Commun. Math. Phys. 125, 91–112 (1989)

Communications in
Mathematical
Physics
© Springer-Verlag 1989

Anderson Localization for the 1-*D* Discrete Schrödinger Operator with Two-Frequency Potential

V. A. Chulaevsky[1] and Ya. G. Sinai[2]

[1] Research Computing Centre, Academy of Sciences of the USSR, SU-142292 Pushchino, Moscow Region, USSR
[2] Landau Institute of Theoretical Physics, Academy of Sciences of the USSR, ul. Kosygina, dom 2, Moscow V-334, USSR

Dedicated to Roland Dobrushin

Abstract. We prove the complete exponential localization of eigenfunctions for the 1-*D* discrete Schrödinger operators with quasi-periodic potentials having two basic frequencies. It is shown also that for such operators there is no forbidden zones in the spectrum, unlike the operators with one basic frequency.

1. Introduction

The phenomenon of the Anderson localization, or exponential decay of eigenfunctions of random self-adjoint operators, has been studied very intensively during the last several years. It is fairly clear now that basic mechanisms of the Anderson localization are essentially the same for differential operators like the Schrödinger operator

$$H_\varepsilon(\alpha) = -\varepsilon \frac{d^2}{dx^2} + V(x,\alpha)\,, \tag{1}$$

α being a random parameter, and for their discrete analogues like

$$(H_\varepsilon(\alpha)\psi)(n) = \varepsilon(\psi(n-1)+\psi(n+1)) + V(n,\alpha)\psi(n)\,. \tag{2}$$

Furthermore, the localization is shown both for "true random" and for almost periodic potentials $V(\cdot,\alpha)$. From the formal point of view, both classes can be considered in a more general context of random potentials having the form

$$V(x,\alpha) = F(T^{-x}\alpha)\,,\qquad \alpha\in\Omega\,,\qquad T^x:\Omega\to\Omega\,,$$

where $\{T^x, x\in\mathbb{Z}\}$ or $\{T^x, x\in\mathbb{R}\}$ is a group of automorphisms of a probability space (Ω,μ) and $F:\Omega\to\mathbb{R}$ is a measurable function. But it is worth emphasizing that such a general approach is not just a formality. On the contrary, it turns out that the language of operator ensembles is adequate for the investigation of a very important phenomenon like tunneling in disordered media due to long-range resonances.

Historically, the Anderson localization was initially proven for random operators whose coefficients at different points were independent or very weakly dependent random variables. We refer to physical papers [1, 2] and mathematical papers [3–6]. In the physical papers some convincing heuristic ideas were proposed while in the mathematical papers the rigorous proofs were given.

In a series of papers by Fröhlich and Spencer [7–9] and by Fröhlich et al. [10] a rigorous mathematical method for the investigation of the localization phenomenon was developed. The essence of that method was a detailed analysis of long-range resonances, and, fortunately, this analysis did not depend crucially on the "true random" type of potentials considered in [7–10]. We mention this because the subject of our paper is rather far from random potentials with independent values, but there is an interesting connection between the main mechanisms leading to exponential localization in strongly stochastic and in almost periodic media.

It was discovered in [11] that for sufficiently small ε and for all E the Lyapunov exponents for the finite difference equation

$$\varepsilon(\psi(n-1)+\psi(n+1))+\cos(n\omega+\alpha)\psi(n)=E\psi(n) \tag{3}$$

(called the almost Mathieu equation) are strictly positive. Later Avron and Simon [12] and independently Figotin and Pastur [13] gave a little more accurate formulation of this fact and proved it rigorously. That was perhaps the first strong argument in favour of Anderson localization for almost periodic operators with nice, bounded potentials. One should note that there exists a special class of unbounded potentials on $\mathbb{Z}^\nu$, $\nu \geqq 1$, for which the Anderson localization can be proved in a relatively simple way. Namely, put

$$V(n,z)=F(z+n_1\omega_1+\ldots+n_\nu\omega_\nu)\ , \qquad n\in\mathbb{Z}^\nu\ , \quad z\in\mathbb{R}\ , \tag{4}$$

and assume that $F(z)$ is a periodic function of period 1, real-valued on $\mathbb{R}$ and meromorphic in some strip $\{|\operatorname{Im} z|<R\}$, $R>0$. Assume also that $F(z)$ has exactly one pole on $[0,1)$ and no poles out of real line. Then it follows from the results by Bellisard et al. [14] that for sufficiently small ε the operator εH_0+V with

$$(H_0\psi)(n)=\sum_{m:\,\|n-m\|=1}\psi(m)\ , \qquad n,m\in\mathbb{Z}^\nu\ ,$$

has only point spectrum with exponentially decreasing eigenfunctions. Moreover, even much more general non-local operators H_0 can be considered in the same way. As a matter of fact, the long-range resonances and tunneling are absent for all such operators. In turn, this is connected with the fact that the function F takes any value exactly at one point on its period. Certainly, such a non-resonant condition is not fulfilled for general quasi-periodic operators, e.g. for almost Mathieu operators. In [15] some new method was developed for analyzing tunneling in resonant quasi-periodic one-dimensional lattices, and the Anderson localization was proved for the almost Mathieu operators. The recent paper by J. Fröhlich, T. Spencer, and P. Wittwer [16] contains results which are very close to the results of [15] and in some respects is more general. However the main ideas of both papers [15, 16] are rather close to each other. In [15] general C^2-functions having one non-degenerate maximum and one non-degenerate minimum on the period were considered, not only $\cos(x)$. In [16] both finite-difference operators with one basic frequency and differential operators with two basic frequencies were studied. Note that due to the dilation group $x\mapsto\lambda x$, $\lambda>0$, acting on $\mathbb{R}$ the spectral properties of differential operators with two basic frequencies are similar to those of discrete single-frequency operators if the spectral parameter E is close to the lower edge of the spectrum.

In our present paper we treat rather general one-dimensional discrete quasi-periodic Schrödinger operators with two basic frequencies,

$$(H_\varepsilon(\alpha)\psi)(n)=\varepsilon(\psi(n-1)+\psi(n+1))+V(\alpha_1+n\omega_1,\ \alpha_2+n\omega_2)\psi(n)\ , \tag{5}$$

where $\alpha\in\mathrm{Tor}^2$ and $\omega=(\omega_1,\omega_2)$ has rationally independent components. The function $V:\mathrm{Tor}^2\to\mathbb{R}$ is assumed to be a "sufficiently non-degenerate" C^2-function (see the exact assumptions below). Following the main ideas of [15], we propose an inductive procedure for construction of eigenfunctions for $H_\varepsilon(\alpha)$ from which we conclude that all eigenfunctions decay exponentially fast for sufficiently small ε and for almost every $\alpha\in\mathrm{Tor}^2$. Often the first step of the proof of Anderson localization is the proof of positivity of the Lyapunov exponents. In our method this step is included implicitly into the whole inductive procedure, so we get the positive lower bound for Lyapunov exponents only at the end of the proof. Note that the positivity of the Lyapunov exponents for some classes of quasi-periodic operators was proved earlier by Herman [17] and by Pastur [18].

An interesting distinction between single-frequency and multi-frequency potentials concerns the regularity of the limiting density of states. One of the results of [15] says that for small ε the support of the density of states is a nowhere dense Cantor set of positive measure and the Lebesgue measure of its complement vanishes as $\varepsilon\to 0$. Moreover, one can deduce from the techniques used in [15] that the density of states has singularities of integrable type $t^{-1/2}$, $t\downarrow 0$, at the edges of each connected interval of the resolvent set (i.e. at the spectral boundaries and at the edges of any spectral gap). We show that for two-frequency potentials the density of states is strictly positive on the segment [inf Spectr $H_\varepsilon(\alpha)$, sup Spectr $H_\varepsilon(\alpha)$]. In other words, the spectrum of a two-frequency operator has no spectral gaps. This fact has a simple geometrical nature and is apparently true for any number of frequencies greater than one, though our proof is still restricted to the case of two frequencies. It follows also from our technique that the density of states (i.e. the derivative of the integrated density of states) has an infinite number of discontinuities on the spectrum though the one-side limits of the density of states exist at any point.

Remark. Recently, R. Johnson informed us that it follows from his results [19] that our assumption $V\in C^2$ is important for the analysis of the density of states. Namely, if we assume that $V\in C^{1-\delta}$ with $\delta>0$, then the support of the density of states might be nowhere dense.

Let us describe now the class of potentials under consideration in more detail. We assume that

I. $V\in C^2(\mathrm{Tor}^2)$ and it has a finite number of non-degenerate critical points β_j, $j=1,\dots,n$. Its values at critical points are different.

II. The level sets $A(V,c)=\{\alpha: V(\alpha)=c\}\subset\mathrm{Tor}^2$ consist of a finite number of closed curves which self-intersect only at critical saddle points.

III. For any $\omega\in[0,1)\times[0,1)$ the level set $\{\alpha: V(\alpha+\omega)-V(\alpha)=0\}$ consists of a finite number of closed curves which self-intersect in a finite number of points. The restriction of V to any of these curves has a finite number of critical points which are degenerate only for vectors ω belonging to a finite number of closed curves or arcs.

The main result of the present paper is the following.

Theorem. *Let* $V:\mathrm{Tor}^2\to\mathbb{R}$ *satisfy the conditions* (I)–(III). *Then for any* $\delta>0$ *there exists a set* $\Omega_\delta\subset[0,1)\times[0,1)$ *of Lebesgue measure less than* δ *and a positive number* $\varepsilon(\delta)$ *such that for any* ε, $|\varepsilon|<\varepsilon(\delta)$, *any* $\omega\notin\Omega_\delta$ *and almost every* $\alpha\in\mathrm{Tor}^2$ *the operator*

$$(H_\varepsilon(\alpha)\psi)(n)=\varepsilon(\psi(n-1)+\psi(n+1))+V(\alpha+n\omega)\psi(n)$$

has purely point spectral measure, and all its eigenfunctions decay exponentially fast. The limiting density of states for $H_\varepsilon(\alpha)$ *does not vanish on the segment* $I=[\inf\mathrm{Spectr}\,H_\varepsilon(\alpha),\ \sup\mathrm{Spectr}\,H_\varepsilon(\alpha)]$.

