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On Existence and Behaviour of Asymptotically Flat Solutions to the Stationary Einstein Equations

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Abstract. We show existence and uniqueness of asymptotically flat solutions to the stationary Einstein equations in $S=\mathbb{R}^3-B_r$, where B_r is a ball of radious $r>0$, when a small enough continuous complex function $\hat{u}$ on ∂S is given. Regularity and decay estimates imply that these solutions are analytic in the interior of S and also at infinity, when suitably conformally rescaled.

Introduction

After a considerable effort a rather clear picture describing the set of stationary, and asymptotically flat vacuum solutions to Einstein equations it is now available. Solutions are represented by a complex scalar field u, and a positive definite metric g_{ab} on a three dimensional manifold S [1–3]. From the four dimensional point of view this manifold is the quotient of space-time with the set of orbits of the killing vector field defining stationarity, the metric is conformally related to the one induced by the space-time metric on S, and u is a given functional of the norm, and the twist of the killing vector field. The equations they satisfy are,

$$(\Delta_g-2R)u=0, \tag{1}$$

$$G_{ab}-2(\nabla_{(a}u\nabla_{b)}u^*-(1+4|u|^2)^{-1/2}\nabla_a|u|^2\nabla_b|u|^2) -g_{ab}(\nabla_c u\nabla^c u^*-(1+4|u|^2)^{-1/2}\nabla_c|u|^2\nabla^c|u|^2)=0, \tag{2}$$

where G_{ab} is the three dimensional Einstein tensor corresponding to g_{ab}. We consider the following asymptotic boundary conditions $u\to 0$, $g_{ab}-e_{ab}\to 0$, as $r\to 0$, where e_{ab} is any flat metric on S, and r is the distance function with respect to it. Given a solution (u, g_{ab}) of the above equations it is possible to reconstruct a unique, stationary, asymptotically flat, maximally extended vacuum space-time.

From local elliptic theory [4, 5] we know that sufficiently smooth solutions (if they exist) are in fact analytic. Furthermore assuming a stronger asymptotic decay than the one above, one can show there exists a conformal factor such that the conformally rescaled fields are also sufficiently smooth and satisfy regular elliptic equations; thus they are also analytic, even at the point representing infinity

[3, 6, 7]. It makes sense then to characterize these solutions by their Taylor expansion at infinity, that is by a set of multipole moments [1, 2, 8, 9]. It turns out that to characterize these solutions it is enough to define multipole moments only for the conformally rescaled field corresponding to u, that is there are no degrees of freedom associated to the three-metric g_{ab} [10, 11].

Unfortunately only a small set of solutions to the above equations, all of them possessing extra symmetries, are explicitly known and so until now we did not know if the asymptotic conditions assumed to obtain the above picture were generic enough so as to allow for the existence of a sufficiently large class of solutions.[1] In other words we did not know how generic was this picture. In order to add weight to the above picture we shall show here the existence of a large class of solutions to the stationary Einstein equations when $S = \{\mathbb{R}^3\text{-}a \text{ smooth ball}\}$. These solutions, which one would like to interpret as the exterior field of stationary compact bodies, are analytic in the interior of S, (even at infinity when suitably conformally transformed) and are uniquely characterized by a continuous complex function $\hat{u}$ on ∂S, the value of u at ∂S, and thus by a harmonic expansion of u at ∂S. To each member of this harmonic expansion there corresponds a unique linearized multipole moment at infinity.

Main Theorem

Existence Theorem. *Fix on ∂S a positive definite metric, h_{ab}, of constant scalar curvature. Then there exists a neighborhood of zero, V, in the Sobolev space*[2] *$H_2(\partial S)$, such that for each $\hat{u} \in V$ there exists a unique, analytic on* int$S$, *and asymptotically flat solution to the stationary vacuum Einstein equations, (u, g_{ab}), with $u|_{\partial S} = \hat{u}$. The solutions, when suitably conformally transformed are analytic even at infinity, and so each of them determines a unique multipole expansion at the point at infinity.*

We partition the proof in a series of lemmata. In the first we show that the stationary equations are equivalent to a reduced elliptic system. This procedure is similar to the one used to reduce the full Einstein equations to a hyperbolic one [12–14].

