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Decay Estimates for Schrödinger Equations

Peter Constantin*

Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637, USA

Abstract. We prove global existence and optimal decay estimates for classical solutions with small initial data for nonlinear nonlocal Schrödinger equations. The Laplacian in the Schrödinger equation can be replaced by an operator corresponding to a non-degenerate quadratic form of arbitrary signature. In particular, the Davey–Stewartson system is included in the the class of equations we discuss.

Introduction

Nonlinear Schrödinger systems arise naturally as envelope equations in the study of water waves ([N, D–S, Z–K]). Their form is

$$i\frac{\partial}{\partial t}u + L_1 u = a|u|^2 u + vu, \tag{0.1}$$

$$L_2 v = L_3(|u|^2), \tag{0.2}$$

where a is real and L_1, L_2, L_3 are quadratic differential operators

$$L_l = g_l^{jk}\frac{\partial^2}{\partial x^j \partial x^k} \tag{0.3}$$

for $l = 1, 2, 3$. The constant real n by n matrices (g_l^{jk}) are invertible but otherwise general. In this paper we assume L_2 to be elliptic; in this case one can solve for v in (0.2) and write the system (0.1), (0.2) as a single equation:

$$i\frac{\partial}{\partial t}u + L_1 u = L(|u|^2)u, \tag{0.4}$$

where $L = aI + (L_2)^{-1}L_3$ is a linear operator which commutes with translations, is real and it is bounded as an operator from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$. The

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initial values of u are given by:

$$u(x,0)=u_0(x). \tag{0.5}$$

If the initial datum u_0 is localized enough then the solution of the free ($L=0$) initial value problem decays in time like $t^{-n/2}$. In this paper we prove that, if u_0 is sufficiently small, smooth and localized then the initial value problem (0.4), (0.5) has a solution which is small, smooth, exists for all t and decays like $t^{-n/2}$. The method of proof is based by the one employed by Klainerman in his study of nonlinear wave equations ([K]). The basic idea due to him is to measure decay in terms of adapted Sobolev spaces in which the usual derivatives are replaced by vector fields which commute with the free equation and are dictated by the geometry of the problem. In the case of the Schrödinger equation the key operators are not derivations but they are conjugated to derivations. More precisely, these operators denoted Λ_j satisfy

$$\Lambda_j=e^{i\psi}\left(2t\frac{\partial}{\partial x^j}\right)e^{-i\psi},$$

where ψ is a real function of x and t which appears in the propagator of the free equation. These operators are familiar objects, both in quantum mechanics ($\Lambda_j=2ipt-ix$) and in the mathematical literature ([H–N–T, G–V, Ka]). They represent the infinitesimal generators of a certain group of symmetries of Schrödinger equation (see, for instance [W]).

In the first section we introduce the Sobolev spaces based on the Lie algebra generated by the Λ_j-s and prove a Klainerman Sobolev lemma. As a direct consequence we obtain the decay estimates for the free equation. In the second section we prove the decay estimates for the nonlinear equation. A minor technical difficulty due to the fact that, in general, L is not bounded in L^∞ is overcome by a fortuitous cancellation and the use of Gagliardo–Nirenberg-type estimates. Global existence (but not decay) of weak solutions to the Davey–Stewartson system corresponding to (0.4) in $n=2$ was proved in ([G–S]).

1. The Free Schrödinger Equation

We consider first the free Schrödinger equation

$$i\frac{\partial u}{\partial t}+P(D)u=0, \tag{1.1}$$

$$u(0,x)=u_0(x), \tag{1.2}$$

where $P(D)$ is given by

$$P(D)=g^{ij}\frac{\partial^2}{\partial x^i\partial x^j}. \tag{1.3}$$

In (1.3) we used the summation convention. The matrix

$$(g^{ij}) \tag{1.4}$$

is a real n by n invertible matrix. Its inverse is denoted

$$(g_{ij}) = (g^{ij})^{-1}. \tag{1.5}$$

We introduce the differential operators Λ_j defined by

$$\Lambda_j = 2t\frac{\partial}{\partial x^j} - ig_{jk}x^k. \tag{1.6}$$

These operators commute with $i(\partial/\partial t) + P(D)$:

