and has no dynamic interpretation. Moreover it excludes a large class of maps.

Mañé managed to drop this condition in his paper [Ma] for general C^2 maps. He considers points whose forward orbits stay away from the set of critical points $C(f)$. The idea of his proof is to construct certain intervals I so that for some $n>0$ the intervals $I, f(I), \ldots, f^{n-1}(I)$ are disjoint and so that $f^n(I)$ is much longer than I. Using a C^2 theorem of Schwartz ($\neq$ Schwarz) he then proves that any compact set K not containing any critical points or non-hyperbolic periodic points is hyperbolic. His proof does not give any way to decide whether or not all periodic points are hyperbolic or not.

In the same direction W. de Melo and I proved that general smooth unimodal maps, having no flat critical points, can have no wandering intervals, see [M.S.]. The main problem is that if one studies orbits which pass close to critical points, then one gets a lot of non-linearity: the bounded non-linearity tools of Schwartz completely break down. So one needs new tools. In [M.S.] it was shown how iterates of a C^2 map f expand or contract cross-ratios of points and how to apply this type of information. The bounds on the contraction of cross-ratios give control on the type of non-linearity that can occur.

Refining techniques of Mañé and [M.S.], this paper gives a very precise description of (piecewise) smooth mappings satisfying the Misiurewicz condition that each critical point of f is either periodic or has a forward orbit which stays away from the critical set. More precisely, there exists a neighbourhood W of $C(f)$ such that

$$\text{(i)} \qquad \begin{cases} \left(\bigcup_{n\geq 1} f^n(C(f))\right) \cap W \subset C(f); \\ f \text{ is not injective,} \end{cases}$$

where $C(f)$ is the set of points which are critical points of f. Of course the condition that $f: M \to M$ is not injective is not very restrictive: if f is injective then the theorems below are either trivially true, trivially false or follow from [He] and [Y].

We will also sometimes need the following condition

(ii) all periodic points of f are hyperbolic and repelling.

Below we will give precise definitions, but let us summarize the main results here already for C^∞ maps. (For C^2 maps we will need to be more precise about what it means for a critical point to be non-flat.)

Theorem. *Let $f: M \to M$ be a C^∞ map without flat critical points. Assume that f*

satisfies condition (i). *Then*

—*the* **period of periodic attractors and non-hyperbolic periodic orbits** *of f is* **uniformly bounded.**

Moreover, if (ii) *holds then*

—*f has an* **absolutely continuous invariant probability measures of positive entropy**;

—*f is* **globally expanding**, i.e., *there exists $\lambda > 1$ and $K > 0$ such that for any maximal interval I_n on which $f^n | I_n$ is a diffeomorphism one has*

$$\frac{|f^n(I_n)|}{|I_n|} \geqq K\lambda^n.$$

Remark. Our proof gives a finite algorithm to check whether condition (ii) is satisfied, provided $\operatorname{dist}(\bigcup_{n\geqq 1} f^n(C(f))\setminus C(f), C(f))$ is known (which is for example the case if all critical points are eventually periodic). This algorithm is sketched during the proof of the theorem.

Statement of Results. Let M be either S^1 or $[0,1]$. Furthermore assume that $f: M \to M$ is a C^2 map. We say that c is a critical point of f if $f'(c) = 0$ and that $c \in C(f)$ if c is a critical point or a boundary point of M. A critical point is said to be **non-flat** if there exists a neighbourhood of c and $2 \leqq k < \infty$, such that f is $C^{\max(3,k)}$ on this neighbourhood and such that the k^{th} derivative at c is non-zero, $f^{(k)}(c) \neq 0$. (It will not suffice for our purposes that f is C^2 near critical points, see the proof of Lemma 1.2.) Clearly this non-flatness condition is satisfied for analytic maps which are not constant.

We should remark that, just as in [Mi], the results we shall state presently also hold for maps which are **piecewise** C^2. In this case the notion of critical point has to be somewhat extended. We shall go into this at the end of this section.

For $p \in M$ let $O(p) = \bigcup_{k\geqq 0} f^k(p)$ be the forward orbit of p. It is said to be a periodic point of period n if the orbit $O(p)$ consists of n points. We say that a periodic point p of period n is **hyperbolic** if $|(f^n)'(p)| \neq 1$. If $f^n(p) = p$ one has $(f^n)'(p) = (f^n)'(f(p)) = \cdots = (f^n)'(f^{n-1}(p))$, and in particular if p is a hyperbolic periodic point then each of the points $f^i(p)$, $i \geqq 0$ is also hyperbolic. So if $|(f^n)'(p)| \neq 1$ we call $O(p)$ a hyperbolic periodic orbit.

Suppose $f^n(p) = p$. If $|(f^n)'(p)| < 1$ then $O(p)$ is said to be a **hyperbolic attracting** periodic orbit. If $|(f^n)'(p)| > 1$ then $O(p)$ is a **hyperbolic repelling** periodic orbit. The **basin** $B(O(p))$ of a periodic orbit $O(p)$ is the set $\{x; f^n(x) \to O(p) \text{ as } n \to \infty\}$. The **immediate basin** $B_0(O(p))$ of $O(p)$ consists of the union of the components of $B(O(p))$ which intersect $O(p)$. We say that periodic orbit $O(p)$ is an **attractor** if $B_0(O(p))$ contains an interval. It is not hard to show that an attracting periodic orbit is either hyperbolically attracting or is non-hyperbolic.

We say that a set K is invariant if $f(K) \subset K$. An invariant set K is called **hyperbolic** if there exist $C > 0$ and $\lambda > 1$ such that for each $x \in K$ one has either $|Df^n(x)| \geqq C \cdot \lambda^n$ for each $n = 0, 1, 2, \ldots$, or $|Df^n(x)| \leqq 1/C \cdot 1/\lambda^n$ for each $n = 0, 1, 2, \ldots$.

Furthermore we say that f satisfies the **Misiurewicz** condition if there exists a neighbourhood W of $C(f)$ such that

$$\text{(i)} \qquad \begin{cases} \left(\bigcup_{n \geq 1} f^n(C(f))\right) \cap W \subset C(f); \\ f \text{ is not injective.} \end{cases}$$

Sometimes we will also need the following **condition**:

(ii) all periodic points of f are hyperbolic and repelling,

or the following weaker version of this:

(ii′) all periodic orbits of f are hyperbolic.

Let us now state the main results of this paper.

Theorem A. "*Periodic attractors have low periods*". *Let $f: M \to M$ be a C^2 map without flat critical points satisfying the Misiurewicz condition* (i). *Then there exists $N < \infty$ such that the minimal period of each periodic attractor or non-hyperbolic orbit is less than N.*

From Theorem A the period of periodic attractors or non-hyperbolic orbits is uniformly bounded. (The boundedness of the period of periodic attractors has recently been proved for general C^2 maps without flat critical points, see [M.M.S.].) Let B_0 be the union of the immediate basins of periodic attractors. From Theorem A it follows that $\text{Clos}(B_0)$ is a finite union of intervals.

Theorem B. "*Hyperbolic structures and quasi-polynomial non-linearity*". *Let $f: M \to M$ be a C^2 map without flat critical points. Assume that f satisfies the Misiurewicz condition* (i) *and also* (ii′). *Then f is* **globally expanding**, i.e., *there exist constants $\lambda > 1$ and $K > 0$ such that for any* **maximal** *interval I_n such that $f^n | I_n$ is a diffeomorphism and $f^n(I_n) \cap B_0 = \emptyset$ one has*

$$(*) \qquad \frac{|f^n(I_n)|}{|I_n|} \geq K \cdot \lambda^n.$$

Also there is a **hyperbolic structure on the set of periodic points**: *there exist constants $\lambda > 1$ and $K > 0$ such that if p is a periodic point of* (*minimal*) *periodic n then*

$$(**) \qquad |Df^n(p)| \geq K \cdot \lambda^n.$$

Moreover, $f^n | I_n$ is **quasi-polynomial** *in a sense which is defined in Proposition 10.1.*

Once we have the estimates from Theorem B it turns out that one can prove the following two results almost immediately.

Theorem C. "*Hyperbolicity, measure and ergodicity*". *Let $f: M \to M$ be a C^2 map without flat critical points. Assume that f satisfies the Misiurewicz condition* (i). *Let K be a compact set such that $f(K) \subset K$ and which does not contain any non-hyperbolic periodic points.*

i) *If $(C(f) \cup B_0) \cap K = \emptyset$ then K is a hyperbolic set.*
ii) *If K is a Borel set with positive Lebesgue measure such that $B_0 \cap K = \emptyset$ and*

$K \neq M$, then $C(f) \neq \emptyset$ and K contains an interval which has at least one critical point in its interior.

Statement i) of Theorem C was already proved by Mañé in [Ma]. Mañé's proof is rather indirect: a certain non-hyperbolic compact invariant set is constructed using the Lemma of Zorn, and then it is shown that this leads to a contradiction. An advantage of our proof is that it is constructive and gives a finite and effective way to check the assumption that K does not contain any non-hyperbolic periodic points.

Theorem D. *"Invariant measures." Let $f: M \to M$ be a C^2 map without flat critical points. Assume that f satisfies the Misiurewicz conditions* (i) *and* (ii). *Then f has an absolutely continuous invariant probability measure of positive entropy.*

From Theorem C the support of each of these absolutely continuous measures is a finite union of intervals. If $C(f) = \emptyset$ then the support of each absolutely continuous invariant measure is equal to S^1 (and in particular there exists just one absolutely continuous invariant measure). If $C(f) \neq \emptyset$ then the support of each absolutely continuous invariant measure contains at least one critical point in the interior of its support (and in particular the number of ergodic components of absolutely continuous invariant measures is at most $C(f)$).

The results stated above also hold for maps which are **piecewise** C^2 and have no flat critical points. Let us define these notions. We say that f is piecewise C^2 if there exists a finite set of points F such that f extends to a C^2 map on the closure of each component of $M \backslash F$ and such that $f'(x) \neq \emptyset$ for all $x \in M \backslash F$. (So the points in F can be discontinuities of f.) In this case we say that $c \in K(f)$ if $c \in F$ or if $c \in \partial M$. We say that f is non-flat at $K(f)$ if for each $c \in K(f)$ and each component U of $M \backslash \{c\}$ there exists $k \in \{1, 2, 3, \ldots\}$ such that $f|U$ is C^{k+1} near c and $(f|U)^{(k)}(c) \neq 0$. (Here a one-sided derivative is meant.) Now replace the Misiurewicz condition (i) by: there exists a neighbourhood W of $K(f)$ such that

$$\text{(i}'\text{)} \qquad \begin{cases} \left(\bigcup_{n \geq 1} f^n(K(f)) \right) \cap W \subset K(f), \\ f \text{ is not injective.} \\ \text{For each } n, f^n \text{ is continuous on a neighbourhood of } \mathrm{Clos}(B_0). \end{cases}$$

Theorems A–D are valid for maps $f: M \to M$ which are piecewise C^2, satisfy (i') and have no flat critical points. The proofs in this more general setting go through without much change if one replaces $C(f)$ everywhere by $K(f)$. In particular the results in this paper imply the results on piecewise C^2 expanding Markov maps from [La.Y.].

Comparison with Results on Collet–Eckmann Maps. After this paper was written, Tomasz Nowicki and I considered C^2 mappings without flat critical points satisfying (ii) and the Collet–Eckmann conditions. These conditions say that there exists $K > 0$ and $\lambda > 1$ such that

$$\text{(C.E.1)} \qquad |Df^n(f(c))| \geqq K\lambda^n, \quad \forall n \geqq 0, \quad \forall c \in C(f),$$

$$\text{(C.E.2)} \qquad n > 0 \quad \text{and} \quad f^n(z) \in C(f) \Rightarrow |Df^n(z)| \geqq K\lambda^n.$$

From Mañé's result follows that, for C^2 maps, Misiurewicz condition (i) implies (C.E.1). So the difference between the Misiurewicz case and the Collet–Eckmann condition is that (i) is replaced by the weaker condition (C.E.1) and that one adds condition (C.E.2). In [N.S.1] and [N.S.2] it was proved that these maps satisfy the assertions of the theorem above.

The proof of the existence of an absolutely continuous invariant measure depends on f being globally expanding. To show that globally expanding maps satisfying the Collet–Eckmann condition have good invariant measures is much more subtle than to prove the corresponding result for globally expanding maps satisfying the Misiurewicz conditions. This is because branches of f^n can be short for Collet–Eckmann maps. It is shown in [N.S.2] that in order to prove the existence of these measures one only needs condition (C.E.1) and the technical condition that there exists $C<\infty$ such that for any $n \geqq 0$ and for any interval I_n for which $f^n|I_n$ is a diffeomorphism one has

$$(*) \qquad \sum_{j=0}^{n-1} |f^j(I_n)| \leqq C.$$

In [N.S.1] it is shown that (*) follows from (C.E.1) and (C.E.2).

On the other hand, to show that a map f is globally expanding (or satisfies (*)), is for essential reasons much harder in the present case. This is because condition (C.E.2) gives a uniform contraction in backward time (and uniform hyperbolicity on the set of periodic points of high period). In the present paper one does not have condition (C.E.2) but only the condition that all periodic orbits are hyperbolic and repelling. This last condition does not give a uniform hyperbolic structure on the set of hyperbolic periodic points. Therefore the proof is much more indirect, and not based on induction.

For unimodal maps satisfying the negative Schwarzian derivative condition T. Nowicki proved that (C.E.1) implies (C.E.2), [No3]. More generally I would like to make the following

Conjecture. Let f be C^∞ and have no flat critical points. If satisfies the (C.E.1) condition and if all periodic orbits of f are hyperbolic and repelling then f also satisfies condition (C.E.2).

Organization of this Paper. Because no assumptions are made on Sf one has no a priori estimates on the nonlinearity of f^n. In Sects. 1, 2, and 3 we give some very general tools which enable us for any $n \geqq 0$ and any smooth map f to get control on the non-linearity of f^n on intervals I_n. These intervals I_n have to have the property that $\sum_{i=0}^{n-1} |f^i(I_n)|$ is universally bounded. In Sects. 4, 5 and 6, the disjointness of certain orbits of intervals and the Misiurewicz condition are combined to show that expansion is big along periodic orbits with high periods an in particular Theorem A is proved. In Sects. 7, 8 and 9 this big expansion along periodic orbits is used to show that $\sum_{i=0}^{n-1} |f^i(I_n)|$ is universally bounded for all $n \geqq 0$ and all intervals I_n such that $f^n|I_n$ is a diffeomorphism and $f^n(I_n) \cap B_0 = \emptyset$. Once this is known, Theorems B–D are quite easy to prove. This is done in Sects. 10–13.

In fact, using the ideas from Sects. 7, 8 it is possible to give a very short proof of the results of [Ma]. This proof will be published in a book entitled "One-dimensional dynamics" which is being written by W. de Melo and myself.

For convenience we will use the following notation: Let (x, y) (respectively $[x, y]$) be the smallest open (respectively closed) interval containing both x and y. The Lebesgue measure of a measurable set of $I \subset M$ is denoted by $|I|$.

Also we will use the following notation. We say that an interval I is **wandering** if $f^i(I) \cap f^j(I) = \emptyset$ for all $0 \leqq i < j$ and if no point of I is contained in the basin of a periodic attractor.

Finally we shall use the convention that $\sigma(t)$ (respectively $O(t)$) denotes a function such that $\sigma(t) \to 0$ as $t \to 0$ (respectively such that $O(t)/t$ is bounded) as $t \to 0$.

1. How Does a Map Distort Cross Ratios?

Later in this paper we need to get good estimates on, for example, the size of $f^{-n}(I)$ for large n and small intervals $I \subset M$. Since f has critical points we cannot hope to get a bound for the non-linearity of f^n. So instead of the affine structure, we will use the projective structure on $\mathbb{R}$. In this section we will make this precise by using the smoothness of f to measure the distortion of the cross-ratio of a pair of intervals.

Let M be either the circle S^1 or the interval $[0, 1]$, and $T \subset M$ be an open interval. Let $g: M \to M$ be a C^1 map with $g|T$ a diffeomorphism onto its image.

1.1 Definition. Let $J \subset T$ be open and bounded intervals such that $T - J$ consists of two non-trivial intervals L and R. Define two cross ratios of intervals as

$$C(T, J) = \frac{|J|\,|T|}{|L \cup J|\,|J \cup R|}, \quad D(T, J) = \frac{|J|\,|T|}{|L|\,|R|}, \tag{1.1}$$

where $|I|$ denotes the length of an interval I. If g is monotone on T we define

$$A(g, T, J) = \frac{C(g(T), g(J))}{C(T, J)}, \quad B(g, T, J) = \frac{D(g(T), g(J))}{D(T, J)}. \tag{1.2}$$

In Sect. 3 it will turn out that monotone maps $g:[0, 1] \to \mathbb{R}$ such that $A(g, T, J) \geqq 1$ (respectively $\geqq c > 0$) for all intervals $J \subset T \subset [0, 1]$ have many properties similar to those of conformal (respectively quasi-conformal) maps in the complex case. The main aim in Sects. 1–9 will be to estimate $A(f^n, T_n, J_n)$ from below for large n and appropriate intervals T_n and J_n. As pointed out in [M.S.] the operators A and B are related to the Schwarzian derivative of f:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{2}{3}\left(\frac{f'''(x)}{f'(x)}\right)^2.$$

In the following two results we give estimates for the distortion of these operators if f is C^2. In many papers on one-dimensional dynamics one only considers maps $f: M \to M$ such that $Sf(x) < 0$ for all $x \in M$. The main motivation for this assumption is that the class of maps with $Sf < 0$ is closed under iteration: if $Sf < 0$ then $Sf^n < 0$ for all $n > 0$. As we will see in the next lemma this implies that f^n expands cross ratios. So if we have assumed that $Sf < 0$ then there would have been no need for most of the results from Sects. 1–9. The next lemma also says that $Sf < 0$ near a non-flat critical point.

1.2 Lemma. *Let $J \subset T$ be intervals such that $T - J$ consists of two non-trivial intervals and $f: T \to \mathbb{R}$ be a C^3 map.*
a) *If $f|T$ is a diffeomorphism and $Sf(x) < 0$ for all $x \in T$ then $A(f,T,J) > 1$ and $B(f,T,J) > 1$.*
b) *If $c \in T$ is a non-flat critical point of f then there exists a neighbourhood U of c such that $Sf(x) < 0$ for every $x \in U - \{c\}$.*

Proof. Statement a) is well known and proved in for example [M.S.]. The proof of b) is elementary. Since f is non-flat at c, there exists $k \geqq 2$ such that $f^{(i)}(c) = 0$ for $i = 1, \ldots, k-1$, $f^{(k)}(c) \neq 0$ and f is $C^{\max(3,k)}$ near c. Therefore there exists $a \neq 0$ such that $f(x) = a \cdot (x-c)^k + \sigma(|x-c|^k)$, $f'(x) = ka \cdot (x-c)^{k-1} + \sigma(|x-c|^{k-1})$, $f''(x) = k(k-1)a \cdot (x-c)^{k-2} + \sigma(|x-c|^{k-2})$. Moreover since f is $C^{\max(3,k)}$, $f'''(x) = k(k-1)(k-2)a \cdot (x-c)^{k-3} + \sigma(|x-c|^{k-3})$. Hence $(f'(x))^2 \cdot Sf(x) = f'(x) f'''(x) - \frac{3}{2}(f''(x))^2 = a^2 \cdot (k^2(k-1)(k-2) - \frac{3}{2}k^2(k-1)^2) \cdot (x-c)^{2k-4} + \sigma(|x-c|)^{2k-4} = a^2 k^2 (k-1)((k-2) - \frac{3}{2}(k-1)) \cdot (x-c)^{2k-4} + \sigma(|x-c|^{2k-4})$. Since $k \geqq 2$, it follows that $Sf(x) < 0$ for x near c (and $x \neq c$). Q.E.D.

Remark. If f is just C^3 outside c, but just C^2 at c and $f^{(2)}(c) \neq 0$ then statement b) of Lemma 1.2 is not true in general. Take for example the function f defined for $x > 0$ by $f(x) = x^2 + \sin(1/x) \cdot x^{9/2}$ and let $f(0) = 0$. Then an explicit calculation shows that f is C^2 on $[0, \infty)$ and that for $x > 0$, $(f'(x))^2 Sf(x) = 2\cos(1/x) \cdot x^{-1/2} - 3/2 \cdot 4 + O(x^{1/2})$. In particular there exists a sequence of points $x_n \downarrow 0$ such that $Sf(x_n) \to \infty$.

The next proposition gives estimates for $A(f,T,J)$, $B(f,T,J)$ also when T is not close to some critical point.

1.3 Proposition. (*Bounded distortion from "projective maps" in the C^2 case.*) *Let $f: M \to M$ be a C^2 map and have no flat critical points. Then there exists a constant $C_0 \in (0, \infty)$ and an increasing function $\sigma: [0, \infty) \to (0, C_0)$ with $\lim_{t \to 0} \sigma(t) = 0$ such that if $T \supset J$ are intervals and $Df(x) \neq 0$ for all $x \in T$ then*

$$A(f,T,J) \geqq \exp\{-|L| \cdot \sigma(|R|)\}, B(f,T,J) \geqq \exp\{-|T| \cdot \sigma(|T|)\}, \tag{1.3}$$

where L and R are the connected components of $T \backslash J$.

Remarks

1. It is really essential in this proposition that f is C^2. It is **not** sufficient that f' is Lipschitz. (Below the proof of Theorem 2.3 an example is given.) This is in contrast with the usual bounded non-linearity results, see for example Proposition 1.4.
2. If in addition f'' is Lipschitz then one can prove under the same assumptions that there exists C_0 such that

$$A(f,T,J) \geqq \exp\{-C_0 |L| \cdot |R|\}, B(f,T,J) \geqq \exp\{-C_0 |T|^2\}, \tag{1.4}$$

see [M.S.]. Although not necessary, these improved estimates make some of the estimates in this paper more explicit.

Proof of Proposition 1.3. Let us prove (1.3) only for the operator A. The proof for the operator B is in fact easier. Let $K_2 = \sup_{x \in M} |Df(x)|$. Let T be an interval such that $f|T$ is a diffeomorphism. Let $J \subset T$ and write $T = [a,d]$, $J = [b,c]$, $L = [a,b]$ and $R = [c,d]$.

Take $\varepsilon > 0$. If $|L|, |R| \geqq \varepsilon$ then $\operatorname{dist}(J, C(f)) \geqq \varepsilon$ and from the mean-value theorem there exists a constant $K_1(\varepsilon) > 0$ such that for any such intervals J and T one has $|f(J)|/|J| \geqq K_1(\varepsilon)$ and $|f(T)|/|T| \geqq K_1(\varepsilon)$. Now

$$A(f,T,J) = \frac{\dfrac{|f(J)|}{|J|}\,\dfrac{|f(T)|}{|T|}}{\dfrac{|f(L\cup J)|}{|L\cup J|}\,\dfrac{|f(R\cup J)|}{|R\cup J|}}. \tag{1.5}$$

Hence if $|L|, |R| \geqq \varepsilon$, then

$$A(f,T,J) \geqq \left(\frac{K_1(\varepsilon)}{K_2}\right)^2.$$

So (1.3) follows if there exists a constant $C_0 < \infty$ and an increasing function

$$\sigma : [0, \operatorname{diam}(M)) \to (0, C_0)$$

with $\lim_{t\to 0} \sigma(t) = 0$ such that for any pair of intervals J and T as above

$$A(f,T,J) - 1 \geqq -|L| \cdot \sigma(|R|). \tag{1.6}$$

From Lemma 1.2b there exists a neighbourhood U of $C(f)$ such that $Sf(x) < 0$ for all $x \in U \setminus C(f)$. For later use assume that each component of U contains a point of $C(f)$. From the non-flatness condition we may also assume that U is chosen sufficiently small so that f' is monotone on each component of $U \setminus C(f)$.

Let us deal with several cases separately.

Case 1. First assume that $T \subset U$.

Then from Proposition 1.2a implies that $A(f,T,J) > 1$ for any $J \subset T$.

So we may from now on assume that T is not completely contained in U. Choose a neighbourhood V of $C(f)$ with $\operatorname{Clos}(V) \subset \operatorname{int}(U)$ and $K_3 > 0$ so that any interval I such that $f|I$ is a diffeomorphism and such that $|f(I)|/|I| \leqq K_3$ is contained in V. Let $K_4 \in (0,1)$ be so that the diameter of each of the components of $U - V$ is at least K_4.

