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Large Field Renormalization. I. The Basic Step of the $\mathbb{R}$ Operation

Tadeusz Balaban*
Department of Mathematics, Boston University, Boston, MA 02215, USA

Abstract. We construct the renormalization operation of the expressions connected with the large field regions. This operation, denoted by $\mathbb{R}$, removes the main obstacle to prove the ultraviolet stability of four-dimensional gauge field theories. The proof will be completed in the second part of this paper.

O. Introduction

Let us repeat briefly why it is necessary to renormalize the large field expressions, and what is a general structure of the operation $\mathbb{R}$. Consider a large plaquette variable in the first step. The restrictions on these variables are the same as in [16] (this refers to References in the paper [I]), so we have $|U(\partial p)-1|\geqq g_0 p_0(g_0)$ for a plaquette $p\in T_1$, where $p_0(g_0)=A_0(\log g_0^{-2})^{p_0}$ with a positive integer p_0. The term in the Wilson action, corresponding to the plaquette p, gives the estimate

$$\exp\left[-\frac{1}{g_0^2}[1-\operatorname{Retr}U(\partial p)]\right]\leqq \exp(-p_0(g_0))=g_0^{A_0(\log g_0^{-2})^{p_0-1}}. \qquad (0.1)$$

For $d<4$ we have $g_0=g\varepsilon^{1/2(4-d)}$, and the bound above can be estimated by an arbitrarily large power of ε. This is enough to control expressions arising in the large field regions surrounding the plaquette p for all steps of the procedure, i.e., until we reach the unit lattice. For $d=4$ the bare coupling constant behaves asymptotically as $(a+b\log\varepsilon^{-1})^{-1/2}$, for $\varepsilon\to 0$, with some positive constants a,b, hence the bound does not give any positive power of ε. It is still small for ε small, and it controls a large number of steps, but this number is a small fraction of the total number of steps. Thus, for some large field regions there is a difficulty in continuing the procedure of [16], the small factor arising from large fields in this region does not control further steps. In such situations we have to change the procedure in order to improve the small factor, i.e., we have to be able to renormalize the expression corresponding to the large field region. There are several possible

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ways of doing it, the one chosen in this paper is closest to the method of G. Gallavotti et al. in [19, 20], and can be described in the simplest way as follows: a large field expression is replaced by the corresponding small field expression in such a way, that integrals of the densities are unchanged. Let us elaborate this description. A density ρ after some number of steps is represented in the form

$$\rho(V) = \sum_Z \rho(Z, V), \tag{0.2}$$

where the sum is over large field regions Z, and V is a gauge field variable. A region Z is decomposed into disjoint subregions $Z', Z'', Z = Z' \cup Z''$, in the following way: Z'' is a union of components of the region Z, for which the small factors connected with large field control some number of next steps, Z' is a union of remaining components, i.e. components for which the corresponding expressions require a renormalization. For such a decomposition we take the density $\rho(Z'', V)$, and we define the operation $\mathbb{R}$ as follows:

$$(\mathbb{R}\rho)(V) = \sum_Z \rho(Z'', V) \frac{\int dV[_{Z'}\rho(Z, V)}{\int dV[_{Z'}\rho(Z'', V)}. \tag{0.3}$$

We will prove that the densities are positive, and the inegration domains in the integrals above are nonempty, hence the denominators are positive, and the operation $\mathbb{R}$ is well defined. It satisfies the basic normalization property

$$\int dV(\mathbf{R}\rho)(V) = \int d\, V\rho(V). \tag{0.4}$$

Consider now the expression on the right-hand side of the definition. It can be written as a double sum over domains Z', Z'', $Z' \subset Z''^c$, and the summation over Z' can be applied to the quotients. The quotients are still small, because some small factors in the regions Z' are left for the densities in the numerator. We localize them, trying to decouple components of Z', i.e., we write a polymer expansion, and then we exponentiate it. Thus, we obtain the representation

$$\sum_{Z' \subset Z''^c} \frac{\int dV[_{Z'}\rho(Z' \cup Z'', V)}{\int dV[_{Z'}\rho(Z'', V)} = \exp \sum_X \mathbb{R}(X, V), \tag{0.5}$$

where the last sum is over X such, that $X \cap Z''^c \neq \emptyset$. Using this representation, we rewrite the definition of the operation $\mathbb{R}$:

$$(\mathbb{R}\rho)(V) = \sum_{Z''} \rho(Z'', V) \exp \sum_X \mathbb{R}(X, V). \tag{0.6}$$

Now the advantages of applying such an operation are clear, the densities on the right-hand side still have enough small factors to control the given number of steps, and the expression in the exponential can be treated in the same way, as the small field effective actions are treated in [I], in particular it can be renormalized in the same way. This renormalization is the necessary renormalization of the expressions connected with the large field regions, and it makes the whole renormalization group procedure convergent, i.e., we can apply all the transformations needed to reach the unit lattice, and we control all the steps of the procedure.

The above description stresses only some general ideas underlying the method used in this paper. The actual procedure is more complicated, and it also differs from the one presented above in some technical aspects, for example in (0.3), (0.5) we take the denominators equal not the integrals of the whole densities $\rho(Z'', V)$, but to the integrals of some parts of these densities. More precisely, we take the parts localized in neighborhoods of the domains Z', so they do not depend on the large field regions Z'', and they are determined by small field effective actions only. These general ideas are very simple and natural. They have many possible variations, and they can be realized in many different ways. Let us mention, that similar ideas were expressed in private conversations by other people, in particular by G. Gallavotti and G. Benfatto, J. Imbrie and D. Brydges, J. Feldman and J. Magnen.

