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Chern-Simons Gauge Theory and Projectively Flat Vector Bundles on $\mathcal{M}_g$

T. R. Ramadas*

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

Abstract. We consider a vector bundle on Teichmüller space which arises naturally from Witten's analysis of Chern-Simons Gauge Theory, and define a natural connection on it. In the case when the gauge group is $U(1)$ we compute the curvature, showing, in particular, that the connection is projectively flat.

1. Introduction

Projectively flat unitary vector bundles on the moduli space of curves are well-known to define conformal field theories [FS]. When such a bundle $\mathscr{W}$ is pulled back to Teichmüller space (which is simply connected), it has a finite-dimensional space $\mathscr{H}_{\mathscr{W}}$ of projectively flat sections with the dimension of $\mathscr{H}_{\mathscr{W}}$ being equal to the rank of $\mathscr{W}$. Moreover, the modular group acts projectively on the space of these sections. In [W], it was argued that the state-space of Chern-Simons gauge theory is, up to projective isomorphism, given by $\mathscr{H}_{\mathscr{W}}$ for a certain projectively flat bundle. The bundle $\mathscr{W}$ is well-known; our aim in this note is to give a purely differential-geometric description of a natural connection on $\mathscr{W}$, with the correct covariance properties under the modular group. The definition draws on a construction from [RSW], and further, makes a certain technical assumption, namely, that the state-vectors of the Chern-Simons theory are normalisable. In the case when the gauge group is $U(1)$, we prove the projective flatness of this connection in a computation which also yields the "central charge." (In this case the assumption of finite norm is trivially true, as we shall see.) It appears that this may complement the deeper treatments of [BN, EMSS] which however involve considerations from conformal field theory.

In Sect. 4 we outline a short proof of projective flatness, again in the $U(1)$ case, which uses the "discrete Heisenberg group."

We are not able, at this point, to prove the projective flatness of the connection for nonabelian groups.

* On leave from the School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India. Work supported by DOE grant DE-FG-02-88ER25066

2. The Bundle $\mathscr{W}$ and the Connection ∇

The arguments in this section are abstract, and meant to provide a concise description of the ideas. In the next section we deal more explicitly with the $U(1)$ case.

Let M denote a compact 2-manifold without boundary (with genus $\mathbf{g} \geqq 2$ – the other cases can be treated with analogous results), G a compact Lie group which can be either $SU(N)$ or $U(1)$, $\mathscr{A}$ the space of connections on the trivial G bundle on M, $\mathscr{A}_F$ the space of flat connections, $\mathscr{A}_F^s$ the space of irreducible flat connections, and $\mathscr{G}$ the group of gauge transformations. Then it is well-known [AB] that $\mathscr{A}_F^s/\mathscr{G}$ is in a natural way a symplectic manifold. In [RSW] a natural hermitian line bundle $\mathscr{L}$ with connection was defined on $\mathscr{A}_F^s/\mathscr{G}$. This does *not* require a complex structure on M, and the curvature Ω of the connection is in fact $i \times$ (the symplectic form).

A choice of conformal structure on M induces a complex structure on $\mathscr{A}_F^s/\mathscr{G}$, and it can be identified (in the case $G = SU(N)$) with the moduli space of stable vector bundles of rank N and trivial determinant on the Riemann surface defined by the conformal structure [NS, AB], or (in the case $G = U(1)$) with the Jacobian. The form Ω is of type $(1, 1)$ with respect to any of these complex structures, and the line bundle $\mathscr{L}$ thus inherits a holomorphic structure.

Recall that the Teichmüller space $T_{\mathbf{g}}$ is defined as the space of conformal structures $\mathscr{C}$ on M modulo the group $\mathscr{D}_0$ of diffeomorphisms of M which can be continuously connected to the identity diffeomorphism. The group $\mathscr{D}_0$ acts trivially on $\mathscr{A}_F/\mathscr{G}$ and in fact the complex structure defined on $\mathscr{A}_F/\mathscr{G}$ does not change when that on M is changed by an element of $\mathscr{D}_0$.