$$\left[i\frac{\partial}{\partial t} + P(D), \Lambda_j\right] = 0. \tag{1.7}$$

One checks easily, via the Fourier transform that

$$\Lambda_j = e^{itP(D)}(-ig_{jk}x^k)e^{-itP(D)}. \tag{1.8}$$

On the other hand, it is clear that the operators Λ_j verify

$$2te^{i\psi}\frac{\partial}{\partial x^j}e^{-i\psi} = \Lambda_j, \tag{1.9}$$

where

$$\psi(x,t) = \frac{g_{ij}x^i x^j}{4t}. \tag{1.10}$$

The operators Λ_j commute. They generate a Lie algebra denoted Λ. For any multi-index

$$m = (m_1, m_2, \dots, m_n)$$

the operator $\Lambda^m = \Lambda_1^{m_1} \cdots \Lambda_n^{m_n}$ satisfies

$$(2t)^{-|m|}\Lambda^m = e^{i\psi}\partial^m e^{-i\psi}, \tag{1.11}$$

where $|m| = m_1 + \cdots + m_n$. We introduce the following notation. If $\mathscr{A}$ is a Lie algebra of (pseudo)differential operators we set, for any integer m

$$[u(x,t)]_{\mathscr{A},m} = \sum_{j=0}^{m}\left(\sum_{|\alpha|=j}(A^\alpha u(x,t))^2\right)^{1/2}, \tag{1.12}$$

where $A^\alpha = A_1^{\alpha_1} \cdots A_l^{\alpha_l}$ and $A_1, \dots, A_l$ are the generators of $\mathscr{A}$. By means of $[u(x,t)]_{\mathscr{A},m}$ we define the generalized $W^{m,p}$ norms for $1 \leqq p \leqq \infty$ by

$$\|u(\cdot,t)\|_{\mathscr{A},m,p} = \left[\int_{\mathbb{R}^n}[u(x,t)]_{\mathscr{A},m}^p dx\right]^{1/p}. \tag{1.13}$$

With these notations we can state a Klainerman Sobolev lemma:

Lemma 1.1. *There exists a constant C_n depending only on the dimension n such that, for every smooth $u(x,t)$,*

$$|u(x,t)| \leqq C_n|t|^{-n/2}\|u(\cdot,t)\|_{\Lambda,[n/2]+1,2}. \tag{1.14}$$

Proof. Let us consider the function $v(x,t)$ defined by

$$v(x,t)=e^{-i\psi}u(x,t)$$

and apply the classical scale-invariant local Sobolev lemma to it:

$$|v(x,t)|\leqq C\sum_{j=0}^{[n/2]+1}R^{j-(n/2)}\left(\sum_{|\alpha|=j}\int_{|y-x|\leqq R}|\partial_y^\alpha v(y,t)|^2dy\right)^{1/2}$$

which holds for any positive R and with C independent of R. Noting that $|v(x,t)|=|u(x,t)|$ and that, in view of (1.11)

$$(2|t|)^{-|\alpha|}|\Lambda^\alpha u(y,t)|=|\partial_y^\alpha v(y,t)|$$

the inequality (1.14) follows by choosing $R=2|t|$.

The estimate (1.11) is singular at $t=0$. One can remove this singularity by augmenting the Lie algebra we are working with. Denoting by $\mathscr{B}$ the Lie algebra generated by the operators $1,\Lambda_j$ and $(\partial/\partial x^j)$ for $j=1,\dots,n$ we obtain

Corollary 1.2. *There exists a constant C_n depending on n only such that every smooth $u(x,t)$ satisfies*

$$|u(x,t)|\leqq C_n(1+|t|)^{-(n/2)}\|u(\cdot,t)\|_{\mathscr{B},[n/2]+1,2}. \tag{1.15}$$

Applying Lemma 1.1 and Corollary 1.2 to derivatives we deduce

Corollary 1.3. *For every positive integer k there exists a constant C_k such that every smooth $u(x,t)$ satisfies*

$$[u(x,t)]_{\Lambda,k}\leqq C_k|t|^{-(n/2)}\|u(\cdot,t)\|_{\Lambda,[n/2]+1+k,2} \tag{1.16}$$

and

$$[u(x,t)]_{\mathscr{B},k}\leqq C_k(1+|t|)^{-(n/2)}\|u(\cdot,t)\|_{\mathscr{B},[n/2]+1+k,2}. \tag{1.17}$$