Case 2. Now assume that $|f(c) - f(a)|/|c-a| \leqq K_3$ and that T is not a subinterval of U.

From the definition of K_3 and V, since we have assumed that T is not completely contained in U, and since $|f(c) - f(a)|/|c-a| \leqq K_3$ we have that $a, b, c \in V$ and $d \notin U$. Therefore there exists $c' \in C(f) \cap U$ such that $|a - c'| < |b - c'| < |c - c'|$. Since we had assumed that the function $t \to f'(t)$ is monotone one each component of $U \setminus C(f)$,

$$\frac{\dfrac{f(c)-f(b)}{c-b}}{\dfrac{f(c)-f(a)}{c-a}} \geqq 1.$$

Hence

$$A(f,T,J)-1=\frac{\dfrac{f(c)-f(b)}{c-b}\,\dfrac{f(d)-f(a)}{d-a}}{\dfrac{f(c)-f(a)}{c-a}\,\dfrac{f(d)-f(b)}{d-b}}-1\geqq\frac{\dfrac{f(d)-f(a)}{d-a}}{\dfrac{f(d)-f(b)}{d-b}}-1$$

$$=\frac{(f(d)-f(a))(d-b)-(f(d)-f(b))(d-a)}{(d-a)(d-b)}\cdot\frac{d-b}{f(d)-f(b)}$$

$$=\frac{(f(b)-f(d))(b-a)+(d-b)(f(b)-f(a))}{(d-a)(d-b)}\cdot\frac{d-b}{f(d)-f(b)}$$

$$=-\frac{(b-a)}{(d-a)}+\frac{(d-b)(f(b)-f(a))}{(d-a)(f(d)-f(b))}$$

$$\geqq-\frac{b-a}{d-a}. \tag{1.7}$$

Since, $a,b,c\in V$ and $d\in U$, we have $|d-a|$, $|d-c|>K_4$ and it follows that

$$\frac{A(f,T,J)-1}{|L|\,|R|}\geqq-\frac{b-a}{d-a}\cdot\frac{1}{(b-a)(d-c)}\geqq-\frac{1}{(d-a)(d-c)}\geqq-\frac{1}{(K_4)^2}.$$

So $(A(f,T,J)-1)/|L|\,|R|$ is bounded away from below. This completes the proof of (1.6) in this case.

Case 3. Now assume that $(|f(d)-f(b)|)/(|d-b|)\leqq K_3$ and that T is not a subinterval of U.

By interchanging the role of L and R one proves as in Case 2 again that $(A(f,T,J)-1)/|L|\,|R|$ is bounded away from below. Again this completes the proof of (1.6) in this case.

Case 4.

$$\frac{|f(c)-f(a)|}{|c-a|}\geqq K_3>0 \quad\text{and}\quad \frac{|f(d)-f(b)|}{|d-b|}\geqq K_3>0.$$

For $x\in M$ such that $x+a\in M$, define $\mu(a,x)$ by

$$f(a+x)=f(a)+\mu(a,x)\cdot x. \tag{1.8}$$

Since f is C^2 the function μ is uniformly continuous. We claim that

$$\Psi(a,x,y)=\begin{cases}\dfrac{\mu(a,x)-\mu(a,y)}{x-y}, & a+x,a+y\in M,\quad x\neq y\\[2ex] 0 & a+x,a+y\in M,\quad x=y\end{cases}$$

is continuous. Indeed, using the convention that $\sigma(\tau)$ stands for some bounded function such that $\sigma(\tau)\to 0$ as $\tau\to 0$, and using (1.8) we have for $x\neq y$,

$$\Psi(a,x,y)=\frac{y\cdot(f(a+x)-f(a))-x\cdot(f(a+y)-f(a))}{x\cdot y\cdot(x-y)}.$$

Since f is C^2, we get from Taylor's theorem

$$\Psi(a,x,y)=\frac{y\cdot\left[f'(a)x+f''(a)\dfrac{x^2}{2}+\sigma(x)x^2\right]-x\cdot\left[f'(a)y+f''(a)\dfrac{y^2}{2}+\sigma(y)y^2\right]}{x\cdot y\cdot(x-y)}$$

$$=\frac{f''(a)}{2}+\frac{(\sigma(x)x-\sigma(y)y)}{x-y}=\frac{f''(a)}{2}+\sigma(x)+\frac{y}{x-y}\cdot(\sigma(x)-\sigma(y)).$$

This proves that $\Psi(a,x,y)$ is continuous on R, where $R=\{(a,x,y);\,a,a+x,a+y\in M$ and $|y|/|x-y|\leqq 1\}$. Moreover, $\Psi(a,x,y)\to f''(a)/2$ as $(x,y)\in R$ and $(x,y)\to(0,0)$. If $|x-y|/|y|\leqq 1$ then write $x=(t+1)y$ and one has

$$\Psi(a,x,y)=\frac{y\left[f(a+y)+f'(a+y)ty+f''(a+y)\dfrac{(ty)^2}{2}+\sigma(ty)(ty)^2-f(a)\right]}{(1+t)ty^3}$$

$$-\frac{(1+t)y[f(a+y)-f(a)]}{(1+t)ty^3}$$

$$=\frac{f(a)-f(a+y)+f'(a+y)y+f''(a+y)\dfrac{ty^2}{2}+\sigma(ty)ty^2}{(1+t)y^2}.$$

Simplifying this, using $f(a)=f(a+y-y)=f(a+y)-f'(a+y)y+f''(a+y)y^2/2+\sigma(y)y^2$, gives

$$\Psi(a,(1+t)y,y)=\frac{f''(a+y)\frac{1}{2}+\sigma(y)+f''(a+y)\dfrac{t}{2}+\sigma(ty)t}{1+t}$$

$$=\frac{f''(a+y)}{2}+\frac{\sigma(y)+\sigma(ty)t}{1+t}.$$

It follows that $(a,t,y)\to\Psi(a,(1+t)y,y)$ is a continuous function on

$$\{(a,t,y);\,a,a+(1+t)y,a+y\in M\}$$

and that $\Psi(a,0,0)=f''(a)/2$. All this together implies that $\Psi(a,x,y)$ is continuous (and even uniformly continuous).

Write $\tilde{b}=b-a$, $\tilde{c}=c-a$ and $\tilde{d}=d-a$. Let us estimate $A(f,T,J)-1$ from below. Since $(|f(c)-f(a)|/|c-a|)\geqq K_3>0$ and $(|f(d)-f(b)|/d-b|)\geqq K_3>0$ one has

$$A(f,T,J)-1=\frac{\dfrac{f(c)-f(b)}{c-b}\cdot\dfrac{f(d)-f(a)}{d-a}-\dfrac{f(c)-f(a)}{c-a}\cdot\dfrac{f(d)-f(b)}{d-b}}{\dfrac{f(c)-f(a)}{c-a}\cdot\dfrac{f(d)-f(b)}{d-b}}$$

$$\geqq\frac{-1}{(K_3)^2}\cdot\left|\frac{f(c)-f(b)}{c-b}\cdot\frac{f(d)-f(a)}{d-a}-\frac{f(c)-f(a)}{c-a}\cdot\frac{f(d)-f(b)}{d-b}\right|$$

$$=\frac{-1}{(K_3)^2}\cdot\left|\left(\frac{\mu(a,c)\cdot\tilde{c}-\mu(a,b)\cdot\tilde{b}}{c-b}\right)\cdot\mu(a,d)-\mu(a,c)\cdot\left(\frac{\mu(a,d)\cdot\tilde{d}-\mu(a,b)\cdot\tilde{b}}{d-b}\right)\right|$$

$$= \frac{-1}{(K_3)^2} \cdot \left| \left(\mu(a,c) + \frac{\mu(a,c)-\mu(a,b)}{c-b} \cdot \tilde{b} \right) \cdot \mu(a,d) - \mu(a,c) \right.$$

$$\left. \cdot \left(\mu(a,d) + \frac{\mu(a,d)-\mu(a,b)}{d-b} \cdot \tilde{b} \right) \right|$$

$$= \frac{-1}{(K_3)^2} \cdot |\tilde{b}| \cdot \left| \frac{\mu(a,c)-\mu(a,b)}{c-b} \cdot \mu(a,d) - \frac{\mu(a,d)-\mu(a,b)}{d-b} \cdot \mu(a,c) \right|.$$

Hence

$$\frac{A(f,T,J)-1}{|L|} \geqq \frac{-1}{(K_3)^2} \cdot \left| \frac{\mu(a,c)-\mu(a,b)}{c-b} \cdot \mu(a,d) - \frac{\mu(a,d)-\mu(a,b)}{d-b} \cdot \mu(a,c) \right|$$

$$= \frac{-1}{(K_3)^2} \cdot | \Psi(a,c,b) \cdot \mu(a,d) - \Psi(a,d,b) \cdot \mu(a,c) |.$$

Since $\mu(a,x)$ and $\Psi(a,x,y)$ are uniformly continuous it follows that the function

$$\sigma(t) = \sup_{\{a,b,c,d;\, |d-c| \leqq t\}} | \Psi(a,c,b) \cdot \mu(a,d) - \Psi(a,b,b) \cdot \mu(a,c) |$$

is monotone increasing, uniformly bounded and $\sigma(t) \to 0$ as $t \to 0$. It follows that

$$A(f,T,J) - 1 \geqq -|L| \cdot \sigma(|R|) \cdot \frac{1}{(K_3)^2}.$$

Again this completes the proof in this case. Since we have dealt with all cases the proof of Proposition 1.3 is completed. Q.E.D.

So f cannot contract the cross-ratio too much. Similarly we will also use that f cannot be too non-linear away from the critical points.

1.4 Proposition. (*Bounded distortion from "linearity".*) *Let f be C^2 and let U be a neighbourhood of the set of critical points $C(f)$. Then there exists $C < \infty$ such that*

a) *for any interval J with $J \cap U = \varnothing$ one has*

$$\frac{|Df(x)|}{|Df(y)|} \leqq \exp\{C \cdot |J|\},$$

for all $x, y \in J$.
b) *for any interval J such that $f|J$ is diffeomorphism and any $x \in J \setminus U$ and one has*

$$|Df(x)| \geqq \exp\{-C \cdot |J|\} \cdot \frac{|f(J)|}{|J|}.$$

Proof. The proof is elementary.

2. The Distortion of Cross-Ratios and Non-linearity under Iterates

In the last section we obtained lower bounds for $A(f,T,J)$, $B(f,T,J)$. In this we also aim to get lower bounds for $A(f^n,T,J)$, $B(f^n,T,J)$ for any n and for appropriate intervals $J \subset T$. (If we had assumed that $Sf(x) < 0$ for all $x \in M$, then one would immediately have $A(f^n, T, J) \geqq 1$ and there would have been no need for this section.)

In this section we prove that f^n cannot contract the cross-ratios C and D too much.

2.1 Theorem. *Let $f: M \to M$ be a C^2 map those critical points are non-flat. Then there exists a bounded increasing function $\sigma: [0, \infty) \to \mathbb{R}_+$ such that $\sigma(t) \to 0$ as $t \to 0$ with the following property. If T is an interval such that f^m is a diffeomorphism on T then:*

$$A(f^m, T, J) \geqq \exp\left\{ -\sigma(\tau) \cdot \sum_{i=0}^{m-1} |f^i(T)| \right\},$$

$$B(f^m, T, J) \geqq \exp\left\{ -\sigma(\tau) \cdot \sum_{i=0}^{m-1} |f^i(T)| \right\}. \tag{2.1}$$

Here $\tau = \max_{i=0,\dots,n-1} |f^i(T)|$.

Proof. Since $A(f^m, T, J) = \prod_{i=0}^{m-1} A(f, f^i(T), f^i(J))$ this theorem is an immediate corollary of Proposition 1.3. Q.E.D.

2.2 Theorem. *Let f be a C^2 map with no flat critical points. There exists a bounded increasing functions $\sigma: [0, \infty) \to \mathbb{R}_+$ with $\sigma(t) \to 0$ as $t \to 0$ with the following property. Let $T \supset J$ be intervals such that $f^n | T$ is a diffeomorphism and such that $T \backslash J$ consists of two components L and R. Then*

$$A(f^n, T, J) \geqq \exp\left\{ -\sigma(\tau) \cdot \sum_{i=0}^{n-1} |f^i(L)| \right\}, \tag{2.2}$$

where $\tau = \max_{i=0,\dots,n-1} |f^i(R)|$.

Proof. From Proposition 1.3 one gets

$$A(f^n, T, J) \geqq \exp\left\{ -\sum_{i=0}^{n-1} (|f^i(L)|) \cdot \sigma(|f^i(R)|) \right\} \geqq \exp\left\{ -\sigma(\tau) \cdot \sum_{i=0}^{n-1} |f^i(L)| \right\}.$$

Q.E.D.

Remark. If f is C^3 there exists $C < \infty$ such that the function $\sigma(t)$ from Theorems 2.1 and 2.2 satisfies $\sigma(\tau) \leqq C\tau$, see the remark below the statement of Proposition 1.3.

The next result tells us roughly speaking the following. Assume that $J \subset T$ are intervals such that $|T| \leqq 2|J|$, $f^n | T$ is a diffeomorphism and $\sum_{i=0}^{n-1} |f^i(J)| \leqq 1$ and let T^1 and T^2 be the components of $T \backslash J$. Then $|f^n(T^1)|$ and $|f^n(T^2)|$ cannot both be much bigger than $|f^n(J)|$.

2.3 Theorem. "*Macroscopic Minimum Principle.*" *Let f be a C^2 map with no flat critical points. Then for every $\rho \in (1, \infty)$, $\rho_1 > 2\rho$ and $S < \infty$ there exists $\tau \in (0, 1)$ with the following property. Take $n > 0$ and let J be an arbitrary interval with*

$$\sum_{i=0}^{n-1} |f^i(J)| \leqq S.$$

Then there exists an endpoint x of J such that for any interval $T \supset J$ having x as one of its boundary points such that i) *$f^n|T$ is a diffeomorphism,* ii) *$|T| \leqq \rho \cdot |J|$ and* iii) *$|f^i(T \backslash J)| \leqq \tau, \forall i = 0, 1, \ldots, n-1$ one has*

$$|f^n(T)| \leqq \rho_1 \cdot |f^n(J)|. \tag{2.3}$$

Proof. Choose $\rho_1 > 2\rho$ and S as above. Let $\sigma(t)$ be the function of Proposition 1.3. Assume that $\tau > 0$ and $\sigma_0 > 0$ be so small that $\sigma(t) \leqq \sigma_0$ for all $t \in [0, \tau]$ and

$$\frac{2\rho - \sigma_0 \cdot S \cdot 2\rho}{1 - \sigma_0 \cdot S \cdot 2\rho} < \rho_1. \tag{2.4}$$

Since $\sigma(t) \to 0$ as $t \to 0$ this is possible.

Take the point γ in the middle of $\operatorname{int}(J) = (a, b)$. Either:

$$\frac{|f^n(b) - f^n(\gamma)|}{|b - \gamma|} \leqq \frac{|f^n(b) - f^n(a)|}{|b - a|} \tag{2.5a}$$

or

$$\frac{f^n(\gamma) - f^n(a)|}{|\gamma - a|} \leqq \frac{|f^n(b) - f^n(a)|}{|b - a|} \tag{2.5b}$$

(or both). Let us assume that (2.5a) holds. Then choose $x = a$ and let $T \supset J$ be an interval having $x = a$ as its boundary point and satisfying i)–iii). Write $T = [a, u]$, $J_0 = (\gamma, b)$, $L = [a, \gamma]$ and $R = [b, u]$. Let ρ_2 be so that

$$|T| = \rho_2 \cdot |J|.$$

Then $\rho_2 \leqq \rho$ and therefore (2.4) implies

$$\frac{2\rho_2 - \sigma_0 \cdot S \cdot 2\rho_2}{1 - \sigma_0 \cdot S \cdot 2\rho_2} < \rho_1. \tag{2.6}$$

Assuming that $\max_{i=0,1,\ldots,n-1} |f^i(R)| \leqq \tau$ we will show that (2.3) holds.

One has $T \backslash J_0 = L \cup R$ and since we are in the case a),

$$\frac{\dfrac{|f^n(J_0)|}{|J_0|}}{\dfrac{|f^n(L \cup J_0)|}{|L \cup J_0|}} = \frac{\dfrac{|f^n(b) - f^n(\gamma)|}{|b - \gamma|}}{\dfrac{|f^n(b) - f^n(a)|}{|b - a|}} \leqq 1. \tag{2.7}$$

Using Proposition 1.3 one has

$$\log A(f^n, T, J_0) \geqq -\sum_{i=0}^{n-1} (|f^i(L)|) \cdot \sigma(|f^i(R)|) \geqq -\sigma_0 \cdot \sum_{i=0}^{n-1} |f^i(L)| \geqq -\sigma_0 \cdot S. \tag{2.8}$$

On the other hand $|J_0 \cup R| / |T| = (2\rho_2 - 1)/2\rho_2$ and therefore, using (2.7),

$$A(f^n, T, J_0) - 1$$

$$= \frac{\dfrac{|f^n(J_0)|}{|J_0|} \dfrac{|f^n(T)|}{|T|}}{\dfrac{|f^n(L \cup J_0)|}{|L \cup J_0|} \dfrac{|f^n(J_0 \cup R)|}{|J_0 \cup R|}} - 1 \leqq \frac{\dfrac{|f^n(T)|}{|T|}}{\dfrac{|f^n(J_0 \cup R)|}{|J_0 \cup R|}} - 1$$

$$\leqq \left\{\frac{|J_0 \cup R|}{|T|} \cdot \frac{|f^n(J)| + |f^n(R)|}{|f^n(R)|}\right\} - 1 \leqq \left\{\left(\frac{2\rho_2 - 1}{2\rho_2}\right) \cdot \left(1 + \frac{|f^n(J)|}{|f^n(R)|}\right)\right\} - 1$$

$$= \frac{1}{2\rho_2}\left\{-1 + (2\rho_2 - 1) \cdot \frac{|f^n(J)|}{|f^n(R)|}\right\}. \tag{2.9}$$

Combining (2.8) and (2.9) one gets

$$\frac{|f^n(R)|}{|f^n(J)|} \leqq \frac{2\rho_2 - 1}{1 - \sigma_0 \cdot S \cdot 2\rho_2},$$

and hence

$$\frac{|f^n(T)|}{|f^n(J)|} = 1 + \frac{|f^n(R)|}{|f^n(J)|} \leqq \frac{1 - \sigma_0 \cdot S \cdot 2\rho_2 + 2\rho_2 - 1}{1 - \sigma_0 \cdot S \cdot 2\rho_2} = \frac{2\rho_2 - \sigma_0 \cdot S \cdot 2\rho_2}{1 - \sigma_0 \cdot S \cdot 2\rho_2}.$$

From (2.6) it follows that this last expression is at most ρ_1. Q.E.D.

Remark. In Theorems 2.1–2.3 it is essential that f is C^2. Indeed, if Df is only Lipschitz then Proposition 1.3 is not valid anymore. Take for example $f(x) = x + x|x|$, $L = [-4\varepsilon, -\varepsilon)$, $J = [-\varepsilon, \varepsilon]$, $R = (\varepsilon, 4\varepsilon]$. Then

$$\frac{B(f, T, J) - 1}{|T|} = \frac{\dfrac{(1+\varepsilon)(1+4\varepsilon)}{(1+5\varepsilon)^2} - 1}{8\varepsilon} = \frac{-5\varepsilon - 21\varepsilon^2}{8\varepsilon(1+5\varepsilon)^2} \to -\frac{5}{8} < 0 \quad \text{as} \quad \varepsilon \to 0.$$

2.4 Corollary. *Let $f: M \to M$ be a C^2 map without flat critical points. If f is unimodal or f satisfies the Misiurewicz condition then f does not have wandering intervals.*

Proof. The corollary is an immediate corollary of the proof in [M.S.] and the theorem above. Q.E.D.

Remark. Corollary (2.4) was first shown for maps with negative Schwarzian derivative and one critical point (without the Misiurewicz condition) by J. Guckenheimer, [Gu1]. Very recently A.M. Blokh and M. Ljubich [Lj] and [B.L.] have shown that C^2 maps $f: M \to M$ without flat critical points (and such that all critical points of f are local maxima or minima) have no wandering intervals. Their proof is based on very precise topological analysis of the dynamics of intervals maps and comined with the analytical tools of [M.S.]. (In [M.M.S.] this result has been generalized to general C^2 maps without flat critical points.)

We also will need a result to deal with the case "away from the critical points." In this case we get bounded distortion from non-linearity.

2.5 Theorem. *Let $f: M \to M$ be a C^2 map and let U be a neighbourhood of $C(f)$. There exists $C_1 < \infty$ with the following property. Let J be an interval and let n be such that $f^n|J$ is a diffeomorphism.*

a) *If $f^i(J) \cap U = \emptyset$, $\forall i = 0, \ldots, n-1$, then*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leqq \exp\left\{C_1 \cdot \sum_{i=0}^{n-1} |f^i(J)|\right\}, \tag{2.10}$$

for all $x, y \in J$.

b) *If* $x \in J$ *and* $f^i(x) \notin U$ *for all* $i = 0, 1, 2, \ldots, n-1$, *then*

$$|Df^n(x)| \geqq \exp\left\{-C_1 \cdot \sum_{i=0}^{n-1} |f^i(J)|\right\} \cdot \frac{|f^n(J)|}{|J|}. \tag{2.11}$$

Proof. The proof of this theorem is an immediate consequence of Proposition 1.4. Q.E.D.

This theorem is the main analytic tool in Mañé's paper [Ma]. As in Lemma 2.1 one can extend bounded non-linearity results to larger intervals. This is formulated in the following theorem, which is due to Schwartz [Sch], see also [Ni].

2.6 Theorem. *Let* $f: M \to M$ *be a* C^2 *map and* U *be a neighbourhood of* $C(f)$. *Then for every* $S < \infty$ *there exists* $\rho > 0$ *and* $C_2 < \infty$ *with the following property. Take* $n < 0$ *and let* J *be an arbitrary with*

$$\sum_{i=0}^{n-1} |f^i(J)| \leqq S \quad \textit{and} \quad f^i(J) \cap U = \varnothing, \quad \forall i = 0, \ldots, n-1.$$

Then for any interval $T \supset J$ *such that* $|T| \leqq (1+\rho) \cdot |J|$ *one has*

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leqq \exp\left\{C_2 \cdot \sum_{i=0}^{n} |f^i(J)|\right\}, \tag{2.12}$$

for all $x, y \in T$.

Proof. See [Sch, M.S.1 or Str2].

3. A Koebe Inequality for Bounded Cross-Ratio Maps

In the last section we got a lower bound for $A(f^n, T, J)$, $B(f^n, T, J)$ provided upper bounds for $\sum_{i=0}^{n-1} |f^i(T)|$ or $\sum_{i=0}^{n-1} |f^i(J)|$ are available. In this section we will show that a lower bound for $A(f^n, T, J)$ and $B(f^n, T, J)$ gives bounds on the type of non-linearity of $f^n | T$. Slightly shorter proofs of the results in this section can be given when one argues by contradiction, but then no explicit estimates are obtained.

3.a. Generalizing the "Minimum Principle." For maps with $S(f|T) < 0$ such that $f|T$ be a diffeomorphism, the derivative of $f|T$ is bounded from below by the derivative of f on ∂T. For C^2 maps which satisfy lower bounds on the cross-ratio operators a similar result is true. This result is the analogue of the maximum principle for conformal mappings.

3.1 "Minimum Principle." *Let* $g: T \to M$ *be a* C^1 *diffeomorphism with* $T = [a, b]$. *Let* $x \in (a, b)$. *If for any* $J^* \subset T^* \subset T$,

$$B(g, T^*, J^*) \geqq C_2 > 0,$$

then

$$|Dg(x)| \geqq C_2^3 \cdot \min(|Dg(a)|, |Dg(b)|).$$

Proof. The proof of this lemma can be found in [M.S.1]. Q.E.D.

Remark. This result is a infinitesimal version of the Macroscopic Minimum Principle from Sect. 2.

3.b. Generalizing the "Koebe Distortion Principle." The next result shows that having good bounds for the cross-ratio operator is almost as good as having bounded distortion. It is the analogue of the Koebe inequality for conformal mappings $f|T$ which gives an estimate of $Df(x)$ for points x such that $f(x)$ stays away from the boundary of $f(T)$. For maps with $S(f)<0$ a version of the corresponding property was first proved and used in [Str1] and reinvented in [Gu2].