Reduction Lemma. *Fix a flat metric, e_{ab}, in S in such a way that it induces h_{ab} on ∂S. Then the stationary vacuum Einstein equations, Eqs. (1, 2), with the boundary condition, $u|_{\partial S} = \hat{u}$, for $(u, g_{ab} - e_{ab}) \in H_{5/2,\,-3/4}(S)$*[3], *$\hat{u} \in H_2(\partial S)$, small enough, are equivalent to the following reduced system for (u, ϕ^{ab}):*

$$E(u, \phi^{ab}) := \Delta_g u + (\text{terms in } \nabla_c \phi^{de}, \phi^{de}, \nabla_c u, u) = 0,$$

$$E^{ab}(u, \phi^{ab}) := g^{cd} \nabla_c \nabla_d \phi^{ab} + (\text{terms in } \nabla_c \phi^{de}, \phi^{de}, \nabla_c u, u) = 0,$$

$$u|_{\partial S} - \hat{u} = 0, \quad \phi|_{\partial S} = 0, \quad \sigma_{ab}|_{\partial S} = 0,$$

$$\dot{\psi}|_{\partial S} = 0, \quad \hat{\psi}^b|_{\partial S} = 0,$$

[1] For a sufficiently large class of solutions we mean one containing all solutions corresponding to physical bodies in equilibrium occupying acompact region of space

[2] Proper definitions of all the spaces we are using, as well as a list of their properties can be found in [15]

[3] The differentiability and decay indices used for the weighted Sobolev spaces are not necessarily the sharpest ones, but rather are taken for definiteness and convenience

where $\phi^{ab} := \sqrt{g}g^{ab} - e^{ab}$, *with* $\sqrt{g}$ *given by* $\varepsilon_{abcd}(g_{ef}) = \sqrt{g}\varepsilon_{abcd}(e_{ef})$, E^{ab} *is obtained by setting, in the expression for* G^{ab} *in terms of* ϕ^{ab} *the combination* $\psi^b := \nabla_a \phi^{ab}$ *to zero,* ∇_a *is the covariant derivative associated to* e_{ab},

$$\phi := e_{ab}\phi^{ab}, \qquad \sigma_{ab}|_{\partial S} := (h_{ac}h_{bd}\phi^{cd} - \tfrac{1}{2}h_{ab}h_{cd}\phi^{cd})|_{\partial S},$$

$$\dot{\psi}|_{\partial S} := (n^a \nabla_a (n_b \psi^b))|_{\partial S},$$

where n^a *is the exterior normal to* ∂S, *and* $\hat{\psi}^b|_{\partial S} := (\psi^a h_a{}^b)|_{\partial S}$.[4]

Proof. First we show that any small enough solution to the reduced system is also a solution to the stationary Einstein equations. Let (u, ϕ^{ab}) be a solution to the reduced system and consider,

$$\nabla_a E^{ab}(u, \phi^{cd}) = 0.$$

It is easy to see that this is an elliptic linear second order equation for ψ^b, $L_{(u,\phi^{cd})}\psi^b = 0$, which is continuous as a map from $(u, \phi^{cd}) \in H_{5/2,-3/4}(S)$, $\psi^b \in H_{3/2,1/4}(S)$, to $H^{\cdot}_{-1/2,9/4}(S)$, and which reduces to the flat Laplacian when $u = \phi^{ab} = 0$. But injectivity is a continuous property for elliptic operators, and since the flat Laplacian is injective for the boundary conditions considered $(\dot{\psi}|_{\partial S} = 0, \hat{\psi}^b|_{\partial S} = 0)$,[5] there exists $\varepsilon > 0$ such that if $\|(u, \phi^{cd})\|_{5/2,-3/4} < \varepsilon$, then $\psi^b = 0$. Thus for these solutions both systems are identical and so they also solve the stationary Einstein equations.