We sketch now the proof of the theorem. The proof goes by induction. For any positive integer $s\geqq1$ we will construct approximate eigenvalues (AEV) and approximate eigenfunctions (AEF) of the operator $H_\varepsilon(\alpha)$. Note that if we have exact EV and exact EF,

$$H_\varepsilon(\alpha)\psi(\alpha)=\lambda(\alpha)\psi(\alpha)\ ,$$

then

$$H_\varepsilon(\alpha+t\omega)T^{-t}\psi(\alpha)=\lambda(\alpha)T^{-t}\psi(\alpha) \tag{6}$$

with $T^tf(n)=f(n-t)$. Thus, we should only construct AEF $\psi(\alpha)$ concentrated "mostly" near the point $n=0$ and, to be definite, to the right side of 0. The identity (6) permits us to construct all the other AEF. The set of all AEF with the described property will be called $\Phi^s(\alpha)$ at the s-th step of induction, and the set of corresponding AEF will be $\Lambda^s(\alpha)$. The multivalued functions Φ^s, Λ^s are the main objects of our construction. We denote the branches of these multivalued functions as $\Phi^s_\ell(\alpha)$, $\Lambda^s_\ell(\alpha)$, $\ell\geqq1$.

Besides the parameter $\varepsilon>0$ we will use a constant $a>0$ and two special sequences of positive real numbers, $a(s)$ and $b(s)$, constructed in two steps. First, define recursively a sequence of integers

$$u_0=4\ ,\quad u_{k+1}=u_k+[\ln u_k]\ ,\quad k>0\ .$$

Then put $a(u_k)=a$, $k\geqq1$, while for any other s we put

$$a(s)=a\exp(\ln 4\,\ln^2 s/s)=a+\delta(s)\ ,\quad \delta(s)\downarrow 0\ as s\to+\infty\ .$$

Now put $b(s)=u_k$ if $u_k\leqq s<u_{k+1}$. Note that $b(s)>s-\ln s$.

For $s=1$ we define a unique branch of AEF $\Phi^1(\alpha)$, namely,

$$\Phi^1_1(\alpha;n)=\delta_{n,0}\ ,$$

and the corresponding AEV is $\Lambda^1_1(\alpha)=V(\alpha)$. A simple calculation shows that

$$F^1_1(\alpha;n)=((H_\varepsilon(\alpha)-\Lambda^1_1(\alpha))\Phi^1_1)(\alpha;n)=\varepsilon(\delta_{n,1}+\delta_{n,-1})\ .$$

In our construction at the s-th step the error

$$F^s_\ell(\alpha;n)=((H_\varepsilon(\alpha)-\Lambda^s_\ell(\alpha))\Phi^s_\ell)(\alpha;n)$$

will be of order $O(\varepsilon^s)$. Thus, in order to construct the next approximations $\chi(\alpha) \in \Phi^2(\alpha)$, we shall use the first order perturbation formulas

$$\Phi_1^2(\alpha) = \Phi_1^1(\alpha) + \sum_{t \neq 0} \frac{(F_1^1(\alpha),\ T^{-t}\Phi_1^1(\alpha - t\omega))}{\Lambda_1^1(\alpha) - \Lambda_1^1(\alpha - t\omega)} T^{-t}\Phi_1^1(\alpha - t\omega) , \tag{7}$$

and

$$\Lambda_1^2(\alpha) = \Lambda_1^1(\alpha) + (F_1^1(\alpha),\ \Phi_1^1(\alpha)) . \tag{8}$$

Remark. It is more convenient for us not to make EV and EF precise as it was done in [15, Sect. 4, Theorems 1 and 2]. Instead, we use only the first order perturbation formulas for the EF and EV in the spectral theory of self-adjoint operators and construct new AEF and AEV omitting higher order terms. More concretely, we mean the following. Consider a self-adjoint operator Q acting in a finite-dimensional Hilbert space $\mathscr{L}$. Assume that there exists a basis in $\mathscr{L}$ consisting of approximate eigenvectors ψ_j with approximate eigenvalues λ_j, i.e.

$$Q\psi_j = \lambda_j \psi_j + \kappa f_j , \quad \|f_j\| \leqq \text{const} , \quad j = 1, \ldots, N .$$

The number κ gives the order of approximation. Assume that for all $j > 1$ we have inequalities $|\lambda_j - \lambda_1| > \Delta$ with κ/Δ being sufficiently small. Then there exists a precise eigenvector φ_1 for Q with the corresponding eigenvalue μ_1 such that

$$\varphi_1 = \psi_1 + \kappa\delta\psi_1 + O(\kappa^2) , \quad \mu_1 = \lambda_1 + \kappa\delta\lambda_1 + O(\kappa^2) ,$$

where

$$\delta\psi_1 = \sum_{j > 1} \frac{(f_1, \psi_j)}{\lambda_1 - \lambda_j} \psi_j , \quad \delta\lambda_1 = (f_1, \psi_1) .$$

The corrections of order $O(\kappa^2)$ can also be written explicitly in terms of perturbation series. However, they become more complicated, and it is difficult to use them in our situation, and moreover, it is even unnecessary. If we construct new AEV $\Lambda_j^2(\alpha)$ and new AEF $\phi_j^2(\alpha)$ which correspond in the above notations to $\tilde{\varphi}_1 = \psi_1 + \kappa\delta\psi_1$, $\tilde{\mu}_1 = \lambda_1 + \kappa\delta\lambda_1$, then one can verify by direct calculations using inductive assumptions that these new AEF and AEV give necessary approximations at the step $s = 2$. This calculation is quite analogous to that leading to the first order perturbation formulas for EF and EV, so we omit the details. A more nontrivial part of this argument is some estimation of C^2-norms of perturbations $\Lambda_j^2(\alpha) - \Lambda_j^1(\alpha)$, $\Phi_j^2(\alpha) - \Phi_j^1(\alpha)$. They are given in the Appendix for a more general situation.

Note that the right-hand side of (7) makes no sense if the denominator of one of the terms is zero while the numerator is non-zero. In such a case we say that the corresponding AEF are in resonance. More exactly, we call a resonance the case when the denominator is sufficiently small in some exact sense rigorously defined below. We call the set of points α where $\Lambda_1^1(\alpha) = \Lambda_1^1(\alpha + t\omega)$ a resonant curve Γ_1^1 and its neighborhood B_1^1 is called a double resonant zone (DRZ).

The series in right-hand side of (7) is in fact a finite sum for any point α since the support of the function $T^{-t}\Phi_1^1(\alpha - t\omega)$ contains one point and the support of F_1^1 consists of two points. Hence,

$$\Phi_1^2(\alpha) = \Phi_1^1(\alpha) + \frac{(F_1^1(\alpha),\ T^1\Phi_1^1(\alpha + \omega)}{\Lambda_1^1(\alpha) - \Lambda_1^1(\alpha + \omega)} + \frac{(F_1^1(\alpha),\ T^{-1}\Phi_1^1(\alpha - \omega))}{\Lambda_1^1(\alpha) - \Lambda_1^1(\alpha - \omega)} .$$

The resonant sets $\{\alpha : \Lambda_1^1(\alpha) = \Lambda_1^1(\alpha+\omega)\}$ and $\{\alpha : \Lambda_1^1(\alpha) = \Lambda_1^1(\alpha-\omega)\}$ play different roles in our construction. Recall that $\Phi^2(\alpha)$ contains, by convention, only those eigenfunctions which are concentrated "mostly" to the right of $n=0$. It is well-known that in the case of resonance of two eigenvalues there exist two eigenfunctions (in our case, two AEF) with slightly different eigenvalues (AEV) which are, in the first approximation, the linear combinations of initial eigenfunctions coming in resonance. So, in the case $\Lambda_1^1(\alpha) = \Lambda_1^1(\alpha-\omega)$ we get two AEF which cannot belong to $\Phi^2(\alpha)$ since their support has an essential part to the left of the point $n=0$. On the other hand, the second resonance leads to AEF which do belong to $\Phi^2(\alpha)$. Hence, the first resonant set should be excluded from the domain of definition of $\Phi^2(\alpha)$. This fact was strongly emphasized in [15]. It is convenient also to exclude the second resonant set from that domain and to define on this set two new AEF $\Phi_2^2(\alpha)$, $\Phi_3^2(\alpha)$ and corresponding AEV $\Lambda_2^2(\alpha)$, $\Lambda_3^2(\alpha)$ which appear due to the splitting of $\Lambda_1^1(\alpha)$ and $\Lambda_1^1(\alpha+t\omega)$. This procedure can be repeated once more and leads to $\Phi^3(\alpha)$, $\Lambda^3(\alpha)$, and so on. The only important difference is that at the step $s \geqq 3$ there may appear points α where three AEV come in resonance, since the graphs of three functions of two real variables can generically have a common point. Multiplicity higher than 3 is obviously an exception, or a degeneration, and we avoid it by changing the vector ω. On each step of induction we exclude from the domains of definition of branches of $\Phi^s(\alpha)$, $\Lambda^s(\alpha)$ all resonant sets, and on the resonant sets of the type

$$\{\alpha : \Lambda_\ell^s(\alpha) = \Lambda_m^s(\alpha+t\omega)\} \ , \qquad t>0 \ ,$$

we define new branches of $\Phi^{s+1}(\alpha)$, $\Lambda^{s+1}(\alpha)$.

At the s-th step of induction we assume that the following inductive assumptions are valid.

For any $t \leqq s$ there exists a set of frequencies Ω_t of Lebesgue measure less than $(b(t))^{-5}$ [with $b(t)$ defined before] such that for any $\omega \notin \bigcup_{t \leqq s} \Omega_t$ the following assumptions hold.