We are now ready to define the vector bundle $\mathscr{W}$ over $T_{\mathbf{g}}$. $\mathscr{W}$ is defined as the bundle whose fibre at any point J of Teichmüller space is the space of holomorphic sections of $\mathscr{L}$, the holomorphic structure on $\mathscr{A}_F^s/\mathscr{G}$ and $\mathscr{L}$ being determined, as outlined above, by J. Let $\mathscr{V}$ denote the trivial vector bundle $T_{\mathbf{g}} \times L^2(\mathscr{L}, \mathscr{A}_F^s/\mathscr{G})$ with fibre the space of square-integrable (with respect to the natural volume element on the symplectic manifold $\mathscr{A}_F^s/\mathscr{G}$ and the hermitian structure on $\mathscr{L}$) sections of $\mathscr{L}$. Assume that the holomorphic sections of $\mathscr{L}$ are in fact square-integrable. (This assumption is necessary in the nonabelian case because the space of stable bundles of zero Chern class is not compact. It does, however have a compactification of a kind which makes this assumption not unreasonable. In the case $G = U(1)$ the Jacobian, of course is compact.) Under this assumption $\mathscr{W}$ is a sub-bundle of $\mathscr{V}$. Since the latter is a unitary bundle with a canonical trivialisation, $\mathscr{W}$ inherits a natural hermitian connection which we denote by ∇. This is our candidate for the projectively flat connection.

Consider now the action of the modular group $\mathscr{T} = \mathscr{D}/\mathscr{D}_0$ on Teichmüller space. By the naturality of the construction of $\mathscr{L}$ (see [RSW] for details) there is also a unitary action on $L^2(\mathscr{L}_k, \mathscr{A}_F^s/\mathscr{G})$ which takes $\mathscr{W}$ to itself. Thus ∇ is clearly preserved under the lifted action of $\mathscr{T}$ on $\mathscr{W}$; and $\mathscr{W}$ descends to $\mathscr{M}_{\mathbf{g}}$ as a bundle with connection.

Imagine now that we knew that the connection ∇ is projectively flat. Since $\mathbf{T}_{\mathbf{g}}$ is contractible there exists over it a hermitian line bundle l with connection such that $\mathscr{W} \otimes l$ is actually flat; further, there exists an action of a central extension (by $U(1)$) of the modular group on l preserving the connection [R, p. 360]. Denoting by

$\mathscr{H}_{\mathscr{W}\otimes l}$ the space of flat sections of $\mathscr{W}\otimes l$ over Teichmüller space, the above action of the modular group defines a projective representation on $\mathscr{H}_{\mathscr{W}\otimes l}$. Any other choice of l yields a projectively equivalent space and representation.

3. The $U(1)$ Case: Projective Flatness

We consider now the case $G=U(1)$. We can now drop the constraint on the genus of M. The space of flat $U(1)$ connections on M modulo gauge transformations is the $2\mathbf{g}$-dimensional torus $J_M \equiv H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$. The construction of the line bundle $\mathscr{L}$ proceeds as follows. Let k be an even integer (this is the integer multiplying the Chern-Simons action in functional integrals). Define the $U(1)$-valued cocycle on $H^1(M,\mathbf{R})\times H^1(M,\mathbf{Z})$

$$\Theta_k(x,u)=\exp(-k\pi i \textstyle\int x \wedge u)$$

and define

$$\mathscr{L}_k = H^1(M,\mathbf{R})\times_{\Theta_k}\mathbf{C},$$

where on the right we mean the quotient of $H^1(M,\mathbf{R})\times\mathbf{C}$ by the equivalence relation

$$(x,z)\cong(x+u,\Theta_k(x,u)z), \qquad u\in H^1(M,\mathbf{Z}).$$

The one form

$$\hat{\omega}_k(x)(y)\equiv -(k\pi i \textstyle\int x\wedge y)$$

defines a connection ω_k on this bundle (i.e., if X is a vector field on $H^1(M,\mathbf{R})$ invariant under $H^1(M,\mathbf{Z})$, and F is a function on $H^1(M,\mathbf{R})$ satisfying $F(x+u)=\Theta_k(x,u)F(x)$ for $u\in H^1(M,\mathbf{Z})$, so is $D_XF(x)=XF(x)+\hat{\omega}(x)(X)F(x)$). The curvature is

$$\Omega_k(x,y)=-2k\pi i\textstyle\int x\wedge y\,.$$

We shall denote by $\mathscr{L}$ the line bundle $\mathscr{L}_k$ for $k=2$. Thus $\mathscr{L}_k=\mathscr{L}^{k/2}$.