Second we show that for any metric g_{ab} close enough to some flat metric, e_{ab}, there exists another flat metric $\tilde{e}_{ab}$ with respect to which $\psi^b(g_{cd}) = \phi|_{\partial S} = \sigma_{ab}|_{\partial S} = 0$. Thus, if (u, g_{ab}) satisfies the stationary Einstein equations it also satisfies the reduced ones. Instead of looking for $\tilde{e}_{ab}$, which is related by a diffeomorphism to e_{ab}, we diffeomorphically transform g_{ab}, keeping e_{ab} fixed. Such diffeomorphism, d, must satisfy,

$$0 = G(d, g_{ab}) := \begin{pmatrix} \nabla_a^* \phi^{ab} \\ {}^*\phi|_{\partial S} \\ D_a^* \sigma^{ab} \end{pmatrix},$$

where ${}^*\phi^{ab} := \sqrt{{}^*g}\,{}^*g^{ab} - e^{ab}$, with ${}^*g^{ab}$ the diffeomorphically transformed g^{ab}, and D_a is the covariant derivative on ∂S associated to h_{ab}.[6] We now use a corollary of the implicit function theorem on Banach spaces [18, 19] to assert, given any g_{ab} in a small enough neighborhood of e_{ab}, the existence of diffeomorphisms, d, satisfying $G(d, g_{ab}) = 0$.

Corollary. *Let* X, Y, Z *be Banach spaces,* U, V *open sets of* X, *and* Y *respectively, and* $G: V \times U \to Z$ *a* C^1 *function. Assume there exist* $x_0 \in U$, *and* $y_0 \in Y$, *such that*

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[4] Where needed we have extended n^a, and therefore h_{ab}, into a neighborhood of ∂S on S using the geodesic equation

[5] See Appendix and [15]

[6] This is equivalent to the existence of a unique, up to translations and rotations, harmonic coordinate system, once ∂S is given. See also [6, 16, 17]

$G(y_0, x_0)=0$, and $D_1G(y_0, x_0)\colon Y\to Z$, the differential of G with respect to the first argument is surjective and has complemented kernel. Then there exist neighborhoods $V_{x_0}\subset V$, and $U_{y_0}\subset U$ around x_0, and y_0 respectively such that for any $x\in U_{x_0}$ there exists at least an $y\in V_{y_0}$ satisfying $G(x, y)=0$.

Proof. It follows from Corollary 2.5.8 and the proof of the implicit function theorem in [18].

We consider $G(d, g_{ab})$ as a function from $Y\times X$ to Z, where $Y=H_{7/2,-7/4}(S)$, $X=H^{\cdot\cdot}_{5/2,-3/4}(S)$, and

$$Z=H^{\cdot}_{3/2,1/4}(S)\times H_2(\partial S)\times\{H^{\cdot}_1(\partial S)/\textit{infinitesimal conformal isometries of } h_{ab}\}.$$

V is chosen, using that $H_{7/2,-7/4}(S)\subset C^1_{1/2}(S)$, to be a small enough neighborhood of the identity map in order to guarantee that all its elements have differentiable inverses. This function satisfies $G(i, e_{ab})=0$, where i is the identity diffeomorphism, is a differentiable function of both arguments, and its differential with respect to the first argument at $(d, g_{ab})=(i, e_{ab})$ is,

$$D_1G_{(i,e_{ab})}\cdot\delta d=\begin{pmatrix}\Delta_e\xi^b\\ \dot{\xi}+k\xi+D_a\hat{\xi}^a\\ D_a(2D^{(a}\hat{\xi}^{b)}-h^{ab}D_c\hat{\xi}^c)\end{pmatrix},$$

where we have identified δd with the vector field it generates, ξ^a, and $\xi=n_a\xi^a|_{\partial S}$, $\dot{\xi}=(n^cV_c\xi)|_{\partial S}$, $k=V_an^a|_{\partial S}$, and $\hat{\xi}^a=h^a{}_b\xi^b|_{\partial S}$.