1_s. There exists on the torus Tor^2 a finite number of non-self-intersecting closed C^2-arcs Γ_j^s, $1 \leqq j \leqq n_1(s) \leqq s^{100}/\ln 1/\varepsilon$, called double resonant arcs, or simply resonant arcs (RA). For each RA Γ_j^s a positive number $t(j)$ is defined which is called the moment of the appearance of Γ_j^s, $1 \leqq t(j) \leqq s$. The curvature of Γ_j^s at any point does not exceed $(t(j))^{200}/\ln 1/\varepsilon$, and its length is less than $s^{-99}/\ln 1/\varepsilon$.

Recall that we have introduced the sequence $a(s)$. The neighborhood B_j^s of Γ_j^s of radius $(a(s))^{t(j)/4}$ is called the zone of double resonance (DRZ), or simply a resonant zone (RZ). Call a RZ B_j^s large at the s-th step iff

$$t(j) \leqq 100 \ln s/\ln 1/\varepsilon \ ,$$

and small otherwise.

Among all pair intersections $B_i^s \cap B_j^s$ there are marked ones which we call triple resonant zones and denote them C_k^s. The appearance of a TRZ C_k^s is also defined, $1 \leqq t(k) \leqq s$. The rule according to which we mark TRZ will be clear from the proof later.

The definition of large and small RZ extends as well to the TRZ with no change. Small TRZ do not intersect small RZ.

The total number $n_2(s,\alpha)$ of RZ containing an arbitrary point $\alpha \in \mathrm{Tor}^2$ is not bigger than $c\ln s/\ln 1/\varepsilon$. The total length of all RA Γ_j^s for which $t(j)=s$ is uniformly bounded in s.

Remarks. 1. In fact, the RA described in (1_s) are short arcs of resonant sets (resonant curves) which appeared at previous steps of induction in the same way as it was explained above for the case of Γ_1^1. However, at the step s we cannot refer to AEV $\Lambda_\ell^{s-1}(\alpha)$ which are responsible for the appearance of Γ_i^s, because the domains of definition of AEV are given in terms of RA and cannot be defined before RA. Thus, we use only local analytic properties of RA, not referring to any AEV, and we define the RA explicitly only at the initial step of induction. For $s=0$, or, more generally, for $s=s_0$, the properties of RA follow from the assumptions (I)–(III) concerning the potential $V(\alpha)$. At all subsequent steps we derive (1_{s+1}) using only the local properties formulated in (1_s)–(6_s) of our inductive procedure.

2. As it was mentioned, we partition the resonant sets into RA having length less than $s^{-99}/\ln 1/\varepsilon$. Our goal here is to get "almost straight" arcs of smooth curves. Indeed, the curvature of Γ_j^s is less than $(t(j))^{60}/\ln 1/\varepsilon \leqq s^{60}/\ln 1/\varepsilon$, so the variation of the direction of tangent vector to any fixed RA is less than s^{-39}.

2_s. At the s-th step of induction a finite number of functions $\Lambda_\ell^s(\alpha)$ is given, $\ell=1,\dots,n_s(s)\leqq cs^{100}/\ln 1/\varepsilon$. Each Λ_ℓ^s is defined on a subset of some DRZ B_j^s or TRZ C_k^s. In the first case we denote $i(l)=i$ and in the second case $k(l)=k$. The nature of the sets which we exclude from $B_{i(\ell)}^s$ or from $C_{k(\ell)}^s$ is already explained. Namely, if Λ_ℓ^s appeared at the moment $r<s$, then at the subsequent steps it can take part in new resonances, i.e. it may happen that for bigger t and for some integers m, u,

$$\{\alpha:\Lambda_\ell^t(\alpha)=\Lambda_m^t(\alpha+u\omega)\}\neq\emptyset\ . \tag{9}$$

Below we introduce a constant $\tilde{c}>0$ and consider only those resonant sets (9) for which $0<|n|<\tilde{c}s/\ln 1/\varepsilon$.

Denote the domain of definition of Λ_ℓ^s as $L_{i(\ell)}^s$ if it appeared in DRZ or as $M_{k(\ell)}^s$ if it appeared in a TRZ. Then

$$L_{i(\ell)}^s=B_{i(\ell)}^s\setminus\bigcup_m\ \bigcup_{|t|\leqq\tilde{c}s/\ln 1/\varepsilon}\{\alpha:\Lambda_\ell^s(\alpha)=\Lambda_m^s(\alpha+t\omega)\}$$

in the first case and

$$M_{k(\ell)}^s=C_{k(\ell)}^s\setminus\bigcup_m\ \bigcup_{|t|\leqq\tilde{c}s/\ln 1/\varepsilon}\{\alpha:\Lambda_\ell^s(\alpha)=\Lambda_m^s(\alpha+t\omega)\}$$

in the second case.

Assume that Λ_m^s appeared at the step $t=t(i(m))$ due to a double resonance of Λ_ℓ^t and Λ_k^t and at the step s it comes in resonance with Λ_n^s. Then we say that this is a double resonance iff

$$100\ln s/\ln 1/\varepsilon \geqq t(i(m))\ .$$

Otherwise we say that Λ_ℓ^s, Λ_m^s, and Λ_n^s are in triple resonance. This convention will be very useful in the sequel.

3_s. $\Lambda_\ell^s\in C^2(L_\ell^s)$. The number of critical points $\beta_{\ell k}^s$ of any function Λ_ℓ^s does not exceed some constant $n_4(v,\omega)$. The absolute value of difference between values of Λ_ℓ^s at different critical points is not less than s^{-10}. At a critical point the absolute value of each eigenvalue of the second differential $D^2\Lambda^s(\beta^s_{\ell k})$ is not less than s^{-20}.

4_s. For any function $f: G\to\mathbb{R}$, $G\subseteq \mathrm{Tor}^2$, define its level sets $A(f,c)=\{\alpha: f(\alpha)=c\}$. Then for each c and $\alpha\in A(\Lambda^s_\ell, c)$ the following inequalities hold

$$c_1 s^{-10}\left(\left(\min_k |c-\Lambda^s_\ell(\beta^s_{\ell k})|\right)^{1/2}+\min_k \|\alpha-\beta^s_{\ell k}\|\right)\leqq \|D\Lambda^s_\ell(\alpha)\|\leqq c_2 \ .$$

5_s. On the domain $L^s_{i(\ell)}$ a normed vector-valued function is defined, $\Phi^s_\ell: L^s_{i(\ell)}\to\ell^2(\mathbb{Z})$ such that

(1) there exists a finite subset $Z(\Phi^s_\ell(\alpha))$ of the lattice called the essential support of $\Phi^s_\ell(\alpha)$ for which

$$0\in Z(\Phi^s_\ell(\alpha))\subset[0, \tilde{c}s/\ln 1/\varepsilon] \ ,$$

$$|\Phi^s_\ell(\alpha;n)|\leqq(a(s)\varepsilon)^{\mathrm{dist}(n, Z(\Phi^s_\ell(\alpha)))} \ ;$$

(2) $(H_\varepsilon(\alpha)\Phi^s_\ell)(\alpha;n)=\Lambda^s_\ell(\alpha)\Phi^s_\ell(\alpha;n)+h^s_\ell(\alpha;n)+f^s_\ell(\alpha;n)$, where $|h^s_\ell(\alpha;n)|\leqq\exp(-7/4\tilde{c}s)$ for all n such that

$$\mathrm{dist}(n, Z(\Phi^s_\ell(\alpha)))<[\tilde{c}s/\ln 1/\varepsilon]$$

and is zero otherwise. The function $f^s_\ell(\alpha;n)$ is non-zero only at those points n where

$$\mathrm{dist}(n, Z(\Phi^s_\ell(\alpha)))=[\tilde{c}s/\ln 1/\varepsilon]$$

or

$$\mathrm{dist}(n, Z(\Phi^s_\ell(\alpha)))=[\tilde{c}s/\ln 1/\varepsilon]+1 \ .$$

This boundary term is quite analogous to the term $\Gamma^{(s)}_{i,\alpha}$ in [15, Sect. 4, Eq. (4.1)] and satisfies the estimates

$$|f^s_\ell(\alpha;n)|\leqq\exp(-\tilde{c}s(1-\kappa)) \ ,$$

where $\kappa>0$ can be chosen arbitrary small. Outside the set

$$\mathrm{dist}(n, Z(\Phi^s_\ell(\alpha)))\leqq[\tilde{c}s/\ln 1/\varepsilon]$$

the function $\Phi^s_\ell(\alpha;n)=0$.

6_s. Put

$$\Phi^s(\alpha)=\bigcup_{\ell:\alpha\in L^s_{i(\ell)}}\{\Phi^s_\ell(\alpha)\} \ .$$

Then for any α the set of vectors

$$\bigcup_{t=-\infty}^{\infty} T^{-t}\Phi^s(\alpha-t\omega)$$

is a basis in $l^2(\mathbb{Z})$, and moreover

$$|(\Phi^s_\ell(\alpha),\ T^t\Phi^s_m(\alpha))-\delta_{t0}\delta_{\ell m}|\leqq(a(s)\varepsilon)^{cs} \ .$$

It is rather easy to see that for $s=1$ and, more generally, for $s=s_0$ with any fixed s_0 the statements (1_s)–(6_s) can be derived from the assumptions (I)–(III) concerning the potential V. In the next sections of the paper we give an inductive construction of

AEF and AEV for arbitrary s and check (1_{s+1})–(6_{s+1}) starting with (1_s)–(6_s). Increasing the initial value $s=s_0$ is possible under appropriately decreasing ε. Thus, choosing ε in further arguments as small as needed, we can deal with s_0 as large as needed. This is important in our construction since we use several monotonically decreasing functions of s which should be sufficiently small. For example, in the inductive assumptions the measure of the set

$$\bigcup_{t=s_0}^{s} \Omega_t$$

can be chosen arbitrarily small by choosing s_0^{-1} and, hence, ε arbitrarily small.

A number of technical statements of [15] is the same for our operators as well. Thus, we concentrate mostly on the technical statements specific for the two-frequency potentials.