3.2 "Koebe Distortion Principle." *For each $C_2, 0<\tau<\frac{1}{4}$ there exists $K<\infty$ with the following property. Let $g:T\to M$ be a C^1 diffeomorphism on some interval T. Assume that for any intervals J^* and T^* with $J^*\subset T^*\subset T$ one has*

$$B(g,T^*,J^*)\geqq C_2>0.$$

For an interval $J^*\subset T$ let L^* and R^* be the components of $T\backslash J^*$. Assume that

$$\frac{|g(L^*)|}{|g(T)|}\geqq\tau \quad \text{and} \quad \frac{|g(R^*)|}{|g(T)|}\geqq\tau.$$

Then

$$(*) \qquad \frac{1}{K}\leqq\frac{|g'(x)|}{|g'(y)|}\leqq K, \quad \forall x,y\in J^*,$$

$$(**) \qquad \frac{1}{K}\cdot\max_{z\in T}|g'(z)|\leqq|g'(x)|, \quad \forall x\in J^*,$$

and for every $x\in L^*\cup J^*$ one has

$$(***) \qquad \frac{|g(x,y)|}{|(x,y)|}\geqq\frac{1}{K}\cdot\frac{|g(T)|}{|T|}, \quad \forall y\in J^*\cup R^*,$$

$$(****) \qquad \frac{|g(x,y)|}{|(x,y)|}\geqq\frac{1}{K}\cdot\frac{|g(L^*\cup J^*)|}{|L^*\cup J^*|}, \quad \forall y\in J^*.$$

Proof. After scaling we can assume that $T=[0,1]$, $g(T)=[0,1]$ and that g is orientation preserving. Let us consider the following operators:

$$B_0(g,\tilde{T})=\frac{|g(\tilde{T})|^2}{|\tilde{T}|^2}\frac{1}{|Dg(\tilde{a})||Dg(\tilde{b})|},$$

where $\tilde{T}=[\tilde{a},\tilde{b}]\subset T$ and

$$B_1(g,\tilde{T},x)=\frac{|Dg(x)|\dfrac{|g(\tilde{T})|}{|\tilde{T}|}}{\dfrac{|g(\tilde{L})|}{|\tilde{L}|}\dfrac{|g(\tilde{R})|}{|\tilde{R}|}},$$

where $\tilde{L}$ and $\tilde{R}$ are the connected components of $\tilde{T} - \{x\}$. Observe that

$$B_0(g, \tilde{T}) = \lim_{\tilde{J} \to \tilde{T}} B(g, \tilde{T}, \tilde{J}).$$

$$B_1(g, \tilde{T}, x) = \lim_{\tilde{J} \to x} B(g, \tilde{T}, \tilde{J}).$$

Hence,

$$B_0(g, \tilde{T}), B_1(g, \tilde{T}, x) \geqq C_2 > 0,$$

for every $x \in \tilde{T} \subset T$.

Step 1. *Let* $\tau' = \frac{1}{2}\tau$. Let $a, b \in T = [0, 1]$ be such that $g(a) = \tau'$, $g(b) = 1 - \tau'$. Let $L = [0, a]$, $J = [a, b]$ and $R = [b, 1]$. (Notice that $J^* \subset J$.) Furthermore let

$$\lambda_1 = \frac{|g(L)|}{|L|}, \quad \rho = \frac{|g(J)|}{|J|}, \quad \lambda_2 = \frac{|g(R)|}{|R|},$$

$$\mu_1 = \frac{|g(L \cup J)|}{|L \cup J|}, \quad \mu_2 = \frac{|g(R \cup J)|}{|R \cup J|}. \tag{3.1}$$

In this step we obtain an estimate for $|g'(a)|$ and $|g'(b)|$ in terms of ρ. First of all, using $B_0(g, J) \geqq C_2$ we get

$$|g'(a)| \cdot |g'(b)| \leqq \frac{1}{C_2} \cdot \rho^2. \tag{3.2}$$

Also $B_1(g, L \cup J, a) \geqq C_2$, and hence

$$|g'(a)| \geqq C_2 \cdot \frac{\lambda_1 \cdot \rho}{\mu_1}. \tag{3.3}$$

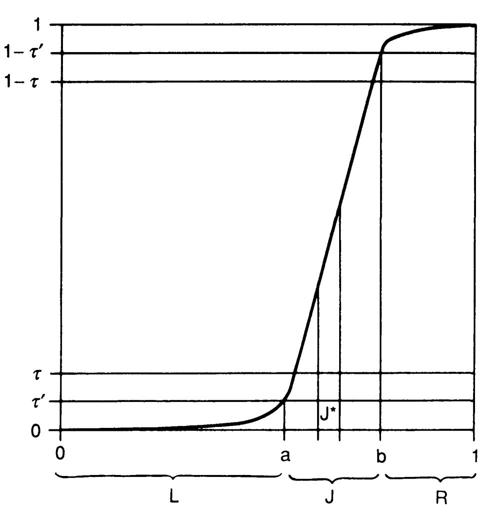

Fig. 1. The intervals L, J and R

Moreover

$$\frac{\lambda_1}{\mu_1}=\lambda_1\cdot\frac{|L\cup J|}{|g(L\cup J)|}=\lambda_1\cdot\frac{|L|+|J|}{1-\tau'}\geqq\lambda_1\cdot\frac{|L|}{1-\tau'}=\frac{\tau'}{1-\tau'}. \tag{3.4}$$

From (3.3) and (3.4) (and the corresponding estimate for b) we get

$$|g'(a)|\geqq C_2\frac{\tau'}{1-\tau'}\rho, \tag{3.5}$$

$$|g'(b)|\geqq C_2\frac{\tau'}{1-\tau'}\rho. \tag{3.6}$$

From (3.2), (3.5) and (3.6) that there exists $K'<\infty$, which only depends on τ' and C_2 and not on g, such that

$$\frac{1}{K'}\rho\leqq|g'(a)|,|g'(b)|\leqq K'\rho. \tag{3.7}$$

Step 2. Let us estimate $|g'(x)|$ for $x\in J$. From Lemma 3.1 one gets a lower bound for $|g'(x)|$:

$$|g'(x)|\geqq(C_2)^3\cdot\min\{|g'(a)|,|g'(b)|\}\geqq\frac{(C_2)^3}{K'}\cdot\rho. \tag{3.8}$$

Here the last inequality follows from (3.7). One obtains an upper bound for $|g'(x)|$ as follows. Let $U=[a,x]$, $V=[x,b]$. Since $U\cup V=J$, $U\cap V=\{x\}$, and $(|g(J)|/|J|)=\rho$, we have either $(|g(U)|/|U|)\leqq\rho$ or $(|g(V)|/|V|)\leqq\rho$. Suppose the former holds. (The second case is similar.) Then using $B_0(g,U)\geqq C_2$ one gets

$$|g'(a)||g'(x)|\leqq\frac{1}{C_2}\left\{\frac{|g(U)|}{|U|}\right\}^2\leqq\frac{1}{C_2}\rho^2. \tag{3.9}$$

With inequality (3.7) this gives

$$|g'(x)|\leqq\frac{\rho^2}{C_2|g'(a)|}\leqq\frac{K'}{C_2}\rho. \tag{3.10}$$

Together with (3.8) this proves that there exists $K''<\infty$ which only depends on τ' and C_2 (and not on g) such that

$$\frac{1}{K''}\rho\leqq|g'(x)|\leqq K''\rho, \tag{3.11}$$

for all $x\in J$. Therefore

$$\left(\frac{1}{K''}\right)^2\leqq\frac{|g'(x)|}{|g'(y)|}\leqq(K'')^2, \tag{3.12}$$

for all $x,y\in J$. Since $J^*\subset J$ inequality $(*)$ follows.

Step 3. Now we prove inequality $(**)$. Let $u\in T$ be so that $\max_{z\in T}|g'(z)|=|g'(u)|$. If $u\in J$ then $(**)$ easily follows from (3.12). So we may assume that $u\notin J$. To be definite assume that $u \in L$. Then using $B_0(g,[u,b]) \geqq C_2$ one gets

$$|g'(u)|\cdot|g'(b)| \leqq \frac{1}{C_2}\cdot\left(\frac{|g[u,b]|}{|[u,b]|}\right)^2 \leqq \frac{1}{C_2}\cdot\left(\frac{1-\tau'}{1-2\tau'}\right)^2\cdot\left(\frac{|g[a,b]|}{|[a,b]|}\right)^2$$
$$= \frac{1}{C_2}\cdot\left(\frac{1-\tau'}{1-2\tau'}\right)^2\cdot\rho^2.$$

Hence, using (3.7), there exists a constant K''' not depending on g but just on C_2 and τ' such that

$$\max_{z\in T}|g'(z)| = |g'(u)| \leqq K'''\cdot\rho. \tag{3.13}$$

Combining (3.11) and (3.13) implies

$$\frac{1}{K}\cdot\max_{x\in T}|g'(z)| \leqq |g'(x)| \quad \text{for all} \quad x\in J \tag{3.14}$$

where $K = K''\cdot K'''$. Since $J^* \subset J, (**)$ follows.

Step 4. Let us now prove $(***)$ and $(****)$. If $|g(x,y)| \geqq \tau' = \frac{1}{2}\tau$ then

$$\frac{|g(x,y)|}{|(x,y)|} \geqq \tau' = \tau'\cdot\frac{|g(T)|}{|T|}$$

and if moreover $(x,y) \subset L^* \cup J^*$,

$$\frac{|g(x,y)|}{|x,y|} \geqq \frac{|g(x,y)|}{|L^*\cup J^*|} \geqq \frac{\tau'}{|L^*\cup J^*|} \geqq \frac{|g(L^*\cup J^*)|}{|L^*\cup J^*|}\cdot\frac{\tau'}{1-\tau'}.$$

On the other hand if $|g(x,y)| \leqq \tau' = \frac{1}{2}\tau$ then since $g(x)\in[0,1-\tau] = [0,1-2\tau']$, $g(y)\in[\tau,1] = [2\tau',1]$ and $\tau < \frac{1}{4}$ this implies $g(x), g(y) \in [\tau', 1-\tau']$ and therefore $[x,y] \subset J$. But then (3.14) and the mean value theorem imply that

$$\frac{|g(x,y)|}{|(x,y)|} \geqq \frac{1}{K}\max_{z\in T}|g'(z)| \geqq \frac{1}{K}\frac{|g(T)|}{|T|}$$

and

$$\frac{|g(x,y)|}{|(x,y)|} \geqq \frac{1}{K}\max_{z\in T}|g'(z)| \geqq \frac{1}{K}\frac{|g(L^*\cup J^*)|}{|L^*\cup J^*|}.$$

So in either case we have proved $(***)$ and $(****)$. Q.E.D.

3.c. Preimages of Sets. The Minimum Principle can be used to prove the following reult. The proof of this result is not difficult and can be found in [N.S.2].

3.3 "Preimage Lemma." *For each C_2 there exists $K < \infty$ with the following property. Let $g: T \to M$ be a C^1 diffeomorphism on some interval $T = [\alpha, \beta]$. Assume that for any intervals J^* and T^* with $J^* \subset T^* \subset T$ one has*

$$B(g, T^*, J^*) \geqq C_2 > 0.$$

Let $\varepsilon > 0$ and $A_\varepsilon \subset M$ a measurable set with $|A_\varepsilon| = \varepsilon$. Let I_α and I_β be the maximal *intervals of length* $\leqq \varepsilon$ *which are contained in* $g(T)$ *and which contain* $g(\alpha)$ *and* $g(\beta)$ *respectively. Then*

$$|g^{-1}(A_\varepsilon)| \leqq K\cdot|g^{-1}(I_\alpha \cap I_\beta)|.$$

Proof. The proof follows from the Minimum Principle, see Lemma 6.1 of [N.S.2]. Q.E.D.

4. Orbits of Intervals with Disjointness Properties

In Sect. 2 it was shown that we could find a lower bound for $B(f^n, T, J)$ provided there is an upper bound for $\sum_{i=0}^{n-1} |f^i(T)|$. If the intervals $f^i(T)$, $i=0,1,2,\ldots,n-1$ are all disjoint then we have a very obvious upper bound: $\sum_{i=0}^{n-1} |f^i(T)| \leqq |M|$. So let us give a sufficient condition for $T, f(T), \ldots, f^{n-1}(T)$ to be disjoint.

Let I and J be subsets of M and $f: M \to M$ some mapping. Let

$$r(n) = \operatorname{card}\{i \mid f^i(I) \cap J \neq \emptyset, 0 \leqq i \leqq n-1\}.$$

We need the following set-theoretic lemma.

4.1 Lemma. *If* $f^n(I) \subset J$ *then each point in* M *is contained in at most* $r(n)$ *of the sets*

$$I, f(I), f^2(I), \ldots, f^{n-1}(I).$$

Proof. Suppose some point x of M is contained in l intervals

$$f^{i(1)}(I), f^{i(2)}(I), \ldots, f^{i(l)}(I),$$

where $0 \leqq i(1) < i(2) < \cdots < i(l) < n$. Let $j = n - i(l)$. Then $f^j(x)$ is contained in the f^j-images

$$f^{j+i(1)}(I), f^{j+i(2)}(I), \ldots, f^n(I)$$

of all these intervals. Since

$$0 < j + i(1) < j + i(2) < \cdots < j + i(l) = n,$$

and $f^j(x) \in f^n(I) \subset J$ this implies that $l \leqq r(n)$. Q.E.D.

From this last lemma we can get the following result. Let p be a repelling periodic point of period n. Let

$$\tilde{n} = \begin{cases} n, & \text{if } Df^n(p) > 0, \\ 2n, & \text{if } Df^n(p) < 0. \end{cases}$$

Let I be the maximal interval such that $p \in I$, $f^n|I$ is a diffeomorphism and $f^n(I) \cap O(p) = \{p\}$. Similarly let $\tilde{I}$ be the maximal interval such that $p \in \tilde{I}$, $f^{\tilde{n}}|\tilde{I}$ is a diffeomorphism and such that $f^{\tilde{n}}(\tilde{I}) \cap O(p) = \{p\}$.

4.2 Lemma. *Let* $p, n, \tilde{n}, I$ *and* $\tilde{I}$ *be as above. Then each point of* M *is contained in at most three of the intervals*

$$I, f(I), \ldots, f^{n-1}(I),$$

and in at most six of the intervals

$$\tilde{I}, f(\tilde{I}), \ldots, f^{\tilde{n}-1}(\tilde{I}).$$

Proof. Let us just prove the second statement. Since $f^{\tilde{n}}(\tilde{I}) \cap O(p) = \{p\}$, we have

$$f^i(\tilde{I}) \cap O(p) = \{f^i(p)\} \tag{4.1}$$

for every $0 \leqq i < n$. Let J be the maximal interval containing $\{p\}$ such that $J \cap O(p) = \{p\}$. Notice that if some interval L intersects J but is not contained in J then it has to contain an endpoint of J which belongs to $O(p)$ (otherwise J would not be maximal). Therefore if $0 \leqq i < \tilde{n}$ and $f^i(\tilde{I}) \cap J \neq \emptyset$ then $f^i(\tilde{I}) \cap \mathrm{Clos}(J) \cap O(p) \neq \emptyset$, and therefore, using (4.1),

$$f^i(p) \in \mathrm{Clos}(J) \cap O(p) \subset \partial J \cup \{p\}.$$

From the choice of $\tilde{n}$ there are at most six i's with $0 \leqq i < \tilde{n}$ for which $f^i(p) \in \partial J \cup \{p\}$. Applying Lemma 4.1 completes the proof. Q.E.D.

5. Branch-Intervals of f^n and Wandering Intervals

We say that an interval I is a **wandering interval** if $I, f(I), f^2(I), \ldots$ are all disjoint and if I is not contained in the basin of a periodic attractor. In [M.S.1] it was shown that C^2 maps satisfying the Misiurewicz condition (i) such that all of its critical points are non-flat cannot have wandering intervals. More recently, based on the analytic techniques in [M.S.1], A. M. Blokh and M. Ljubich, [Lj] and [B.L.2], have shown that general C^2 interval and circle maps without flat critical points (and such that all critical points are local extrema) cannot have wandering intervals. In this section we will show that for large n, many intervals I_n exist which are extremely small and such that $|f^n(I_n)|$ is not too small. Later this will be used to show that f is globally expanding. The conclusions in this section are based on the non-existence of wandering intervals and on the Misiurewicz condition, without using properties related to the smoothness of the map f.

As before the **basin** $B(K)$ of an invariant set $K, f(K) \subset K$, is the set

$$B(K) = \{x; f^n(x) \to K \text{ as } n \to \infty\}.$$

The union of the components $B_0(K)$ of $B(K)$ containing points of K will be called the **immediate basin** of K. Notice that $f(B(K)) \subset B(K)$. We say that a periodic point is a (possibly one-sided) **attractor** if $B_0(O(p))$ contains an interval. Let B be the **basin of periodic attractors** and B_0 be the **immediate basin of periodic attractors**. More precisely, $B = B(A)$ and B_0 is the union of the components of $B(A)$ which contain points of A, where A is the set of all periodic attractors of f. Let I be a component of B_0. Then for some $k, f^k(I) \subset I$. Moreover if $I \cap \partial M = \emptyset$ then $f^k(\partial I) \subset \partial I$ and in particular one of the boundary points of I is a fixed point of f^k and the other boundary point I is either a fixed point of f^k or mapped by f^k on the first boundary points. If $I \cap \partial M$ consists of one point then the other boundary point of I is a fixed point of f^k.

If $M = [0, 1]$, by extending f to a slightly bigger interval, we may assume that

$$f(\partial M) \subset \partial M,$$

and so at least one of the boundary points of M is periodic with period $\leqq 2$. Without loss of generality we may also assume that the periodic point(s) of f in ∂M are hyperbolic. (Notice that we can choose this extension in such a way that all points in the interior of the bigger interval will eventually be mapped into the original interval. From this it follows that it suffices to prove Theorems A–D for the extended map.) From now on we will make these assumptions if $M=[0,1]$. We say that I_n is a **branch-interval of** f^n if I_n is a maximal interval for which $f^n|I_n$ is a diffeomorphism.

If $M=S^1$ then, in order to make sure that branch-intervals of f^n either coincide or are disjoint, we have to be a bit more precise. If $\#C(f)>0$, then choose and *fix* some arbitrary point $x_0 \in C(f)$. If $C(f)=\emptyset$ then, since f is not a circle diffeomorphism, $|\deg(f)|>1$ and we can choose some fixed point $x_0 \in S^1$ of f. Then I_n is a **branch-interval** of f^n if it is a maximal interval such that $f^n|I_n$ is a diffeomorphism and $x_0 \notin f^n(I_n)$. Notice that if I_n is a branch-interval of f^n then

$$x_0 \notin f^i(I_n), \quad \forall i=0,1,\dots,n-1. \tag{5.1}$$

In fact if $C(f)=\emptyset$ this is true since x_0 is a fixed point of f. If $C(f)\neq\emptyset$ this holds since $f^n|I_n$ is a diffeomorphism and since $x_0 \in C(f)$.

Similarly we say I_n is a $*$**-branch-interval** for f^n if it is a maximal interval such that I_n is contained in a branch-interval of f^n and such that furthermore $f^n(I_n)\cap \mathrm{Clos}(B_0)=\emptyset$.

For simplicity of notation let $C_+(f)=C(f)\cup\partial M$ if $M=[0,1]$ and $C_+(f)=C(f)\cup\{x_0\}$ if $M=S^1$. (Remember that we had assumed that $f(\partial M)\subset\partial M$.) Notice that I_n is a branch-interval of f^n if and only if it is a maximal interval with the property that $\mathrm{int}(f^i(I_n))\cap C_+(f)=\emptyset$ for $i=0,\dots,n-1$. So from the assumption on f it follows that either

$$f^j(C_+(f))\subset f^j(C(f))\cup\partial M, \quad \forall j\geqq 0 \tag{5.2a}$$

or

$$C(f)=\emptyset \quad\text{and}\quad f^j(C_+(f))=\{x_0\}, \quad \forall j\geqq 0. \tag{5.2b}$$

5.1 Lemma. *Assume that $f\colon M\to M$ is not injective and has no wandering intervals. For each $\delta>0$ there exist $k_0, l_0 \in \mathbb{N}$ such that for any interval T such that $|T|\geqq\delta$ and such that $f^n|T$ is a diffeomorphism for all $n\geqq 1$, there exist $1\leqq k\leqq k_0$, $0\leqq l\leqq l_0$ and an interval L such that $f^k|L$ is a diffeomorphism, $f^k(L)\subset L$, $f^l(T)\subset L$ (and therefore each point of T is in the basin of a fixed point of $f^k|L$ or $f^{2k}|L$).*

Proof. If T is contained in the basin of a periodic attractor then this lemma is trivially true. So assume that T is not (completely) contained in the basin of a periodic attractor.

First we claim that there exist $k_0<\infty$ and $l_0<\infty$ such that for any interval T as above, and which is not (completely) contained in a basin of a periodic attractor, there exist $k\in\{1,2,\dots,k_0\}$ and $l\in\{0,1,\dots,l_0\}$ such that $f^l(T)\cap f^{l+k}(T)\neq\emptyset$. Indeed, otherwise there exist a sequence of intervals T_i with $|T_i|\geqq\delta$, and $n(i)\to\infty$ such that $f^l(T_i)\cap f^m(T_i)=\emptyset$ for all $0\leqq l$, $m\leqq n(i)$ with $l\neq m$. By taking a subsequence we get an interval T such that $T_i\supset T$ for infinitely many i's and therefore $f^l(T)\cap f^m(T)=\emptyset$ for all $l,m\geqq 0$, $l\neq m$. But since T is not contained in the basin of a periodic attractor this implies that T would be a wandering interval, a contradiction.

So there exists k_0 and l_0 such that for any interval T which is as above and not contained in the basin of a periodic attractor, there are integers l and k with $1 \leqq l \leqq l_0$, $0 < k \leqq k_0$ such that $f^l(T) \cap f^{l+k}(T) \neq \emptyset$. Write $T_0 = f^l(T)$. Then $f^{jk}(T_0) \cap f^{(j+1)k}(T_0) \neq \emptyset$, $\forall j \geqq 0$. Hence $L = \bigcup_{j \geqq 0} f^{jk}(T_0)$ is an interval and f^k maps L diffeomorphically into itself. Since f is not injective, L is a proper subinterval of M. The lemma follows. Q.E.D.

Remark. It is not hard to give a finite algorithm which, given a map f as in Lemma 5.1, finds an upper bound for k_0 and l_0.

5.2 Lemma. *Assume that $f: M \to M$ is not injective and has no wandering intervals. For each $\delta > 0$ there exist $k_1 \in \mathbb{N}$ and $\delta' \in (0, \delta)$ with the following property. Let T be an interval with $|T| \geqq \delta$ and $f^n | T$ a diffeomorphism. Suppose that one of the following holds*:

a) $f^n(T) \cap B_0 = \emptyset$;
b) *all periodic orbits of f of period $\leqq k_1$ are hyperbolic and $f^n(T)$ is not (completely) contained in B_0*;
c) *T contains a periodic point of period greater than k_1 and $f^{k_1}(T)$ contains no periodic point of period less than k_1*;
d) $n \leqq 7 \cdot k_1$.

Then

$$|f^n(T)| \geqq \delta'.$$

Proof. Let k_0 and l_0 be the integers from Lemma 5.1 corresponding to δ. Let $k_1 = 2k_0 + l_0$. Assume by contradiction that we can take a sequence of intervals T_i satisfying a), b) or c) and integers $n(i) \to \infty$ such that for every $i \geqq 0$, $|T_i| > \delta$, $f^{n(i)} | T_i$ is a diffeomorphism and such that $\lim_{i \to \infty} |f^{n(i)}(T_i)| = 0$. By taking a subsequence we may assume that there exists a limit T of T_i such that $|T| \geqq \delta$, $f^n | T$ a diffeomorphism for all $n \geqq 0$, and finally $|f^{n(i)}(T)| \to 0$ for some sequence $n(i) \to \infty$.

From Lemma 5.1 it follows that there exists an interval L and $k \leqq k_0$, $l \leqq l_0$ such that $f^k | L$ maps L diffeomorphically into itself, $f^k(L) \subset L$, and such that $T' = f^l(T) \subset L$. By assumption

$$|f^{n(i)-l}(T')| \to 0 \quad \text{as} \quad i \to \infty. \tag{5.3}$$

This implies that T^i is contained in the immediate basin of an attracting fixed point of $f^k: L \to L$. Hence $f^l(T) = T' \subset B_0$ and we get a contradiction if a) holds.