1. The Basic Step of the Operation $\mathbb{R}$

In this section we describe in detail the fundamental part of the $\mathbb{R}$-operation. For simplicity of notation we consider the k^{th} density, instead of $k+1^{\text{st}}$. Each term in the expansion (2.18) [III] has a large field region Λ_k^c. It is a union of connected components. We consider components of almost the minimal possible size. Each renormalization step adds at least ten layers of MR_k-cubes, hence the size must be greater than 20 MR_k. Passing to the next step it is usually rescaled by L^{-1}, but in some steps the number R_k decreases by the factor L^{-1}, and adding ten new layers of MR_k-cubes we get a region with a size greater than 40 MR_k. It is easy to see that the minimal size is approximately equal to $42(L/(L-1))MR_k$. We consider components of sizes smaller than, or equal to $100\,MR_k$. More precisely, we consider the class of components such that each satisfies the following two properties:

(i) it is contained in a cube of the size $100\,MR_k$,
(ii) in the preceding N renormalization steps no new large field regions were created inside this component, and the previous regions contained in it satisfy the condition (i) on the corresponding scales.

According to our rule of construction of the large field regions, for such a component all the regions connected with the last N steps are rectangular parallelepipeds. It will simplify some geometric considerations in the future. Conditions on N will be formulated in constructions of this section. Let us denote the union of the above class of components by Z. For simplicity we denote intersections of the regions Z_j with Z by Z_j also, hence Z_k is identified with Z. We write the intersections explicitly only if it may lead to a misunderstanding. Thus we write the factorization property (2.19) [III],

$$\mathbb{T}_k(Z_k) = \mathbb{T}_k(Z_k \cap Z^c)\mathbb{T}_k(Z) = \mathbb{T}_k(Z_k \cap Z^c) \prod_{i=1}^{m} \mathbb{T}_k(X_i), \tag{1.1}$$

where $Z = \bigcup_{i=1}^{m} X_i$ is the decomposition into disjoint components. In this section we do not make any changes in the operation $\mathbb{T}_k(Z_k \cap Z^c)$, therefore we will usually omit it in the formulas.

We consider $\mathbb{T}_k(Z)\exp A_k$, and we use the above described simplified notation. Using the conditions (i), (ii), and the factorization property (2.22) [III], we write

$$\mathbb{T}_k(Z)\exp A_k = \chi_k(\Omega_k^{\sim 4}) \prod_{j=k-1}^{h} \mathbb{T}^{(j)}(Z_{j+1})\chi_h(\Omega\backslash\Omega_{h+1}^{\sim})\mathbb{T}_h(Z_h)\exp A_k, \tag{1.2}$$

where $h = k - N$, and we have written explicitly the first and the last characteristic functions in the product of the last N one-step operations. These operations are given by the formula (2.21) [III], in which the functions ζ, χ have the simplest form, namely by the condition (ii) no large field characteristic functions are included in them. We write these functions now, because we have to compare them with other characteristic functions. Thus $\zeta(\Omega_{j+1}^c \cap Z_{j+1})\chi(\Omega_{j+1} \cap Z_{j+1})$ is equal to the product of the following seven groups of functions:

$$\chi\left(\left\{\sup_{p\subset\square^{\sim}} |U_{j,\square}(V_j, \partial p) - 1| < \varepsilon_j(L^{k-j}\eta)^2\right\}\right) \tag{1.3}$$

for $\square \subset (\Omega_j^{\sim 4}\backslash\Omega_{j+1}^{\sim}) \cap Z$,

$$\chi\left(\left\{\sup_{p\subset\square'^{\sim}} |U_{j+1,\square'}(V_{j+1}, \partial p) - 1| < \varepsilon_{j+1}(L^{k-j-1}\eta)^2\right\}\right) \tag{1.4}$$

for $\square' \subset (\Omega_{j+1}^{\sim 4}\backslash\Omega_{j+2}^{\sim}) \cap Z$,

$$\chi(\{|V_j(y,x) - 1| < \varepsilon_j\}) \tag{1.5}$$

for $y \in (\Omega_{j+1}^{\sim 3}\backslash\Omega_{j+1}^{\sim})^{(j+1)} \cap Z$, $x \in B(y)$, $x \neq y$,

$$\frac{1}{z}\exp\left[-\frac{1}{g_j^2}[1 - \mathrm{Re\,tr}\, V_j(y,x)]\right] \tag{1.6}$$

for $y \in (\Omega_{j+1}^{\sim 3}\backslash\Omega_{j+1})^{(j+1)} \cap Z$, $x \in B(y)$, $x \neq y$,

$$\chi\left(\left\{\sup_{b\in(\square'^{\sim 2})^{(k)*}} |V_j(b)(V_{\square'}^{(j)}(b))^{-1} - 1| < 2\delta_j\right\}\right) \tag{1.7}$$

for $\square' \subset (\Omega_{j+1}^{\sim 2}\backslash\Omega_{j+1}) \cap Z$,

$$\chi\left(\left\{\sup_{b\in(\square'^{\sim 2})^{(k)*}} |\exp ig_jA_j(b)V_{Z_{j+1}\backslash Z_j}^{(j)}(b)(V_{\square'}^{(j)}(b))^{-1} - 1| < 2\delta_j\right\}\right) \tag{1.8}$$

for $\square' \subset (\Omega_{j+1}\backslash\Lambda_{j+1}^{\sim}) \cap Z$,

$$\chi\left(\left\{\sup_{b\in(\square'^{\sim 2})^{(k)*}} |A_j(b)| < g_j^{-1}\delta_j\right\}\right) \tag{1.9}$$

for $\square' \subset (\Lambda_{j+1}^{\sim}\backslash\Lambda_{j+1}) \cap Z$.

The cubes $\square$ in (1.3) are the LM_2R_j-cubes of the partition of the lattice $T_{L^{-j}}$, or the $L^{-(k-j)}LM_2R_j$-cubes of the lattice T_η, and the cubes $\square'$ in (1.4), (1.7)–(1.9) are the LM_2R_{j+1}-cubes of the partition of the lattice $T_{L^{-(j+1)}}$.