Thus to complete the proof we need only to show that the above linear map has complemented kernel and is surjective. This map satisfies the hypothesis of the Main Elliptic Theorem of the Appendix. Since its kernel and the one of its adjoint, $DG'_{(i,e_{ab})}\colon Y\to X'$, consist of C^∞ functions, it is easy to see that in the first case it consists only of the infinitesimal isometries of e_{ab}, since this space is finite-dimensional it is complemented, and in the second only of the infinitesimal conformal isometries of h_{ab}, which have been factored out, but then $\mathrm{Range}\,DG_{(i,e_{ab})}=Y$, and so surjectivity is also asserted.

Finally we show that if $\hat{\sigma}_{ab}\in H^{\cdot\cdot}_2(\partial S)$ is trace and divergence free, then $\hat{\sigma}_{ab}=0$. To see this, conformally transform $h_{ab}=\Omega^2f_{ab}$, $\hat{\sigma}^{ab}=\Omega^{-4}\sigma^{ab}$, using $\Omega=\dfrac{2}{1+r^2}$, so that f_{ab} is a flat metric in $\mathbb{R}^2$. Since $\hat{\sigma}^{ab}$ is trace free its divergence is invariant under the above conformal transformation, therefore σ^{ab} is also trace and divergence free. But then in a cartesian (with respect to f_{ab}) coordinate system the components of σ^{ab} satisfy the Laplacian equation, $\Delta_f\sigma^{\mu\nu}=0$, and so they are constant or grow asymptotically. In any case they always give rise to singular tensors when conformally transformed back to ∂S. Thus $\hat{\sigma}^{ab}$ must vanish. □

Strictly speaking the diffeomorphism we have used on the second part of the above proof can move the sphere around, therefore one should take ∂S to be contained in a open region where the stationary solution exists.

Note that although the tensor density ϕ^{ab} is background (e_{ab}) dependent, the metric tensor it generates, g_{ab}, is not. The background metric is only an artifact to render the equations elliptic and thus to establish existence.

We now use the fact that the reduced system is elliptic in the sense of Hörmander, [20, 21][7] which includes the Lopatinskiĭ-Šapiro conditions [22, 23] for the boundary equations, to establish existence of solutions to the reduced system.

Existence Lemma. *There exists* $\varepsilon>0$, *such that for each complex function* $\hat{u}$ *with* $\|\hat{u}\|_{2,\partial S}<\varepsilon$ *there exists a unique* $(u,\phi^{ab})\in H_{5/2,-3/4}(S)$ *solution to the reduced system, Eqs.* (1, 2).

Proof. Consider the reduced system as an implicit function, $F((u,\phi^{ab}),\hat{u})=0$, for (u,ϕ^{ab}) in terms of $\hat{u}$. As a function from

$$Y\times X:=(H_{5/2,-3/4}(S)\times \ddot{H}_{5/2,-3/4}(S))\times H_2(\partial S)$$

to

$$Z:=H_{1/2,5/4}(S)\times \ddot{H}_{1/2,5/4}(S)\times H_2(\partial S)\times H_2(\partial S)\times \ddot{H}_2(\partial S)\times H_0(\partial S)\times \dot{H}_1(\partial S)$$

it is differentiable and furthermore $F(0,0),0)=0$. Thus if we could show that $D_1F_{(0,0),0)}:(\delta u,\delta\phi^{ab})\in Y\to Z$, the differential of F with respect to the first entry, is an isomorphism then the implicit function theorem [18, 19] would assert the existence of $\varepsilon>0$, and of a unique, continuously differentiable function $(u(\hat{u}),\phi^{ab}(\hat{u}))$ defined for all $\|\hat{u}\|_{2,\partial S}<\varepsilon$ such that $F((u(\hat{u}),\phi^{ab}(\hat{u})),\hat{u})=0$. This would complete the proof.