2. An Inductive Construction of Non-Resonant AEF

First of all, let us consider the simplest case where

$$[\tilde{c}s/\ln 1/\varepsilon] = [\tilde{c}(s+1)/\ln 1/\varepsilon] \ .$$

In this case we can put

$$\Gamma_j^{s+1} = \Gamma_j^s \ , \quad B_j^{s+1} = B_j^s \ , \quad L_\ell^{s+1} = L_\ell^s \ ,$$
$$n_1(s+1) = n_1(s) \ , \quad n_2(s+1, \alpha) = n_2(s, \alpha)$$

and

$$\Lambda_\ell^{s+1} = \Lambda_\ell^s \ , \quad \Phi_\ell^{s+1} = \Phi_\ell^s \ , \quad Z(\Phi_\ell^{s+1}(\alpha)) = Z(\Phi_\ell^s(\alpha)) \ .$$

Then the conditions (1_{s+1})–(6_{s+1}) follow easily from (1_s)–(6_s).

Consider now the case where at the $(s+1)$-th step we have

$$d(s+1) = [\tilde{c}(s+1)/\ln 1/\varepsilon] > d(s) \ ,$$

but the following non-resonant condition holds:

$$\begin{aligned} |\Lambda_\ell^s(\alpha) - \Lambda_m^s(\alpha + t\omega)| &> R_{\ell m}(\alpha, t) \\ &= [s^{10} + (\operatorname{dist}(Z(\Phi_\ell^s(\alpha))), Z(\Phi_m^s(\alpha + t\omega)))]^{-1} \end{aligned}$$

for all m and t, $d(s) < t \leqq d(s+1)$. This strict inequality holds, obviously, in some open neighborhood of the point α. Therefore, in this neighborhood we can apply the lemma on non-resonant deformation (see Appendix, Lemma 1) and get new AEF and AEV, namely,

$$\Phi_\ell^{s+1}(\alpha) = \Phi_\ell^s(\alpha) + \sum_m \sum_{0<t\leqq d(s+1)} \frac{(F_\ell^s(\alpha),\ T^t\Phi_m^s(\alpha+t\omega))}{\Lambda_\ell^s(\alpha) - \Lambda_m^s(\alpha+t\omega)}\ T^t\Phi_m^s(\alpha+t\omega)$$

and

$$\Lambda_\ell^{s+1}(\alpha) = \Lambda_\ell^s(\alpha) + (F_\ell^s(\alpha),\ \Phi_\ell^s(\alpha)) \ .$$

Now using the exponential decay estimates (5_s) we construct in the considered neighborhood of α a new AEF Φ_ℓ^{s+1} (or, more precisely, the next approximation of the existing branch Φ_ℓ^s) which vanishes outside the set

$$\{n \in \mathbb{Z} : \operatorname{dist}(n, Z(\Phi_\ell^s(\alpha))) \leqq [\tilde{c}(s+1)/\ln 1/\varepsilon] \ .$$

If it is non-zero outside this set, then we cut it off in order to satisfy (5_{s+1}). Using the argument given in [15, Sect. 7, p. 899] one can show that either

$$|\Phi_\ell^{s+1}(\alpha; n+1)| \leqq c|\varepsilon| \, |\Phi_\ell^{s+1}(\alpha; n)|$$

or

$$1/4 \leqq |\Phi_\ell^{s+1}(\alpha; n+2)/\Phi_\ell^{s+1}(\alpha; n)| \leqq 4 \ .$$

Thus, we can use the inductive decay estimates of Φ_ℓ^s and either obtain an inductive estimate for the AEF at the next step $(s+1)$ using the first inequality or restrict ourselves to a weaker estimate

$$|\Phi_\ell^{s+1}(\alpha; n+2)| \leqq 4 |\Phi_\ell^{s+1}(\alpha; n)| \ .$$

This argument does not depend on (1_{s+1})–(6_{s+1}). So, we can iterate it t times and obtain for $t \leqq \ln s$, the following estimate:

$$|\Phi_\ell^{s+1}(\alpha; n)| \leqq 4^{\ln s} (a\varepsilon)^{\operatorname{dist}(n, Z(\Phi_\ell^s(\alpha))) - \ln s} \leqq (a(s)\varepsilon)^{\operatorname{dist}(n, Z(\Phi_\ell^s(\alpha)))}$$

with $a(s) \leqq a \exp(\ln 4/\varepsilon \ln s/s) \leqq a + \delta(s)$, $\delta(s) \downarrow 0$ as $s \to \infty$.

Remark. Certainly, this argument cannot be applied infinitely many times. Therefore, we should use another argument for $t > \ln s$. We follow here the strategy of [15, Sect. 7] and prove the needed estimate for $t = [\ln s] + 1$ using ergodic properties of the shift transformation $\alpha \mapsto \alpha + \omega$ and inductive assumptions concerning resonant arcs appeared at previous steps of induction. This will be made in Sect. 4.

3. An Inductive Construction of Resonant AEF

Suppose for some $t \in [d(s)+1, d(s+1)]$ the following set is nonempty:

$$\{\alpha : |\Lambda_\ell^s(\alpha) - \Lambda_m^s(\alpha + t\omega)| \leqq R_{\ell m}^{s+1}(\alpha; t)\} \ .$$

Without loss of generality we can assume that the set

$$\{\alpha : \Lambda_\ell^s(\alpha) = \Lambda_m^s(\alpha + t\omega)\} \neq \emptyset \ , \tag{10}$$

otherwise the resonance of Λ_ℓ^s and Λ_m^s disappears outside of a set of frequencies of Lebesgue measure less than s^{-9}. Assume that for all k and $r \in [d(s)+1, d(s+1)]$,

$$|\Lambda_\ell^s(\alpha) - \Lambda_k^s(\alpha + r\omega)| \geqq R_{\ell k}^{s+1}(\alpha; r) \ ,$$

$$|\Lambda_m^s(\alpha + t\omega) - \Lambda_k^s(\alpha + r\omega)| \geqq R_{mk}^{s+1}(\alpha; t-r) \ .$$

Then we apply the lemma on the resonance splitting (see Appendix) and construct new AEF concentrated on the set

$$\{n : \operatorname{dist}(n, Z(\Phi_\ell^s(\alpha))) \geqq d(s) + d(s+1)\} \ .$$

Then we cut these AEF so that the diameter of ES would not exceed $[\tilde{c}(s+1)/\ln 1/\varepsilon]$. Introduce some ordering of the sets (10) and denote by $\Gamma_{n_1(s)+i}^{s+1}$ the set having the number i in this ordering. Define $B^{s+1}_{n_1(s)+i}$ in the way it was done in (1_s) and put $t(n_1(s)+i)=s+1$, i.e. we regard $\Gamma^{s+1}_{n_1(s)+i}$ and $B^{s+1}_{n_1(s)+i}$ as appeared at the step $s+1$. By definition, all the new DRZ are small at the current step. Using the lemma on resonant splitting we construct in each DRZ also a pair of new AEV $\Lambda^{s+1}_{\ell'}$, $\Lambda^{s+1}_{m'}$ with $i(m')=i(\ell')=n_1(s)+i$ and AEF $\Phi^{s+1}_{\ell'}$, $\Phi^{s+1}_{m'}$. Similarly, we define new TRZ and triplets of AEV and AEF in each TRZ.

Estimate now the total number of new DRZ for typical ω. All possible intersections of RZ at the current step can be classified as follows:

1. intersection with a large RZ,
2. intersection of a small TRZ with a small RZ,
3. intersection of two small DRZ.

Note that all intersections of type (2) can be removed by excluding a set of frequencies of Lebesgue measure less than s^{-6} (see Fig. 1). Indeed, it follows from the

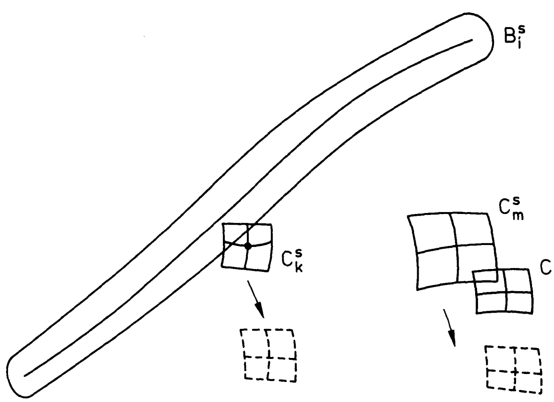

Fig. 1. Removing intersections of small TRZ with small RZ

definition of RZ that the width of any small DRZ is less than $(a\varepsilon)^{(100\ln s/\ln 1/\varepsilon)/4} \leqq s^{-24}$. Let us perturb the vector ω by a small vector $\delta\omega$. Then for any t, $0<|t|<\text{const}\, s$, the perturbation of the point $\alpha+t\omega$ of the torus has the same direction as $\delta\omega$ and is not less than $\|\delta\omega\|$. This is true also for the relative shift vector for two functions $f(\alpha+t_1\omega)$, $g(\alpha+t_2\omega)$ with $t_1, t_2 \leqq \text{const}\, s$. According to the inductive assumption (4_s) [or more precisely, all assumptions (4_r) for $s_0 \leqq r \leqq s$] the existing AEV appeared at prevous steps not too close to critical points of corresponding AEV. Thus, the

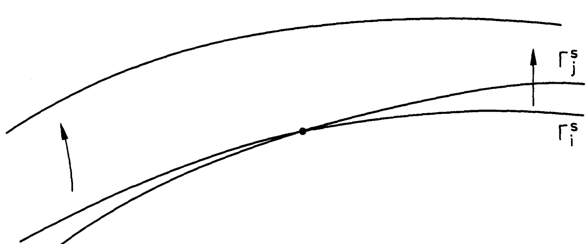

Fig. 2. Removing intersections of almost tangent RA

perturbation of ω leads to the relative shift of RZ which is not less than $\|\delta\omega\|$. In the case of intersection of two small TRZ the direction of the vector is not important, and in the case of intersection of a small TRZ with a small DRZ one should pick $\delta\omega$ orthogonal to the tangent vector to the corresponding RA for DRZ at some point [the position of that point is not important since RA is short enough, due to the inductive assumption (1_s)]. Summing over all relevant TRZ and DRZ, and taking into account the length or diameter estimates for RA and TRZ, we come to the needed estimate of the set of frequencies which is to be excluded.