A direct consequence of these considerations is

Theorem 1.3. *Let $u(x,t)$ be a solution of (1.1), (1.2). Then*

$$|u(x,t)|\leqq C|t|^{-(n/2)}\|u_0\|_{\mathscr{X},[n/2]+1,2}, \tag{1.18}$$

where $\mathscr{X}$ is the Lie algebra generated by the operators of multiplication by $x^j, j=1,\dots,n$. More generally,

$$[u(x,t)]_{\Lambda,k}\leqq C_k|t|^{-(n/2)}\|u_0\|_{\mathscr{X},[n/2]+1+k,2}$$

and

$$[u(x,t)]_{\mathscr{B},k}\leqq C_k(1+|t|)^{-(n/2)}\|u_0\|_{\mathscr{B}_0,[n/2]+1+k,2}, \tag{1.19}$$

where $\mathscr{B}_0$ is the Lie algebra generated by the operators $1,(\partial/\partial x_j)$ and multiplication by x_j for $j=1,\dots,n$.

Proof. We apply Lemma 1.1 (respectively Corollary 1.3) to $u(x,t)$. In view of (1.8) and the fact that the Schrödinger equation (1.1) preserves L^2 norms (1.18) (respectively (1.19)) follow.

We end this section with a Gagliardo–Nirenberg lemma for the operators Λ_j:

Lemma 1.4. *For any pair of positive integers* $0<j<m$ *there exists a constant* C *such that every smooth function* $u(x,t)$ *satisfies*

$$\sum_{|\beta|=j} \| \Lambda^\beta u(\cdot,t) \|_{L^{(2m/j)}(\mathbb{R}^n)} \leqq C \| u(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}^{1-(j/m)} \left(\sum_{|\alpha|=m} \| \Lambda^\alpha u(\cdot,t) \|_{L^2(\mathbb{R}^n)} \right)^{j/m}. \quad (1.20)$$

Proof. The inequality (1.20) in which the differential operators Λ_j are replaced by $(\partial/\partial x^j)$ is a well-known Gagliardo–Nirenberg inequality. We apply to $v(x,t)=e^{-i\psi}u(x,t)$; the estimate (1.20) follows from (1.11) and the scale invariance of the usual Gagliardo–Nirenberg inequality.

2. Nonlinear Nonlocal Schrödinger Equations

The equation we investigate is

$$i\frac{\partial u}{\partial t} + P(D)u = L(|u|^2)u, \quad (2.1)$$

$$u(x,0) = u_0(x), \quad (2.2)$$

where $P(D)$ is given in (1.3) and the linear operator L is real, t-independent and commutes with translations:

$$[L, *] = 0, \quad (2.3)$$

where we denote by $*$ the operation of taking the complex conjugate and

$$\left[L, \frac{\partial}{\partial x^j} \right] = 0. \quad (2.4)$$

Moreover we assume that L is bounded in $L^p(\mathbb{R}^n)$ for $1<p<n$:

$$\| Lf \|_{L^p(\mathbb{R}^n)} \leqq C_p \| f \|_{L^p(\mathbb{R}^n)}. \quad (2.5)$$

Such equations arise in water wave theory ([D–S, Z–K, N]) and, typically, L is a product of Riesz transforms. We start with a Leibniz rule:

Lemma 2.1. *For any multi-index* α *the formula*

$$\Lambda^\alpha(L(|u|^2)v) = \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta!\gamma!\delta!}(L(\Lambda^\beta u(\Lambda^\gamma u)^*)\Lambda^\delta v) \quad (2.6)$$

holds.

Proof. In (2.6) we denoted $\alpha! = \alpha_1! \cdots \alpha_n!$. The proof is done by induction on $|\alpha|$ and follows easily from the observations

$$\Lambda_j(ab) = \left(2t\frac{\partial}{\partial x^j} a \right) b + a\Lambda_j b$$

and

$$2t\frac{\partial}{\partial x^j}(ab^*) = (\Lambda_j a)b + a(\Lambda_j b)^*.$$

For every non-negative integer m we denote by F_m the integral

$$F_m = \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |\Lambda^\alpha u|^2 dx. \tag{2.7}$$

If u solves (2.1) then F_m satisfies

$$\frac{1}{2}\frac{d}{dt}F_m = \operatorname{Im} \sum_{|\alpha|=m} \int_{\mathbb{R}^n} \Lambda^\alpha(L(|u|^2)u)(\Lambda^\alpha u)^* dx. \tag{2.8}$$