If b) holds then $f^k: L \to L$ has only hyperbolic fixed points, and therefore (5.3) implies that $\mathrm{Clos}(T')$ is contained in the basin of some (hyperbolic) attracting fixed point of $f^k | L$. But this would imply $f^l(T) = T' \subset \mathrm{int}(B_0)$, a contradiction.

If c) holds then notice that L contains at least one periodic point of period k and no periodic points of other periods. Since T_i contains a periodic point of period greater than $k_1 = 2k_0 + l_0$ and since $f^l(T_i) \to f^l(T) \subset L$, it follows that $f^l(T_i)$ contains one of the boundary points a of L (and $a \notin \partial M$). But by assumption, $f^l(T_i)$ contains no periodic points of period less than k_1. Since $a \notin \partial M$, $f^k(a)$ has period k and we obtain a contradiction.

If d) holds then the result is trivial. Q.E.D.

Remark. Again there exists a finite algorithm which, for each $\delta > 0$, gives a lower bound for δ'.

The following corollary tells us that we can shrink branch-intervals I_n of f^n so that $|f^j(I_n)|$ is not too big for all $0 \leqq j \leqq n$ and so that at the same time $|f^n(I_n)|$ is not too small.

5.3 Corollary. *Assume that $f: M \to M$ is not injective and has no wandering intervals. Take $\delta > 0$ and let $k_0 \in \mathbb{N}$ and $\delta' \in (0, \delta)$ be the corresponding numbers from Lemmas 5.1 and 5.2 respectively. Take an interval I_n such that $f^n | I_n$ is a diffeomorphism and such that $|f^n(I_n)| \geqq \delta$. Let $I'_n \subset I_n$ be a maximal interval such that $|f^j(I'_n)| \geqq \delta$ for all $0 \leqq j \leqq n$. Assume one of the following holds:*

a) $f^n(I'_n) \cap B_0 = \emptyset$;
b) $f^n(I'_n)$ *is not (completely) contained in* B_0 *and all periodic orbits of f of period $\leqq k_0$ are hyperbolic.*

Then

$$|f^n(I'_n)| \geqq \delta'.$$

Proof. By maximality $|f^j(I'_n)| = \delta$ for some $0 \leqq j \leqq n$. Taking $T = f^j(I'_n)$ the result follows from Lemma 5.2. Q.E.D.

In the next two results we will require that f is C^2, has no flat critical points and satisfies the Misiurewicz condition (i). Then Corollary 2.4 implies that f has no wandering intervals. (The Misiurewicz condition (i) implies that f is not injective and therefore we can apply Theorem 5.2 to f.) If the Misiurewicz condition (i) holds then (5.2) implies that we can choose $\delta_0 > 0$ such that if $I = [x, y]$ is a (non-trivial) interval then

$$x \in \bigcup_{k \geqq 0} f^k(C_+(f)), \quad y \in C_+(f) \Rightarrow |I| \geqq 4 \cdot \delta_0. \tag{5.4}$$

For later use let N_0 be so that $c \in C(f)$ and $f^i(c) \in C(f)$ implies that either $i \leqq N_0$ or that c has period $\leqq N_0$. If $C(f) \neq \emptyset$ we choose neighbourhoods $U_0 \subset V_0 \subset W_0$ of $C(f)$ such that each component of W_0 contains precisely one point of $C(f)$, such that each component of $W_0 \backslash V_0$, $V_0 \backslash U_0$, $U_0 \backslash C(f)$ has at least length δ_0 and such that

$$f^n(C(f)) \cap W_0 \subset C(f), \quad \forall n > 0. \tag{5.5a}$$

Moreover, choose these neighbourhoods (and $\delta_0 > 0$) so that if c is a non-periodic point of f such that $f^i(c) = c' \in C(f)$ for some $i > 0$ then

$$f^i \text{ maps a component of } I \backslash \{c\} \text{ diffeomorphically onto a component of } I \backslash \{c'\} \tag{5.5b}$$

for $I = U_0, V_0$ or W_0. Because $i \leqq N_0$ this last condition can easily be satisfied. (Condition (5.5b) is later needed to take care of additional complications that arise when $\bigcup_{i\geq 1} f^i(C(f))\cap C(f)\neq\emptyset$.) Let $\delta_0'\in(0,\delta_0)$ be equal to the number δ' corresponding to $\delta=\delta_0$ from Lemma 5.2. We will keep these numbers δ_0,δ_0' fixed throughout the remainder of this paper.

In the next corollary we will show that images under f^n of branch-intervals of f^n cannot be too small.

5.4 Corollary. *Assume that $f:M\to M$ is C^2, has no flat critical point and satisfies the Misiurewicz condition* (i). *Furthermore assume that all periodic orbits of f are hyperbolic. Let I_n be a branch-interval of f^n such that $f^n(I_n)$ is not completely contained in B_0. Then $|f^n(I_n)|\geqq\delta_0'$.*

If additionally $\mathrm{Clos}(B_0)$ *consists of at most a finite number of intervals then there exists a number $\tilde{\delta}_0'>0$ such that $|f^n(I_n)|\geqq\tilde{\delta}_0'$ for every $*$-branch-interval I_n of f^n.*

Proof. Let us first prove the result if I_n is a branch-interval. Let $\partial I_n=\{a_n,b_n\}$. From the maximality of I_n there exists $i\leqq j<n$ such that $f^i(a_n)\in C_+(f)$ and $f^j(b_n)\in C_+(f)$. If $i=j$ then $f^j(I_n)$ contains two distinct points of $C_+(f)$. Hence $|f^j(I_n)|\geqq\delta_0$. If $i<j$ then $f^j(a_n)\in\bigcup_{k\geq 1} f^k(C_+(f))$ and also $f^j(b_n)\in C_+(f)$. Again from the choice of δ_0 this implies that $|f^j(I_n)|\geqq\delta_0$. From Lemma 5.2 it follows that in both cases $|f^n(I_n)|\geqq\delta_0'$.

Let us now prove the result for $*$-branch-intervals of f^n. So suppose that $\mathrm{Clos}(B_0)$ consists of a finite number of intervals. Let N be a multiple of the period of each of the periodic points in $\mathrm{Clos}(B_0)$. Let I be the finite union of intervals such that each boundary points of $\mathrm{Clos}(B_0)$ is contained in precisely one component of I and such that I is the maximal set in $M\setminus\mathrm{Clos}(B_0)$ such that f^N is a diffeomorphism on each component of I. If for $c\in C(f)$ there exists an integer $i\in\mathbb{N}$ and a one-sided neighbourhood J of c such that $f^i(c)\in I$, $f^i|J$ is a diffeomorphism, and $f^i(J)$ is contained in I and contains the boundary point of I which is in $\mathrm{Clos}(B_0)$, then let $i(c)$ be the minimal such integer. Choose $\tilde{\delta}_0\in(0,\delta_0)$ such that the distance between endpoints of $\mathrm{Clos}(B_0)$, $C(f)$ and $\{f^{i(c)}(c);\ c\in C(f)$ such that $i(c)$ exists$\}$ is at least $\tilde{\delta}_0$. Let $\tilde{\delta}_0'\in(0,\tilde{\delta}_0)$ be equal to the number δ' corresponding to $\delta=\tilde{\delta}_0$ from Lemma 5.2.

Now let $I_n=(a_n,b_n)$ be a $*$-branch-interval of f^n. Then from maximality there exist $0\leqq i,j<n$ such that $f^i(a_n)\in C_+(f)\cup\partial(\mathrm{Clos}(B_0))$, $f^j(a_n)\in C_+(f)\cup\partial(\mathrm{Clos}(B_0))$. If $f^i(a_n), f^j(a_n)\in C_+(f)$ then the proof goes as in the case that I_n is a branch-interval of f^n. If $f^i(a_n)$, $f^j(b_n)\in\partial\mathrm{Clos}(B_0))$ then $f^n(a_n)$, $f^n(b_n)\in\partial(\mathrm{Clos}(B_0))$ and $|f^n(I_n)|\geqq\tilde{\delta}_0\geqq\tilde{\delta}_0'$. Now assume $f^i(a_n)\in C_+(f)$ and $f^j(b_n)\in\partial(\mathrm{Clos}(B_0))$. If $j\leqq i$ then $f^i(b_n)\in\partial(\mathrm{Clos}(B_0))$, and since $f^i(a_n)\in C_+(f)$, $|f^i(I_n)|\geqq\tilde{\delta}_0$. Using Lemma 5.2, $|f^n(I_n)|\geqq\tilde{\delta}_0'$. Finally if $i<j$ then from the choice of $\tilde{\delta}_0$ and since $f^j(b_n)\in\partial(\mathrm{Clos}(B_0))$ one gets $|f^j(I_n)|\geqq\tilde{\delta}_0$. Again using Lemma 5.2 one gets $|f^n(I_n)|\geqq\tilde{\delta}_0'$. Q.E.D.

Now we will show that branch-invervals containing critical values of f have images which are not too small in "both directions."

5.5 Corollary. *Assume that f satisfies the Misiurewicz conditions* (i), *has no wandering intervals and that f and that all periodic orbits of f are hyperbolic and let δ_0' be as above. Take $c\in C(f)$, $n\geqq 0$, and the branch-interval I_n of f^n containing*

$f(c)$. If $f^n(I_n)$ is not (completely) contained in B_0 then one has

$$|f^n(I_n^i)| \geqq \delta_0', \quad i=1,2;$$

here I_n^i are the components of $I_n \backslash f(c)$.

If, additionally, $\mathrm{Clos}(B_0)$ *consists of at most a finite number of intervals then let $\tilde{\delta}_0' > 0$ be the number from Corollary 5.4. Then for any $*$-branch-interval I_n of f^n containing $f(c)$ one has $|f^n(I_n^i)| \geqq \tilde{\delta}_0'$, $i=1,2$ for every $*$-branch-interval I_n. Here I_n^1, I_n^2 are the components of $I_n \backslash f(c)$.*

Proof. Let us just prove the corollary if I_n is a branch-interval of f^n. Consider for example $I_n^1=(a_n,b_n)=(a_n,f(c))$. Then there exists $0 \leqq i < n$ such that $f^i(a_n) \in C_+(f)$. Since $f^i(a_n) \in C_+(f)$ and $f^i(b_n)=f^{n+1}(c) \in \bigcup_{j \geqq 1} f^j(C(f))$ one gets $|f^i(I_n^1)| \geqq \delta_0$. From Lemma 5.2 the result follows for $I_n^1=(a_n,f(c))$. The proof for I_n^2 is the same. Q.E.D.

5.6 Remark. In the results 5.1–5.3, the assumption that f is not injective and does not have wandering intervals can be replaced by the assumption that f is C^2, has no flat critical points and satisfies the Misiurewicz condition.

Proof. In [M.S.1] it was shown that C^2 maps satisfying the Misiurewicz condition and having no flat critical points, cannot have wandering intervals. Q.E.D.

6. The Proof of Theorem A: The Finiteness of the Period of Attractors

In this section we prove that expansion along periodic orbits increases as the period increases. In later sections we sharpen this in an essential way.

6.1 Theorem. *Let f be a C^2 map such that all critical points of f are non-flat. Furthermore suppose that f satisfies the Misiurewicz condition* (i). *Then there exists a sequence K_n with $K_n \to \infty$ as $n \to \infty$ such that if p is a periodic point and n the period of p, then*

$$|Df^n(p)| \geqq K_n. \tag{6.1}$$

Let p be a periodic point of period n. Then choose $k=n$ if $Df^n(p) \geqq 0$ and $k=2n$ if $Df^n(p)<0$. Then $Df^k(p) \geqq 0$. Let J be a maximal interval containing p such that $f^k|J$ is a diffeomorphism and such that $f^k(J) \cap O(p)=\{p\}$. From Lemma 4.2 we know that each point of M is contained in at most six of the intervals $J, f(J), \ldots, f^{k-1}(J)$ and therefore

$$\sum_{i=0}^{k-1} |f^i(J)| \leqq 6 \cdot |M|.$$

Notice that this disjointness also implies that for $0 \leqq i < k$, $f^i(J)$ contains no periodic points of period less than $(k-i)/6$.

First we will prove two lemmas related to the results from Sect. 5.

6.2 Lemma. *For each $\delta > 0$ there exists $k_0 < \infty$ such that for any interval J as above $|J| \geqq \delta$ implies $k \leqq k_0$. In particular there exists a sequence K_k' with $K_k' \to \infty$ as $k \to \infty$*

such that for any k and any interval J as above

$$\frac{1}{|J|} \geqq K_k'. \tag{6.2}$$

Proof. Let $k_0, l_0 \in \mathbb{N}$ be the integers from Lemma 5.1 corresponding to δ. Let $|J| \geqq \delta$. From Lemma 5.1 there exists $0 \leqq l \leqq l_0$ and $1 \leqq \tilde{k} \leqq k_0$ such that $f^l(J)$ is contained in an interval L and $f^{\tilde{k}}|L$ maps L diffeomorphically into itself; in particular each periodic point in L has at most period k_0. Since $f^l(p)$ is a periodic point of period k and since $f^l(p) \in f^l(J) \subset L$ it follows that $k \leqq k_0$. Q.E.D.

6.3 Lemma. *Let $\delta > 0$ and $\delta' \in (0, \delta)$ be the number corresponding to δ from Lemma 5.2. Then for any interval J as above and any J_* for which $p \in J_* \subset J$ and with $|f^i(J_*)| \geqq \delta$ for some $0 \leqq i \leqq k$ one has $|f^k(J_*)| \geqq \delta'$.*

In particular, if $|f^k(J)| \geqq \delta$, then for any maximal interval $J_* \subset J$ such that $p \in J_*$ and $|f^i(J_*)| \leqq \delta$ for all $i = 0, 1, \ldots, k$, one has $|f^k(J_*)| \geqq \delta'$.

Proof. Let k_1 be the number from Lemma 5.2 corresponding to δ. If $k - i \leqq 7k_1$ then Lemma 5.2d implies $|f^k(J_*)| \geqq \delta'$. If $k - i > 7k_1$ then $f^{i+k_1}(J_*)$ contains no periodic point of period less than $(k-(i+k_1)/6) > k_1$. Since $f^i(J_*)$ contains a periodic point of period $k \geqq k_1$ Lemma 5.2c implies again $|f^k(J_*)| \geqq \delta'$. Q.E.D.

Next we state and prove a lemma which gives sufficient conditions for $(|f^k(J)|)/|J|$ to be big for large k. Let δ_0 and δ_0' be the numbers which are chosen in Sect. 5 (above Corollary 5.4).

6.4 Lemma. *For each f as above, there exists a function $\rho''(t)$ such that $\rho''(t) \to \infty$ as $t \to 0$ with the following property. Let J and k be as above and let J^i be the components of $J \backslash \{p\}$. Let T be an interval containing p such that $f^k|T$ is a diffeomorphism and such that for the components T^i of $T \backslash \{p\}$ one has $T^i \supset J^i$, $f^k(J^i) \supset T^i$ and $|f^k(T^i)| \geqq \delta_0$ for $i = 1, 2$. Then*

$$\frac{|f^k(J)|}{|J|} \geqq \rho''(|J|). \tag{6.3}$$

Proof. If $|f^k(J)| \geqq \frac{1}{4}\delta_0$, then

$$\frac{|f^k(J)|}{|J|} \geqq \frac{\frac{1}{4}\delta_0}{|J|}. \tag{6.4}$$

So for the remainder of the proof assume that $|f^k(J)| \leqq \frac{1}{4}\delta_0$. Let $L = T^1 \backslash J^1$ and $R = T^2 \backslash J^2$. Since $f^k(J^i) \supset T^i$,

$$\frac{|f^k(J)|}{|J|} \geqq \frac{|T|}{|J|}. \tag{6.5}$$

Choose $\tau(\rho) \in (0, \frac{1}{4}\delta_0)$ corresponding to Theorem 2.3 for $S = 6 \cdot |M|$, $\rho = |T|/|J|$ and $\rho_1 = 1 + 2\rho$. From (6.5),

$$\frac{|f^k(J)|}{|J|} \geqq \rho. \tag{6.6}$$

Let $\tau'(\rho)\in(0,\tau(\rho))$ be less or equal to the number $\delta'\in(0,\delta)$ corresponding to $\delta=\tau(\rho)$ from Lemma 6.3. We may assume that $\rho\to\tau'(\rho)$ is non-increasing. Since $|f^k(T^i)|\geqq\delta_0$ for $i=1,2$, and since we have assume that $|f^k(J)|\leqq\frac{1}{2}\delta_0$, one has $|f^k(L)|$, $|f^k(R)|\geqq\frac{1}{2}\delta_0\geqq\tau(\rho)$. Hence from Lemma 6.3 we can shrink T such that still $T\supset J$ and such that

$$|f^i(L)|,\ |f^i(R)|\leqq\tau(\rho),\quad \forall i=0,1,\ldots,k$$

and

$$|f^k(L)|,\ |f^k(R)|\geqq\tau'(\rho). \tag{6.7}$$

Since $\sum_{i=0}^{k-1}|f^i(J)|\leqq 6\cdot|M|=S$ we get from Theorem 2.3 either $|f^k(L)|/|f^k(J)|=(|f^k(T)|/|f^k(J)|)-1\leqq 2\rho$ or $(|f^k(R)|/|f^k(J)|)\leqq 2\rho$. From (6.7) we get in either case $|f^k(J)|\geqq(\tau'(\rho)/2\rho)$ and therefore

$$\frac{|f^k(J)|}{|J|}\geqq\frac{\tau'(\rho)}{2\rho}\cdot\frac{1}{|J|}. \tag{6.8}$$

So, from (6.8),

$$\frac{\tau'(\rho)}{2\rho}\geqq\sqrt{|J|}\Rightarrow\frac{|f^k(J)|}{|J|}\geqq\frac{1}{\sqrt{|J|}}. \tag{6.9}$$

But since $\tau'(\rho)>0$ and $\tau'(\cdot)$ is non-increasing, there exists a function $\tilde{\rho}:\mathbb{R}_+\to\mathbb{R}_+$ such that $\tilde{\rho}(t)\to\infty$ as $t\to 0$ and such that $(\tau'(\rho)/2\rho)\leqq\sqrt{|J|}$ implies $\rho\geqq\tilde{\rho}(|J|)$. In particular inequality (6.6) gives that

$$\frac{\tau'(\rho)}{\rho}\leqq\sqrt{|J|}\Rightarrow\frac{|f^k(J)|}{|J|}\geqq\tilde{\rho}(|J|). \tag{6.10}$$

Combining (6.4), (6.9) and (6.10) one has

$$\frac{|f^k(J)|}{|J|}\geqq\min\left(\tilde{\rho}(|J|),\frac{1}{\sqrt{|J|}},\frac{\frac{1}{4}\delta_0}{|J|}\right).$$

This finishes the proof of Lemma 6.4. Q.E.D.

Proof of Theorem 6.1. If $C(f)\neq\emptyset$, then take the neighbourhoods U_0, V_0 and W_0 from Sect. 5. If $C(f)=\emptyset$ simply take $U_0=V_0=\emptyset$.

Let O be a periodic orbit with period n. Without loss of generality we may assume that the period of O is bigger than $\#C(f)$ and therefore that $O\cap C(f)=\emptyset$. Let $k=n$ or $2n$ as before. Then $Df^k(p)>0$. We will subdivide the proof of Theorem 6.1 in some cases.

Proof of Theorem 6.1 if $O\cap U_0=\emptyset$. Consider a periodic orbit O such that $O\cap U_0=\emptyset$ and take some point $p\in O$. Let J^1 and J^2 be the components of $J\backslash\{p\}$. Lemma 4.2 gives

$$\sum_{i=0}^{k-1}|f^i(J)|\leqq 6\cdot|M|.$$

From this and since $O \cap U_0 = \emptyset$, we can apply Theorem 2.5b and there exists $K < \infty$ such that for any periodic point $p \in O$, as above,

$$|Df^k(p)| \geqq \frac{1}{K} \cdot \max\left(\frac{|f^k(J^1)|}{|J^1|}, \frac{|f^k(J^2)|}{|J^2|}, \frac{|f^k(J)|}{|J|}\right). \tag{6.11}$$

Case I. Let us first assume that $C(f) \neq \emptyset$ (and that $O \cap U_0 = \emptyset$). Of course $Df^k(p)$ is the same for each $p \in O$. So we may estimate $Df^k(p)$ at a convenient point p in the orbit O. Since $C(f) \neq \emptyset$, we assume that p is "closest to $C(f)$," i.e., that p is chosen on the orbit O such that there exists $c \in C(f)$ such that

$$(c, p) \cap O = \emptyset. \tag{6.12}$$

Let J^1 be the component of $J \setminus \{p\}$ such that $f^k(J^1)$ contains points from (c, p). We claim that

$$|f^k(J^1)| \geqq \delta_0'.$$

Indeed, from the maximality of J either the interval $\operatorname{Clos}(f^k(J^1))$ contains another point of $O(p)$ and therefore, from (6.12) and the definition of J^1, $f^k(J^1) \supset (c, p)$ or there exists $0 \leqq i < k$ such that $\operatorname{Clos}(f^i(J^1)) \cap C(f) \neq \emptyset$. In the first case $|f^k(J^1)| \geqq \delta_0 \geqq \delta_0'$, because $O(p) \cap U_0 = \emptyset$ and because each component of $U_0 \setminus C(f)$ has length $\geqq \delta_0$. In the second case this gives $|f^i(J^1)| \geqq \delta_0$ and therefore from Lemma 6.3, $|f^k(J^1)| \geqq \delta_0'$. Hence (6.11) implies that

$$|Df^k(p)| \geqq \frac{1}{K} \cdot \frac{|f^k(J^1)|}{|J^1|} \geqq \frac{1}{K} \cdot \frac{\delta_0'}{|J^1|}. \tag{6.13}$$

From Lemma 6.2 it follows that $|Df^k(p)| \to \infty$ as $k \to \infty$. This completes the proof of Theorem 6.1 in the case that $C(f) \cap U_0 = \emptyset$ and $C(f) \neq \emptyset$.

Case II. Let us now assume that $C(f) = \emptyset$. In this case f is an immersion of the circle with degree $\geqq 2$ (or $\leqq -2$). In this case there is no uniform lower bound for $|f^k(J)|$ and we cannot use the same argument as in the previous case. Define for $t > 0$,

$$\rho'''(t) = \min\left(\frac{\frac{1}{4}|M|}{t}, \rho''(t)\right), \tag{6.14}$$

where ρ'' is the function from Lemma 6.4. Clearly $\rho'''(t) \to \infty$ as $t \downarrow 0$. Let T be the interval containing p such that $f^k | T$ is a diffeomorphism and such that for the components T^i of $T \setminus \{p\}$ one has $|f^k(T^i)| = \frac{1}{2}|S^1|$ for $i = 1, 2$. Let J^i be the components of $J \setminus \{p\}$, and let J^1 be the interval such that J^1 and T^1 are on the same side of $\{p\}$. From Lemma 6.2 it follows that (in the case that $C(f) = \emptyset$) the proof of Theorem 6.1 is completed once we show that for any periodic point p of period k, and any J as above,

$$\frac{|f^k(J)|}{|J|} \geqq \rho'''(|J|). \tag{6.15}$$

If $|f^k(J)| \geqq \frac{1}{4}|M|$, then

$$\frac{|f^k(J)|}{|J|} \geqq \frac{\frac{1}{2}|M|}{|J|} \geqq \rho'''(|J|) \tag{6.16}$$

and the required estimate holds. So assume that $|f^k(J)| \leqq \frac{1}{4}|M|$. Let T be the interval containing p such that $f^k|T$ is a diffeomorphism and such that for the components T^i of $T \setminus \{p\}$ one has $|f^k(T^i)| = \frac{1}{2}|S^1|$ for $i=1,2$. As before let J^i be the components of $J \setminus \{p\}$ so that J^i, T^i are on the same side of p. Since $|f^k(J)| \leqq \frac{1}{4}|M|$ and $|f^k(T)| \geqq \frac{1}{2}|S^1|$, this implies $J_i \subset T_i$. By definition of J^i and since $C(f) = \emptyset$, $\mathrm{Clos}(f^k(J^i))$ contains a periodic point $p^i \in O(p)$, $p^i \neq p$. (Unless $O(p) = \{p\}$ in which case there is nothing to prove.) Since $p^i \in \mathrm{Clos}(f^k(J^i))$, and only one point of the orbit $O(p)$ can be contained in T (since f is a circle immersion and $f^k|T$ is a diffeomorphism),

$$f^k(J^i) \supset T^i.$$

So applying Lemma 6.4 gives

$$\frac{|f^k(J)|}{|J|} \geqq \rho''(|J|). \tag{6.17}$$

Combining (6.14), (6.16) and (6.17) inequality (6.15) follows. Thus the proof of Theorem 6.1 is completed in the case that $C(f) = \emptyset$.