We first show injectivity, namely $D_1F_{((0,0),0)}(\delta u,\delta\phi^{ab})=0\Rightarrow(\delta u,\delta\phi^{ab})=0$.

Decomposing $\delta\phi^{ab}$ at the boundary as $\delta\phi^{ab}=\tau n^an^b+2n^{(a}\hat{\phi}^{b)}+\frac{\sigma}{2}\hat{h}^{ab}+\hat{\sigma}^{ab}$, with $0=\hat{\phi}^an_a=\hat{\sigma}^{ab}n_a=\hat{\sigma}^{ab}h_{ab}$, we obtain

$$0=D_1F_{((0,0),0)}\cdot(\delta u,\delta\phi^{ab})=\begin{pmatrix} \Delta_e\delta u \\ \Delta_e\delta\phi^{ab} \\ \delta u|_{\partial S} \\ \tau+\sigma \\ \hat{\sigma}^{ab} \\ D^2\left(\tau+\frac{\sigma}{2}\right)-\frac{k^2}{2}\left(\tau-\frac{\sigma}{2}\right)+\frac{k}{2}\dot{\sigma} \\ \dot{\hat{\phi}}^b+\frac{3}{2}k\hat{\phi}^b+\frac{1}{2}D^b\sigma \end{pmatrix}.$$

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[7] Hörmander ellipticity of the system for points not at the boundary follows from usual ellipticity. That is one considers for each one of these points the constant coefficients differential system on $\mathbb{R}^3$ obtained by defining its coefficients as the value of those of the principal part of the system at that point. One then shows that there is no bounded solution to the constant coefficients system of the form $u_l(x)=C_l\exp(ik_jx^j)$ with some $k_j\neq0$. For points at the boundary one considers for each one of them the constant coefficients differential system on ${}^+\mathbb{R}^3$, obtained by defining its coefficients as the value of those of the principal part of the operators at that point (now taking into account the boundary operators!). One then shows that there is no bounded solution to it of the form $u_l=C_l\exp(i\hat{k}_j\hat{x}^j)\omega(\tau)$, with some $\hat{k}_j\neq0$, where τ is the coordinate defining ${}^+\mathbb{R}^3$ as $\tau\leqq0$, and the $\hat{x}^j$ are normal coordinates to τ

Using the first and third lines of the above equation we get

$$0=\int_S \delta u \Delta_e \delta u \, dV_e = \int_S V_c \delta u V^c \delta u \, dV_e .$$

Therefore $\delta u = \text{cte}$, but the constant functions are not in $H_{5/2,-3/4}(S)$ and so $\delta u = 0$. Similarly the trace of the second line, and the fourth line imply $e_{ab}\delta\phi^{ab}=0$, and so $\dot\sigma+\dot\tau=0$. Thus the last two lines become,

$$\frac{1}{2}D^2\tau-\frac{3}{4}k^2\tau-\frac{k}{2}\dot\tau=0, \qquad \dot{\hat\phi}^b+\frac{3}{2}k\hat\phi^b-\frac{1}{2}D^b\tau=0.$$

Using these two equations we now obtain,

$$0=\int_S \delta\phi_{ab}\Delta_e\delta\phi^{ab}\,dV_e=\int_S V_c\delta\phi_{ab}V^c\delta\phi^{ab}\,dV_e+\int_{\partial S}\left(\frac{3}{2}k\hat\phi_a\hat\phi^a+\frac{1}{k}D_a\tau D^a\tau+\frac{k}{4}\tau^2\right)dV_h .$$

Since $k>0$ the right-hand side is positive definite, and so we conclude $\delta\phi^{ab}=0$ which asserts injectivity.

