Commun. Math. Phys. 128, 191–196 (1990)

Communications in
Mathematical Physics

$N=2$ Super Yang-Mills Theory in Projective Superspace

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Abstract. We construct $N=2$ Yang-Mills theory in projective superspace by exploiting the analogy to Ward's twistor construction of self-dual Yang-Mills fields.

In a series of papers [1,2] we have developed a formalism for describing $N=2$, $d=4$ (or equivalently, $N=1$, $d=6$ or $N=4$, $d=2$) supersymmetry, useful for studying scalar multiplets off shell. This *projective* superspace adjoins to the usual $N=2$ superspace a complex coordinate, which can be viewed as a coordinate on $CP(1)$, and parametrizes the $N=1$ subspaces of $N=2$ superspace (see below). This development largely parallels that of harmonic superspace [3,4] which instead of using a projective coordinate on $CP(1)$ uses spinor harmonic analysis on S^2. Though the two approaches are presumably essentially equivalent, the projective approach is concerned with analytic properties on the Riemann sphere, while the harmonic approach focuses on group theoretical properties under $SU(2)$ (acting on S^2). We have concentrated entirely on classical properties of various scalar multiplets and non-linear σ-models, whereas the harmonic superspace methods have also been applied to super Yang-Mills and supergravity systems, as well as quantum calculations [4]. Recently a harmonic approach to self-dual Yang-Mills theory, based on the harmonic superspace formulation of $N=2$ super Yang-Mills theory, has been proposed [5]. This article also discusses the relation to Ward's construction of self-dual Yang-Mills fields [6,7]. Here we give a projective superspace description of $N=2$ supersymmetric Yang-Mills theory which is completely analogous to Ward's twistor construction of self-dual Yang-Mills. We find our approach simpler and more direct than that of [4], but we have not carried our program as far, since we have not found the unconstrained prepotential for the Yang-Mills field.

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** Work supported in part by NSF grant No. PHY 85-07627

The usual $N=2$, $d=4$ superspace has, in addition to the spacetime coordinate $x^{\alpha\dot\alpha}$, anticommuting coordinates $\theta^{\alpha a}, \bar\theta^{\dot\alpha}_a$ where $\alpha, \dot\alpha$ are left and right-handed Weyl spinor indices and a is an internal $SU(2)$-isospinor index [8, 9, 10]. The spinor derivatives

$$D_{\alpha a}=\frac{\partial}{\partial\theta^{\alpha a}}+\frac{i}{2}\bar\theta^{\dot\alpha}_a\frac{\partial}{\partial x^{\alpha\dot\alpha}},\qquad \bar D^a_{\dot\alpha}=\frac{\partial}{\partial\bar\theta^{\dot\alpha}_a}+\frac{i}{2}\theta^{\alpha a}\frac{\partial}{\partial x^{\alpha\dot\alpha}}, \tag{1}$$

satisfying the supersymmetry algebra

$$\{D_{\alpha a}, \bar D^b_{\dot\beta}\}=i\delta^b_a\partial_{\alpha\dot\beta},\qquad \{D_{\alpha a}, D_{\beta b}\}=\{\bar D^a_{\dot\alpha}, \bar D^b_{\dot\beta}\}=0. \tag{2}$$

We obtain projective superspace by introducing a projective isospinor $(1, \zeta)$

$$\nabla_\alpha\equiv D_{1\alpha}+\zeta D_{2\alpha},\qquad \bar\nabla_{\dot\alpha}\equiv\bar D^2_{\dot\alpha}-\zeta\bar D^1_{\dot\alpha},\qquad \Delta_\alpha\equiv D_{2\alpha}-\frac{1}{\zeta}D_{1\alpha},\qquad \bar\Delta_{\dot\alpha}\equiv\bar D^1_{\dot\alpha}+\frac{1}{\zeta}\bar D^2_{\dot\alpha}. \tag{3}$$