Combining Cases I and II it follows that Theorem 6.1 holds for periodic orbits O such that $O \cap U_0 = \emptyset$. Q.E.D.

Now we will consider periodic orbits O with period k such that $O \cap U_0 \neq \emptyset$.

Proof of Theorem 6.1 for Periodic Orbits O such that $O \cap U_0 \neq \emptyset$. Take k' so big that if a point in $C(f)$ is in the basin of a periodic attractor then the period of this periodic attractor is at most k'. Without loss of generality we may assume that O is a periodic orbit whose period is bigger than $2k'$. Define k to be the period or twice the period of O as before. Since $k \geqq 2k'$, for each $p \in 0$ one has $Df^k(p) \neq 0$ and therefore, from the choice of k, that $Df^k(p) > 0$.

Of course $Df^k(p)$ is the same for each $p \in O$. So we may estimate $Df^k(p)$ at a convenient point in the orbit O. If there exists no $c \in C(f)$ and $i > 0$ such that $f^i(c) \in C(f)$ then simply choose some point $p \in U_0 \cap O$. In the general case we claim that there are two possibilities:

a) one can choose $p \in O \cap U_0$ such that it is impossible to find $i > 0$, $p' \in O$, $c, c' \in C(f)$ and segments (p', c'), (p, c) such that f^i maps (p', c') diffeomorphically onto (p, c), or
b) for each $p \in O$ there exists $c \in C(f)$ such that f^k maps (p, c) diffeomorphically into itself.

Indeed assume that $p \in O \cap U_0$ and does not satisfy a). Then let i be the maximal number $0 < i \leqq k$ such that there exists $p' \in O$, $c, c' \in C(f)$ and segments (p', c'), (p, c) such that f^i maps (p', c') diffeomorphically onto (p, c). If $i = k$ then we are in case b). If $i < k$ then one gets from (5.5b) that $p' \in O \cap U_0$. So replacing p by p', one has that the new point p satisfies the conditions in a). This completes the proof of the claim.

Now condition b) contradicts our assumption that $k \geqq 2k'$. So we may assume that p is as in case a). Let T be the maximal interval containing p such that $f^k|T$ is a diffeomorphism. Therefore $\mathrm{Clos}(f^k(T))$ contains two critical values of f^k. Now the Misiurewicz condition (5.5a) implies that all critical values of f^k are outside

W_0 or coincide with one of the critical points $U_0 \cap C(f)$. If the last possibility occurs then there exists $0 < i < k$ and $a \in \partial T$ such that $f^i(a) \in C(f)$ and $f^k(a) \in C(f)$. This clearly contradicts the fact that we are in case a). It follows that both endpoints of $f^k(T)$ lie outside W_0 and, since $f^k(p) = p \in U_0$, that $f^k(T) \supset W_0$.

Let $T^1 \supset J^1$ and $T^2 \supset J^2$ be the two components of $T \backslash \{p\}$. From the maximality of J it follows that either

$$T^i = J^i, \tag{6.18a}$$

or,

$$\partial f^k(J^i) \text{ contains a point of } O(p) \backslash \{p\} \text{ say } p^i. \tag{6.18b}$$

If (6.18a) holds then one has $f^k(J^i) \supset J^i = T^i$ because otherwise the critical point of $f^k | \mathrm{Clos}(T^i)$ would be attracted to a periodic point with period k. But since $k > 2k'$ this is impossible. If (6.18b) holds then $f^k(J^{i)} = (p, p^i)$, and since $f^k | (p, p^i)$ cannot be monotone (otherwise f would be a circle homeomorphism) $(p, p^i) \supset T^i$. So in either case one has

$$f^k(J^i) \supset T^i. \tag{6.19}$$

Since both endpoints of $f^k(T) \supset W_0$ and $f^k(p) = p \in U_0$, $f^k(T^1)$ and $f^k(T^2)$ both contain a component of $V_0 \backslash U_0$. In particular $|f^k(T^i)| \geqq \delta_0$, $i = 1, 2$. From Lemma 6.4 it follows that

$$\frac{|f^k(J)|}{|J|} \geqq \rho''(|J|). \tag{6.20}$$

Now let

$$\min_{i=1,2} \frac{|f^k(J^i)|}{|J^i|} = \frac{|f^k(J^2)|}{|J^2|}. \tag{6.21}$$

If (6.21) holds for J^1 instead of J^2 then we proceed similarly. Let R be the component of $T \backslash J^2$ which is contained in T^2. From Lemma 4.2 one has that

$$\sum_{i=0}^{k-1} |f^i(J^1)| \leqq \sum_{i=0}^{k-1} |f^i(J)| \leqq 6 \cdot |M|. \tag{6.22}$$

Let

$$C = \exp(-\sigma(|M|) \cdot 6 \cdot |M|) > 0,$$

where σ is the minimum of the functions from Theorems (2.1) and (2.2). (In particular, C is independent of J and k.) From (6.22), the choice of $C > 0$ and Theorem 2.2, it follows that $A(f^k, T', J') \geqq C$ for all intervals $J' \subset T'$ such that $f^k | T'$ is a diffeomorphism and such that one of the components of $T' \backslash J'$ is contained in J^1. In particular if we take $T' = R \cup J$, $J' = J^2$, $L' = J^1$ and $R' = R$, then

$$A(f^k, T', J') \geqq C. \tag{6.23}$$

On the other hand,

$$\begin{aligned} A(f^k, T', J') &= \frac{|f^k(J')|}{|J'|} \cdot \frac{|f^k(T')|}{|f^k(R' \cup J')|} \cdot \frac{|L' \cup J'|}{|f^k(L' \cup J')|} \cdot \frac{|R' \cup J'|}{|T'|} \\ &\leqq \frac{|f^k(J^2)|}{|J^2|} \cdot \frac{|f^k(T')|}{|f^k(R' \cup J')|} \cdot \frac{|L' \cup J'|}{|f^k(J)|} \cdot \frac{|R' \cup J'|}{|T'|}. \end{aligned} \tag{6.24}$$

In this last line, the second factor is at most $|M|/\delta_0$ since $|f^k(R' \cup J')| = |f^k(T^2)| \geqq \delta_0$ and the last is at most one. That is,

$$A(f^k, T', J') \leqq \frac{|M|}{\delta_0} \cdot \frac{|f^k(J^2)|}{|J^2|} \cdot \frac{|J|}{|f^k(J)|}.$$

This and (6.23) gives

$$\frac{|f^k(J^2)|}{|J^2|} \geqq \frac{C\delta_0}{|M|} \cdot \frac{|f^k(J)|}{|J|}. \tag{6.25}$$

In particular from (6.21) and (6.20),

$$\min_{i=1,2} \frac{|f^k(J^i)|}{|J^i|} \geqq \frac{C\delta_0}{|M|} \cdot \frac{|f^k(J)|}{|J|} \geqq \frac{C\delta_0}{|M|} \cdot \rho''(|J|). \tag{6.26}$$

Now we can finish the proof of Theorem 6.1. Take a p, J^1, J^2, J^2 and k as above. As before from (6.22) and the choice of C it follows that

$$B(f^k, T^*, J^*) \geqq C$$

for all $J^* \subset T^* \subset J$. Therefore Lemma 3.1 implies that for every $a \in J^1$ and every $b \in J^2$ one has

$$|Df^k(p)| \geqq C^3 \cdot \min\{|Df^k(a)|, |Df^k(b)|\}. \tag{6.27}$$

Using this, the mean-value theorem and inequalities (6.26) one gets

$$|Df^{k(n)}(p_n))| \geqq \frac{C^4\delta_0}{|M|} \cdot \rho''(|J|).$$

From this and Lemma 6.2 it follows that the proof of Theorem 6.1 is completed. Q.E.D.

Sketch of Algorithm. From the proof above it follows that there exists an algorithm which gives lower bounds for the functions ρ'' and ρ'''. In particular it follows that there exists an algorithm which gives some ε such that if we choose N so big so that $|J| < \varepsilon$, for every interval J as above corresponding to periodic orbits of period $k \geqq N$, and if every periodic orbit of f of period $k \leqq N$ is hyperbolic then all periodic orbits of f are hyperbolic. In particular there exists a finite algorithm to check whether all periodic orbits of f are hyperbolic and repelling.

6.5 Corollary. *Let $f: M \to M$ be a C^2 map satisfying the Misiurewicz condition* (i). *Then the closure of the immediate basins of f,* Clos(B_0), *consists of a finite union of intervals. Furthermore there is a neighbourhood V of $C(f)$ such that for each component V_i of V either*

—*all periodic points of f in V_i are hyperbolic and repelling, or*
—$V_i \subset B_0$.

Proof. Theorem 6.1 tells us that the period of all attracting periodic orbits is uniformly bounded. From this it follows that Clos(B_0) consists of a finite union of intervals. Also the uniform bound on the period of attractors implies that if there exists a critical point which is accumulated by attracting or non-hyperbolic periodic points then this critical points is also periodic. But this contradicts the Misiurewicz condition. Q.E.D.

6.6 Corollary. *Let f be an analytic map satisfying the Misiurewicz condition* (i) *from above. Then f has only a finite number of non-hyperbolic or attracting periodic points. (Recently this corollary has been proved for arbitrary analytic maps $f: M \to M$, see* [M.M.S.].)

Proof. Let $N < \infty$ be such that all attracting or periodic points of f have period less than N. (This N exists from Theorem 6.1.) If there are infinitely many such points then they are all fixed points of $f^{N!}$. Since f is analytic this implies that $f^{N!} = \mathrm{id}$. This implies that f is a diffeomorphism contradicting the Misiurewicz condition (i). Q.E.D.

7. Compatible Intervals

As before we say that I_n is a $*$-branch-interval for f^n if I_n is a maximal interval such that $f^n|I_n$ is a diffeomorphism and such that $f^n(I_n) \cap B_0 = \emptyset$. We know from Sect. 6 that Close(B_0) consists of a finite union of intervals. We want to show that there exists a constant $S < \infty$ such that for any $*$-branch-interval I_n of f^n we have $\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$. In this section we will simplify this question by showing that it suffices to consider special intervals I_n. In Proposition 7.1 we will give a condition which gives a uniform bound for $\sum_{i=0}^{n-1} |f^{n-i-1}(I_n)| = \sum_{i=0}^{n-1} |f^i(I_n)|$. This condition does **not** require that $|f^{n-i}(I_n)|$ goes down exponentially with i. Instead this condition requires roughly speaking that there exists a constant $\lambda < 1$ such that for any $*$-branch interval I_n for f^n and any $0 \leqq i < j < n$ such that $f^i(I_n) \subset f^j(I_n)$ one has $|f^i(I_n)| \leqq \lambda \cdot |f^j(I_n)|$. Clearly this is not quite enough in order to show that $\sum_{i=0}^{n-1} |f^i(I_n)|$ is uniformly bounded: one also needs to be able compare the length of intervals $f^i(E), f^j(E) \subset f^k(E)$ for which $f^i(E) \cap f^j(E) = \emptyset$.

To estimate $\sum_{i=0}^{n-1} |f^{n-1-i}(I_n)| = \sum_{i=0}^{n-1} |f^i(I_n)|$ it will be useful to have that the iterates of intervals can be split up in disjoint groups of nested intervals. More precisely introduce the following notations. We say that an interval E is ***m*-compatible** if E is open and if the following three conditions are satisfied.

i) $f^m|E$ is a diffeomorphism;
ii) $f^i(E) \cap f^j(E) \neq \emptyset$, for some $i < j \leqq m$, implies that $f^i(E) \subset f^j(E)$;
iii) $f^m(E) \cap B_0 = \emptyset$.

Similarly we say that E is **strongly *m*-compatible** if E is m-compatible, and if moreover

iv) $f^i(E), f^j(E) \subset f^k(E)$, for some $i < j \leqq k \leqq m$, implies that there exists $\tilde{E} \supset f^i(E)$ which is $(j-i)$-compatible such that $f^{j-i}(\tilde{E}) = f^k(E)$.

Let $U_0 \subset V_0 \subset W_0, \delta_0$ and $\delta'_0 \in (0, \delta_0)$ be as in Sect. 5. As in Sect. 6 we will subdivide

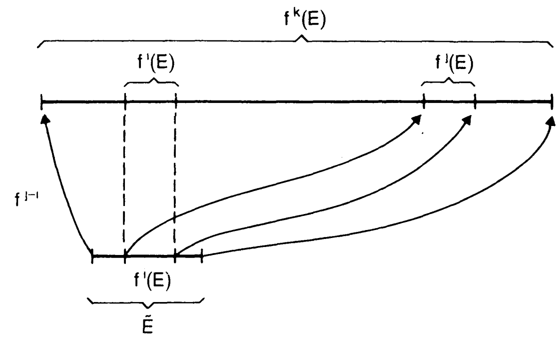

Fig. 2. Compatible intervals

intervals in two cases: those whose orbit stays away from $C(f)$ and those whose orbit comes close to $C(f)$. To be more precise, we will introduce two new conditions. If I is contained in a $*$-branch-interval of f^m then we say that I satisfies **condition** A_m if

$$f^i(I) \cap U_0 = \emptyset, \quad \text{for all} \quad 0 \leqq i \leqq m-1.$$

We say that I satisfies **condition** B_m if

$$f^m(I) \text{ is contained in some component of } V_0 \backslash C(f)$$

and if there exists no $0 \leqq i < m$ such that $f^i(E) \subset V_0 \backslash C(f)$ and such that f^{m-i} maps the component of $V_0 \backslash C(f)$ containing $f^i(E)$ diffeomorphically onto the component of $V_0 \backslash C(f)$ containing $f^m(E)$. (This last condition is needed when there exists $i > 0$ such that $f^i(C(f)) \cap C(f) \neq \emptyset$.)

7.1 Proposition. *Let $f: M \to M$ be a C^2 map without flat critical points satisfying the Misiurewicz condition* (i). *Then for each $\lambda \in (0,1)$ and $\delta \in (0, \delta_0')$ there exists $S < \infty$ with the following property. Assume that for any strongly m-compatible interval E satisfying either condition A_m or condition B_m and such that $|f^i(E)| \leqq \delta$ for all $i = 0, \ldots, m$ one has that $0 \leqq i < j \leqq k \leqq m$ and*

$$f^i(E), f^j(E) \subset f^k(E)$$

implies that

$$|f^i(E)| \leqq \lambda \cdot |f^j(E)|. \tag{7.1}$$

Then for each $*$-branch-interval I_n of f^n one has

$$\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S. \tag{7.2}$$

Remark. The proof of Proposition 7.1 will show that for each $\delta \in (0, \delta_0')$ and each $\lambda \in (0,1)$ one can give an effective algorithm which gives an upper bound for S.

For the proof of this proposition we need three lemmas. We also recall that a $*$-branch-interval I_n for f^n is a maximal interval such that $f^n | I_n$ is a diffeomorphism, $f^n(I_n) \cap B_0 = \emptyset$ and if $M = S^1$ such that also $f^n(I_n) \cap \{x_0\} = \emptyset$.

In the next lemma we give conditions for a $*$-branch-interval to be compatible.

7.2 Lemma. *Let I_n be a $*$-branch-interval of f^n and $0 \leqq m \leqq n$.*

a) *If for each $a \in \partial I_n$ there exists $k \in \mathbb{N}$ such that $m \leqq k \leqq n$ and $f^k(a) \in \mathrm{Clos}(B_0) \cup C_+(f)$ then I_n is m-compatible.*
b) *If $\mathrm{Clos}(f^i(I_n)) \cap C(f) = \emptyset$ for all $0 \leqq i \leqq m-1 \leqq n-1$ then I_n is m-compatible.*

Proof. Let us first prove a). Assume by contradiction that for some $0 \leqq i \leqq j \leqq m$ $f^i(I_n) \cap f^j(I_n) \neq \emptyset$ and $f^i(I_n) \not\subset f^j(I_n)$. Then there exists $a \in \partial I_n$ such that $f^j(a)$ is contained in $f^i(I_n)$. Let k be so that $m \leqq k \leqq n$ and $f^k(a) \in \mathrm{Clos}(B_0) \cup C_+(f)$. Then $f^{k-j+i}(I_n)$ contains $f^k(a) \in \mathrm{Clos}(B_0) \cup C_+(f)$ (in its interior). Since $k-j+i<n$ this either implies that $f^n | I_n$ is not a diffeomorphism, $x_0 \in f^n(I_n)$ or that $f^n(I_n) \cap \mathrm{Clos}(B_0) \neq \emptyset$. This contradicts that I_n is $*$-branch-interval for f^n.

Let us now prove b). If $\mathrm{Clos}(f^i(I_n)) \cap C(f) = \emptyset$ for all $0 \leqq i \leqq m-1 \leqq n-1$ then the maximality of the interval I_n implies that the condition in the statement of Lemma 7.2a is satisfied. Q.E.D.

Let us now give conditions for an interval to be strongly compatible.

7.3 Lemma. *Assume f satisfies the Misiurewicz condition* (i) *and has no wandering intervals. Let I_n be a $*$-branch-interval of f^n. Assume that for that some $0 \leqq m \leqq n$, I_n* satisfies *$|f^m(I_n)| < \delta'_0$ and condition A_m or condition B_m. Then I_n is m-strongly compatible.*

Proof. Case I. Assume that I_n satisfies condition A_m. Since $\mathrm{Clos}(f^i(I_n)) \cap C(f) \subset \mathrm{Clos}(f^i(I_n)) \cap U_0 = \emptyset$, for $0 \leqq i \leqq m-1$, Lemma 7.2b implies that I_n is m-compatible.

Let us show that I_n is m-strongly compatible. So assume that $f^i(I_n), f^j(I_n) \subset f^k(I_n)$ for some $i<j \leqq k \leqq m$. Write $E = f^i(I_n)$ and take $\tilde{E} \supset E = f^i(I_n)$ to be the maximal interval such that (i), $f^{j-i} | \tilde{E}$ is a diffeomorphism and (ii) such that $f^{j-i}(\tilde{E}) \subset f^k(I_n)$. If $f^{j-i}(\tilde{E}) \neq f^k(E)$ then it follows from the maximality of $\tilde{E}$ that $\mathrm{Clos}(f^l(\tilde{E})) \cap C(f) \neq \emptyset$ for some $0 \leqq l < j-i$. Thus, $\tilde{E} \supset E$ and $f^l(E) \cap U_0 = \emptyset$ implies $|f^l(\tilde{E})| \geqq \delta_0$. Since $f^{j-i} | \tilde{E}$ is a diffeomorphism, $f^{j-i}(\tilde{E}) \subset f^k(I_n)$, $m-k<n-k$, $f^n | I_n$ is a diffeomorphism and $m-k+j-i>l$ one has that $f^{m-k+j-i-l} | f^l(\tilde{E})$ is a diffeomorphism. $f^n(I_n) \cap B_0 = \emptyset$ implies $f^{m-k+j-i-l}(f^l(\tilde{E})) \cap B_0 = \emptyset$. Hence we can apply Lemma 5.2a and obtain $|f^{m-k+j-i}(\tilde{E})| \geqq \delta'_0$. But also $|f^{m-k+j-i}(\tilde{E})| < |f^{m-k}(f^k(I_n))| = |f^m(I_n)| < \delta'_0$. Thus we have proved by contradiction that $f^{j-i}(\tilde{E}) = f^k(I_n)$ and also that $\mathrm{Clos}(f^l(\tilde{E})) \cap C(f) = \emptyset$ for all $0 \leqq l < j-i$. From Lemma 7.2b it follows that $\tilde{E}$ is $(j-i)$-compatible. Therefore I_n is m-strongly compatible.

Case II. Assume that I_n satisfies condition B_m. Suppose by contradiction that I_n is not m-compatible. Then the assumption of Lemma 7.2b is certainly not satisfied. Then for some $0 \leqq i < m$ and some $a \in \partial I_n$, one has $f^i(a) \in C(f)$. But then $f^m(a) \in \bigcup_{l \geqq 1} f^l(C(f))$ and also $f^m(I_n) \subset V_0$. Since $\bigcup_{r \geqq 1} f^r(C(f)) \cap V_0 \subset C(f)$ this implies that $f^m(a) \in C(f)$. It follows that the condition in Lemma 7.2a is satisfied. It follows that I_n is m-compatible.

Let us show that I_n is also m-strongly compatible. Again write $E = f^i(I_n)$ and take $\tilde{E} \supset E = f^i(I_n)$ to be the maximal interval such that (i). $f^{j-i} | \tilde{E}$ is a diffeomorphism and (ii) such that $f^{j-i}(\tilde{E}) \subset f^k(I_n)$. If $f^{j-i}(\tilde{E}) \neq f^k(E)$ then it follows from the maximality of $\tilde{E}$ that $\mathrm{Clos}(f^l(\tilde{E})) \cap C(f) \neq \emptyset$ for some $0 \leqq l < j-i$. But then $f^{j-i}(\tilde{E})$ contains a point of $\bigcup_{r\geqq 1} f^r(C(f))\subset(M\setminus W_0)\cup C(f)$ and therefore $f^{m-k+j-i}(\tilde{E})\subset f^{m-k}(f^k(I_n))=f^m(I_n)$ also contains a point of $\bigcup_{r\geqq 1} f^r(C(f))$. Since $f^m(I_n)$ is contained in some component of $V_0\setminus C(f)$ this is impossible. Again we have proved by contradiction that $f^{j-i}(\tilde{E})=f^k(I_n)$ and $\mathrm{Clos}(f^l(\tilde{E}))\cap C(f)=\emptyset$ for all $0\leqq l<j-i$. Again Lemma 7.2 implies that $\tilde{E}$ is $(j-i)$-compatible and therefore I_n is m-compatible. Q.E.D.

Now we will reduce the problem of whether $\sum_{i=0}^{n-1}|f^i(I_n)|$ is uniformly bounded for all $*$-branch-intervals I_n of f^n to one where we just have to consider strongly compatible intervals satisfying either condition A_m or condition B_m.

7.4 Lemma. *Let $f:M\to M$ be C^2 without flat critical points and satisfy the Misiurewicz* (i) *condition. Take U_0, V_0, δ_0 and δ_0' as above. For each $0<\delta<\delta_0'$ and $S<\infty$ there exists $\tilde{S}<\infty$ such that the following holds. Assume that for each m-strongly compatible interval E such that $|f^i(E)|\leqq\delta$ for all $i=0,\dots,m$, which satisfies condition A_m or condition B_m one has*

$$\sum_{i=0}^{m-1}|f^i(E)|\leqq S. \tag{7.3}$$

Then for any $*$-branch-interval I_n of f^n one has

$$\sum_{i=0}^{n-1}|f^i(I_n)|\leqq\tilde{S}. \tag{7.4}$$

Proof. Let $1<N<\infty$ be so that for any interval T and any $m\geqq 0$ with $|T|\geqq\delta$, $f^m|T$ is a diffeomorphism and $f^m(T)\cap B_0=\emptyset$ one has $m\leqq N$. This number N exists from Lemma 5.1. (As we noted before, the assumptions of Lemma 5.1 are satisfied for any C^2 Misiurewicz maps without flat critical points.) Hence $0\leqq i\leqq n$, $|f^i(I_n)|\geqq\delta$ implies $n-i\leqq N$. Let m be the largest integer such that $0\leqq m\leqq n-N$ and such that $f^m(I_n)\cap U_0\neq\emptyset$. If there exists no such m then define $m=-1$. From the fact that every component of $V_0\setminus U_0$ has length at least δ_0, since $f^m(I_n)\cap U_0\neq\emptyset$ and since $|f^i(I_n)|\leqq\delta<\delta_0'<\delta_0$ for $0\leqq i\leqq n-N$ it follows that $f^m(I_n)\subset V_0$. Moreover $f^n|I_n$ is a deffeomorphism. Therefore, and since $m<n$, $f^m(I_n)\cap C(f)=\emptyset$. So $f^m(I_n)$ is contained in some component of $V_0\setminus C(f)$. Let m' be the smallest number such that $0\leqq m'\leqq m$, such that $f^{m'}(E)\subset V_0\setminus C(f)$ and such that $f^{m-m'}$ maps the component of $V_0\setminus C(f)$ containing $f^{m'}(E)$ diffeomorphically onto the component of $V_0\setminus C(f)$ containing $f^m(E)$. (If there exists no $i>0$ such that $f^i(C(f))\cap C(f)\neq\emptyset$ then $m'=m$.) It follows that I_n satisfies condition $B_{m'}$. From the choice of m, $f^i(I_n)\cap U_0=\emptyset$ for $m<i\leqq n-N$ and so $f^{m+1}(I_n)$ satisfies condition $A_{n-N-m-1}$. Since $|f^i(I_n)|\leqq\delta<\delta_0'$ for all $i=0,1,\dots,n-N$, Lemma 7.3 implies that I_n and $f^{m+1}(I_n)$ are respectively m'- and $(n-N-m-1)$-strongly compatible.