As a first step of the $\mathbb{R}$-operation we try to do a part of the integrations in the operations $\mathbb{T}^{(j)}(Z_j)$ in (1.2), namely the integrations with respect to the variables localized in a neighborhood of the boundary ∂Z_j. There are two reasons for doing this. The first is that the functions (1.5)–(1.9) are not gauge invariant with respect to gauge transformations of the variables V_j, and we would like to choose a convenient gauge fixing for the integration with respect to V_j restricted to Γ_j in (1.2). The second, more important reason, is that we would like to get a new, much smaller large field region Z, in order to be able to fit the constructions of this section to the inductive assumptions on the effective action. To specify the integrations we define a new sequence of the large field regions. Let us recall that the domains $\Omega_j^{\sim n}$ are unions of $L^{-(k-j)}MR_j$-cubes of the lattice T_η. Take the smallest positive integer N_0 such, that $L^{-N_0+1}MR_{k-N_0+1}=M$. It is easy to see that either there is exactly one such integer, or there are two. We assume that $N>N_0$, in fact it will become clear later that N is much greater than N_0. Define

$$\begin{aligned} Z_k'' &= (Z_{k-N_0}')^{\sim 3} = (\Omega_{k-N_0+1}^{\sim 5})^c \cap Z, \\ Z_{k-N_0+1}'' &= (Z_{k-N_0}')^{\sim} = (\Omega_{k-N_0+1}^{\sim 7})^c \cap Z, \end{aligned} \tag{1.10}$$

and complete these two sets to a sequence $Z_k'', Z_{k-1}'', \dots, Z_{k-N_0+2}'', Z_{k-N_0+1}''$ in such a way that the complements of these sets form an admissible sequence of domains based on partitions into M-cubes in the corresponding scales. Thus Z_k'', $Z_k''\backslash Z_{k-1}''$ are unions of M-cubes of the lattice T_η, and $Z_k''^c$, Z_{k-1}'' are separated by one layer of M-cubes. Similarly, Z_{k-1}'', $Z_{k-1}''\backslash Z_{k-2}''$ are unions of $L^{-1}M$-cubes of this lattice, and $Z_{k-1}''^c$, Z_{k-2}'' are separated by one layer of $L^{-1}M$-cubes, and so on. Next, define

$$\begin{aligned} Z_j'' &= (\Omega_j^{\sim 5})^c \cap Z \quad \text{for} \quad j=k-N_0,\, k-N_0-1, \dots, k-N+1=h+1, \\ Z_j'' &= Z_j \quad \text{for} \quad j=h,\, h-1, \dots, 1. \end{aligned} \tag{1.11}$$

This is the new sequence of the large field regions. We define new domains Ω_j'' by

$$\begin{aligned} \Omega_j'' &= (Z_j''^c \cap Z) \cup (\Omega_j \cap Z^c) \quad \text{for} \quad j=k,\, k-1, \dots, h+1, \\ \Omega_j'' &= \Omega_j \quad \text{for} \quad j=h,\, h-1, \dots, 1. \end{aligned} \tag{1.12}$$

The sequence $\{\Omega_j''\}$ is an admissible sequence, and the corresponding generating set is denoted by $\mathbb{B}_k''$. Thus

$$\begin{aligned} &\mathbb{B}_k'' = \{\Gamma_j''\}, \quad \Gamma_j'' = (\Omega_j''\backslash\Omega_{j+1}'')^{(j)}, \quad j=k-1,\dots,1, \\ &\Gamma_k'' = (\Omega_k'')^{(k)}, \quad \Gamma_0'' = \Omega_1''^c. \end{aligned} \tag{1.13}$$

By the definition all components of the domains Z_j'' are also rectangular parallelepipeds for $j=h,\, h+1,\dots,k$.

We would like to do the integrations in $\mathbb{T}^{(j)}(Z_{j+1})$ with respect to the variables localized in $Z_{j+1}\backslash Z_{j+1}''$, $j=h,\dots,k-1$. It is most convenient to do the integrations successively in this order, starting with $j=h$, because then we have almost the same situation as in the one-step renormalization transformations, and we can use the results of the third section. In fact, the situation is simpler than there, and we have to make only few comments.

Each integration gives a new background field, connected with some determining sets. We describe now this sequence of determining sets. We start with $\mathbb{B}_k$, which we write as

$$\mathbb{B}_k^{(0)} = \mathbb{B}_k = (\mathbb{B}_k \cap (\Omega_k \cup Z^c)) \cup \{\Gamma_{k-1},\dots,\Gamma_{h+1},\Gamma_k\} \cup (\mathbb{B}_k \cap \Omega_h^c \cap Z). \tag{1.14}$$

Here, and in the subsequent formulas, the symbols Γ_j mean the intersections of these sets with the region Z. The first integration is with respect to the variables localized in $Z_{h+1}\backslash Z''_{h+1}$. The relevant part of the integration is with respect to the variables V_h on $(\Omega^c_{h+1}\backslash Z''_{h+1})\cap Z$, restricted by the conditions $\bar{V}_h = V_{h+1}$. In effect, the set Γ_h is resplaced by the two sets $(\Gamma_h\cap Z''^c_{h+1})^{(1)}$, $\Gamma_h\cap Z''_{h+1}$. The first set is combined with Γ_{h+1}, and the union can be written as $(\Omega^c_{h+2}\backslash Z''_{h+1})^{(h+1)}$. Hence the next determining set is defined by

$$\begin{aligned}\mathbb{B}^{(1)}_k = (\mathbb{B}_k\cap(\Omega_k\cup Z^c))\cup\{\Gamma_{k-1},\dots,\Gamma_{h+2},(\Omega^c_{h+2}\backslash Z''_{h+1})^{(h+1)},\Gamma_h\cap Z''_{h+1}\}\\ \cup(\mathbb{B}_k\cap\Omega^c_h\cap Z),\end{aligned}\tag{1.15}$$