Second we prove surjectivity. Note that $D_1F_{((0,0),0)}\cdot(\delta u,\delta\phi^{ab})$ satisfies the hypothesis of the Main Elliptic Theorem of the Appendix, (in particular Hörmander ellipticity), thus it has an index, and it only depends on the principal parts of the operators. We now prove that this index vanishes. This together with the above injectivity result implies that the co-dimension of the range is zero, and so surjectivity. To find the index we consider a family of nested spheres (with respect to the flat metric e_{ab}) and decompose as before $\Delta_e\delta\phi^{ab}$ into equations for τ, σ, $\hat\phi^b$, and $\hat\sigma^{ab}$, now defined for the whole family of spheres. Since the index is an invariant – it only depends on the principal part of the equations – one can continuously distort the equations without changing it. We distort them by setting to zero k, the value of the scalar curvature of the spheres, in several places where it appears in such a way as to decouple the equations.[8] It is easy to show then that the resulting map, which consists of several decoupled Laplacians is an isomorphism, and therefore the vanishing of its index. □

We now study regularity for the solutions whose existence has been asserted.

Regularity Lemma. *Let S' be strictly contained in S. Then $(u,\phi^{ab})\in C^\omega(S')$, and therefore $(u,g_{ab})\in C^\omega(S')$. Furthermore there exists an analytic conformal factor, Ω approaching zero asymptotically as r^2 such that $(\tilde u,\tilde g_{ab})=(\Omega^{-1/2}u,\Omega^2 g_{ab})$ can be extended analytically to the conformally completed space, that is when the point at infinity is added to S'.*

Proof. By considering some of the u's and ϕ^{cd}'s in the expressions for the operators $E(u,\phi^{cd})$, and $E^{ab}(u,\phi^{cd})$ as coefficients we arrive to a coupled, linear elliptic system for the remaining u's and ϕ^{cd}'s. If $(u,\phi^{ab})\in H_{s,-3/4}(S')$ then such system approaches Δ_e with differentiability s, and rate -1. Furthermore since $C^j_i=0$ we can apply the Regularity Theorem of the Appendix to conclude that $(u,\phi^{ab})\in H_{s+1,-3/4}(S')$. Starting with $s=\frac{5}{2}$ and iterating the procedure we conclude $(u,\phi^{ab})\in H_{s',-3/4}(S')$ for

[8] One is not deforming the spheres into planes, which would result in a discontinuous distortion of the Laplacian, but rather setting to zero some terms not in the principal parts of the differential operators

any s', and so $(u, \phi^{ab}) \in H_{\infty, -3/4} \subset C^{\infty}_{3/4-\varepsilon}(S')$, $\varepsilon > 0$. Local analyticity follows now from this result and Morrey's theorem [4, 5]. Analyticity at infinity follows from the above results and the work of Beig and Simon [10]. Although Beig and Simon assumed a faster decay at infinity than the one we have so far obtained here it is not difficult to see that their results are still valid assuming this slower decay. Alternatively one can, by writing the reduced equations as,

$$\Delta_e u = F(u, \nabla_c u, \phi^{ef}, \nabla_c \phi^{ef}, \nabla_c \nabla_d \phi^{ef}),$$

$$\Delta_e \phi^{ab} = F^{ab}(u, \nabla_c u, \phi^{ef}, \nabla_c \phi^{ef}, \nabla_c \nabla_d \phi^{ef}),$$

by using the decay estimates obtained see that $(F, F^{ab}) \in C^{\infty}_{7/2-\varepsilon}$, and by applying the Asymptotic Behaviour Theorem of the Appendix, conclude that (u, ϕ^{ab}) do in fact decay as it was assumed by Beig and Simon. □

This completes the proof of the Main Theorem.

Discussion

We have asserted the existence and uniqueness of solutions to the stationary vacuum Einstein equations, when a sufficiently small – with respect to a Sobolev norm – complex function, $\hat{u}$, on a sphere is given. Thus the freedom on the boundary data is the same as the one available from a multipole moment expansion at infinity. It is clear that each $\hat{u}$ giving rise to a solution determines a unique multipole set. Does there exist a norm on the multipole set such that each small enough multipole set gives rise to a unique solution to the stationary equations and thus determines a unique $\hat{u}$?