[Note that $\bar\nabla(\bar\Delta)$ is *not* the complex conjugate of $\nabla(\Delta)$]. Under isospin rotations of the internal $SU(2)$, ζ transforms as the complex coordinate under rotations of the Riemann sphere. The algebra of the derivatives in (3) is

$$\begin{aligned}&\{\nabla_\alpha, \nabla_\beta\}=\{\nabla_\alpha, \bar\nabla_{\dot\beta}\}=\{\bar\nabla_{\dot\alpha}, \bar\nabla_{\dot\beta}\}=0,\\ &\{\Delta_\alpha, \Delta_\beta\}=\{\Delta_\alpha, \bar\Delta_{\dot\beta}\}=\{\bar\Delta_{\dot\alpha}, \bar\Delta_{\dot\beta}\}=0,\\ &\{\nabla_\alpha, \Delta_\beta\}=\{\bar\Delta_{\dot\alpha}, \bar\nabla_{\dot\beta}\}=0,\\ &\{\nabla_\alpha, \bar\Delta_{\dot\beta}\}=\{\Delta_\alpha, \bar\nabla_{\dot\beta}\}=2i\partial_{\alpha\dot\beta}.\end{aligned} \tag{4}$$

Observe that the ∇'s (idem Δ's) form a graded abelian subalgebra that corresponds to an $N=1$ subsuperspace. These subalgebras are preserved by the involution R which composes the antipodal map $\zeta\to-\zeta^{-1}$ with complex conjugation:

$$R(\nabla)=-\zeta^{-1}\bar\nabla,\qquad R(\Delta)=\zeta\bar\Delta. \tag{5}$$

R defines a real structure on projective superspace. The basic superfields we work with are annihilated by *all* the ∇'s, and thus effectively live in an $N=1$ subsuperspace (cf. chiral fields in ordinary $N=1$ superspace). Different superfields are defined by their ζ-dependence.

In [2] we introduced a number of scalar superfields. The multiplet that we use here is an analytic multiplet defined by

$$\nabla\eta=\bar\nabla\eta=0,\qquad \eta=\sum_{i=0}^{\infty}\eta_i\zeta^i. \tag{6}$$

Substituting in (3) this implies

$$D_2\eta_i=-D_1\eta_{i+1},\qquad \bar D_2\eta_i=\bar D^1\eta_{i-1}, \tag{7a}$$

$$\Rightarrow D_1\eta_0=D^\alpha_1 D_{1\alpha}\eta_1=0. \tag{7b}$$

This means that, as $N=1$ superfields, η_0 is antichiral, η_1 is antilinear and all other η_i's are unconstrained. We also have the conjugate field, defined to be

$$\bar{\eta} \equiv R(\eta), \qquad \eta = \sum_{i=0}^{\infty} \bar{\eta}_i \left(-\frac{1}{\zeta} \right)^i . \tag{8}$$

In [2] we use these multiplets to describe supersymmetric nonlinear σ-models on hyperkähler manifolds. In particular, the free action for this multiplet is

$$I = \int dx \oint \frac{\zeta d\zeta}{2\pi i} \Delta^2 \bar{\Delta}^2 \bar{\eta} \eta . \tag{9}$$

By analogy to the $N=1$ supersymmetric Yang-Mills transformations of chiral superfields we define $N=2$ gauge transformations on η

$$\eta' = e^{i\lambda} \eta , \tag{10}$$

where η is in some representation of the gauge group. The gauge parameter λ is chosen to preserve the constraints (6), which simply means that it also satisfies (6). Note in particular that λ is analytic in ζ. The conjugate field $\bar{\eta}$ transforms with the conjugate parameter $\bar{\lambda}$ which is analytic in ζ^{-1}. Again, in analogy to $N=1$ supersymmetric Yang-Mills, to make the action (9) invariant under local gauge transformations we introduce a superfield V that "converts" $\bar{\lambda}$-transformations to λ-transformations:

$$e^{V'} = e^{i\bar{\lambda}} e^{V} e^{-\lambda} . \tag{11}$$

Here V is chosen to satisfy

$$\begin{gathered} \nabla V = \bar{\nabla} V = 0 , \qquad R(V) = V , \\ \Rightarrow V = \sum_{-\infty}^{+\infty} v_i \zeta^i , \qquad v_{-i} = (-)^i \bar{v}_i . \end{gathered} \tag{12}$$

These constraints imply that v_0 is real and that v_i satisfy the analog of (7a); however, they do not imply any $N=1$ constraints analogous to (7b), and thus as $N=1$ superfields the v_i are unconstrained. The gauge invariant action constructed using V is

$$\int dx \oint \frac{\zeta d\zeta}{2\pi i} \Delta^2 \bar{\Delta}^2 \bar{\eta} e^V \eta . \tag{13}$$