Note that a diffeomorphism σ of M induces a symplectic automorphism of J_M, which lifts to a connection-preserving automorphism of $\mathscr{L}_k$. It is also clear that $\mathrm{Diff}_0 M$ acts trivially. Suppose given a diffeomorphism σ of M. Let $\tilde{\sigma}$ be the induced map on J_M. From the definition of $\mathscr{L}_k$ it is clear that there is a natural lift of $\tilde{\sigma}$ to $\mathscr{L}_k$ which preserves the connection ∇, and preserves the inner product on sections of $\mathscr{L}_k$.

Note that up to now we have not used any complex structure on M. A complex structure on a surface is given by any endomorphism J of the cotangent bundle of M such that $J^2=-1$. Any such J defines a complex structure $\tilde{J}$ on the vector-space $H^1(M,\mathbf{R})$ in a standard way (in short: represent the real 1-cohomology by forms α satisfying $d\alpha=0$ and $dJ\alpha=0$, and define $\tilde{J}$ by $\alpha\mapsto\tilde{J}(\alpha)\equiv J\alpha$). The torus J_M becomes the Jacobian $\mathscr{J}$. The curvature form Ω is of type (1,1) with respect to $\tilde{J}$, $\Omega(\tilde{J}x,\tilde{J}y)=\Omega(x,y)$; also

$$\langle x,y\rangle\equiv\frac{i}{2\pi}\Omega(x,\tilde{J}y)$$

defines an inner product on $H^1(M,\mathbf{R})$ and a translation invariant Kähler metric on $\mathscr{J}$.

We can now define the bundle $\mathscr{W}$: its fibre at any J in $T_{\mathbf{g}}$ is defined to be the space of holomorphic sections of $\mathscr{L}_k$ with respect to the complex structure induced by J.

We are now ready to compute the curvature $\mathscr{F}_\nabla$ of ∇. Let us first define ∇ explicitly. Let X be a vector field on $T_{\mathbf{g}}$, s a section of $\mathscr{W}$ over $T_{\mathbf{g}}$. We define

$$\nabla_X(s) = P \circ X(s),$$

where P is the orthogonal projection $L^2(J_M, \mathscr{L}_k) \to H^0(\mathscr{J}, \mathscr{L}_k)$ and $X(s)$ is the action of the vector field on s, the latter regarded as a function from $T_{\mathbf{g}}$ to the vector space $L^2(J_M, \mathscr{L}_k)$.

The holomorphic sections of $\mathscr{L}_k$ are in fact the ground states of the Laplacian on sections of $\mathscr{L}_k$. This is because of the identity

$$\Delta_{\omega_k} = 2\bar{\partial}^*_{\omega_k}\bar{\partial}_{\omega_k} + \pi \times (\dim J_M).$$

Note that the lowest eigenvalue is independent of the complex structure, so in fact (see Sect. 4) is the degeneracy of the ground state, so that $\mathscr{W}$ is really a vector bundle. Let H denote the operator $\Delta_{\omega_k} - \pi \times (\dim \mathscr{J})$; then H^{-1} is well-defined on the orthogonal complement of $H^0(\mathscr{J}, \mathscr{L}_k)$, and the connection ∇ can be expressed as:

$$\nabla_X(s) = X(s) + H^{-1}X(H)s.$$

Note that $X(H)s$ is orthogonal to $H^0(\mathscr{J}, \mathscr{L}_k)$, for, given v any other section of $\mathscr{W}$, we have $(v, X(H)s) = X((v, Hs)) - (X(v), Hs) - (v, HX(v)) = 0$, where we use the fact that H is selfadjoint and annihilates s and v. We check that this is indeed a unitary connection on $\mathscr{W}$:

$$\begin{aligned} H\{\nabla_X(s)\} &= H\{X(s) + H^{-1}X(H)s\} \\ &= X(Hs) \\ &= 0 \end{aligned}$$

and $(\nabla_X(v), s) + (v, \nabla_X(s)) = X(v, s)$.