Remark. We will use the arguments of the described type in the sequel but omit the details.

Consider now intersections of type (1). The number of pair intersections of large RZ can be easily estimated by means of the inductive assumptions (1_s) and does not exceed

$$\tfrac{1}{2}\,(d(s+1)-d(s))n_1^2(100\ln s/\ln 1/\varepsilon) \leqq \text{const}\ln^{200} s/\ln^4 1/\varepsilon\ .$$

Note that an intersection of a small DRZ with any RZ, $B_i^s \cap B_j^s$ or $B_i^s \cap C_k^s$, means that the corresponding AEV $\Lambda_\ell^s(\alpha)$, $i(\ell)=i$, is in a triple resonance, by definition. Therefore, these intersections do not contribute to the number of new DRZ, but do contribute to new TRZ. The same is true for all intersections of type (3). Dividing new resonant curves into arcs of length $(s+1)^{-99}$ or less, and taking summation over $t\in[1,s+1]$, we get the inductive estimate for the number of new RA and DRZ.

Analyse now the smoothness of new RA. As it was mentioned above, these arcs appear inside the DRZ B_j^s which, in turn, appeared not later than at the step $t=100\ln s/\ln 1/\varepsilon$.

Therefore, at all subsequent steps of induction, the corresponding AEF $\Phi_\ell^s(\alpha)$ undergo only non-resonant deformations. Hence, the C^2-norm of $\Lambda_\ell^s-\Lambda_\ell^t$, $i(\ell)=j$, is less than

$$\sum_{r=t}^{s}(a'\varepsilon)^r \leqq \text{const}\,(a'\varepsilon)^t\ .$$

Taking into account the inductive assumptions (1_t)–(6_t), $1\leqq t\leqq t(j)$, we conclude that the absolute value of each eigenvalue of $D^2\Lambda_\ell^s(\beta_{\ell k}^s)$ at any critical point $\beta_{\ell k}^s$ is greater than $\ln^{-9} s\ln 1/\varepsilon$ and less than $(a\varepsilon)^{-2t(j)}<s^{200}$, if ε is small enough.

Consider two possible cases.

1. $\Lambda_1^s(\alpha)$ and $\Lambda_1^s(\alpha+t\omega)$ are in resonance, $d(s)+1\leqq t\leqq d(s+1)$. Then, applying the lemma on resonant splitting to $\Phi_1^s(\alpha)$, $T^t\Phi_1^s(\alpha+t\omega)$, we construct new AEF $\Phi_{\ell'}^{s+1}$, $\Phi_{m'}^{s+1}$. Recall that for any t $\Lambda_1^t(\alpha)$ stands for that branch of $\Lambda^t(\alpha)$ which never took part in resonances, by our convention. So, its domain of definition does not contain any RZ. Applying the lemma on non-resonant deformation to Λ_1^t at each step t, $s_0\leqq t\leqq s$, we see that in the domain of Λ_1^s the C^2-norm of the difference $\Lambda_1^s-\Lambda_1^1$ is less than $c\varepsilon$ with the constant depending only on V. Thus, (1_{s+1}) follows in this case from (I)–(III) and from perturbation estimates for Λ_1^t. Note also that the total number of new critical points in RZ appeared in the neighborhood of the new RA is uniformly bounded in s: for ε small this estimate depends essentially on V. Furthermore, due to the assumption (III) concerning $V(\alpha)$, we can avoid degeneracies of new critical points in the case in question for a.e. ω. It is not difficult

to see that the EV of $D^2 \Lambda_\ell^{s+1}(\beta_{\ell k}^{s+1})$ corresponding to the normal vector of Γ_j^{s+1} at $\beta_{\ell k}^{s+1}$ is always exponentially large with respect to s. Indeed, consider the vertical section of the graph of the AEV $\Lambda_\ell^{s+1}(\alpha)$ after resonant splitting, by the normal plane to the resonant arc Γ_j^{s+1} at $\beta_{\ell k}^{s+1}$. Then the needed lower estimate follows from the explicit first order perturbation formula for a pair of resonant eigenvalues of a self-adjoint operator. It is quite analogous to that used in [15, Sect. 4, Theorem 2, (c_3)], so we omit the details.

The minimal eigenvalue of the second differential of an AEV appeared earlier in a resonance, i.e. Λ_ℓ^s with $\ell > 1$, can be estimated for typical ω as follows. If for some m the point $\beta_{\ell k}^s + t\omega$ lies in a s^{-9}-neighborhood of $\beta_{\ell k}^s$, then we remove such a degeneracy by omitting a set of frequencies of Lebesgue measure less than $c' s^{-18}$. Since $V(\alpha)$ has only a finite number of critical points with distinct values, we can remove all such degeneracies by excluding a set of ω of Lebesgue measure less than $c'' s^{-18}$. If $\beta_{\ell k}^s$ is a saddle point, an analogous procedure allows us to remove all degeneracies related to the fact that the absolute value of the indefinite quadratic form $D^2 \Lambda_\ell^s(\beta_{\ell k}^s)$ on the vector $t\omega/\|t\omega\|$ is less than $(s+1)^{-9}$. For all remaining ω the point $\beta_{\ell k}^s$ is non-degenerate due to the assumption (III) concerning the function $V(\alpha)$.

2. Suppose at least one of the functions Λ_ℓ^s, Λ_k^s coming into resonance appeared before as a result of a resonance. To be definite, assume that

$$t(i(\ell)) \geqq t(i(k)) \ , \quad i = i(\ell) \ , \quad j = i(k) \ , \quad t = t(i)$$

(see Fig. 3). Assume also that Λ_ℓ^s appeared in a double resonance of $\Lambda_{m_1}^t$ and $\Lambda_{m_2}^t$.

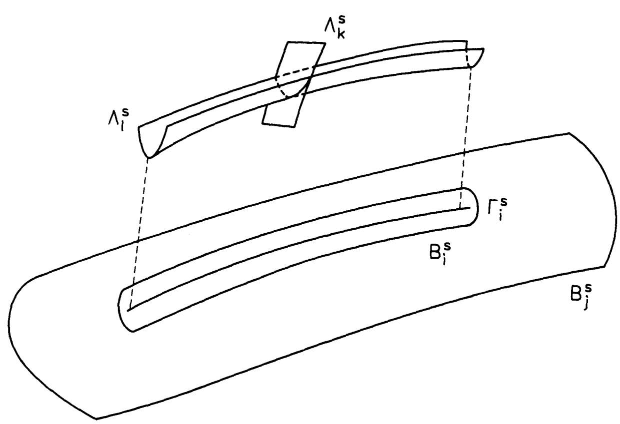

Fig. 3. Resonance of with appeared in DRZ; $i(k) = j$ $i(\ell) = i$

The case of a triple resonance is quite similar and even more simple although the notation is somewhat cumbersome. Pick an arbitrary point α' on Γ_j^s and linearize $\Lambda_{m_1}^t$ and $\Lambda_{m_2}^t$ at α':

$$\Lambda_{m_1}^t(\alpha) = \Lambda_{m_1}^t(\alpha') + D\Lambda_{m_2}^t(\alpha')(\alpha - \alpha') + \delta^2 \Lambda_{m_1}^t(\alpha) = \tilde{\Lambda}_{m_1}^t(\alpha) + \delta^2 \Lambda_{m_1}^t(\alpha)$$

with

$$\|\delta^2 \Lambda_{m_1}^t\|_{C^2} \leqq \text{const} \ln \ln s \ln^{-1} 1/\varepsilon \, \|\alpha - \alpha'\|^2 \ ,$$

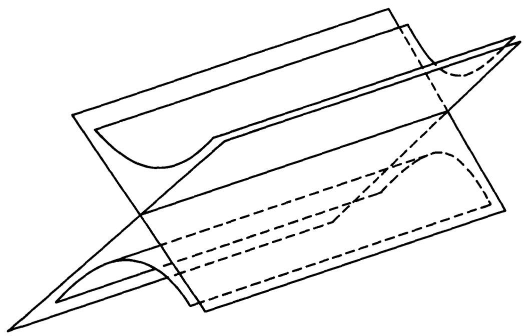

Fig. 4. Double resonant splitting of linearrized AEV

which hold since $\Lambda_{m_1}^t$ did not undergo any resonance after the step $100 \ln t/\ln 1/\varepsilon$. Similarly

$$\Lambda_{m_2}^t(\alpha) = \tilde{\Lambda}_{m_2}^t(\alpha) + \delta^2 \Lambda_{m_2}^t(\alpha) \ .$$

Define a function $\tilde{\Lambda}_\ell^s$ by analogy with Λ_ℓ^s but use instead of $\Lambda_{m_1}^t$, $\Lambda_{m_2}^t$ coming into the resonance, their linearizations. The graph of $\tilde{\Lambda}_\ell^s$ is then a part of an algebraic surface of the second degree which can be decomposed into a one-parameter family of parallel straight lines. Taking into account the width of B_i^t equal to $(a\varepsilon)^{t/4}$, we see that up to higher order terms the function $\tilde{\Lambda}_\ell^s$ coincides with Λ_ℓ^s inside a rectangle with one side parallel to the tangent vector to Γ_i^s at α' and of length $t^{20}(\alpha\varepsilon)^{t/4}$ while the other side has length $2(a\varepsilon)^{t/4}$.

If Λ_ℓ^s appeared due to a triple resonance, then up to higher order terms Λ_ℓ^s coincides within a circle of radius $t^{20}(a\varepsilon)^{t/4}$ with a third order algebraic function $\tilde{\Lambda}_\ell^s$, the graph of which is a "smoothed" trihedral angle (see Fig. 5).