In the second integration we integrate with respect to the variables V_{h+1} on $(\Omega^c_{h+2}\backslash Z''_{h+2})\cap Z$. In effect, the set $(\Omega^c_{h+2}\backslash Z''_{h+1})^{(h+1)}$ is replaced by the two sets $(\Omega^c_{h+2}\backslash Z''_{h+2})^{(h+2)}$, $(Z''_{h+2}\backslash Z''_{h+1})^{(h+1)} = \Gamma''_{h+1}$. The first set is combined with Γ_{h+2}, and the union can be written as $(\Omega^c_{h+3}\backslash Z''_{h+2})^{(h+2)}$. This determines the set $\mathbb{B}^{(2)}_k$. In general, the set $\mathbb{B}^{(j-h)}_k$ is defined by

$$\begin{aligned}\mathbb{B}^{(j-h)}_k = (\mathbb{B}_k\cap(\Omega_k\cup Z^c))\cup\{\Gamma_{k-1},\dots,\Gamma_{j+1},(\Omega^c_{j+1}\backslash Z''_j)^{(j)},\Gamma''_{j-1},\dots,\Gamma''_{h+1},\Gamma_h\cap Z''_{h+1}\}\\ \cup(\mathbb{B}_k\cap\Omega^c_h\cap Z),\end{aligned}\tag{1.16}$$

for $j = h+1,\dots,k-1$. For $j = k$ we replace the sequence in the curly bracket above by $(\Omega^c_k\backslash Z''_k)^{(k)}$, $\Gamma''_{k-1},\dots,\Gamma''_{h+1}$, $\Gamma_h\cap Z''_{h+1}$. The first set in this sequence is equal to $\Omega^c_k\cap\Gamma''_k$, and we combine it with the set Γ_k in $\mathbb{B}_k\cap(\Omega_k\cup Z^c)$. The union is equal to Γ''_k, hence

$$\mathbb{B}^{(k-h)}_k = \mathbb{B}^{(N)}_k = \mathbb{B}''_k.\tag{1.17}$$

Take any of the determining sets defined above. The gauge field V_j is defined on the whole j^{th} component of the set, therefore the collection of the restrictions of these fields defines a gauge field on the determining set, analogously to the field V defined on $\mathbb{B}_k$. We denote these fields by V also, in fact they are defined by the same equality in (2.10) [III]. The determining set $\mathbb{B}^{(n)}_k$ and the gauge field V define the function

$$U^{(n)}_k = U^{(n)}_k(V) = U(\mathbb{B}^{(n)}_k, V).\tag{1.18}$$

These configurations are background fields in the successive integrations.

We need also a sequence of determining sets, and the corresponding sequence of functions, localized in the region Z. The determining sets $\mathbb{B}^{(n)}_k(Z)$ are defined as above, by the equalities (1.14)–(1.17), but the set $\mathbb{B}_k\cap(\Omega_k\cup Z^c)$ is replaced by $\mathbb{B}_k(Z)\cap\Omega_k$. Thus, according to the definition (2.14) [III], they may be defined as

$$\mathbb{B}^{(n)}_k(Z) = \mathbb{B}^{(n)}_k\cup\mathbb{B}_k(Z).\tag{1.19}$$

The corresponding functions are defined by the gauge fields V restricted to Ω^c_k, and by $M_{B_k(Z)}(Q^{s*}_k V_k)$ restricted to $Z\cap\Omega_k$, thus

$$U^{(n)}_{k,Z} = U^{(n)}_{k,Z}(V) = U(\mathbb{B}^{(n)}_k(Z),\quad (M^{\cdot}(Q^{s*}_k V_k)\restriction_{Z\cap\Omega_k}, V\restriction_{Z\cap\Omega^c_k})).\tag{1.20}$$

These functions play an important role in subsequent definitions of characteristic functions, and in other constructions. For the last functions in the sequences

(1.18)–(1.20) we introduce the following notations

$$U_k^{(N)} = U_k'', \quad U_{k,Z}^{(N)} = U_{k,Z}''. \tag{1.21}$$

We do the first integration with respect to the variables A_h, V_h restricted to $Z_{h+1} \backslash Z_{h+1}''$. We follow the procedure of Sect. 3 [III], so we introduce at first new characteristic functions in such a way that the function restricting the fluctuation field does not depend on the background field. We start with the decomposition of unity $1 = \chi_k^{(0)} + (1 - \chi_k^{(0)})$, where

$$\chi_k^{(0)} = \chi\left(\left\{ |U_{k,Z}^{(0)}(\partial p) - 1| < (1 - \beta\tfrac{1}{2})\varepsilon_h(L^{k-h}\eta)^2 \text{ for } p \in \Omega_h \backslash \Omega_{h+1} \right\}\right). \tag{1.22}$$

More precisely, we introduce this decomposition in each component of Z separately. In components with the function $1 - \chi_k^{(0)}$ we have introduced new large fields, therefore they do not satisfy the condition (ii), and we exclude them from the region Z. They are not changed by the subsequent operations, and we treat them in the same way as the remaining large field regions. We denote the union of components with functions $\chi_k^{(0)}$ by Z again. The restrictions in the function (1.22) imply that the characteristic functions (1.3) for $j = h$ and $\square \subset (\Omega_{h+1}^{\sim})^c \backslash Z_h$ are equal to 1. Thus the function $\chi_h(\Omega_h \backslash \Omega_{h+1}^{\sim})$ in (1.2) is replaced by $\chi_k^{(0)}\chi_h(Z_h \cap \Omega_h)$. We introduce the next decomposition of unity $1 = \chi_k^{(1)} + (1 - \chi_k^{(1)})$ in components of Z, where

$$\chi_k^{(1)} = \chi\left(\left\{ |U_{k,Z}^{(1)}(\partial p) - 1| < \left(1 - \beta\left(\frac{1}{2} + \frac{1}{2^2}\right)\right)\varepsilon_h(L^{k-h}\eta)^2 \text{ for } p \in Z_{h+1}'' \cap \Omega_h, \right.\right.$$
$$\left.\left. \cdot |U_{k,Z}^{(1)}(\partial p) - 1| < (1 - \beta\tfrac{1}{2})\varepsilon_{h+1}(L^{k-h-1}\eta)^2 \text{ for } p \in \Omega_{h+2}^c \backslash Z_{h+1}'' \right\}\right). \tag{1.23}$$

We redefine Z again removing these components, for which the functions $1 - \chi_k^{(1)}$ were introduced. For components of Z the restrictions in (1.23) imply that all the functions (1.4) for $j = h$ are equal to 1. To remove $\chi_k^{(0)}$, and the other characteristic functions in (1.5)–(1.8), we have to introduce restrictions on the fluctuation field. This, and the other operations connected with the integration, will be discussed in a general case.