One would expect that it would be possible to assert existence not only for solutions close to Minkowski space-time but also for those close to Schwarzschild space-times of arbitrary mass. This is indeed the case, since by applying a constant conformal transformation to any of our solutions with nonvanishing mass we obtain solutions with arbitrarily large mass[9]. This corresponds to letting the sphere ∂S get bigger and bigger, i.e. having smaller and smaller constant curvature, while keeping the same $\hat{u}$.

Along the same line as above it should be possible to establish existence for the interior problem, namely when S is taken to be a ball, or the space within two concentric spheres. This should be relevant in order to perform an analysis similar to the one in [14], but for quantum cosmologies.

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Appendix

Here we list several definitions and theorems used in this work, which although known are very difficult to find in the literature.

[9] I thank Bernd Schmidt for this argument

Definition (Bartnik [17])[10]. A linear elliptic operator in an n dimensional asymptotically flat manifold is said to have differentiability, $s > \frac{n}{2}$, and to be asymptotic to, Δ_e, for some flat metric e_{ab} $\left(\text{at rate } \tau > -\frac{n}{2}\right)$, if it can be expressed as,

$$P(u) = A_i^{jab} \nabla_a \nabla_b u_j + B_i^{jb} \nabla_b u_j + C_i^j u_j ,$$

with ∇_a the covariant derivative associated to e_{ab}, and A_i^{jab}, B_i^{jb}, and C_i^j tensor fields satisfying,

$$\| A_i^{jab} - I_i^j e^{ab} \|_{s,\tau} + \| B_i^{jb} \|_{s-1,r+1} + \| C_i^j \|_{s-2,\tau+2} < \infty .$$

Note that the Sobolev imbedding theorems imply that A_i^{jab} is Hölder continuous, and $|A^{ab} - Ie^{ab}| = o\left(r^{-\tau-\frac{n}{2}}\right)$ [15]. For this work it is enough to consider the case $\tau = -1$.

Main Elliptic Theorem. *Let $u_j \to P_i^j u_j$, $i,j, = 1, \ldots, N$ be an operator and asymptotic to Δ_e, at rate -1, on $S = \mathbb{R}^3 - B_1$ and let $u_j \to B_i^{(k)j} u_j$, $i,j,k = 1, \ldots, N$ be boundary operators on ∂S. Then if the coefficients of the whole system are smooth and the whole system is elliptic in the sense of Hörmander* [20, 21] *we have;*

i) *The map*

$$(P,B): \prod_{j=1}^{N} H_{s,\delta}(S) \to \prod_{i=1}^{N} H_{s-2,\delta+2}(S) \times \prod_{k=1}^{N} H_{s-2-m_k}(\partial S) ,$$

for $s > 2$, $\delta \neq p + \frac{1}{2}$, $p \in Z$, and where m_k denotes the order of $B^{(k)}$, is Fredholm. That is, dim Kernel $< \infty$, co-dim Range $< \infty$, *and $i = \dim \mathrm{Ker}(P,B) - \text{co-dim Range}(P,B)$ is an invariant.*

ii) *For any $u \in H_{s,\delta}(S)$,*

$$\| u - \mathrm{Ker}(P,B) \|_{s,\delta,S} \leqq C \left\{ \| P(u) \|_{s-2,\delta+2,S} + \sum_{k=1}^{N} \| B^{(k)}(u) \|_{s-1/2-m_k,\partial S} \right\} ,$$

where C depends only on (P,B), and

$$\| u - \mathrm{Ker}(P,B) \|_{s,\delta,S} = \inf \{ \| u - w \|_{s,\delta,S} : w \in \mathrm{Ker}(P,B) \} .$$

iii) $\mathrm{Range}(P,B)^{\perp} \subset C^{\infty}(S)$.