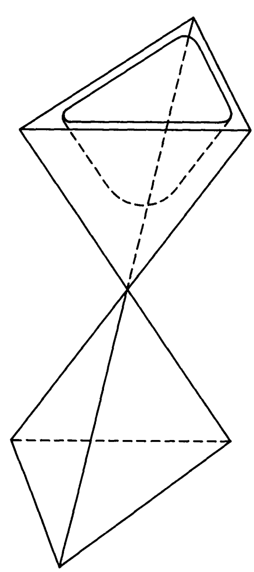

Fig. 5. Triple resonant splitting of linearized AEV

If Λ_ℓ^s appeared at the step $t(i(k))>1$, then the same arguments are applicable to it as well. If $k=1$, then Λ_k^s did not come into a resonance, and $\tilde{\Lambda}_k^s$ is linear. So, in order to estimate the correction term $\delta^2\Lambda_\ell^s(\alpha)$, one can use the initial C^2-norm estimate of $V(\alpha)=\Lambda_1^1(\alpha)$ by some constant. Anyway, the graphs of $\tilde{\Lambda}_\ell^s$ and $\tilde{\Lambda}_k^s$ can only have isolated tangency points removable by an arbitrary small perturbation of ω. Without any loss of generality, we can assume that the RA for $\tilde{\Lambda}_\ell^s$ and $\tilde{\Lambda}_k^s$ lies outside of all the s^{-10}-neighborhoods of critical points of those AEV. Hence, the norm of the first differential of each function on the RA is greater than $s^{-10}/\ln^{10}s \geqq s^{-11}$.

Furthermore, for a set of frequencies of measure greater than $1-s^{-9}$, the norm of $D\tilde{\Lambda}_\ell^s(\alpha)-D\tilde{\Lambda}_k^s(\alpha)$ on the RA is greater than s^{-10}. Otherwise the tangent planes to the graphs of $\tilde{\Lambda}_\ell^s$ and of $\tilde{\Lambda}_k^s$ are almost parallel, and their intersection is removable by excluding a set of frequencies of Lebesgue measure less than s^{-9}.

For $\tilde{\Lambda}_\ell^s(\alpha)$, $\tilde{\Lambda}_k^s(\alpha+t\omega)$ the resonant set

$$\{\alpha : \tilde{\Lambda}_\ell^s(\alpha)=\tilde{\Lambda}_k^s(\alpha+t\omega)\}$$

is a piece of an algebraic curve since $\tilde{\Lambda}_\ell^s(\alpha)$ and $\tilde{\Lambda}_k^s(\alpha+t\omega)$ are functionally independent in the considered neighborhood of the resonant set. Denote this curve as $\gamma(u)$, where u is the length parameter along the curve. Let $v(u)=\dot{\gamma}(u)/\|\dot{\gamma}(u)\|$ be a family of unit tangent vectors to $\gamma(u)$. Then we have

$$\begin{cases} \tilde{\Lambda}_\ell^s(\gamma(u))\,[v(u)]=D\tilde{\Lambda}_k^s(\gamma(u)+t\omega)\,[v(u)] \\ (v(u),v(u))=1 \ . \end{cases}$$

Recall that if $\Lambda_\ell^s(\alpha)$ and $\Lambda_k^s(\alpha+t\omega)$ are in a double resonance, then they have appeared before the step $100\ \ln s/\ln 1/\varepsilon$, otherwise this resonance would be triple, by convention. Therefore, we can use the inductive estimates of $D^2\Lambda_\ell^s$, $D^2\Lambda_k^s$ which are close to $D^2\Delta_\ell^{t(i(\ell))}$, $D^2\Lambda_k^{t(i(k))}$ with $t(i(\ell))$, $t(i(k))<100\ln s/\ln 1/\varepsilon$ The smoothness of $D\tilde{\Lambda}_\ell^s$ and of $D\tilde{\Lambda}_k^s$, the inequality

$$\|D\tilde{\Lambda}_\ell^s(\alpha)-D\tilde{\Lambda}_k^s(\alpha+t\omega)\|\geqq s^{-10}$$

and the C^2-norm estimates of $\tilde{\Lambda}_\ell^s-\Lambda_\ell^s$, $\tilde{\Lambda}_k^s-\Lambda_k^s$ yield the existence and smoothness of the solution to the above approximate system of equations, as well as of the solution to the "exact" system for "true" RA $\gamma'(u)$ for the AEV Λ_ℓ^s, Λ_k^s. We also have

$$\|v(u)\|\leqq 2s^{10}(a\varepsilon)^{-t(i)}\ , \qquad t(i)\leqq 100\ln s/\ln 1/\varepsilon \ .$$

Thus, $\|v(u)\|\leqq(s+1)^{200}$, which completes the inductive estimate of the curvature of the new RA.

The intersection of a new RZ with a small TRZ may occur only for a set of frequencies of measure less than

$$(a\varepsilon)^{\frac{1}{4}100\ln s/\ln 1/\varepsilon}+c''(a\varepsilon)^{s/4} \ .$$

Thus, the desired estimate of the measure follows from the inductive estimates of the number of RZ and of their lengths.

Remark. In the study of new RZ, we focus mainly on DRZ, their number, geometry,and on properties of the corresponding AEV. The reason is that the TRZ do not exist independently, in addition to the DRZ. They are parts of DRZ where the analytic properties of the graphs of AEV change, and this is perhaps the only important difference. Moreover, once a TRZ appears, it is less "dangerous" from point of view of later resonances which will occur when the given TRZ, small at the step s, will become a large one at the step $\exp(\ln 1/\varepsilon s/100)=\varepsilon^{-s/100}$. Indeed, the main difficulties come from those α where the first differential of an AEV is small by norm, and in TRZ the graph of new AEV is less degenerate than it can be in DRZ.

Arguments analogous to the ones given above show that for "typical" ω (i.e. for all ω except for a set of measure which is negligible at the step $s+1$) new RZ appear inside large RZ. This leads to the inductive upper bound for $n_2(s+1,\alpha)$ (the number of RZ containing the point α at the step $s+1$). All remaining statements in (2_{s+1}) describe, in fact, the procedure of constructing new AEV.

Let us investigate now the differential properties of new AEV. The fact that $\Lambda_\ell^{s+1}\in C^2(L_\ell^{s+1})$ follows immediately from the lemma on the resonant splitting. If $\Lambda_1^s(\alpha)$ and $\Lambda_1^s(\alpha+t\omega)$ come into resonance, then for sufficiently small ε the number of critical points is uniformly bounded in s. If at least one of the AEV appeared in earlier resonances, then the number of critical points admits an upper bound independent of V and ω since up to C^2-small corrections both AEV in the new RZ are given by algebraic functions considered above. Recall that these algebraic functions are obtained by replacement of the "true" AEV coming into resonance by their linearizations. Since gradients of all AEV are uniformly bounded in norm (only the second derivatives can be large) the relative shift of these linear approximations due to the change of ω leads to the shift of values of new AEV at any critical point. It is not difficult to verify using the boundedness of gradients that for a set of frequencies of measure greater than $1-s^{-6}$ all the differences between values of new AEV at different critical points are greater than $(s+1)^{-10}$.

The EV of $D^2\Lambda_\ell^{s+1}(\beta_{\ell k}^{s+1})$ corresponding to the normal vector to Γ_i^{s+1}, $i=i(\ell)$, at $\beta_{\ell k}^{s+1}$ grows by absolute value as s grows; this was already mentioned above. So, we need only to estimate from below the second EV in the case of double resonance. Since $\beta_{\ell k}^{s+1}$ lies in a DRZ, it did not appear in any RZ after the step $100\ln(s+1)/\ln 1/\varepsilon$. Thus, Λ_ℓ^s satisfies (3_r), (4_r) for $r\leqq 100\ln(s+1)/\ln 1/\varepsilon$, or, more precisely, (3_r), (4_r) with changed values of constants: $c_1'=c_1/2$, $c_2'=2c_2$.

We can use the condition (III) concerning the potential $V(\alpha)$ and the recursive estimate of the C^2-norm of the difference $\Lambda_1^s(\alpha+t\omega)-\Lambda_1^1(\alpha+t\omega)$ within the domain of $\Lambda_1^s(\alpha+t\omega)$. This implies the needed estimate of the smallest EV of the second differential in the case when Λ_1^s comes into resonance with its own shift. For other resonances we have already obtained algebraic approximations of new AEV. Outside a set of frequencies of Lebesgue measure less than $2s^{-s}$ the curvature of any curve from $A(\Lambda_m^s(\cdot+t\omega),\ \Lambda_m^s(\beta_{mk}^{s+1}+t\omega))$ at the point β_{mk}^{s+1} is greater than $\mathrm{const}\, s^{-10}$ and differs from the curvature of the corresponding curve from $A(\Lambda_\ell^s(\cdot st,\Lambda_\ell^s(\beta_{mk}^{s+1}))$ at the intersection point more than in s^{-9}. Use again the Taylor expansions for $\Lambda_\ell^s(\alpha)$, $\Lambda_m^s(\alpha+t\omega)$ at $\beta_{\ell k}^{s+1}$ up to terms of second order. At the critical point of new AEV the gradients of two resonant AEV are either parallel or anti-parallel (see Fig. 6). The direct calculation of the curvature of the new resonant curve (broken line in Fig. 6) for the quadratic approximants of resonant AEV in terms of their Taylor coefficients shows that at such an intersection for all ω outside a set of Lebesgue measure less than s^{-7}, the curvature is greater

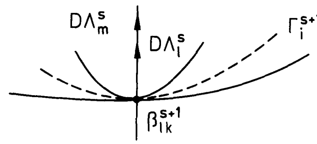

Fig. 6. Non-degenerate RA near new critical point

than const $s^{-9} > s^{-10}$. Furthermore, the gradients of considered AEV coming into resonance at β_{lk}^{s+1} are orthogonal to the new resonant curve, so the absolute value of the second derivative of the restriction of Λ_ℓ^s (or, equivalently, of Λ_m^s) to the new curve is greater than $s^{-20} > (s+1)^{-20}$ due to the inductive assumption of non-degeneracy of critical points (we refer to (3_r), (4_r) mentioned above). This completes the proof of (3_{s+1}).