From the definition of N_0 (and the choice of V_0 in (5.5b)) there are two possibilities: a) $m-m'\leqq N_0$ or b) the critical point c in the boundary of the component of $V_0\setminus C(f)$ which contains $f^{m'}(E)$ is periodic (with period $N_2\leqq N_0$). Because $f^m(E)\subset V_0\setminus B_0$, and because of the choice of V_0 in (5.5c) case b) this implies that all the iterates $f^{m_0}(E),\dots,f^m(E)$ are disjoint ($f^{m-m'}$ maps the segment $(c,f^{m'}(E))$ connecting c and $f^{m'}(E)$ is diffeomorphically onto the segment $(c,f^m(E))$ and the point c is a periodic point of f).

It follows that in both cases $\sum_{i=m'}^{m}|f^i(E)|\leqq N_0\cdot|M|$. Hence

$$\sum_{i=0}^{n-1}|f^i(I_n)|\leqq\sum_{i=0}^{m'}|f^i(I_n)|+\sum_{i=m'+1}^{m}|f^i(I_n)|+\sum_{i=m+1}^{n-N}|f^i(I_n)|+\sum_{i=n-N+1}^{n-1}|f^i(I_n)|$$
$$\leqq\sum_{i=0}^{m'}|f^i(I_n)|+N_0\cdot|M|+\sum_{i=0}^{n-N-m-1}|f^i(f^{m+1}(I_n))|+N\cdot|M|.$$

From (7.3) and since I_n and $f^{m+1}(I_n)$ are respectively m'- and $(n-N-m-1)$-strongly compatible the last inequality completes the proof of the lemma. Q.E.D.

Proof of Proposition 7.1. From Lemma 7.4 it follows that if suffices to prove that we can find $S<\infty$ such that

$$\sum_{i=0}^{m-1}|f^l(E)|\leqq S \tag{7.5}$$

holds for all strongly m-compatible intervals E with $|f^i(E)|\leqq\delta$ which satisfy either condition A_m or condition B_m.

So choose the set $I\subset\{0,\dots,m-1\}$ such that $\bigcup_{l\in I}f^l(E)$ contains $\bigcup_{i=0}^{m-1}f^i(E)$ and that there is no smaller subset $\tilde{I}\subset I$ with this property. Then from the minimality of I and the fact that E is m-compatible one gets that the intervals $f^l(E)$, $l\in I$ are all disjoint, i.e.

$$\sum_{l\in I}|f^l(E)|\leqq|M|. \tag{7.6}$$

Moreover each interval $f^i(E)$ is contained in one of the intervals $f^l(E)$, $l\in I$. So

$$\sum_{0\leqq l\leqq m}|f^l(E)|\leqq\sum_{l\in I}\Bigg(\sum_{\{i;f^i(E)\subset f^l(E)\}}|f^i(E)|\Bigg). \tag{7.7}$$

Let $\{i;f^i(E)\subset f^l(E)\}=\{i_1,i_2,\dots,i_k)$ and $i_1<i_2<\dots<i_k$. For each $l\in I$ one gets from (7.1), $|f^{i_j}(E)|\leqq\lambda\cdot|f^{i_{j+1}}(E)|$, $j=1,2,\dots,k-1$ and therefore

$$\sum_{\{i;f^i(E)\subset f^l(E)\}}|f^i(E)|\leqq\Bigg(\sum_{i\geqq 0}\lambda_i\Bigg)\cdot|f^l(E)|=\frac{1}{1-\lambda}\cdot|f^l(E)|. \tag{7.8}$$

Combining (7.6)–(7.8) one gets (7.5). This completes the proof of this proposition. Q.E.D.

8. Expansion of the Return-Map on Strongly Compatible Intervals

In this section we will check the conditions of Proposition 7.1 for strongly compatible intervals. We will prove the following proposition.

8.1 Proposition. *Let $f:M\to M$ be a C^2 map having no flat critical points. Assume that f satisfies the Misiurewicz conditions* (i) *and* (ii′). (*Condition* (ii′) *requires that* *all periodic orbits of f are hyperbolic.) Then there exist $\delta \in (0, \delta_0)$ and $\lambda \in (0, 1)$ such that for any strongly m-compatible interval E satisfying either condition A_m or condition B_m and such that $|f^i(E)| \leqq \delta$, $\forall 0 \leqq i \leqq m$, one has the following. If for some $i < j \leqq k \leqq m$ one has*

$$f^i(E), f^j(E) \subset f^k(E), \tag{8.1}$$

then

$$|f^i(E)| \leqq \lambda \cdot |f^j(E)|. \tag{8.2}$$

Remark. From the proof of this proposition it follows that there exists an effective algorithm which gives an upper bound $\lambda_0 \in (0, 1)$ for λ.

In the proof of Proposition 8.1 we need to distinguish between the case where E satisfies condition A_m or condition B_m.

8.a. E Satisfies Condition A_m. Let E be a m-compatible interval such that for some neighbourhood U of $C(f)$ one has

$$f^i(E) \cap U = \emptyset, \quad \forall i = 0, \ldots, m-1. \tag{8.3}$$

Assume $f^i(E) \subset f^j(E)$. In Sect. 6 we proved that the expansion along periodic orbits grows as the period of these periodic orbits grows. In the following lemma we will use this to get expansion for the map $f^{j-i}|f^i(E): f^i(E) \to f^j(E)$.

8.2 Lemma. *Let $f: M \to M$ be C^2. Then there exists a sequence of numbers K_k such that $K_k \to \infty$ as $k \to \infty$ with the following property. Let E be an r-compatible interval satisfying (8.3) such that*

$$E \subset f^r(E),$$
$$f^l(E) \cap f^r(E) = \emptyset, \quad l = 0, \ldots, r-1,$$

then

$$|Df^r(x)| \geqq K_r, \quad \forall x \in E. \tag{8.4}$$

Proof of Lemma 8.2. Since $f^l(E) \cap f^r(E) = \emptyset$ for all $l = 0, 1, 2, \ldots, r-1$ it follows from Lemma 4.1 that $E, f(E), \ldots, f^{r-1}(E)$ are disjoint and therefore

$$\sum_{l=0}^{r-1} |f^l(E)| \leqq |M|. \tag{8.5}$$

Now remark that formulas (8.3) and (8.5) imply that we can apply Theorem 2.5 and get a constant $K < \infty$ such that

$$\frac{|Df^r(x)|}{|Df^r(y)|} \leqq K, \quad \forall x, y \in E. \tag{8.6}$$

Since $f^r(E) \supset E$, $f^r: E \to f^r(E)$ has a fixed point p_r. Moreover the disjointness implies this point p_r is a periodic point of minimal period r. According to Theorem 6.1 there exists a sequence $\tilde{K}_k$ (which only depends on f) such that $\tilde{K}_k \to \infty$ as $k \to \infty$ such that

$$|Df^r(p_r)| \geqq \tilde{K}_r.$$

This and (8.6) implies that (8.4) holds with $K_r = \tilde{K}_r / K$. Q.E.D.

8.b. E Satisfies Condition B_m. Take neighbourhoods $U_0 \subset V_0 \subset W_0$ of $C(f)$, $\delta_0 > 0$ and $\delta_0' > 0$ as before such that each of the components of $U_0 \backslash C(f)$, $W_0 \backslash V_0$ and $V_0 \backslash U_0$ has at least length δ_0 and such that

$$f^n(C_+(f)) \cap W_0 \subset C_+(f)$$

for all $n \geqq 1$.

8.3 Lemma. *Let $f: M \to M$ be a C^2 map without flat critical points satisfying the Misiurewicz condition* (i) *and* (ii′). *Let U_0, V_0, W_0 be as above. There exists $\delta_1 > 0$ with the following property. Assume E is a r-compatible interval such that E satisfies condition B_r, $f^i(E) \cap f^r(E) = \varnothing$ for all $0 \leqq i < r$ and $|f^i(E)| \leqq \delta_1$, for all $i = 0, \ldots, r$. Then there exists a neighbourhood F of* Clos(E) *such that $f^r|F$ is a diffeomorphism and such that for both components F_1 and F_2 of $F \backslash E$ one has*

$$|f^r(F_1)| = |f^r(F_2)| = |f^r(E)| \tag{8.7}$$

and

$$|f^i(F_1)|, |f^i(F_2)| \leqq 4 \cdot |f^i(E)|, \quad \forall i = 0, 1, \ldots, r. \tag{8.8}$$

Proof. Let I_r be the branch-interval of f^r containing E. By maximality, both boundary points of $f^r(I_r)$ are in $\bigcup_{n \geqq 1} f^n(C_+(f)) \subset (M \backslash W_0) \cup C(f)$. If $f^r(I_r) \in C(f)$ then from the maximality of I_r, these exists $0 \leqq i < r$ such that $f^i(\partial I_r) \in C(f)$. But then f^{r-i} maps the component of $V_0 \backslash C(f)$ which has a non-empty intersection with $f^i(I_r)$ diffeomorphically onto the component of $V_0 \backslash C(f)$ which contains $f^r(E)$. It follows that $f^i(E)$ is also contained in a component of $V_0 \backslash C(f)$. So $f^r(I_r) \in C(f)$ contradicts the assumption that E satisfies condition B_r. It follows that $f^r(I_r) \supset W_0$. So choosing δ_1 sufficiently small gives that there exists an interval $F, E \subset F \subset I_r$ so that $f^r|F$ is a diffeomorphism and such that for both components F_1 and F_2 of $F \backslash E$ one has $|f^r(F_1)| = |f^r(F_2)| = |f^r(E)|$.

Let us prove that $|f^i(F_1)| \leqq 4 \cdot |f^i(E)|$ for all $i = 0, 1, \ldots, r-1$. The corresponding statement for F_2 is proved similarly. Choose $\tau \in (0, \frac{1}{2}\delta_0)$ such that for the function $\sigma(t)$ of Theorem 2.2, $\exp\{-\sigma(t) \cdot |M|\} \geqq \frac{2}{3}$, for all $t \in (0, \tau)$. Since $\lim_{t \to 0} \sigma(t) = 0$ this is possible. Take $\tau' \in (0, \tau)$ be equal to the number δ' corresponding to $\delta = \tau$ from Lemma 5.2. For the moment choose $\delta_1 \in (0, \frac{1}{2}\delta_0)$. Later on we may have to shrink δ_1 further but we will keep τ fixed throughout the remainder of the proof. Assume that E satisfies $|f^i(E)| \leqq \delta_1$ for all $i = 0, 1, \ldots, r$.

Let T^* be the component of $I_r \backslash F_2$ containing $F_1 \cup E$ and let $L^* = T^* \backslash (F_1 \cup E)$. Since $|f^r(F_1)| = |f^r(E)| \leqq \delta_1 \leqq \frac{1}{2}\delta_0$, $f^r(E)$ is contained in V_0 and since the length of each of the component of $W_0 \backslash V_0$ is at least δ_0 one has $f^r(E \cup F_i) \subset W_0$; in fact $\operatorname{dist}(f^r(E \cup F_i), \partial W_0) \geqq \frac{1}{2}\delta_0$. Therefore, since $f^r(L^*)$ contains critical values of f^r and therefore points of $M \backslash W_0$, it follows that $|f^r(L^*)| \geqq \frac{1}{2}\delta_0$. In particular $|f^r(L^*)| \geqq \tau$.

We may need to shrink T^* slightly on one side. More precisely choose T with $E \cup F_1 \subset T \subset T^*$ such that for $L = T \backslash (F_1 \cup E)$ one has

$$|f^r(L)| \geqq \tau', \quad |f^j(L)| \leqq \tau, \quad j = i, \ldots, r-1.$$

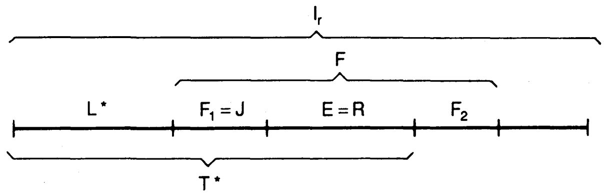

Fig. 3. The intervals I_r, T^*, L^*, F_i and E

Indeed, since f has only hyperbolic periodic points we can apply Corollary 5.3b and it follows that this is possible. For simplicity write

$$L = T \backslash (F_1 \cup E), \quad R = E, \quad J = F_1.$$

Since $f^l(E) \cap f^r(E) = \emptyset$ for all $l = 0, 1, \ldots, r-1$ it follows from Lemma 4.1 that $E, f(E), \ldots, f^{r-1}(E)$ are disjoint and therefore

$$\sum_{l=0}^{r-1} |f^l(R)| = \sum_{l=0}^{r-1} |f^l(E)| \leqq |M|.$$

Therefore from Theorem 2.2 one gets that for each $i = 0, 1, \ldots, r-1$,

$$A(f^{r-i}, f^i(T), f^i(J)) \geqq \exp\{-\sigma(t) \cdot |M|\},$$

where $t = \max_{l=i,\ldots,r-1} |f^i(R)| \leqq \tau$. From the choice of τ this gives

$$A(f^{r-i}, f^i(T), f^i(J)) \geqq \tfrac{2}{3}. \tag{8.9}$$

Using this notation we have from (8.7)

$$A(f^{r-i}, f^i(T), f^i(J)) = \frac{|f^r(T)|}{|f^r(L \cup J)|} \cdot \frac{|f^r(J)|}{|f^r(R \cup J)|} \cdot \frac{|f^i(R \cup J)|}{|f^i(J)|} \cdot \frac{|f^i(L \cup J)|}{|f^i(T)|}$$

$$\leqq \frac{|f^r(T)|}{|f^r(L \cup J)|} \cdot \frac{1}{2} \cdot \frac{|f^i(R \cup J)|}{|f^i(J)|} \cdot 1. \tag{8.10}$$

Using $|f^r(L)| \geqq \tau'$ and $|f^r(E)| \leqq \delta_1$, one gets

$$\frac{|f^r(T)|}{|f^r(L \cup J)|} \leqq 1 + \frac{|f^r(R)|}{\tau'} = 1 + \frac{|f^r(E)|}{\tau'} \leqq \frac{\tau' + \delta_1}{\tau'}. \tag{8.11}$$

Combining (8.9)–(8.11) gives

$$\frac{|f^i(R \cup J)|}{|f^i(J)|} \geqq \frac{4}{3} \cdot \frac{\tau'}{\tau' + \delta_1}.$$

Hence for δ_1 is sufficiently small

$$\frac{|f^i(E)|}{|f^i(F_1)|} = \frac{|f^i(R)|}{|f^i(J)|} \geqq \frac{\tau' - 3\delta_1}{3(\tau' + \delta_1)} > \frac{1}{4}.$$

This proves (8.8). Q.E.D.

Now we will prove that the return-map on compatible intervals is expanding.

8.4 Lemma. *Under the assumptions of Lemma 8.3, there exists a sequence* $K_k \to \infty$ *and* $\delta_1 > 0$ *such that for any m-compatible interval E for which* $f^m(E)$ *is contained in a component of* $V_0 \setminus C(f)$, *and for which* $|f^i(E)| \leqq \delta_1$ *for all* $i = 0, 1, \ldots, m$, *the following holds. If for some* $0 \leqq r \leqq m$

$$E \subset f^r(E)$$

and

$$f^l(E) \cap f^r(E) = \emptyset, \quad l = 0, 1, \ldots, r-1,$$

then one has

$$|Df^r(x)| \geqq K_r, \quad \forall x \in E. \tag{8.13}$$

Proof. Choose δ_1 as in Lemma 8.3. We will need the following.

Claim. Assume that E satisfies condition B_r. For δ_1 sufficiently small, there exists $K < \infty$ (which is independent of E and r) such that for any interval E as above,

$$\frac{|Df^r(x)|}{|Df^r(y)|} \leqq K, \quad \forall x, y \in E. \tag{8.14}$$

Proof of Claim. Furthermore $|f^i(E)| \leqq \delta_1$ for every $i = 0, 1, \ldots, m$. It follows from Lemma 8.3 that there exists an interval $F \supset E$ such that the two components F_1 and F_2 of $F \setminus E$ satisfy

$$|f^r(F_1)| = |f^r(F_2)| = |f^r(E)|, \tag{8.15}$$

$f^r | F$ is a diffeomorphism and

$$|f^l(F)| \leqq (4 + 4 + 1) \cdot |f^l(E)|, \quad l = 0, \ldots, r. \tag{8.16}$$

From Lemma 4.1, and $f^l(E) \cap f^r(E) = \emptyset$ for all $l = 0, 1, \ldots, r-1$, one gets that the intervals $E, f(E), \ldots, f^r(E)$ are disjoint and so, using (8.16),

$$\sum_{l=0,\ldots,r-1} |f^l(F)| \leqq 9 \cdot |M|. \tag{8.17}$$

Therefore Theorem 2.1 can be applied and one gets a constant $C_3 > 0$ which only depends on f (and not on E and r), such that $B(f^r, F^*, J^*) \geqq C_3$ for all intervals $J^* \subset F^* \subset F$. From (8.15) and the Koebe Distortion Principle 3.2 it follows that one has bounded non-linearity on $f^r | E$. More precisely there exists $K < \infty$ which only depends on $C_3 > 0$ (and not E and r) such that (8.14) holds. This proves the claim.

Let us now continue with the proof of the lemma.

Case I. Assume that E satisfies condition B_r. Since $f^l(E) \cap f^r(E) = \emptyset$ for all $l = 0, 1, \ldots, r-1$ the fixed point $p_r \in f^i(E)$ of $f^r: E \to f^r(E)$ has minimal period r. From Theorem 6.1 there exists a sequence $\tilde{K}_k \to \infty$ as $k \to \infty$ such that

$$|Df^r(p_r)| \geqq \tilde{K}_r. \tag{8.18}$$

From this and (8.14) one gets

$$|Df^r(x)| \geqq K_r, \quad \forall x \in E, \tag{8.19}$$

where $K_r = \tilde{K}_r/K$.

Case II. If $f^r(E)$ does not satisfy condition B_r then $r < m$ and we proceed as follows. Let m_0 be the smallest integer with $r < m_0 \leqq m$ such that $f^{m_0}(E)$ satisfies condition B_{m_0}. Since the intervals $E, f(E), \ldots, f^m(E)$ are either disjoint or the one with the smaller index is contained in the one with the larger index, this implies that $f^l(E) \cap f^{m_0}(E) = \emptyset$, for all $l = r, r+1, \ldots, m_0 - 1$. This and Lemma 4.1 implies that $f^r(E), f^{r+1}(E), \ldots, f^{m_0-1}(E)$ are disjoint. Since $f^{m_0-r}(f^r(E)) = f^{m_0}(E) \subset V_0$ we can apply the claim for the map $f^{m_0-r}|f^r(E)$. In particular

$$\frac{|Df^{m_0-r}(x)|}{|Df^{m_0-r}(y)|} \leqq K, \quad \forall x, y \in f^r(E) \supset E. \tag{8.20}$$

Now

$$(f^{m_0-r}|f^r(E)) \circ (f^r|E) = (f^r|f^{m_0-r}(E)) \circ (f^{m_0-r}|E).$$

Apply the chain-rule to this. Then (8.20) implies that

$$\inf_{x \in E} |Df^r(x)| \geqq \frac{1}{K^2} \inf_{x \in f^{m_0-r}(E)} |Df^r(x)|. \tag{8.21}$$

Since $E \subset f^r(E)$ one has $f^{m_0-r}(E) \subset f^{m_0-r}(f^r(E)) = (f^{m_0}(E)$. So we can apply Case I and get

$$|Df^r(x)| \geqq K_r, \quad \forall x \in f^{m_0-r}(E). \tag{8.22}$$

This and (8.21) prove

$$|Df^r(x)| \geqq \frac{K_r}{K^2}, \quad \forall x \in E. \quad \text{Q.E.D.}$$

8.c. Conclusion of the Proof of Proposition 8.1.

Proof of Proposition 8.1. Assume that E is a strongly m-compatible interval E such that $|f^i(E)| \leqq \delta$, $\forall 0 \leqq i < m$ and such that either

$$f^i(E) \cap U_0 = \emptyset \quad \text{for all} \quad i = 0, 1, \ldots, m-1, \tag{8.23a}$$

or

$$f^m(E) \subset V_0. \tag{8.23b}$$

We need to show that if for some $0 \leqq i < j \leqq k \leqq m$ one has $f^i(E), f^j(E) \subset f^k(E)$ then

$$|f^i(E)| \leqq \lambda \cdot |f^j(E)|.$$

Let K_k be a sequence of numbers tending to infinity as $k \to \infty$ so that the estimates (8.4) from Lemma 8.2 and (8.13) from Lemma 8.4 are both satisfied for this sequence of K_k. Choose $N < \infty$ so large such that $K_k \geqq 2$ for all $k \geqq N$. Then let

$$P_N = \{q; q \text{ is a repelling periodic orbit with period } l \leqq N\}.$$

Denote the (minimal) period of a periodic orbit $q \in P_N$ by $\operatorname{per}(q)$ and define

$$\mu_N = \frac{1 + \min\{|Df^{\operatorname{per}(q)}(q)|; q \in P_N\}}{2}.$$

All periodic orbits of f are hyperbolic and the period of periodic points in P_N is uniformly bounded and therefore $\mu_N > 1$. (Otherwise f would have a non-hyperbolic periodic point.) Let $\delta'_2 \in (0, \frac{1}{2}\delta_0)$ be equal to the number δ' corresponding to $\delta = \delta_2$ from Lemma 5.2.b where $\delta_2 = \frac{1}{2}\delta_0$. Choose δ such that $0 < \delta < \min(\frac{1}{2}\delta'_0, \delta'_2)$ and such that for every $q \in P_N$, and for every x in a δ neighbourhood of q

$$|Df^{\operatorname{per}(q)}(x)| \geqq \mu_N > 1.$$

Let $\lambda = \max\{1/\mu_N, \frac{1}{2}\} < 1$.

Let $0 \leqq i < j \leqq k \leqq m$ be such that $f^i(E), f^i(E) \subset f^k(E)$. We may assume that j is the smallest number, with $i < j \leqq k \leqq m$, such that $f^j(E) \subset f^k(E)$. Since E is strongly m-compatible there exists a $(j-i)$-compatible interval $\tilde{E} \supset f^i(E)$ with $f^{j-i}(\tilde{E}) = f^k(E)$.

If (8.23a) holds then $f^i(E) \cap U_0 = \emptyset$, $\forall i = 0, 1, \ldots, m-1$. Now let $U \subset U_0$ be a neighbourhood of $C(f)$ such that each component of $U_0 \backslash U$ has length $\frac{1}{2}\delta_0$. So if $f^l(\tilde{E}) \cap U \neq \emptyset$ for some $0 \leqq l < j-i$ then, since $f^l(E) \cap U_0 = \emptyset$, $f^l(\tilde{E})$ contains a component of $U_0 \backslash U$ and therefore $|f^l(\tilde{E})| \geqq \frac{1}{2}\delta_0$. Using Lemma 5.2b and from the choice of δ this implies $|f^k(E)| = |f^{j-i}(\tilde{E})| \geqq (\frac{1}{2}\delta_0)' > \delta$. Since $|f^k(E)| \leqq \delta$ this gives a contradiction. Thus we have shown that $f^l(\tilde{E}) \cap U = \emptyset$ for all $l = 0, 1, \ldots, j-i-1$. Since $\tilde{E}$ is $(j-i)$-compatible, $\tilde{E} \subset f^{j-i}(\tilde{E}) = f^k(E)$ and the minimality of j implies $f^l(\tilde{E}) \cap f^{j-i}(\tilde{E}) = \emptyset$ for all $l = 0, 1, \ldots, j-i-1$. So we can apply Lemma 8.2 on $f^{j-i}|\tilde{E}$.