In view of (2.6) the right-hand side of (2.8) is a sum for $|\alpha|=m$ and $\alpha=\beta+\gamma+\delta$ of terms of the form

$$\frac{\alpha!}{\beta!\gamma!\delta!}\operatorname{Im} \int_{\mathbb{R}^n} L(\Lambda^\beta u(\Lambda^\gamma u)^*)\Lambda^\delta u(\Lambda^\alpha u)^* dx. \tag{2.9}$$

The term (2.9) corresponding to $\beta=\gamma=0$ equals zero. If $\beta=\delta=0$ or $\gamma=\delta=0$ then we estimate (2.9) by

$$\left| \int_{\mathbb{R}^n} L(u(\Lambda^\alpha u)^*)u(\Lambda^\alpha u)^* dx \right| \leqq C \int_{\mathbb{R}^n} |u|^2 |\Lambda^\alpha u|^2 dx.$$

The rest of the terms (2.9) have $0<|\delta|<m$. In these terms we apply a Hölder inequality raising the last term to the second power, the term involving $\Lambda^\delta u$ to the power $(2m/|\delta|)$ and the term involving L to the power $q=2(1-(|\delta|/m))^{-1}$. Using the boundedness of L in L^q spaces and the Gagliardo–Nirenberg inequality (1.20) with $j=|\delta|$ we majorize (2.9) by

$$C\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1-(j/m)} F_m^{(m+j)/2m} \left[\int_{\mathbb{R}^n} |\Lambda^\beta u|^q |\Lambda^\gamma u|^q dx \right]^{1/q}.$$

In the integral above we use a Hölder inequality with powers $(2m/q|\beta|)$ and $(2m/q|\gamma|)$ (their inverse add up to one) and again the Gagliardo–Nirenberg inequality (1.20). The end product of these estimates is the fact that all terms (2.9) can be majorized by

$$C\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^2 F_m,$$

and consequently we established the inequality

$$\frac{d}{dt}F_m \leqq C\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^2 F_m. \tag{2.10}$$

Clearly, the same argument applies to G_m, where

$$G_m = \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx, \tag{2.11}$$

and yields the analogue of (2.10). Defining

$$E_m = \sum_{k=0}^{m} (F_k + G_k) \tag{2.12}$$

we proved:

Lemma 2.2. *For every non-negative integer m there exist a constant C_m such that* *every smooth solution of (2.1) satisfies*

$$\frac{d}{dt} E_m \leqq C_m \| u(\cdot, t) \|^2_{L^\infty(\mathbb{R}^n)} E_m. \tag{2.13}$$

The next thing is to observe that, in view of (1.14) and the usual Sobolev lemma

$$\| u(\cdot, t) \|^2_{L^\infty(\mathbb{R}^n)} \leqq C(1 + |t|)^{-n} E_{[n/2]+1}. \tag{2.14}$$

Consequently, if $[n/2] + 1 \leqq m$ then the ordinary differential equality

$$\frac{d}{dt} E_m \leqq C_m (1 + |t|)^{-n} E_m^2 \tag{2.15}$$

holds. If $2 \leqq n$ we deduce that

$$E_m (1 - K_m E_m(0)) \leqq E_m(0). \tag{2.16}$$

Using (2.16) and standard arguments we can prove:

Theorem 2.3. *For every $n, 2 \leqq n$ and m satisfying $[n/2] + 1 \leqq m$ there exist positive numbers ε_m and C_m such that, if $u_0(x)$ satisfies*

$$\sum_{|\alpha| \leqq m} \int_{\mathbb{R}^n} (|\Lambda^\alpha u_0(x)|^2 + |\partial^\alpha u_0(x)|^2) dx \leqq \varepsilon_m \tag{2.17}$$

then the solution $u(x, t)$ of (2.1), (2.2) exists for all t and satisfies

$$\sum_{|\alpha| \leqq m} \int_{\mathbb{R}^n} (|\Lambda^\alpha u(x, t)|^2 + |\partial^\alpha u(x, t)|^2) dx \leqq C_m \varepsilon_m, \tag{2.18}$$

and

$$|u(x, t)| \leqq (C_m \varepsilon_m)^{1/2} (1 + |t|)^{-(n/2)}. \tag{2.19}$$

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