We assume, that after n integrations the only characteristic functions which remain, and which are connected with these integrations, are $\chi_k^{(n)}\chi_h(Z_h \cap \Omega_h)$. The definition of $\chi_k^{(n)}$ is rather complicated, and it has a different form for $j = h + n \leqq k_0 = k - N_0$, and for $j > k_0$. In the first case the integration regions are disjoint, and the functions are defined by generalizations of (1.22) and (1.23). In the second case some integration regions overlap, more exactly the regions with indices greater than k_0, and for these we change the regularity conditions by a factor, which is a power of some number, the power being proportional to a number of overlapping regions. We write the definition in the second case,

$$\chi_k^{(n)} = \chi\left(\left\{ |U_{k,Z}^{(n)}(\partial p) - 1| < (1 - \beta(1 - 2^{-(j-h+1)}))\varepsilon_h(L^{k-h}\eta)^2 \text{ for } p \in Z_{h+1}'' \cap \Omega_h, \right.\right.$$
$$\cdot |U_{k,Z}^{(n)}(\partial p) - 1| < (1 - \beta(1 - 2^{-(j-h)}))\varepsilon_{h+1}(L^{k-h-1}\eta)^2 \text{ for } p \in Z_{h+2}'' \backslash Z_{h+1}'',$$

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$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta(1-2^{-(j-k_0+1)}))\varepsilon_{k_0}(L^{k-k_0}\eta)^2 \text{ for } p\in Z''_{k_0+1}\backslash Z''_{k_0},$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta(1-2^{-(j-k_0)}))\varepsilon_{k_0+1}(L^{k-k_0-1}\eta)^2 \text{ for } p\in Z''_{k_0+2}\backslash Z''_{k_0+1},$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta(1-2^{-(j-k_0-1)}))L_0^2\varepsilon_{k_0+2}(L^{k-k_0-2}\eta)^2 \text{ for } p\in Z''_{k_0+3}\backslash Z''_{k_0+2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta(1-2^{-2}))L_0^{2(j-k_0-2)}\varepsilon_{j-1}(L^{k-j+1}\eta)^2 \text{ for } p\in Z''_j\backslash Z''_{j-1},$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta\tfrac{1}{2})L_0^{2(j-k_0-1)}\varepsilon_j(L^{k-j}\eta)^2 \text{ for } p\in \Omega^c_{k_0+1}\backslash Z''_j,$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta\tfrac{1}{2})L_0^{2(j-k_0-2)}\varepsilon_j(L^{k-j}\eta)^2 \text{ for } p\in \Omega_{k_0+1}\backslash \Omega_{k_0+2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdot|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta\tfrac{1}{2})L_0^2\varepsilon_j(L^{k-j}\eta)^2 \text{ for } p\in \Omega_{j-2}\backslash \Omega_{j-1},$$
$$\left.\left.\cdot|U^{(n)}_{k,Z}(\partial p)-1|<c(1-\beta\tfrac{1}{2})\varepsilon_j(L^{k-j}\eta)^2 \text{ for } p\in \Omega_{j-1}\backslash \Omega_{j+1}\right\}\right), \tag{1.24}$$

where $c=1$ for $j<k$, and $c=3$ for $j=k$. We choose the number β satisfying $0<\beta\leqq 1/2$, but not too small, e.g., we can take $\beta=1/2$. The number L_0 satisfies $2\leqq L_0<1/2L$, e.g., we can take $L_0=(1/2)(L-1)$. The definition (1.24) is so complicated because of the needs of this inductive construction, but basic properties of $\chi^{(n)}_k$ are simple. They will be described for $n=N$, i.e., for $j=k$.

We assume further that the effective action $A^{(n)}_k$, obtained after the n integrations, depends on the background field $U^{(n)}_k$, and has the form (2.23) [III], with some new boundary terms only, i.e., some new terms in $\mathbb{B}_k$. Now we will analyze the next, $n+1^{\text{st}}$ integration. We integrate with respect to the variables A_j, V_j localized in $Z_{j+1}\backslash Z''_{j+1}$. It is a part of the operation $\mathbb{T}^{(j)}(Z_{j+1})$, and we have to consider the following integral

$$\begin{aligned}&\int dA_j\Gamma_{Z_{j+1}\cap\Omega_{j+1}}\chi(Z_{j+1}\cap\Omega_{j+1})\exp[-\tfrac{1}{2}\langle A_j, C^*\Delta^{(j)}_{A^c_{j+1}}CA_j\rangle]\\ &\cdot\int dV_j\Gamma_{\Omega^c_{j+1}\backslash Z''_{j+1}}\delta(\bar{V}_jV^{-1}_{j+1})\zeta(\Omega^c_{j+1})\chi^{(n)}_k\exp A^{(n)}_k,\end{aligned} \tag{1.25}$$

where $n=j-h$, and the quadratic form in the exponential is written explicitly in (2.21) [III]. In this integral we introduce the decomposition of unity $1=\chi^{(n+1)}_k+(1-\chi^{(n+1)}_k)$ for each component of Z, and we exclude from Z the components with the large field functions $1-\chi^{(n+1)}_k$. The function $\chi^{(n+1)}_k$ does not depend on integration variables, and we have to consider the integral (1.25) multiplied by $\chi^{(n+1)}_k$. In this integral the gauge fixing terms (1.6) were introduced on the domain $\Omega^{\sim 3}_{j+1}\backslash\Omega_{j+1}$, and they constitute a part of the function $\zeta(\Omega^c_{j+1})$. All the expressions and the functions in (1.25) are invariant with respect to gauge transformations v defined on the set $((\Omega^{\sim 3}_{j+1})^c\backslash Z''_{j+1})\cap T^{(j)}$, and equal to 1 on the intersection of this set with $T^{(j+1)}$. Using the Faddeev–Popov procedure we introduce the gauge fixing terms (1.5), (1.6) on this set, thus we get the gauge fixing terms on the whole integration region $(\Omega^c_{j+1}\backslash Z''_{j+1})\cap T^{(j)}$. Next, we introduce restrictions on an approximate fluctuation field. Define the configuration