If (8.23b) holds then $\tilde{E}$ is $(j-i)$-compatible and $f^{j-i+m-k}(\tilde{E}) = f^{m-k}(f^k(E)) = f^m(E) \subset V_0$ and so we can apply Lemma 8.4 on $f^{j-i}|\tilde{E}$.

So in either case from the choice of N one has $|Df^{j-i}(x)| \geqq K_{j-i} \geqq 2$ for all $x \in \tilde{E}$ if $j-i \geqq N$. It follows that

$$|f^i(E)| \leqq \tfrac{1}{2}|f^{j-i}(f^i(E))| = \tfrac{1}{2} \cdot |f^j(E)|, \quad \text{if} \quad (j-i) \geqq N. \tag{8.24}$$

Now assume that $(j-i) < N$. Then there exists a periodic point q of period $j-i$ in $\tilde{E}$. Since $|f^i(E)| \leqq \delta, i = 0, \ldots, m-1$ one has from the choice of δ and μ_N

$$|f^j(E)| = |f^{j-i}(f^i(E))| \geqq \mu_N \cdot |f^i(E)|, \quad \text{if} \quad (j-i) < N. \tag{8.25}$$

Combining (8.24) and (8.25) and using the definition of λ we get

$$|f^i(E)| \leqq \lambda \cdot |f^j(E)|. \quad \text{Q.E.D.}$$

9. Cross Ratios are Bounded from below on *-Branch-Intervals

Combining Propositions 7.1 and 8.1 we get the following result.

9.1 Proposition. *Let f be a C^2 map satisfying the Misiurewicz conditions* (i) *and* (ii′) *and without flat critical points. Then there exists a constant $S < \infty$ with the following property. Let I_n be a *-branch-interval for f^n. Then*

$$\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S. \tag{9.1}$$

Using Theorem 2.1 this implies the following result.

9.2 Theorem. *Let f be a C^2 map satisfying the Misiurewicz conditions* (i) *and* (ii′) *and without flat critical points. Then there exists a constant $C>0$ such that for every $*$-branch-interval I_n of f^n and for all intervals $J_n \subset T_n \subset I_n$,*

$$A(f^n, T_n, J_n), B(f^n, T_n, J_n) \geqq C. \tag{9.2}$$

We should emphasize that if we had assumed from the start that $Sf<0$ then we would immediately have obtained that $A(f^n, T_n, J_n)$, $B(f^n, T_n, J_n) \geqq 1$ and most of the previous sections would be superfluous. The reason for Theorem 9.2 is that it enables us to apply the Schwarz and Koebe Distortion Principle even for high iterates of f.

Remark. For a given map f one can give an effective algorithm to give a lower bound for C. This follows from the remarks after Propositions 7.1 and 8.1.

10. f^n is Quasi-Polynomial on $*$-Branch-Intervals of f^n

Let us show that Theorem 9.2 implies that we have very good control on the non-linearity of $f^n|I_n$.

10.1. Proposition. *Let $f:M \to M$ be a C^2 map satisfying the Misiurewicz conditions* (i) *and* (ii′) *and without flat critical points. Then the restriction of f^n to $*$-branch-intervals is quasi-polynomial in the following sense. There exist $0 \leqq l' < \infty$ and a constant $K<\infty$ such that for any $n \geqq 0$ and any $*$-branch-interval $I_n=(a_n, b_n)$ of f^n there exist $1 \leqq l, \hat{l} \leqq l'$ such that*

$$\frac{1}{K}\cdot\frac{|f^n(I^n)|}{|I_n|^l}\cdot|(a_n,x)|^{l-1} \leqq |Df^n(x)| \leqq K\cdot\frac{|f^n(I_n)|}{|I_n|^l}\cdot|(a_n,x)|^{l-1}, \tag{10.1a}$$

for all $x \in I_n$ with $|(a_n,x)| \leqq \frac{1}{2}|I_n|$,

$$\frac{1}{K}\cdot\frac{|f^n(I_n)|}{|I_n|^{\hat{l}}}\cdot|(x,b_n)|^{\hat{l}-1} \leqq |Df^n(x)| \leqq K\cdot\frac{|f^n(I_n)|}{|I_n|^{\hat{l}}}\cdot|(x,b_n)|^{\hat{l}-1}, \tag{10.1b}$$

for all $x \in I_n$ with $|(x,b_n)| \leqq \frac{1}{2}|I_n|$.

Remark. One can give an effective algorithm to estimate K. This follows since there exists an effective estimate which gives a lower bound of C, see Sect. 9, and from the proof below.

Proof. As before there exists N_0 such that if there exists $i>0$ such that if $c \in C(f)$ and $f^i(c) \in C(f)$ then either c is periodic or $i<N_0$. Suppose that each of the critical points of $f, \dots, f^{N_0}$ is at most of order l'. Then there exists $\theta>0$ such that for each $c \in C(f)$, each $0 \leqq i \leqq N_0$ there exists some $1 \leqq l \leqq l'$, such that if (c, v) is an interval such that $f^i|(c,v)$ is a diffeomorphism then

$$\theta \leqq \frac{|f^i(c,v)|}{|(c,v)|^l} \leqq \frac{1}{\theta}, \tag{10.2}$$

$$\theta \cdot \mathrm{dist}(v, C(f))^{l-1} \leqq |Df^i(v)| \leqq \frac{1}{\theta} \cdot \mathrm{dist}(v, C(f))^{l-1}, \tag{10.3}$$

Write $I_n = (a_n, b_n)$. By maximality of $*$-branch-intervals, there exists $0 \leqq k < n$, so that $f^k(a_n) \in C_+(f) \cup \partial B_0$. Assume that k is the smallest integer with this property. Let us distinguish three cases.

Case 1. $C(f) \neq \emptyset$ and there exists $0 \leqq k < n$ such that $f^k(a_n) \in C(f)$. Write $c = f^k(a_n)$. If c is periodic then, because f is C^2, c would be a periodic attractor contradicting the assumption that I_n is a $*$-branch-interval. So assume that c is not periodic. (We should notice here that if f is only piecewise C^2 then c could be a repelling periodic point. If this happens then proceed as in Case 3 below). Let i be the minimal number such that $f^j(c) \neq C(f)$ for all $j \geqq i$. Then $i < N_0$.

Let I_k be the $*$-branch-interval of f^k containing I_n. From the Misiurewicz condition and the fact that $\mathrm{Clos}(B_0)$ consists of a finite number of intervals not containing points of $C(f)$ in its boundary, it follows that there exists $\delta_0' > 0$ such that for any n, k and any $*$-interval-branch $I_n = (a_n, b_n)$ with $f^k(a_n) \in C(f)$, the interval $f^k(I_k)$ contains a δ_0'-neighbourhood of $f^k(a_n) \in C(f)$, see Corollary 5.4. Now $A(f^k, T', J'), B(f^k, T', J') \geqq C > 0$ for all $J' \subset T' \subset I_k = T^*$. Let $K_0 < \infty$ be the constant from the Koebe Distortion Principle corresponding to C and $\tau = \rho_0$, where $\rho_0 = \min(\frac{1}{2}, \delta_0'/3|M|)$. Now take ρ such that

$$0 < \rho < \min\left(\frac{\theta^2}{K_0 \cdot 2^{l'}}, \rho_0\right).$$

From the above remarks it follows that we can choose $T' \supset I_n$ such that $f^k|T'$ is a diffeomorphism, $T' \backslash I_n$ consists of one interval, contains a_n in its interior and satisfies

$$\frac{|f^k(T' \backslash I_n)|}{|f^k(T')|} = \rho_0. \tag{10.4}$$

Take $T^* = T'$, $J^* \cup R^* = I_n$ and $L^* = T' \backslash I_n$ such that

$$\frac{|f^k(R^*)|}{|f^k(T^*)|} = \rho_0, \frac{|f^k(L^*)|}{|f^k(T^*)|} = \rho_0, \frac{|f^k(I_n)|}{|f^k(T^*)|} = 1 - \rho_0. \tag{10.5}$$

(Here the last two equalities follow from (10.4).) From the Koebe Distortion principle and (10.5) follows

$$\frac{1}{K_0} \cdot \frac{|f^k(I_n)|}{|I_n|} \leqq |Df^k(x)| \leqq K_0 \cdot \frac{|f^k(I_n)|}{|I_n|}, \quad \forall x \in J^*. \tag{10.6}$$

Moreover, if $x \in I_n = (a_n, b_n)$ and $(|f^k(a_n, x)|/|f^k(I_n)|) \leqq \frac{1}{2}$ then $(|f^k(x, b_n)|/|f^k(T^*)|) \geqq \frac{1}{2}(|f^k(I_n)|/|f^k(T^*))$, and using (10.4) this is at least $\frac{1}{2}(1 - \rho_0) \geqq \frac{1}{4} \geqq \rho_0$.'

Hence

$$x \in I_n \quad \text{and} \quad \frac{|f^k(a_n, x)|}{|f^k(I_n)|} \leqq \tfrac{1}{2} \Rightarrow x \in J^*. \tag{10.7}$$

Since $I_n = J^* \cup L^* = (a_n, b_n)$, one gets from the Koebe inequality (****)

$$\frac{1}{K_0} \cdot \frac{|f^k(I_n)|}{|I_n|} \leqq \frac{|f^k(a_n, x)|}{|(a_n, x)|}, \quad \forall x \in I_n.$$

Furthermore, from the mean-value theorem, the Koebe inequality (**) and (10.6) one has

$$\begin{aligned}\frac{|f^k(a_n, x)|}{|(a_n, x)|} &\leqq \max_{z \in T^*} |Df^k(z)| \leqq K_0 \cdot \max_{z \in J^*} |Df^k(z)| \\ &\leqq (K_0)^2 \cdot \frac{|f^k(I_n)|}{|I_n|}, \quad \forall x \in I_n.\end{aligned}$$

Together this gives

$$\frac{1}{(K_0)^2} \cdot \frac{|f^k(I_n)|}{|I_n|} \leqq \frac{|f^k(a_n, x)|}{|(a_n, x)|} \leqq (K_0)^2 \cdot \frac{|f^k(I_n)|}{|I_n|}, \quad \forall x \in I_n. \tag{10.8}$$

One gets from (10.6) and (10.7) that for each $x \in I_n$ such that $(|f^k(a_n, x)|/|f^k(I_n)|) \leqq \frac{1}{2}$ one has

$$\frac{1}{K_0} \cdot \frac{|f^k(I_n)|}{|I_n|} \leqq |Df^k(x)| \leqq K_0 \cdot \frac{|f^k(I_n)|}{|I_n|}. \tag{10.9}$$

Because $f^k(a_n) \in C(f)$, and $i \leqq N_0$, (10.2) and (10.3) give that there exists l, $0 \leqq l \leqq l'$ such that for all $x \in I_n$,

$$\theta \leqq \frac{|f^{k+i}(a_n, x)|}{|f^k(a_n, x)|^l}, \frac{|f^{k+i}(I_n)|}{|f^k(I_n)|^l} \leqq \frac{1}{\theta}, \tag{10.10}$$

$$\theta \cdot |f^k(a_n, x)|^{l-1} \leqq \frac{|Df^{k+i}(x)|}{|Df^k(x)|} \leqq \frac{1}{\theta} \cdot |f^k(a_n, x)|^{l-1}, \tag{10.11}$$

Finally take the *-branch-interval I_{n-k-i} of f^{n-k-i} which contains $f^{k+i}(a_n) \in f(C(f)) \backslash C(f)$. Corollary 5.5 implies that $f^{n-k-i}(I_{n-k-i})$ contains a δ-neighbourhood of $f^n(a_n)$ and we can apply the Koebe Distortion Principle exactly as before. Apply the Koebe inequality (****) for $f^{n-k-i}|I_{n-k-i}$. This gives that for all $x \in I_n$

$$\begin{aligned}\frac{|f^n(a_n, x)|}{|f^{k+i}(a_n, x)|} &= \frac{|f^{n-k-i}(f^{k+i}(a_n, x))|}{|f^{k+i}(a_n, x)|} \geqq \frac{1}{K_0} \frac{|f^{n-k-i}(f^{k+i}(I_n))|}{|f^{k+i}(I_n)|} \\ &= \frac{1}{K_0} \frac{|f^n(I_n)|}{|f^{k+i}(I_n)|}.\end{aligned} \tag{10.12}$$

Similarly, from the Koebe inequality for all $x \in I_n$ with $(|f^n(a_n, x)|/|f^n(I_n)|) \leqq \frac{1}{2}$ one has

$$\frac{1}{K_0}\cdot\frac{|f^n(I_n)|}{|f^{k+i}(I_n)|}\leqq\frac{|Df^n(x)|}{|Df^{k+i}(x)|}\leqq K_0\cdot\frac{|f^n(I_n)|}{|f^{k+i}(I_n)|}. \tag{10.13}$$

Applying (10.10) twice to (10.12) gives

$$\left(\frac{|f^k(a_n,x)|}{|f^k(I_n)|}\right)^l\leqq\frac{1}{\theta^2}\frac{|f^{k+i}(a_n,x)|}{|f^{k+i}(I_n)|}\leqq\frac{K_0}{\theta^2}\frac{|f^n(a_n,x)|}{|f^n(I_n)|},\quad \forall x\in I_n. \tag{10.14}$$

Since

$$0<\rho<\frac{\theta^2}{2^l K_0}$$

inequality (10.14) implies

$$\frac{|f^n(a_n,x)|}{|f^n(I_n)|}\leqq\rho\Rightarrow\frac{|f^k(a_n,x)|}{|f^k(I_n)|}\leqq\frac{1}{2}. \tag{10.15}$$

Hence we can apply (10.9) for all points $x\in I_n$ for which $(|f^n(a_n,x)|/|f^n(I_n)|)\leqq\rho$. Multiplying (10.9), (10.11) and (10.13) and using (10.15) gives

$$\begin{aligned}\frac{\theta}{(K_0)^2}\frac{|f^n(I_n)|}{|I_n|}\cdot\frac{|f^k(I_n)|}{|f^{k+i}(I_n)|}\cdot|f^k(a_n,x)|^{l-1}&\leqq|Df^n(x)|\\ &\leqq\frac{(K_0)^2}{\theta}\frac{|f^n(I_n)|}{|I_n|}\cdot\frac{|f^k(I_n)|}{|f^{k+i}(I_n)|}\cdot|f^k(a_n,x)|^{l-1},\end{aligned}$$

for all $x\in I_n$ with $(|f^n(a_n,x)|/|f^n(I_n)|)\leqq\rho$. Using (10.8) this gives

$$\begin{aligned}\frac{\theta}{(K_0)^{2l}}\frac{|f^n(I_n)|}{|I_n|}\cdot\frac{|f^k(I_n)|^l}{|f^{k+i}(I_n)|}\cdot\left(\frac{|(a_n,x)|}{|I_n|}\right)^{l-1}&\leqq|Df^n(x)|\\ \frac{(K_0)^{2l}}{\theta}\frac{|f^n(I_n)|}{|I_n|}\cdot\frac{|f^k(I_n)|^l}{|f^{k+i}(I_n)|}\cdot\left(\frac{|(a_n,x)|}{|I_n|}\right)^{l-1}&,\end{aligned}$$

for all $x\in I_n$ with $(|f^n(a_n,x)|/|f^n(I_n)|)\leqq\rho$. Finally using (10.10), this last inequality gives

$$\frac{\theta^2}{(K_0)^{2l}}\frac{|f^n(I_n)|}{|I_n|^l}\cdot|(a_n,x)|^{l-1}\leqq|Df^n(x)|\leqq\frac{(K_0)^{2l}}{\theta^2}\frac{|f^n(I_n)|}{|I_n|^l}\cdot|(a_n,x)|^{l-1}, \tag{10.16}$$

for all $x\in I_n$ with $(|f^n(a_n,x)|/|f^n(I_n)|)\leqq\rho$. Similarly there exists $1\leqq\hat{l}\leqq l'$ such that

$$\frac{\theta^2}{(K_0)^{2\hat{l}}}\frac{|f^n(I_n)|}{|I_n|^{\hat{l}}}\cdot|(x,b_n)|^{\hat{l}-1}\leqq|Df^n(x)|\leqq\frac{(K_0)^{2\hat{l}}}{\theta^2}\frac{|f^n(I_n)|}{|I_n|^{\hat{l}}}\cdot|(x,b_n)|^{\hat{l}-1}, \tag{10.17}$$

for all $x\in I_n$ with $(|f^n(x,b_n)|/|f^n(I_n)|)\leqq\rho$. By integrating these inequalities one gets that there exists a constant $K_1<\infty$ such that if $x\in I_n$ and $\operatorname{dist}(x,\partial I_n)\leqq 1/K_1|I_n|$ then $(|f^n(a_n,x)|/|f^n(I_n)|),(|f^n(x,b_n)|/|f^n(I_n)|)\leqq\rho$. In particular, from (10.16) and (10.17) there exists a constant $K_2<\infty$ (independent of n and I_n) such that if $|f^n(a_n,x)|/|f^n(I_n)|$ or $|f^n(x,b_n)|/|f^n(I_n)|$ is equal to ρ then $(1/K_3)(|f^n(I_n)|/|I_n|)\leqq$

$|Df^n(x)| \leqq K_3(|f^n(I_n)|/|I_n|)$. Using this, and the Koebe inequality (*) one gets

$$\frac{1}{K_0K_2}\frac{|f^n(I_n)|}{|I_n|} \leqq |Df^n(x)| \leqq K_0K_2\frac{|f^n(I_n)|}{|I_n|}, \tag{10.18}$$

for all $x, y \in I_n$ such that $(|f^n(a_n, x)|/|f^n(I_n)|), (|f^n(x, b_n)|/|f^n(I_n)|) \geqq \rho$. Since $\mathrm{dist}(x, \partial I_n) \leqq 1/K_1|I_n|$ implies (10.16) or (10.17) and

$$\frac{|(x, b_n)|^{\hat{l}-1}}{|I_n|^{\hat{l}-1}}, \frac{|(a_n, x)|^{l-1}}{|I_n|^{l-1}},$$

are bounded and bounded away from zero for $\mathrm{dist}(x, \partial I_n) \geqq (1/K_1)|I_n|$, combining (10.16)–(10.18) proves (10.1).

Case 2. $C(f) = \emptyset$ and $f^k(a_n) \in C_+(f) = \{x_0\}$. In this case one can use Theorem 2.5 since $\sum_{i=0}^{n-1} |f^i(I_n)|$ uniformly bounded and so there exists $K_0 < \infty$ such that

$$\frac{|Df^n(x)|}{|Df^n(y)|} \leqq K_0, \quad \forall x, y \in I_n. \tag{10.19}$$

This finished the proof of Proposition 10.1 in this case.

Case 3. $f^k(a_n) \in \partial(\mathrm{Clos}(B_0)) \cup \partial M$ and $f^i(a_n) \notin C(f) \cup \mathrm{Clos}(B_0)$ for all $0 \leqq i \leqq k$. In particular either $f^k(\mathrm{Clos}(I_n)) \cap \partial(\mathrm{Clos}(B_0)) \neq \emptyset$ or $k = 0$ and $a_n \in \partial M$. If $M = [0, 1]$ in order to streamline our argument, it is convenient at this point to extend $f:[0, 1] \to [0, 1]$ to a C^2 map $f: \mathbb{R} \to \mathbb{R}$. In order to complete the proof of Proposition 10.1 we need the following lemma.

10.2 Lemma. *There exist* $\delta > 0$, $S' < \infty$ (*which are independent of* n *and* I_n) and an interval $I'_n = (a'_n, b_n) \supset (a_n, b_n) = I_n$ *such that* $f^n | I'_n$ *is a diffeomorphism,*

$$|f^n(I'_n \backslash I_n)| \geqq \delta \tag{10.20}$$

and such that

$$\sum_{i=0}^{n-1} |f^i(I'_n)| \leqq S'. \tag{10.21}$$

Proof of Lemma 10.2. From Sect. 6 and since f has only hyperbolic periodic orbits, it follows that there exists at most a finite number of periodic attractors and $\mathrm{Clos}(B_0) \cap C(f) = \emptyset$. In particular B_0 consists of a finite number of intervals and the boundary of each of these intervals (except possibly if this boundary is in ∂M) consists of periodic or eventually periodic points, which are hyperbolic (all periodic orbits are hyperbolic from assumption (ii')) and have uniformly bounded period. Furthermore by assumption $f(\partial M) \subset \partial M$ and the periodic points in ∂M are hyperbolic. So we can choose $\varepsilon > 0$ so small so that there exists $C > 0$ and $\kappa > 1$ such that for each $n \geqq 0$ and each x, y in one component of $\mathrm{Clos}(B_0)$ with $x \in \partial(\mathrm{Clos}(B_0)) \backslash (\partial M \cup C(f))$ one has

$$Df^n | [x, y] \text{ a diffeomorphism and } |f^n(x) - f^n(y)| < \varepsilon \Rightarrow |Df^n(y)| \geqq C \cdot \kappa^n. \tag{10.22a}$$

Similarly if $x \in \partial M \setminus C(f)$ and $(y,x) \cap M = \emptyset$ then

$$Df^n|[x,y] \text{ a diffeomorphism and } |f^n(x) - f^n(y)| < \varepsilon \Rightarrow |Df^n(y)| \geqq C \cdot \kappa^n. \quad (10.22b)$$

(Here we use the extension of f to $\mathbb{R}$.) Moreover these exists $\delta \in (0, \varepsilon)$ such that if $c \in C(f)$ then

$$i_0 \geqq 0, \quad f^{i_0}(c) \in \operatorname{int}(B_0) \Rightarrow \operatorname{dist}(f^i(c), \partial(\operatorname{Clos}(B_0))) \geqq \delta, \quad \forall i \geqq i_0. \quad (10.23)$$

Since $f^i(a_n) \notin C(f)$ for all $i = 0, 1, \ldots, k-1$, there exists an interval $I'_n = (a'_n, b_n)$ strictly containing $(a_n, b_n) = I_n$ such that $f^n | I'_n$ is a diffeomorphism, either $f^k(I'_n \setminus I_n) \subset \operatorname{Clos}(B_0)$ or $f^k(I'_n \setminus I_n) \cap M = \emptyset$ (if $f^k(a_n) \in \partial M$), $f^i(I'_n) \cap B_0 = \emptyset$, $\forall i = 0, 1, \ldots, k-1$ and using (10.23), $|f^n(I'_n \setminus I_n)| > \delta$. In particular I'_n is contained in a *-branch-interval of f^{k-1}. Now shrink $I'_n \supset I_n$ so that

$$|f^n(I'_n \setminus I_n)| = \delta < \varepsilon. \quad (10.24)$$

It follows from (10.22) that

$$\sum_{i=k}^{n-1} |f^i(I'_n \setminus I_n)| \leqq \frac{1}{C} \cdot \left(\sum_{i \geqq 0} \kappa^{-i} \right) \cdot |f^n(I'_n \setminus I_n)| \leqq \frac{1}{C} \cdot \frac{\kappa}{\kappa - 1} \cdot |M|. \quad (10.25)$$

Moreover, I'_n is contained in a *-branch-interval for f^{k-1} one has

$$\sum_{i=0}^{k-1} |f^i(I'_n)| \leqq S. \quad (10.26)$$

Combining (10.25), (10.26) and $\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$ gives

$$\sum_{i=0}^{n-1} |f^i(I'_n)| \leqq 2S + \frac{1}{C} \cdot \frac{\kappa}{\kappa - 1} \cdot |M|. \quad (10.27)$$

This finishes the proof of this lemma. Q.E.D.

Conclusion of the Proof of Proposition 10.1 in Case 3. Let $\delta > 0$ be the constant from Lemma 10.2. Choose $\rho \in (0, \min(\delta/3|M|), 1/2))$. From Lemma 10.2, we have

$$\sum_{i=0}^{n-1} |f^i(I'_n)| \leqq S',$$

and therefore there exists $C' > 0$ such that for all intervals I_n as above, $A(f^n, T_n, J_n)$, $B(f^n, T_n, J_n) \geqq C'$ for all $J_n \subset T_n \subset I'_n$. Let $K_0 < \infty$ be the constant from the Koebe Distortion Principle 3.2 corresponding to C and $\tau = \rho$.