As discussed below, v_0 is essentially an $N=1$ Yang-Mills prepotential and v_1 contains an $N=1$ chiral superfield (the $N=2$ partner of v_0). Because of the constraints (12) it is not obvious how to construct gauge covariant derivatives from V (the procedure directly analogous to the $N=1$ construction fails). Instead we factor e^V as follows:

$$e^V = e^{V_-} e^{V_0} e^{V_+} , \qquad V_- = \sum_{i=1}^{\infty} \bar{V}_i \left(-\frac{1}{\zeta} \right)^i , \qquad V_+ = \sum_{i=1}^{\infty} V_i \zeta^i . \tag{14}$$

For the abelian case, $V_i = v_i$, but in general this factorization is difficult to perform. Note that $\nabla V_+ \neq 0$ etc. From (12) it follows that

$$\begin{aligned} 0 = e^{-V}(\nabla e^V) &= e^{-V_+} e^{-V_0} (e^{-V_-} \nabla e^{V_-}) e^{V_0} e^{V_+} + e^{-V_+} e^{-V_0} (\nabla e^{V_0} e^{V_+}) \\ \Rightarrow e^{-V_-}(\nabla e^{V_-}) &= e^{V_0} e^{V_+} (\nabla e^{-V_+} e^{-V_0}) . \end{aligned} \tag{15}$$

Note that

$$\Gamma_- \equiv e^{-V_-}(\nabla e^{V_-}) = \sum_{i=0}^{\infty} \Gamma_{i-}\zeta^{-i},$$

whereas

$$\Gamma_+ \equiv e^{V_0}e^{V_+}(\nabla e^{-V_+}e^{-V_0}) = \sum_{i=0}^{\infty} \Gamma_{i+}\zeta^{i}.$$

However, (15) says that $\Gamma_- = \Gamma_+$, and hence

$$\Gamma_{0+} = \Gamma_{0-} \equiv \Omega, \qquad \Gamma_{i+} = \Gamma_{i-} = 0, \qquad i \geqq 1. \tag{16}$$

Thus

$$e^{-V_-}\nabla e^{V_-} = D_1 + \zeta D_2 + \Omega \equiv \mathscr{D}_1 + \zeta \mathscr{D}_2, \tag{17}$$

and it is natural to identify Ω as the $\mathscr{D}_1$ spinor gauge connection. This implies that we are in a gauge where the $\mathscr{D}_2$ connection vanishes.

So far we have constructed a connection but we must identify what representation it transforms under. To do this we substitute (14) into (11) and find

$$e^{V'_-} = e^{i\bar{\lambda}}e^{V_-}e^{-i\bar{\lambda}_0}, \qquad e^{V'_0} = e^{i\bar{\lambda}_0}e^{V_0}e^{-i\lambda_0}, \qquad e^{V'_+} = e^{i\lambda_0}e^{V_+}e^{-i\lambda}. \tag{18}$$

This implies

$$\delta\Omega = e^{i\bar{\lambda}_0}D_1 e^{-i\bar{\lambda}_0}, \tag{19}$$

which is the correct transformation for a connection when the gauge parameter is $\bar{\lambda}_0$. Note that (7a) implies $D_2\bar{\lambda}_0 = 0$, consistent with the observation that the $\mathscr{D}_2$ connection vanishes. Equation (18) allows us to go to $\bar{\lambda}$, λ_0, or λ representation using e^{V_-}, e^{V_0} or $e^{V_0}e^{V_+}$. Similarly, we can find the $\bar{\mathscr{D}}$ connection from the relation $e^{-V}(\bar{\nabla}e^{V}) = 0$; in the $\bar{\lambda}_0$ representation

$$e^{-V_-}\bar{\nabla}e^{V_-} = \bar{D}^2 - \zeta\bar{D}^1 + \bar{\Omega} \equiv \bar{\mathscr{D}}^2 - \zeta\bar{\mathscr{D}}^1, \tag{20}$$

and hence the $\bar{\mathscr{D}}^1$ connection vanishes. In $N=1$ terms, we are in an antichiral representation. We have thus constructed the gauge connections. The gauge action is well known [10]:

$$I = \int dx (D_1)^2 (D_2)^2 \,\mathrm{Tr}(W^2), \qquad W \equiv \tfrac{1}{2}\{\bar{\mathscr{D}}^{1\dot{\alpha}}, \bar{\mathscr{D}}^2_{\dot{\alpha}}\}. \tag{21}$$

This completes our construction of $N=2$ Yang-Mills theory. As noted above, in the non-abelian case we can relate the components V_i that enter in the action to the components v_i that satisfy simple constraints only via a series expansion.