The uniform boundedness of $\|D\Lambda_k^{s+1}(\alpha)\|$ follows from (4_s), the lemma on resonant splitting and the lemma on non-resonant deformation applied to those steps t when Λ_k^t does not come into resonances. The first inequality of (4_{s+1}) follows from (3_{s+1}).

4. Exponential Decay of AEF

The most important part of (5_{s+1}) is the upper bound for an AEF outside its ES. To establish this bound we can proceed by analogy with [15, Sect. 7, Lemma 4]. However, the proof of a statement analogous to Lemma 4 of [15] should be slightly modified in our case.

Recall that (1_s)–(6_s) imply (5_t) for $t \leqq \ln s + s$. Therefore, it suffices to show that for a point α the frequency of entering RZ during the time interval $[s, s+\ln s]$ is less than $\varepsilon^{1/4}$. The Lebesgue measure of all RZ appearing at the step t does not exceed $(a\varepsilon)^{t/8}$ and the total number of them is less than t^{100}. These RZ are $(a\varepsilon)^{t/4}$-neigh-

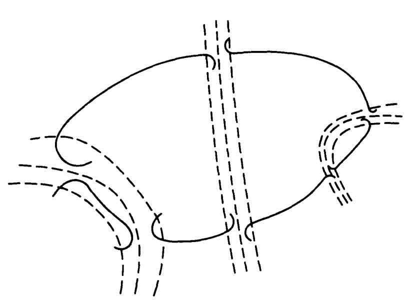

Fig. 7. Intersections of a level set with RZ

borhoods of either RA or of intersection points of two RA. The curvature of the RA at step t is less than t^{200}, so, we can decompose the new resonant set into a union of less than t^{1000} components each contained in a circle of radius $(a\varepsilon)^{t/8}$. We will essentially use the ergodic properties of the shift transformation on the torus $T_\omega : \alpha \mapsto \alpha + \omega$. For a circle D of arbitrary radius $R < 1$ on the torus the frequency with which the trajectory

$$\{\alpha, T_\omega \alpha, \ldots, T_\omega^{r-1} \alpha\}$$

comes into this circle can be estimated as follows:

$$r^{-1}N_r(\alpha, D, R) \leqq r^{-1} + 2\pi R^2 \ , \qquad r \geqq r_0 \ , \qquad R < R_0 \ .$$

The first term corresponds to at least two visits to the circle during the time interval $[0, t_0]$. We avoid this by excluding a set of having measure less than $2\pi R_0^2$. For all $t \leqq t_0$ the double points are removed simply by making choice of a smaller ε, if the RA have no points of triple intersection. The latter follows from inductive assumptions (for a.e. ω). Thus, for $t > t_0$ at the step t the frequency for the point α increases less than on $\varepsilon^{1/2}(1 + \text{const}\, \varepsilon^{s/10}) < \varepsilon^{1/4}$ for all ω except for a set of measure less than s^{-9}. Summing over t, $s \leqq t \leqq s + \ln s$, we conclude that for the set of frequencies of measure greater than $1 - \text{const}\, s^{-9}$ and any α, the frequency of entering the RZ is less than $\varepsilon^{1/2}(1 + \text{const}\, \varepsilon^{s/10}) < \varepsilon^{1/4}$. Proceeding now as in [15, Sect. 7, 8] we complete the proof of the inductive upper bound of AEF decay outside its ES.

5. Completeness of the Set of AEF

At each step of induction we get a countable set of finite-dimensional subspaxes $\mathscr{L}(\alpha, t)$ of the form

$$\mathscr{L}(\alpha, t) = \mathbb{C}\,[T^t \Phi_\ell^s(\alpha + t\omega) \ , \qquad \alpha \in L_\ell^s] \ ,$$

each having dimension less than $\text{const} \cdot s$. For $|t_1 - t_2| > 2\,\text{const} \cdot s$ the subspaces $\mathscr{L}(\alpha, t_1)$ and $\mathscr{L}(\alpha, t_2)$ are orthogonal, and in each $\mathscr{L}(\alpha, t)$ the system of vectors $T^t \Phi_\ell^s(\alpha + t\omega)$ is a basis with

$$|(\Phi_\ell^s(\alpha),\ \Phi_m^s(\alpha)) - \delta_{\ell m}| \leqq (a\varepsilon)^{c's} \ .$$

This follows from the construction of AEF with the help of the first order perturbation formula and the inductive estimates of decay of AEF. Renumber all vectors $T^t \Phi_\ell^s(\alpha + t\omega)$ by integers in such a way that for any $t < r$ the label of $T^t \Phi_\ell^s(\alpha + t\omega)$ would be less than the label of any $T^r \Phi_m^s(\alpha + r\omega)$. Let the n-th vector in this numeration be $\Psi_n^s(\alpha)$. Then we can define an operator $W^s(\alpha): T^n \Phi_1^1 \mapsto \Psi_n^s(\alpha)$ and decompose it into the sum $W^s = U^s + G^s$, where U^s is unitary with

$$(U^s(\alpha) T^n \Phi_1^1,\ T^k \Phi_1^1) = 0 \ , \qquad |n - k| \geqq \text{const} \cdot s \ ,$$

and the norm of $G^s(\alpha)$ is less than $(a\varepsilon)^{c's}$. Furthermore,

$$|(G^s(\alpha) T^n \Phi_1^1,\ T^k \Phi_1^1)| \leqq (a\varepsilon)^{c''s} \ .$$

Thus, for ε small enough the following series converges in the strong operator topology:

$$\sum_{\ell=0}^{\infty} ((-U^s(\alpha))^{-1} G^s(\alpha))^\ell \ ,$$

yielding a well-defined representation for $(W^s(\alpha))^{-1}$

$$(W^s(\alpha))^{-1} = \sum_{\ell=0}^{\infty} (-U^s(\alpha))^{-1} G^s(\alpha))^\ell (U^s(\alpha))^{-1} \ .$$

This means that any basic function in $l^2(\mathbb{Z})$ concentrated on a one-point set is contained in the minimal closed subspace containing the vectors $T^t\phi_\ell^s(\alpha+t\omega)$. This implies the completeness of the system $\{T^t\phi_\ell^s(\alpha+t\omega),\ t\in\mathbb{Z}\}$.

6. Absence of Spectral Gaps

At the intial step of induction the function $\Lambda_1^1(\alpha)=\mathrm{V}(\alpha)$ has a connected range $I=[\min V(\alpha), \max V(\alpha)]$. The distribution of its values posesses a piecewise smooth density $K_1^1(\lambda)$ since $V\in C^2(\mathrm{Tor}^2)$ with a finite number of non-degenerate critical points. At the boundary points of I it has discontinuities of the first order and vanishes out of I. At each subsequent step from the domains of some branches of multivalued function Λ^s the new RZ are excluded as well as sets obtained from RZ by some shift on the torus. We have already explained this process in the Introduction. On the other hand, the new branches appear on new RZ. Each new branch is defined either on the $(a\varepsilon)^{(s+1)/4}$-neighborhood of new RA or on the intersection of two such domains. The distribution function for the new $\Lambda_\ell^{s+1}(\alpha)$ also posesses a density, $K_\ell^{s+1}(\lambda)$, since all Λ_ℓ^{s+1} have only non-degenerate critical points. From the exponential bound on the domain of new AEV and from the polynomial bound on the minimal EV of $D^2\Lambda_\ell^{s+1}(\beta_{\ell k}^{s+1})$ we conclude that

$$0\leqq K_\ell^{s+1}(\lambda)\leqq(a\varepsilon)^{(s+1)/s}/(s+1)^{-30}\leqq(a\varepsilon)^{(s+1)/10} \ .$$

Summing over all the branches of Λ^t and over all t, $1\leqq t\leqq s+1$, we get the following upper bound on the density of the distribution function for the multivalued function $\Lambda^{s+1}(\alpha)$:

$$0\leqq K^{s+1}(\lambda)=\sum_\ell K_\ell^{s+1}(\lambda)\leqq\sum_{s=1}^{\infty}(a\varepsilon)^{(s+1)/10}s<\infty \ .$$

So, the limit

$$K(\lambda)=\lim_{s\to\infty}K^s(\lambda)$$

exists and is uniformly bounded. However, it is easy to see that $K(\lambda)$ is nowhere continuous on its support because $K_\ell^s(\lambda)$ has a discontinuity of the first order at any point $\lambda_{\ell k}^s=\Lambda_\ell^s(\beta_{\ell k}^s)$, while later the magnitude of this discontinuity is not less than one half of its initial value, if ε is chosen small enough.

Let us show that the range of

$$\Lambda(\alpha)=\lim_{s\to\infty}\Lambda^s(\alpha)$$

is connected and inside this set $K(\lambda)$ does not vanish. We assume that $K(\lambda)$ vanishes at some point and will come to the contradiction.

Let Δ be any maximal interval where $K(\lambda)=0$ or sufficiently small interval containing a point where $K(\lambda)=0$. For any $\delta>0$ there is such $s(\delta)$ that for $s>s(\delta)$

$$\mathrm{mes}\{\alpha:\Lambda_\ell^s(\alpha)\in\Delta \text{ at least for one value of } 1\}<\delta \ . \tag{11}$$

Let Δ be of the form $\Delta=[b,d]$. The set (11) is bounded by a union of piecewisely smooth curves obtained by a small deformation (for small ε) of initial curves from the level sets $A(V,b)$, $A(V,d)$ out of those arcs of the curves which enter into RZ.