Let now X and Y be two vector fields on $\mathscr{T}_{\mathbf{g}}$. We can now compute $(s, \mathscr{F}_\nabla(X, Y)s)$ and find $(s, \mathscr{F}_\nabla(X, Y)s) = (H^{-1}X(H)s, H^{-1}Y(H)s) - (H^{-1}Y(H)s, H^{-1}X(H)s)$.

We need an explicit expression for $X(H)$. Choose a standard basis $\{e_i | i = 1, \ldots, 2\mathbf{g}\}$ for the integral cohomology of M. Let ∂_i denote the co-ordinate vector fields on J_M, and define the (translation-invariant) tensor fields $\tilde{J}^j_i$, Ω_{ij} and K^j_i by $\tilde{J}\partial_i = \tilde{J}^j_i\partial_j$, $\Omega_{ij} = \Omega(\partial_i, \partial_j)$ and $K^j_i = \langle \partial_i, \partial_j \rangle$. Note that $K_{ij} = \frac{i}{2\pi}\tilde{J}^k_i\Omega_{jk}$, and if we define K^{ij} and Ω^{ij} as inverses of K_{ij} and Ω_{ij} respectively we have $K^{ij} = 2\pi i\tilde{J}^i_k\Omega^{kj}$. (Repeated indices are summed.) Denote by D_i the covariant derivative with respect to the coordinate vector field ∂_i. We have then $[D_i, D_j] = \Omega_{ij}$, $\Delta_{\omega_k} = -K^{ij}D_iD_j$ and $[D_i, H] = 4\pi iJ^j_iD_j$. For any w, we have $[H, D_i] = -4\pi iJ^j_iD_jw$, yielding

$$\begin{aligned} HP^{\pm j}_iD_jw &= -4\pi iP^{\pm j}_iJ^k_jD_kw + P^{\pm j}_iD_jHw \\ &= \mp 4\pi P^{\pm j}_iD_jw + P^{\pm j}_iD_jHw, \end{aligned}$$

where $P^{\pm j}_i = \frac{1}{2}(\delta^j_i \pm iJ^j_i)$. From this we conclude:

i) If w is a ground-state of H, $P_i^{+j}D_jw=0$, $P_i^{-j}D_jw=D_iw$, which is just the statement that w is a holomorphic section of $\mathscr{L}_k$, and D_jw is an eigenstate with eigenvalue 4π.

ii) Again, if w is a ground-state. $X(H)w=X(K^{ij})D_iD_jw$ is an eigenstate of H with eigenvalue 8π. (It is à priori a linear combination of states with eigenvalue 0 and with eigenvalue 8π, on the other hand it is orthogonal to the ground-states.)

The expression for the curvature now becomes

$$(s, \mathscr{F}_V(X,Y)s)=\frac{1}{(8\pi)^2}X(K^{ij})Y(K^{kl})((D_iD_js, D_kD_ls)-(D_kD_ls, D_iD_js)).$$