$$V^{(j)}_Z=M^j(U^{(j+1-h)}_{k,Z}), \tag{1.26}$$

and the characteristic function

$$\chi'_j=\chi(\{|V_j(b)(V^{(j)}_Z(b))^{-1}-1|<2\delta'_j \text{ for } b\in(\Omega^c_{j+1}\backslash Z''_{j+1})^{(j)*}\}), \tag{1.27}$$

where $\delta'_j = g_j A_1 p_1(g_j)$, $p_1(g_j) = (\log g_j^{-2})^{p_1}$, and $p_1 < p_0$; other conditions on p_1 will be formulated later. Introduce the decomposition of unity $1 = \chi'_j + (1 - \chi'_j)$, for each component of Z, under the integral (1.25), and exclude from Z the components with the large field functions $1 - \chi'_j$. In the remaining region, which we denote again by Z, we obtain the integral (1.25) multiplied by $\chi_k^{(n+1)}$, and with the function χ'_j inside. Finally, let us recall that the functions $\chi_{\square'}^{(j)}$, given by (1.9), are present in $\chi(Z_{j+1} \cap \Omega_{j+1})$ for $\square' \subset Z_{j+1} \backslash Z_{j+1}^{\sim -1}$. We complete them to the whole domain $Z_{j+1} \cap \Omega_{j+1}$, introducing the decomposition of unity

$$1 = \prod_{\square' \subset \Omega_{j+1} \backslash \Omega_{j+1}^{\sim -1}} \chi_{\square'}^{(j)} + \left(1 - \prod_{\square' \subset \Omega_{j+1} \backslash \Omega_{j+1}^{\sim -1}} \chi_{\square'}^{(j)}\right) \tag{1.28}$$

for each component of Z, and excluding from Z the components with the large field functions. Let us notice that all the characteristic functions introduced above depend on the field variables localized in the corresponding components of the large field region Z_k. This is an important part of the inductive assumption for the effective density, more precisely for the operation $\mathbb{T}_k(Z_k)$. It is also clear how this operation has been changed by the above decompositions, but we will not attempt to give a systematic description of all possible cases; there are too many of them, and we need only general properties of the operation.

Now we will prove that the restrictions introduced by the new characteristic functions imply that the functions (1.3), (1.4), (1.5), (1.7), (1.8), $\chi_k^{(n)}$ are equal to 1. More precisely, we have

$$\begin{aligned}
&\chi_k^{(n+1)} \chi(Z_{j+1} \cap \Omega_{j+1}) \left(\prod_{\square' \subset \Omega_{j+1} \backslash \Omega_{j+1}^{\sim -1}} \chi_{\square'}^{(j)} \right) \zeta(\Omega_{j+1}^c) \chi'_j \chi_k^{(n)} \\
&\cdot \left(\prod_{y \in ((\Omega_{j+1}^{\sim 3})^c \backslash Z''_{j+1})^{(j+1)}} \prod_{x \in B(y), x \neq y} \chi(\{ | V_j(x,y) - 1 | < \varepsilon_j \}) \frac{1}{z} \right. \\
&\qquad \left. \cdot \exp\left[-\frac{1}{g_j^2} [1 - \operatorname{Re} \operatorname{tr} V_j(x,y)] \right] \right) = \chi_k^{(n+1)} \left(\prod_{\square' \subset Z_{j+1} \cap \Omega_{j+1}} \chi_{\square'}^{(j)} \right) \chi'_j \\
&\cdot \left(\prod_{y \in (\Omega_{j+1}^c \backslash Z''_{j+1})^{(j+1)}} \prod_{x \in B(y), x \neq y} \frac{1}{z} \exp\left[-\frac{1}{g_j^2} [1 - \operatorname{Re} \operatorname{tr} V_j(x,y)] \right] \right)
\end{aligned} \tag{1.29}$$

for $j+1 < k$, and for $j+1 = k$ the right-hand side above is multiplied by $\chi_k(\Omega_k^{\sim 4})$.

Let us start with a function (1.3). The cube $\square$ is contained in $\Omega_j^{\sim 4} \backslash \Omega_{j+1}^{\sim}$, hence $\square^{\sim 4} \subset \Omega_{j-1} \backslash \Omega_{j+1}$. On this domain the configuration $U_k^{(n)}$ satisfies the last inequality in (1.24) (with $c = 1$), and the constraints $M^j(U_k^{(n)}) = V_j$. The same constraints are satisfied by $U_{j,\square}$ on $\square^{\sim 3}$. Reasoning as in the part of Sect. 3 [III] between the formulas (3.6), (3.8), we get

$$U_{j,\square} = \left(\exp i\xi \mathbb{H}_{j,\square} \left(-\frac{1}{i} \log \left[M^j(U_k^{(n)}) (M^j(Q_j^{s*} V_j))^{-1} \right] \right) U_k^{(n)} \right)^{u_{j,\square}^{-1}} . \tag{1.30}$$

The field in the argument of the function $\mathbb{H}_{j,\square}$ is equal to 0 on almost the whole cube $\square^{\sim 4}$, except a boundary layer of the width $2M_1$ in the lattice T_ξ, $\xi = L^{-j}$. On this boundary the field can be bounded by $22 d^2 \varepsilon_j$, by the argument leading to the estimate (1.65) in [14]. The exponential decay property of $\mathbb{H}_{j,\square}$ implies that on the cube $\square^{\sim}$ this function and its covariant derivatives can be bounded by $B_3\exp(-\delta 2M_2R_j)22d^2\varepsilon_j < (\beta/10)\varepsilon_j$. Estimating $\partial U_{j,\square}$ as in (3.8) [III] we obtain

$$|U_{j,\square}(\partial p)-1| < \left(1+\frac{2\beta}{10}\varepsilon_j\xi\right)\left(1-\beta\frac{1}{2}\right)\varepsilon_j\xi^2+\frac{2\beta}{10}\varepsilon_j\xi^2\left(1+\frac{4\beta}{10}\varepsilon_j\right)$$

$$< \left(1-\frac{\beta}{2}+\frac{2\beta}{10}L^{-2}+\frac{4\beta}{10}\right)\varepsilon_j\xi^2 < \varepsilon_j(L^{k-j}\eta)^2 \quad \text{for} \quad p\subset\square^{\sim}. \qquad (1.31)$$