From Lemma 10.2 it follows that we can choose T' such that $I_n \subset T' \subset I'_n$, $T' \setminus I_n$ consists of one interval, contains a_n and satisfies

$$\frac{|f^n(T' \setminus I_n)|}{|f^n(T')|} \geqq \rho. \quad (10.28)$$

Take $T^* = T'$, $R^* \cup J^* = I_n$ and $L^* = T' \setminus I_n$ such that

$$\frac{|f^n(R^*)|}{|f^n(T^*)|} \geqq \rho. \quad (10.29)$$

From (10.28) and (10.29) it follows that we can apply the Koebe inequality $(*)$ and one gets for all $x \in J^*$,

$$\frac{1}{K_0} \cdot \frac{|f^n(I_n)|}{|I_n|} \leqq |Df^n(x)| \leqq K_0 \cdot \frac{|f^n(I_n)|}{|I_n|}.$$

From (10.28) and (10.29) one has that $x \in I_n = (a_n, b_n)$ and $(|f^n(a_n, x)|/|f^n(I_n)|) \leqq \rho$ implies $(|f^n(x, b_n)|/|f^n(I_n)|) \geqq \rho$ (here we use $\rho < \frac{1}{2}$) and therefore $x \in J^*$. Hence for all $x \in I_n = (a_n, b_n)$ such that $(|f^n(a_n, x)|/|f^n(I_n)|) \leqq \rho$,

$$\frac{1}{K_0} \cdot \frac{|f^n(I_n)|}{|I_n|} \leqq |Df^n(x)| \leqq K_0 \cdot \frac{|f^n(I_n)|}{|I_n|}. \tag{10.30}$$

Moreover, from the Koebe inequality $(*)$ there exists $K' < \infty$ such that

$$|Df^n(x)| \leqq K' \cdot \frac{|f^n(I_n)|}{|I_n|}, \quad \text{for all} \quad x \in I_n$$

such that

$$\frac{|f^n(a_n, x)|}{|f^n(I_n)|}, \quad \frac{|f^n(x, b_n)|}{|f^n(I_n)|} \geqq \rho. \tag{10.31}$$

From (10.30), the corresponding statement for (x, b_n) and inequality (10.31), Proposition 10.1 follows. Q.E.D.

11. The Exponential Decay of the Length of $*$-Branch-Intervals I_n as $n \to \infty$

In this section we prove Theorem B. From Theorem 9.2 we have that there is a constant $S < \infty$ such that

$$\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$$

for all $*$-branch-intervals I_n of f^n. In this section we are in the position to improve this result. We will show that these intervals go exponentially fast to zero, i.e., there are constants $C'' > 0$ and $\kappa < 1$ such that $|f^i(I_n)| \leqq (1/C'') \cdot \kappa^{n-i}$. Since $f^i(I_n)$ is contained in a $*$-branch-interval of f^{n-i} it suffices to show that

$$|I_n| \leqq \frac{1}{C''} \cdot \kappa^n,$$

for all $n \geqq 0$ and all $*$-branch-intervals of f^n. The next result finishes the proof of Theorem B.

11.1 Theorem. *Let $f: M \to M$ be a C^2 map without flat critical points and satisfying the Misiurewicz condition* (i) *and* (ii′). *Then there are constants $C'' > 0$ and $\kappa < 1$ such that for any $n \geqq 0$ and any $*$-branch-interval I_n of f^n one has*

a)

$$\frac{|f^n(I_n)|}{|I_n|} \geqq C'' \frac{1}{\kappa^n}$$

and therefore

$$|I_n| \leqq \frac{1}{C''} \cdot \kappa^n \cdot |M|,$$

b) *for every periodic point of (minimal) period n one has*

$$|Df^n(p)| \geqq C'' \cdot \frac{1}{\kappa^n}.$$

Proof. Since f satisfies Misiurewicz conditions (i) and (ii′) it follows from Sect. 6 that B_0 consists of at most a finite number of intervals. From Corollary 5.4 there exists $\delta' > 0$ such that for any $*$-branch-interval I_n one has $|f^n(I_n)| \geqq \delta'$. Lemma 5.1 then gives that there exists $N < \infty$ (independent of n and I_n) such that either $f^{n+N}|I_n$ is not a diffeomorphism or $f^{n+N}(I_n) \cap B_0 \neq \emptyset$. In particular there exists $i \leqq N$ and $x \in \operatorname{int}(I_n)$ such that $f^{n+i}(x) \in C(f) \cup B_0$. Choose $0 < i \leqq N$ minimal with respect to this property. So I_n is a $*$-branch-interval for f^{n+i} but not for f^{n+i+1}. Let I_{n+i+1} be one of the $*$-branch-intervals of f^{n+i+1} in I_n. First we will prove the following.

Claim. There exists $\varepsilon > 0$ which is independent of n and I_n such that

$$|f^{n+i}(I_n \backslash I_{n+i+1})| \geqq \varepsilon.$$

Proof of Claim. The boundary points of $f^{n+i}(I_n)$ are in $\bigcup_{j \geqq 1} f^j(C_+(f)) \cup \partial(\operatorname{Clos}(B_0))$. From this, the Misiurewicz conditions (i) and the fact that $\operatorname{Clos}(B_0)$ consists of a finite union of intervals whose boundary points are eventually periodic and since $f(\partial M) \subset \partial M$, it follows that there exists ε such that

$$\begin{aligned} f^{n+1}(\operatorname{int}(I_n)) \cap C(f) \neq \emptyset \Rightarrow f^{n+i}(I_n) &\text{ contains an} \\ \varepsilon \text{ neighbourhood of } C(f) \cap f^{n+i}&(\operatorname{int}(I_n)) \end{aligned} \tag{11.1}$$

and, as in Lemma 10.2, also such that

$$f^{n+1}(\operatorname{int}(I_n)) \cap B_0 \neq \emptyset \Rightarrow |f^{n+i}(I_n) \cap B_0| \geqq \varepsilon. \tag{11.2}$$

Since $f^j(\operatorname{int}(I_{n+i+1})) \cap (C(f) \cup B_0) = \emptyset$ for all $0 \leqq j < n+i+1$, we get from (11.1) and (11.2) that $|f^{n+i}(I_n \backslash I_{n+i+1})| \geqq \varepsilon$. This finishes the proof of the Claim.

Let us now prove statement a) of Theorem 11.1. Since I_n is a $*$-branch interval for f^{n+i}, inequality (10.1) gives a universal constant $K < \infty$ such that

$$|Df^{n+i}(x)| \leqq K \cdot \frac{|f^{n+i}(I_n)|}{|I_n|}, \quad \forall x \in I_n.$$

It follows that

$$\frac{|f^{n+i}(I_n \backslash I_{n+i+1})|}{|I_n \backslash I_{n+i+1}|} \leqq K \cdot \frac{|f^{n+i}(I_n)|}{|I_n|}.$$

In particular from the Claim

$$\frac{|I_n \backslash I_{n+i+1}|}{|I_n|} \geqq \frac{1}{K} \cdot \frac{|f^{n+i}(I_n \backslash I_{n+i+1})|}{|f^{n+i}(I_n)|} \geqq \frac{\varepsilon}{|M| \cdot K}.$$

Hence there exists $\tilde{\kappa} < 1$ such that

$$\frac{|I_{n+i+1}|}{|I_n|} \leqq \tilde{\kappa}. \tag{11.3}$$

Since any $*$-branch-intervals I_k of f^k and I_l of f^l, with $k \leqq l$, are either disjoint or I_l is contained in I_k, it follows that for each $*$-branch-interval I_{n+j} of f^{n+j} which has non-empty intersection with I_n and each $j \geqq 1$

$$|I_{jN}| \leqq \tilde{\kappa}^j \cdot |I_n|. \tag{11.4}$$

In particular for any $*$-branch-interval I_n of f^n,

$$|I_n| \leqq \kappa^n \cdot \frac{|M|}{\tilde{\kappa}}, \tag{11.5}$$

where $\kappa = (\tilde{\kappa})^{1/N}$. Moreover, from Corollary 5.4 there exists $\delta' > 0$ such that for any $*$-branch-interval I_n for f^n one has $|f^n(I_n)| \geqq \delta'$. So (11.5) implies

$$\frac{|f^n(I_n)|}{|I_n|} \geqq \frac{\tilde{\kappa} \cdot \delta'}{|M|} \cdot \kappa^{-n}.$$

In particular there exists $C'' > 0$ such that

$$\frac{|f^n(I_n)|}{|I_n|} \geqq C'' \cdot \kappa^{-n}. \tag{11.6}$$

This finishes the proof of statement a) of the theorem.

Let us now prove statement b) of Theorem 11.1. Let $U_0 \subset V_0 \subset W_0$ be neighbourhoods $C(f)$ such that, as before, $f^n(C(f)) \cap V_0 \subset C(f)$ and such that each of the components of $V_0 \setminus U_0$ has length δ_0. From Theorem A we may assume that p is a repelling periodic point. Let n be the period of p and let I_n be the $*$-branch-interval of f^n containing p and I_n^1 and I_n^2 the components of $I_n \setminus \{p\}$. Notice that $\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$. If $O(p) \cap U_0 = \emptyset$ then from Proposition 1.4 one gets that there exists $C > 0$, independent of n and p, such that

$$|Df^n(p)| \geqq \exp(-C \cdot S) \cdot \frac{|f^n(I_n)|}{|I_n|}. \tag{11.7}$$

On the other hand if $O(p) \cap U_0 \neq \emptyset$ then choose $p \in U_0$. As before $\partial f^n(I_n) \in M \setminus V_0$. In particular, $|f^n(I_n^i)| \geqq \delta$. It follows from the Minimum Principle 3.1, $\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$ and (11.5), that there exists a universal constant $C > 0$ such that

$$\begin{aligned} |Df^n(p)| &\geqq \exp(-3 \cdot C \cdot S) \cdot \min_{1,2} \frac{|f^n(I_n^i)|}{|I_n^i|} \\ &\geqq \exp(-3 \cdot C \cdot S) \cdot \delta \cdot \min_{1,2} \frac{1}{|I_n^i|} \\ &\geqq \exp(-3 \cdot C \cdot S) \cdot \delta \cdot \frac{\tilde{\kappa}}{|M|} \cdot \kappa^{-n}. \end{aligned} \tag{11.8}$$

This finishes the proof of this Theorem 11.1. Q.E.D.

12. Hyperbolicity, Measure and an Alternative Proof of Mañé's Result

In this section we shall prove Theorem C.

12.1 Theorem. *Let $f: M \to M$ be a C^2 map such that all its critical points are non-flat. Assume that f satisfies the Misiurewicz condition* (i). *Let $\mathscr{K}$ be a compact set such that $f(\mathscr{K}) \subset \mathscr{K}$ and which does not contain any non-hyperbolic periodic points. If $(C(f) \cup B_0) \cap \mathscr{K} = \emptyset$ then $\mathscr{K}$ is a hyperbolic set.*

Proof. Let us first prove that we may assume that all periodic orbits of f are hyperbolic. (For this we will not use that $\mathscr{K} \cap C(f) = \emptyset$.) Let NH be the set of non-hyperbolic periodic orbits of f. By assumption $NH \cap \mathscr{K} = \emptyset$. From Theorem A, the orbits in NH have uniformly bounded period. Therefore NH is compact and $NH \cap C(f) = \emptyset$. So we can choose a neighbourhood W of NH such that $W \cap \mathscr{K} = \emptyset$ and $W \cap C(f) = \emptyset$. The assertion of the theorem does not depend on $f \mid W$, and therefore we may change $f \mid W$ arbitrarily as long as we keep $f \mid (M \setminus W)$ unchanged. Since f is a diffeomorphism on each component of W, it is very easy to find a C^2 map g such that $g \mid (M \setminus W) = f \mid (M \setminus W)$, such that all periodic orbits of g are hyperbolic and such that $g \mid W$ is a diffeomorphism on each component of W. Therefore g coincides with f on a neighbourhood of $\mathscr{K}$ and g also satisfies the assumptions of the theorem.

So without loss of generality we assume that all periodic orbits of f are hyperbolic. Let U be a neighbourhood of $C(f)$ consisting of a finite number of components such that $U \cap \mathscr{K} = \emptyset$. If $C(f) = \emptyset$ let $U = C_+(f) = \{x_0\}$,

$$\mathscr{K}_n = \{x; f^i(x) \notin (U \cup \operatorname{Clos}(B_0)), \forall_i = 0, \ldots, n-1\}.$$

Since $\mathscr{K} \cap U = \emptyset$ and since $\mathscr{K}$ is forward invariant one has $\mathscr{K} \subset \mathscr{K}_n$ for all $n \geqq 0$. From Proposition 9.1 there exists $S < \infty$

$$\sum_{i=0}^{n-1} |f^i(I_n)| \leqq S$$

for any component of I_n of K_n (such a component is contained in a $*$-branch-interval of f^n). So from part b of Theorem 2.5, for each $x \in I_n$,

$$|Df^n(x)| \geqq \frac{|f^n(J_n)|}{|J_n|},$$

where J_n is the $*$-branch interval of f^n containing I_n. From Corollary 5.4 there exists a constant $\tilde{\delta}'_0$ such that $|f^n(J_n)| \geqq \tilde{\delta}'_0$. It follows that

$$|Df^n(x)| \geqq \frac{\tilde{\delta}'_0}{|J_n|},$$

and from Theorem 11.1 one has

$$|J_n| \leqq C'' \cdot \kappa^n \cdot |M|.$$

From the last two inequalities we get that there exists a constant $C''' > 0$ (which does not depend n and x) such that

$$|Df^n(x)| \geqq C''' \cdot \frac{1}{\kappa^n} \tag{12.1}$$

for each $x \in \mathscr{K}_n \supset \mathscr{K}$. Theorem 12.1 follows. Q.E.D.

Let us now prove that any invariant Borel set $\mathscr{K}$ either has Lebesgue measure zero or contains an interval.

12.2 Theorem. *Let $f: M \to M$ be a C^2 map such that all its critical points are non-flat. Assume that f satisfies the Misiurewicz condition* (i). *Let $\mathscr{K}$ be a compact Borel set such that $f(\mathscr{K}) \subset \mathscr{K}$ which does not contain any non-hyperbolic periodic points. If $B_0 \cap \mathscr{K} = \varnothing$, $\mathscr{K} \neq M$ and $\mathscr{K}$ is Borel set with positive Lebesgue measure then $C(f) \neq \varnothing$ and $\mathscr{K}$ contains a segment in M which contains at least one critical point in its interior.*

Proof. As in the proof of Theorem 12.1 we may assume that all periodic orbits of f are hyperbolic. Assume $\mathscr{K}$ is an invariant set as above with positive Lebesgue measure. Take a density point x' of $\mathscr{K}$. Since $\bigcup_{n \geqq 0} f^{-n}(C(f) \cup \partial(\mathrm{Clos}(B_0)))$ is countable we may assume that $f^n(x') \notin C(f) \cup \mathrm{Clos}(B_0)$ for all $n \geqq 0$. For each $n > 0$ let I_n be the $*$-branch-interval of f^n containing x'. From Corollary 5.4, $|f^n(I_n)| \geqq \tilde{\delta}'_0$. Let $I'_n \subset I_n$ such that $f^n(I_n \backslash I'_n)$ consists of two components each of which has length equal to $|f^n(I'_n)|$. Then

$$|f^n(I'_n)| \geqq \tfrac{1}{3}\tilde{\delta}'_0. \tag{12.2}$$

From Theorem 9.2 there exists a constant $C > 0$ such that $B(f^n, T^*, J^*) \geqq C$ for all $J^* \subset T^* \subset I_n$. Therefore it follows from the Koebe inequality $(*)$ that there exists $K' < \infty$ such that

$$\frac{1}{K'} \leqq \frac{|Df^n(x)|}{|Df^n(y)|} \leqq K', \quad \forall x, y \in I'_n. \tag{12.3}$$

From the forward invariance of $\mathscr{K}$ and (12.3) it follows that

$$\frac{|f^n(I'_n) \backslash \mathscr{K}|}{|f^n(I'_n)|} \leqq \frac{|f^n(I'_n \backslash \mathscr{K})|}{|f^n(I'_n)|} \leqq K' \cdot \frac{|I'_n \backslash \mathscr{K}|}{|I'_n|} \leqq \frac{|I'_n \backslash \mathscr{K}|}{|I'_n|} \to 0, \quad \text{as} \quad n \to \infty, \tag{12.4}$$

since x' is a density point of $\mathscr{K}$. From (12.2) it follows that there exists a sequence $n_i \to \infty$ and an interval J_0 of length $\geqq \frac{1}{3}\tilde{\delta}'_0$ such that $f^{n_i}(I'_{n_i}) \to J_0$. Using (12.4) this implies that $|J_0 \backslash \mathscr{K}| = 0$. It follows that $\mathscr{K} \supset J_0$. Since $f^n(\mathscr{K}) \cap B_0 = \varnothing$ for all $n \geqq 0$, Lemma 5.1 implies that there exists $i \geqq 0$ such that either

—$f^i | J_0$ is not injective and therefore $\mathscr{K} \supset f^i(\mathscr{K}) \supset M$, or
—$\mathrm{int}(\mathscr{K}) \cap C(f) \supset \mathrm{int}(f^i(J_0)) \cap C(f) \neq \varnothing$.

This completes the proof of this theorem. Q.E.D.

Theorem C follows from Theorems 12.1 and 12.2. Let us show that Mañé's results of [Ma] easily follow as a byproduct.

12.3 Theorem. [*Ma*]. *Let $f:M \to M$ be a C^2 map which is not a diffeomorphism. Let $\mathscr{K}$ be a compact forward invariant set with $(C(f) \cup B_0) \cap \mathscr{K} = \emptyset$. Then*

(i) *there exists $N < \infty$ such that all periodic orbits of f in $\mathscr{K}$ which are non-hyperbolic or attracting have period less than N.*

Moreover, if $\mathscr{K}$ does not contain non-hyperbolic periodic points then

(ii) *$\mathscr{K}$ is hyperbolic;*
(iii) *$\mathscr{K}$ has Lebesgue measure zero or $\mathscr{K}$ contains an interval.*

Proof. If $\mathscr{K} = \emptyset$ then there is nothing to prove. So assume that $\mathscr{K} \neq \emptyset$. Take a neighbourhood U of $C(f)$ such that $U \cap \mathscr{K} = \emptyset$. The statements about $\mathscr{K}$ do not depend on $f|U$. So as long as we keep $f|(M \backslash U)$ unchanged we may change $f|U$ arbitrarily. So choose a neighbourhood $V \subset \mathrm{Clos}(V) \subset \mathrm{int}(U)$ and a map g such that $g|(M \backslash U) = f|(M \backslash U)$ such that

a) g is C^2 and C^3 on V;
b) all critical points of g are turning points, are contained in V and are quadratic; moreover, g maps each of these turning points into $\mathscr{K}$ or into some periodic point by g (it follows that g satisfies the Misiurewicz condition (i));

Theorem 6.1 implies that there exists $N < \infty$ such that all non-hyperbolic or attracting periodic orbits have period $\leqq N$. In particular since $\mathscr{K} \cap U = \emptyset$, and f and g coincide outside U, statement *i*) follows. Now we can assume also that

c) all periodic points of g are hyperbolic.

We may assume c) because once we constructed a map as in a) and b) then the (minimal) period of non-hyperbolic periodic points is uniformly bounded and, by a small perturbation near these non-periodic orbits, one can ensure that all periodic points become hyperbolic. Since these non-hyperbolic periodic points are contained outside a neighbourhood of $\mathscr{K}$ the statements about $\mathscr{K}$ do not depend on this perturbation.

It follows that $\mathscr{K} \subset \{x; g^n(x) \notin U, \forall n \geqq 0\}$ and that g is a map satisfying the conditions of Theorems 12.1 and 12.2. Q.E.D.

13. The Existence of Absolutely Continuous Invariant Measures

13.1 Theorem. *Let f be a C^2 map without flat critical points. Assume that f satisfies the Misiurewicz condition* (i) *and all periodic points of f are hyperbolic and repelling. Then f has an absolutely continuous invariant probability measure of positive entropy.*

Proof. Let $|A|$ be the Lebesgue measure of a measurable set A. According to a theorem of Dowker and others, see [Fo], in order to show that there exists an absolutely continuous measure it suffices to show that there exists $\gamma > 0$ and $C < \infty$ such that for any measurable set $A \subset M$ and any $n \geqq 0$ one has

$$|f^{-n}(A)| \leqq C \cdot |A|^{\gamma}. \tag{13.1}$$

The measure is defined by taking a weak-limit of the measures

$$\mu_N(A) = \frac{1}{N} \sum_{i=0}^{N-1} |f^{-n}(A)|.$$

Since f has only repelling periodic points, from Corollary 5.4 there exists $\delta > 0$ such that for each branch-interval $I_n = (a_n, b_n)$ of f^n one has $|f^n(I_n)| \geqq \delta$. Take a set A of Lebesgue measure ε and let I_α, I_β be the maximal intervals in $f^n(I_n)$ of length $\leqq \varepsilon$ such that I_α contains $f^n(a_n)$ and I_β contains $f^n(b_n)$. From the Preimage Lemma 3.3 it follows that there exists a universal constant $K < \infty$ such that

$$|f^{-n}(A) \cap I_n| \leqq K \cdot \{|(f^{-n}(I_\alpha \cup I_\beta)) \cap I_n|\}. \tag{13.2}$$

Let l be the maximum of the orders of f at the critical points. Integrating the inequalities (10.1a) one gets that there exists $K' < \infty$ such that if we take $(a_n, x) = f^{-n}(I_\alpha) \cap I_n$,

$$\frac{\varepsilon}{|f^n(I_n)|} = \frac{|f^n(a_n, x)|}{|f^n(I_n)|} \geqq \frac{1}{K} \cdot \left(\frac{|(a_n, x)|}{|I_n|} \right)^l = \frac{1}{K} \cdot \left(\frac{|(f^{-n}(I_\alpha) \cap I_n|}{|I_n|} \right)^l.$$

Similarly for I_β. Since $|f^n(I_n)| \geqq \delta$ this gives

$$|(f^{-n}(I_\alpha)) \cap I_n|, \quad |(f^{-n}(I_\beta)) \cap I_n \leqq \left(\frac{\varepsilon \cdot K}{\delta} \right)^{1/l} \cdot |I_n|. \tag{13.3}$$

Since this holds for all branch-intervals I_n, and M is the union of f^n branch-intervals, (13.2) and (13.3) imply (13.1).

Denote the invariant measure constructed in this way by μ. Now we show that the entropy $h^\mu(f)$ of this measure μ is positive. Indeed let ξ be the partition of M generated by $C_+(f)$, and let $\xi_n = \xi \vee f^{-1}(\xi) \vee \cdot \vee f^{-n+1}(\xi)$. By definition,

$$h_\mu(f) \geqq \lim_{n \to \infty} \frac{H(\xi_n)}{n},$$

where

$$H(\xi_n) = - \sum_{I \in \xi_n} m(I) \log m(I). \tag{13.4}$$

Of course the elements of this partition ξ_n are the branch-intervals for f^n. From Sect. 11 there exists $C' < \infty$ and $\rho \in (0, 1)$ such that for all $n \geqq 0$,

$$|I_n(x)| \leqq C' \cdot \rho^n. \tag{13.5}$$

From (13.1) and (13.5)

$$m(I_n(x)) \leqq C \cdot |I_n(x)|^\gamma \leqq C \cdot C' \cdot \rho^{\gamma n}.$$

Letting $C'' = C \cdot C'$, one gets from this and (13.4),

$$H(\xi_n) \geqq \sum_{I \in \xi_n} m(I) \cdot (n\gamma \log(\rho) + \log(C'')) \geqq (n \cdot \gamma \cdot \log(\rho) + \log(C'')). \tag{13.6}$$

It follows that

$$h_\mu(f) \geqq \gamma \cdot \log(\rho) > 0. \quad \text{Q.E.D.}$$

Corollary. *Let μ be a ergodic component of such an absolutely continuous measure. Then Pesin-Rohlin formula*

$$h_\mu(f) = \int_M \log|f'|\,d\mu$$

*holds. For μ-*a.e. *x one has that*

$$\frac{\log|(f^n)'(x)|}{n}$$

converges to $h_\mu(f) > 0$.

Proof. This corollary follows immediately from the ergodicity of μ, $h_\mu(f) > 0$ and [Le]. Q.E.D.

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