Sketch of Proof. To prove the above theorem we follow Bartnik's proof of Proposition 1.11 in [17] which establishes a similar result but for manifolds without boundaries. We depart from that proof in the following two points:

i) To prove the analog to Proposition 1.6 in [17] instead of starting with the usual local Gärding estimate, we start with the following one which reaches the boundary: There exists $C > 0$ such that for all $u \in H_s(\bar{S})$, $\partial S \subset \bar{S} \subset S$, compact,

$$\| u \|_{s,\bar{S}} \leqq C \left\{ \| P(u) \|_{s-2,\bar{S}} + \sum_{k=1}^{N} \| B^{(k)}(u) \|_{s-1/2-m_k,\partial S} \right\} .$$

[10] Note the different convention for the Sobolev indices in [17]

This estimate can be deduced following the steps on the proof of Lemma 10.5.1 in [21] where the above inequality is proven for the case $N=1$.

ii) To prove the analog to Theorem 1.10 in [17] we take the patch function equal to 1 in a compact neighborhood of S containing ∂S. □

Remark. Partial proofs of the above theorem can be found in [15, 24–27]. We suspect that the theorem still remains valid if the smoothness condition on the coefficients is relaxed, that is if only certain differentiability, $s' > \frac{n}{2}$, where $n = \dim S$, is required for P, and that

$$B_i^{(k)j} = D_i^{(k)jab} \nabla_a \nabla_b + E_i^{(k)jb} \nabla_b + F_i^{(k)j},$$

with $D_i^{(k)jab} \in H_{s'-1/2,\delta S}$, $E_i^{(k)jb} \in H_{s'-m_k+1/2,\delta S}$, $F_i^{(k)j} \in H_{s'-m_k-1/2,\delta S}$, and $D_i^{(k)jab} n_a = E_i^{(k)jb} n_b = 0$.

Regularity Theorem (Choquet-Bruhat and Christodoulou [15]). *Let P be an elliptic operator of differentiability $s > \frac{3}{2}$, asymptotic to Δ_e at rate -1, and with $C_i^j = 0$, defined in $S = \mathbb{R}^3 - B_1$. Then for any $S' \subset S$, $S' \cap \partial S = \emptyset$, there exists $C > 0$ such that if $u \in H_{2,\delta}(S)$, and $P(u) \in H_{s'-2,\delta+2}(S')$, $s' \leqq s+1$, then,*

$$\|u\|_{s',\delta,S'} \leqq C\{\|P(u)\|_{s'-2,\delta+2,S'} + \|u\|_{0,\delta,S'}\}.$$

The proof of the above theorem is an immediate consequence of the local elliptic theory, see for example [28], and Theorem 6.1 in [15].

Asymptotic Behaviour Theorem (Meyers [29], Beig and Simon [9]). *Let $u \in C^\infty_\varepsilon(\mathbb{R}^3)$, $\varepsilon > 0$, such that $\Delta_e u := f \in C^\infty_{q+\varepsilon}(\mathbb{R}^3)$, $q \in N$, $q \geqq 3$. Then,*

$$u = \sum_{l=0}^{q-3} \sum_{m=-l}^{l} \frac{C_l}{r^{l+1}} Y_{l,m}(\vartheta, \varphi) + v,$$

where $v \in C^\infty_{q-2+\varepsilon}(\mathbb{R}^3)$.

Proof. Using Green's function we have,

$$u = \int_{\mathbb{R}^3} \frac{f(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV.$$

To estimate v one uses the following inequality valid for all $p \in N$ [30, 31],

$$\left| \frac{1}{\sqrt{1+\omega^2 - 2\omega\xi}} - \sum_{l=0}^{p-1} \omega^l P_l(\xi) \right| \leqq \frac{2p+1}{\sqrt{1+\omega^2-2\omega\xi}} \omega^p,$$

and the fact that the n^{th} derivative of u satisfies the Poisson equation with a source decaying as $r^{-q-n-\varepsilon}$. □

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