Thus the functions (1.3) in the product on the left-hand side of (1.29) are equal to 1. The same holds for the functions (1.4), because the argument above applies to an arbitrary j, hence for $j+1$, except when $j+1=k$. In this case we leave the functions (1.4) as $\chi_k(\Omega_k^{\sim 4})$ on the right-hand side of (1.29). For the functions (1.5) we use the inequality (3.9) [III], only with k replaced by j, and the configuration $V_{\square'}^{(k)}$ replaced by $V_Z^{(j)}$. From the restrictions (1.27) we get the bound $O(1)\delta'_j<\varepsilon_j$, hence the functions (1.5) are equal to 1 also.

Now we consider the functions (1.7), (1.8). We start with the second, more difficult case. At first, notice that the restrictions introduced by the functions $\chi_{\square'}^{(j)}$ in (1.29) imply the bound

$$|\exp ig_jA_j(b)V_{Z_{j+1}\setminus Z_j}^{(j)}(b)(V_{\square'}^{(j)}(b))^{-1}-1| < \delta_j+|V_{Z_{j+1}\setminus Z_j}^{(j)}(b)-V_{\square'}^{(j)}(b)|. \qquad (1.32)$$

The configuration $V_{Z_{j+1}\setminus Z_j}^{(j)}$ is defined on $Z_{j+1}^{\sim -1}\setminus Z_j^{\sim}$ by

$$V_{Z_{j+1}\setminus Z_j}^{(j)} = M^j(U(\mathbb{B}(Z_{j+1}\setminus Z_j)\cup\mathbb{B}_k, M^{\cdot}(Q^{s*}V)), \qquad (1.33)$$

where the determining set was defined in (2.14) [III]. In fact in the above case it is simply given by $\mathbb{B}_{j+1}(Z_{j+1})\lceil_{\Omega_{j+1}}\cup\mathbb{B}_j(Z_j^c)\lceil_{\Omega_{j+1}^c}$, and the configuration $Q^{s*}V$ is equal to $Q_{j+1}^{s*}V_{j+1}$ on $\Omega_{j+1}\cap Z_{j+1}$, and to $Q_j^{s*}V_j$ on $\Omega_{j+1}^c\cap Z_j^c$. Now we represent the configuration $U(\cdot)$ in the standard way, as in (1.30)

$$\begin{aligned}
&U(\mathbb{B}(Z_{j+1}\setminus Z_j)\cup\mathbb{B}_k, M^{\cdot}(Q^{s*}V))\\
&\quad = U(\mathbb{B}(Z_{j+1}\setminus Z_j)\cup\mathbb{B}_k, [M^{\cdot}(Q^{s*}V)(M^{\cdot}(U_k^{(n+1)}))^{-1}]M^{\cdot}(U_k^{(n+1)}))\\
&\quad = \left(\exp i\xi\mathbb{H}\left(\mathbb{B}(Z_{j+1}\setminus Z_j)\cup\mathbb{B}_k, -\frac{1}{i}\log[M^{\cdot}(U_k^{(n+1)})(M^{\cdot}(Q^{s*}V))^{-1}]\right)U_k^{(n+1)}\right)^{u_j^{-1}}.
\end{aligned} \qquad (1.34)$$

This implies

$$V_{Z_{j+1}\setminus Z_j}^{(j)} = (\tilde{M}^j(\exp i\xi\mathbb{H})M^j(U_k^{(n+1)}))^{u_j^{-1}} = \exp i\mathbb{H}^{(j)}V_Z^{(j)}, \qquad (1.35)$$

where

$$\exp i\mathbb{H}^{(j)}(b) = u_j^{-1}(b_-)\exp i\tilde{Q}_j(\xi\mathbb{H}, b)\bar{R}^j(u_j(b_+)). \qquad (1.36)$$

From the equalities and bounds (106)–(108), (159)–(163) in [12] we obtain

$$|\exp i\mathbb{H}^{(j)}(b)-1| \leq O(1)L^{\cdot} \sup_{B^j(b_-)\cup B^j(b_+)} |\mathbb{H}|. \qquad (1.37)$$

Consider now the field in the argument of the function $\mathbb{H}$. On the domain $\Omega_{j+1} \cap Z_{j+1}$, except a boundary layer of the width $2LM_1$ at the boundary ∂Z_{j+1}, this field is equal to 0. On the boundary layer it can be bounded by $22d^2\varepsilon_{j+1}$. Similarly, on the domain $\Omega_j^c \cap Z_j^c$, except a boundary layer of the width $2M_1$ at the boundary ∂Z_j^c, the field is equal to $(1/i)\log[V_j(V_Z^{(j)})^{-1}]$, hence it can be bounded by $4\delta_j'$, as it follows from the restrictions in (1.27). On the boundary layer it can be bounded by $22d^2\varepsilon_j$. From this, and the exponential decay property, it follows that

$$\begin{aligned} |\mathbb{H}| &< B_3(4\delta_j' + \exp(-\delta MR_{j+1})22d^2\varepsilon_{j+1} + \exp(-\delta 6LMR_{j+1})22d^2\varepsilon_j) \\ &\leqq B_3(4\delta_j' + \exp(-\delta MR_{j+1})44d^2(1+\beta_0)\varepsilon_j) \\ &= B_3\left(4\frac{p_1(g_j)}{p_0(g_j)} + \exp(-\delta MR_{j+1})44d^2(1+\beta_0)\frac{A_0}{A_1}\right)\delta_j \end{aligned} \tag{1.38}$$