The width of RZ at the step t is $2(a\varepsilon)^{t/4}$. Excluding a set of measure vanishing as $\varepsilon\to 0$ we can guarantee that for any c no arc from the initial level sets $A(V,c)$ of length $10a\varepsilon$ or less intersects a RZ at the first step of induction. Since the angle between a given curve and a RA intersecting it decreases not faster than s^{-10}, the length of arc entering the RZ is less than $2(a\varepsilon)^{s/4}s^{10}$. Hence, it remains always a part of the arc of length greater than

$$10a\varepsilon-\sum_{s>s_0}(2a\varepsilon)^{s/4}>5a\varepsilon>0$$

outside all the RZ. Without loss of generality assume that the arc lies strictly inside $\{\alpha:\Lambda_\ell^s(\alpha)\in\Delta\}$. Then for some constants $b>0$, $c>0$,

$$\operatorname{mes}\{\alpha:\Lambda_\ell^s(\alpha)\in[\lambda-c,\lambda+c]\}\geqq 5ab\varepsilon\ ,$$

where $\lambda\in\Delta$ because together with any point of the arc outside of a RZ this set contains a segment of an orthogonal line to this arc at the given point. This follows from the polynomial upper bound of curvature of the RA at the step s.

Thus, we see that the density of the distribution function for $\Lambda(\alpha)$ is uniformly bounded from below by a positive constant. This completes the proof.

Appendix

In the formulations of Lemmas 1 and 2 we use the notations of the Sect. 1.

Lemma 1. *Let the* AEF $\Phi_\ell^s(\alpha)$ *and the corresponding* AEV $\Lambda_\ell^s(\alpha)$ *satisfy the following condition: for any* $t\in[d(s)+1,\,d(s+1)]$ *with* $d(s)=[\tilde{c}s/\ln 1/\varepsilon]$ *and my* $m\neq 1$,

$$|\Lambda_\ell^s(\alpha)-\Lambda_m^s(\alpha+t\omega)|>R_{\ell m}(\alpha,t)=(s^{10}+(\operatorname{dist}(Z(\Phi_\ell^s(\alpha)),Z(T^t\Phi_t^s(\alpha+t\omega)))^{10})^{-1}\ .$$

Then there exists an AEF $\Phi_\ell^{s+1}(\alpha)$ *with an* AEV $\Lambda_\ell^{s+1}(\alpha)$, *namely,*

$$\Phi_\ell^{s+1}(\alpha)=\Phi_\ell^s(\alpha)+\sum_{m\neq\ell}\sum_{t=0}^{d(s+1)}\frac{(F_\ell^s(\alpha),\,T^t\Phi_m^s(\alpha+t\omega))}{\Lambda_\ell^s(\alpha)-\Lambda_m^s(\alpha+t\omega)}\,T^t\Phi_m^s(\alpha+t\omega)$$

and

$$\Lambda_\ell^{s+1}(\alpha)=\Lambda_\ell^s(\alpha)+(F_\ell^s(\alpha),\,\Phi_\ell^s(\alpha))\ ,$$

satisfying (3_{s+1}), (5_{s+1}), (6_{s+1}) *and such that*

(i) *the domain of* Φ_ℓ^{s+1}, Λ_ℓ^{s+1} *coincides with that of* Φ_ℓ^s, Λ_ℓ^s

(ii)

$$\|\Phi_\ell^{s+1}-\Phi_\ell^s\|_{C^2}<\exp(-3/4\tilde{c}s)\ ,$$

$$\|\Lambda_\ell^{s+1}-\Lambda_\ell^s\|_{C^2}<\exp(-3/4\tilde{c}s)\ .$$

We call this statement the lemma on non-resonant deformation.

Lemma 2. *Assume that the* AEV $\Lambda_{\ell_k}(\alpha)$, $2\leqq k\leqq N$, $N\leqq 3$, *satisfy the following conditions:*

$$|\Lambda_{\ell_1}(\alpha)-\Lambda_{\ell_j}(\alpha+t_j)|\leqq R_{\ell_1\ell_j}(\alpha;t_j)\ ,\quad 2\leqq j\leqq N\ ; \tag{A1}$$

$$|\Lambda_{\ell_j}(\alpha)-\Lambda_m(\alpha+t\omega)|>R_{\ell_j m}(\alpha;t),\ 2\leqq j\leqq N,\ m\notin\{\ell_j,1\leqq j\leqq N\}\ . \tag{A2}$$

Define in the N-dimensional Hilbert space with the orthogonal normed basis $e_1, \ldots, e_N$ *a self-adjoint operator* $Q(\alpha)$ *with the following matrix elements:*

$$Q_{ij}(\alpha) = (\Phi^s_{\ell_i}(\alpha), H_\varepsilon(\alpha) T^t \Phi^s_{\ell_j}(\alpha + t_j\omega)) .$$

Let $\lambda_j(\alpha)$ *be the eigenvalue of* $Q(\alpha)$ *and*

$$\chi_j(\alpha) = \sum_{k=1}^{N} c_{jk}(\alpha) e_k$$

be its normed eigenfunctions. Then there exists the AEF $\Phi^{s+1}_{\ell_k}$ *and the* AEV $\Lambda^{s+1}_{\ell_k}$, $1 \leqq k \leqq N$ *defined on the domain where* (A1), (A2) *hold and have the following form:*

$$\Phi^{s+1}_{\ell_i}(\alpha) = \sum_{j=1}^{N} c_{ij}(\alpha) T^{t_j} \Phi^s_{\ell_j}(\alpha)$$
$$+ \sum_{j=1}^{N} c_{ij}(\alpha) \sum_{m \neq \ell_1, \ldots, \ell_N} \sum_{t=0}^{d(s+1)} \frac{(F^s_{\ell_j}(\alpha + t_j\omega), T^t \Phi^s_m(\alpha + t\omega))}{\Lambda^s_{\ell_j}(\alpha + t_j\omega) - \Lambda^s_m(\alpha + t\omega)} T^t \Phi^s_m(\alpha + t\omega) ,$$
$$\Lambda^{s+1}_{\ell_i}(\alpha) = \lambda_i(\alpha) .$$

These AEF *and* AEV *satisfy the indictive assumptions* (3_{s+1}), (5_{s+1}), (6_{s+1}).

This statement will be called the lemma on resonant splitting.

Remarks. 1. The new AEF and AEV in both cases are constructed with the help of the first order perturbation formulas, and the one uses the cut-off in order to satisfy the condition that the diameter of ES is bounded by const s. This procedure is quite similar to that used in [15, Sect. 4]. The only difference with [15] is that we do not construct the exact EF and EV but only the approximate ones. Therefore, we can omit the seond order correction terms. Then it remains to study the local analytic properties of new AEF and AEV as functions of the variables α_1, α_2. This can be done by the term-by-term differentiation of the equalities which determine new AEF, AEV in the above lemmas. One should use also the inductive estimates of the decay of AEF (which gives the upper bound for the numerators in the above-mentioned formulas) and the estimates of polynomial type for the decay of denominators corresponding to non-resonant terms. The resonant terms are studied with the help of an auxiliary spectral problem as was done in [15, Sect. 4, Theorem 2].

2. Further analysis of new AEV and AEF is given in Sects. 2–4.

Acknowledgements. It is a pleasure to thank T. Spencer, E. Dinaburg, S. Zhitomirskaya for reading the manuscript and making many useful remarks leading to the essential improvement of the text. We also thank R. A. Johnson for communicating his results [19] prior to publication. One of us (V.Ch.) is also grateful to J. Bellissard for fruitful discussions of [19].

References

1. Mott, N., Twose, W.D.: Adv. Phys. **10**, 107 (1961)
2. Borland, R.E.: The nature of the electronic states in disordered one-dimensional systems. Proc. R. Soc. London A **274**, 529 (1963)

3. Goldsheid, I.Ya., Molchanov, S.A., Pastur, L.A.: A pure point spectrum of the stochastic one-dimensional Schrödinger equation. Funkt. Anal. Appl. **11**, 1 (1977)
4. Kunz, H., Souillard, B.: Sur le spectre des opérateurs aux différences finies aléatoires. Commun. Math. Phys. **78**, 201 (1980)
5. Carmona, R.: Exponential localization in one dimensional disordered systems. Duke Math. J. **49**, 191 (1982)
6. Simon, B., Taylor, M., Wolf, T.: Some rigorous results for the Anderson model. Phys. Rev. Lett. **54**, 1589 (1985)
7. Fröhlich, J., Spencer, T.: Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151 (1983)
8. Fröhlich, J., Spencer, T.: A rigorous approach to Anderson localization. Phys. Rep. **103**, 9 (1984)
9. Fröhlich, J., Spencer, T.: Mathematical theory of Anderson localization, Proc. IAMP Conf. Boulder. Physica **124**A, 303 (1984)
10. Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. **101**, 21 (1985)
11. Aubry, S., André, G.: Analyticity breaking and Anderson localization in incommensurate lattices. Ann. Isreal Phys. Soc. **3**, 133 (1980)
12. Avron, J., Simon, B.: Almost periodic Schrödinger operators. II. The density of states. Duke Math. J. **50**, 369 (1983)
13. Figotin, A.L., Pastur, L.A.: The positivity of Lyapunov exponent and the absence of absolutely continuous spectrum for the Almost-Mathieu equation, J. Math. Phys. **25**, 774 (1984)
14. Bellissard, J., Lima, R., Scoppola, E.: Localization in ν-dimensional incommensurate structures. Commun. Math. Phys. **88**, 207 (1983)
15. Sinai, Ya. G.: Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential, J. Stat. Phys. **46**, 861 (1987)
16. Fröhlich, J., Spencer, T., Wittwer, P.: Localization for a class of one-dimensional quasi-periodic Schrödinger operators. Preprint, 1987
17. Herman, M.: Une methode pour minorer des exposants de Lyapunov. Preprint, Palaiseau, 1982
18. Pastur, L.A.: Lower bounds of the Lyapunov exponent for some finite-difference equations with quasi-periodic coefficients. In: Operators in functional spaces and problemes of the theory of functions, Kiev: Naukova Dumka 1987 (Russian)
19. Johnson, R.A.: Private communication

Communicated by A. Jaffe

Received January 25, 1989