We now examine the gauge transformations (18) in more detail. We study the abelian case, or equivalently, the lowest order in the expansion in fields and parameters in the nonabelian case. In this limit, $v_i = V_i$, and the gauge transformation reduces to

$$\delta V = i(\bar{\lambda} - \lambda) \Rightarrow \delta v_0 = i(\bar{\lambda}_0 - \lambda_0), \qquad \delta v_i = -i\lambda_i, \qquad i \geqq 1. \tag{22}$$

In $N=1$ terms, the components of the gauge parameter λ are unc unconstrained except for λ_0 and λ_1, which are antichiral and antilinear respectively. This means that all the components v_i can be gauged away except v_0, v_1, and $\bar{v}_1$. The component v_0 transforms like the usual $N=1$ gauge prepotential, and the gauge transformation of v_1 can be used to gauge away all but the anti-chiral field strength $(D_1)^2 v_1$. This is precisely the correct $N=1$ superfield content of $N=2$ super Yang-Mills theory.

We now describe some partial $N=2$ *supersymmetric* gauge choices. The basic idea is to truncate the ζ expansion of V by gauging away all but a finite number of its components while insisting that $N=2$ supersymmetry is maintained. Choosing

$$V=\sum_{-n}^{n} \zeta^i v_i\,, \qquad n>2\,, \tag{23a}$$

$N=2$ supersymmetry ($\nabla V=\bar{\nabla} V=0$) implies

$$\bar{D}^1 v_n=(\bar{D}^1)^2 v_{n-1}\,. \tag{23b}$$

The remaining gauge transformations are generated by

$$\lambda=\sum_{0}^{n} \zeta^i \lambda_i \Rightarrow \bar{D}\lambda_n=(\bar{D}^1)^2 \lambda_{n-1}\,. \tag{24}$$

For $n=2$, $\lambda_1=\lambda_{n-1}$ and thus satisfies $(\bar{D}^1)^2\lambda_1=(D_1)^2\lambda_1=0$; however, (23) implies that v_1 is also constrained:

$$(\bar{D}^1)^2 v_1=0\,. \tag{25}$$

In both cases, the remaining gauge freedom is precisely what is needed to gauge away all but the physical fields. The further possibility of $n=1$ above leads to a supersymmetric Landau gauge for the $N=1$ vector multiplet v_0, but unfortunately gives a higher derivative theory for the chiral multiplet.

The construction of $N=2$ Yang-Mills theory that we have presented is a direct transcription of the Ward construction of self-dual Yang-Mills fields [6, 7] into superspace. For completeness, we include a brief summary of the relevant aspects. The observation that the two theories are closely related was made in [5].

Ward begins by writing the self-dual Yang-Mills equations as

$$[D_\alpha, D_\beta]=0\,, \qquad D_\alpha \equiv D_{\alpha\dot{1}}+\zeta D_{\alpha\dot{2}}\,, \tag{26}$$

where $D_{\alpha\dot{\alpha}}$ is the gauge covariant derivative. He then considers a group element g (corresponding to e^V above) that satisfies

$$\partial_\alpha g=0\,, \qquad \partial_\alpha \equiv \partial_{\alpha\dot{1}}+\zeta\partial_{\alpha\dot{2}}\,. \tag{27}$$

Finally, he factorizes g:

$$g=fh^{-1}\,, \tag{28}$$

where f is regular around $\zeta\to\infty$, h is regular around $\zeta\to 0$, and there exists some region where both are regular. Substituting (28) into (27), one finds

$$D_\alpha=f^{-1}\partial_\alpha f=h^{-1}\partial_\alpha h\,. \tag{29}$$

This is analogous to Eq. (15). The regularity conditions on f and h ensure that D_α is at most linear in ζ, and thus (26) can be used to define the connection. This is analogous to (17). We conclude our brief discussion of the relation between the two constructions by emphasizing that despite the strict formal analogy between the two, the physical Yang-Mills field in the $N=2$ multiplet is completely unconstrained.

Acknowledgements. It is a pleasure to thank the ITP at Stony Brook and at Stockholm for hospitality.

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Communicated by L. Alvarez-Gaumé

Received July 3, 1989