Let us now compute the expression $Q_{ij,kl}\equiv(D_iD_js,\ D_kD_ls)$. We have (the inner product being defined to be $\mathbf{C}$-linear in the first argument),

$$\begin{aligned}
Q_{ij,kl}&=(D_iD_js, D_kD_ls)\\
&=(D_iP_j^{-p}D_ps, D_kD_ls)\\
&=P_j^{-p}(D_iD_ps, D_kD_ls)\\
&=P_j^{-p}(\Omega_{ip}s, D_kD_ls)+P_j^{-p}(D_pD_is, D_kD_ls)\\
&=P_j^{-p}\Omega_{ip}(s, D_kD_ls)-P_j^{-p}(D_is, D_pD_kD_ls)\\
&=P_j^{-p}\Omega_{ip}(s, D_kD_ls)-P_j^{-p}(D_is, \Omega_{pk}D_ls)-P_j^{-p}(D_is, D_kD_pD_ls)\\
&=P_j^{-p}\Omega_{ip}(s, D_kD_ls)+P_j^{-p}\Omega_{pk}(D_is, D_ls)+P_j^{-p}\Omega_{pl}(D_is, D_ks)\\
&\quad-P_j^{-p}(D_is, D_kD_lD_ps)\\
&=P_j^{-p}\Omega_{ip}(s, D_kD_ls)-P_j^{-p}\Omega_{pk}(s, D_iD_ls)-P_j^{-p}\Omega_{pl}(s, D_iD_ks).
\end{aligned}$$

We can similarly compute $(s, D_qD_rs)=+P_r^{+s}\Omega_{qs}(s,s)$ which yields

$$Q_{ij,kl}=\{+P_j^{-p}P_l^{+s}\Omega_{ip}\Omega_{ks}-P_j^{-p}P_l^{+s}\Omega_{kp}\Omega_{is}+P_j^{-p}P_k^{+s}\Omega_{ip}\Omega_{ls}\}(s,s).$$

Finally,

$$(s, \mathscr{F}_V(X,Y)s)=\frac{1}{(8\pi)^2}X(J_a^j)Y(J_b^l)(P_l^{-a}P_j^{+b}-P_l^{+a}P_j^{-b})(s,s).$$

This proves projective flatness, since we have shown that the curvature is a multiple of the identity. Note also that the curvature is independent of k, and a universal expression independent of $\mathbf{g}$. The *first Chern form*, however depends on k since the rank of $\mathscr{W}$ is $k^{\mathbf{g}}$. Note that the curvature is of type $(1,1)$.

4. The Heisenberg Group and Projective Flatness; Conclusion

One can give a short proof (which, however, does not yield the central charge) of projective flatness using the "discrete Heisenberg group". The importance of this group in this context was stressed to me by R. Bott and M. S. Narasimhan.

The following facts about the line bundle $\mathscr{L}_k$, considered as a holomorphic line bundle on $\tilde{J}$, can be established easily [M1]:

i) It is ample.

ii) $\dim H^0(\tilde{J}, \mathscr{L}_k)=k^{\mathbf{g}}$.

We restrict ourself to the case $k=2$, for simplicity. Consider the abelian group $H^1(M, \mathbf{Z}/2\mathbf{Z})$. The cup product defines a central extension (which we denote by $\mathscr{Q}$) of $H^1(M, \mathbf{Z}/2\mathbf{Z})$ by $\mathbf{Z}/2\mathbf{Z}$. One can check, using the expression for $\hat{\omega}_2$ and Ω_2, that the action of $H^1(M, \mathbf{Z}/2\mathbf{Z})$ on J_M (i.e. translation by elements of order 2) lifts to a connection-preserving action of $\mathscr{Q}$ on $\mathscr{L}$ such that the nontrivial central element acts by -1. $\mathscr{Q}$ acts by unitary transformations on $L^2(J_M, \mathscr{L})$, and, if we choose a complex structure on M, this action leaves $\mathscr{W}$ invariant. In fact the representation on $\mathscr{W}$ is irreducible [M2]: one can check this directly by a dimension count. One can check from the definitions that the action on $\mathscr{W}$ commutes with $\mathscr{F}_V$, and this proves that $\mathscr{F}_V$ is a multiple of the identity.

A similar proof probably works for the case $SU(N)$, $k=1$. As D. Freed pointed out to me, it fails when $k \geqq 2$.

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Since this paper was written, I have learned of related works by N. Hitchin and S. Axelrod, S. Della Pietra and E. Witten, both reported at the Second Meeting on "Links between Geometry and Physics," Schloß Ringberg, April 1989.

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