on $\square'^{\sim 2}$, for $\square' \subset \Omega_{j+1}^{\sim} \backslash \Lambda_{j+1}^{\sim}$. The quotient on the right-hand side of the last equality is a negative power of $\log g_j^{-2}$, hence the coefficient at δ_j can be arbitrarily small, also after multiplying it by $O(1)L$ from (1.37), if γ is sufficiently small. We assume that it is so small that the expression on the left-hand side of (1.37) can be bounded by $1/2\delta_j$. From this we obtain

$$|V^{(j)}_{Z_{j+1}\backslash Z_j}(b) - V^{(j)}_{\square'}(b)| < \tfrac{1}{2}\delta_j + |V^{(j)}_{\square'}(b) - V^{(j)}_Z(b)|. \tag{1.39}$$

The configuration $V^{(j)}_{\square'}$ is equal to $M^j(U_{j+1,\square'})$. For $U_{j+1,\square'}$ we have the representation (1.30), with j, n, $\square$, and ξ replaced by $j+1$, $n+1$, $\square'$, and $L^{-1}\xi$. This representation implies

$$V^{(j)}_{\square'} = (\tilde{M}^j(\exp iL^{-1}\xi\mathbb{H}_{j+1,\square'})M^j(U_k^{(n+1)}))^{u^{-1}_{j+1,\square'}} = \exp i\mathbb{H}^{(j)}_{\square'} V^{(j)}_Z, \tag{1.40}$$

where

$$\exp i\mathbb{H}^{(j)}_{\square'}(b) = u^{-1}_{j+1,\square'}(b_-)\exp i\tilde{Q}_j(L^{-1}\xi\mathbb{H}_{j+1,\square'}, b)\bar{R}^j(u_{j+1,\square'}(b_+)). \tag{1.41}$$

For the configuration (1.41) we have again the bound (1.37), with $\mathbb{H}$ replaced by $\mathbb{H}_{j+1,\square'}$ on the right-hand side, and without the factor L. The function $\mathbb{H}_{j+1,\square'}$ is bounded on $\square'^{\sim 2}$ by $B_3\exp(-\delta LM_2R_{j+1})22d^2\varepsilon_{j+1}$, hence

$$\begin{aligned} |V^{(j)}_{\square'}(b) - V^{(j)}_Z(b)| &= |\exp i\mathbb{H}^{(j)}_{\square'}(b) - 1| \\ &< O(1)B_3\exp(-\delta LM_2R_{j+1})22d^2(1+\beta_0)\frac{A_0}{A_1}\delta_j \leqq \tfrac{1}{2}\delta_j \end{aligned} \tag{1.42}$$

on $\square'^{\sim 2}$. The inequalities (1.32), (1.39), and (1.42) imply that the functions (1.8) in the product in (1.29) are equal to 1. For the functions (1.7) the argument is simpler. We have

$$\begin{aligned} |V_j(b)(V^{(j)}_{\square'}(b))^{-1} - 1| &\leqq |V_j(b)(V^{(j)}_Z(b))^{-1} - 1| + |V^{(j)}_{\square'}(b) - V^{(j)}_Z(b)| \\ &< 2\delta_j' + \tfrac{1}{2}\delta_j < 2\delta_j \quad \text{on} \quad \square'^{\sim 2}, \end{aligned} \tag{1.43}$$

for $\square' \subset \Omega_{j+1}^{\sim 2} \backslash \Omega_{j+1}$. Here we have used the restrictions from (1.27), and the estimate (1.42). The above inequality implies that the functions (1.7) are equal to 1 also.

It remains to be proven that the function $\chi_k^{(n)}$ in (1.29) is equal to 1. Consider the configuration $U^{(n)}_{k,Z}$. Representing the field V_j on $\Omega^c_{j+1}\backslash Z''_{j+1}$ as $[V_j(V^{(j)}_Z)^{-1}]V^{(j)}_Z$, we have

$$U^{(n)}_{k,Z}=\left(\exp i\eta\mathbb{H}^{(n)}_{k,Z}\left(\frac{1}{i}\log[V_j(V^{(j)}_Z)^{-1}]\right)U^{(n+1)}_{k,Z}\right)^{u^{-1}}. \tag{1.44}$$

The field in the argument of the function $\mathbb{H}^{(n)}_k$ is bounded by $4\delta'_j$, and is localized in the domain $\Omega^c_{j+1}\backslash Z''_{j+1}$, therefore the following estimate holds:

$$\sup_{B^i(y)} L^i\eta|\mathbb{H}^{(n)}_{k,Z}|,\ \sup_{B^i(y)}(L^i\eta)^2|\nabla^\eta_{U^{(n+1)}_k}\mathbb{H}^{(n)}_{k,Z}|\leqq B_3\exp(-\delta d(y,\Omega^c_{j+1}\backslash Z''_{j+1}))4\delta'_j. \tag{1.45}$$

Here $d(\cdot,\cdot)$ is the scaled distance, y is a point of the i^{th} component of the determining set $\mathbb{B}^{(n)}_k$. Using the representation (1.44), and the above estimate, we obtain

$$\begin{aligned}|U^{(n)}_{k,Z}(\partial p)-1|&\leqq|U^{(n+1)}_{k,Z}(\partial p)-1|(1+\tfrac{1}{2}\eta|\mathbb{H}^{(n)}_{k,Z}|(\partial p))\\&\quad+\eta^2|(D^\eta_{U^{(n+1)}_{k,Z}}\mathbb{H}^{(n)}_{k,Z})(p)|+\tfrac{1}{2}\eta^2(|H^{(n)}_{k,Z}|(\partial p))^2\\&\leqq|U^{(n+1)}_{k,Z}(\partial p)-1|(1+8B_3L^{-i}\exp(-\delta d(y,\Omega^c_{j+1}\backslash Z''_{j+1}))\delta'_j)\\&\quad+9B_3\frac{A_1}{A_0}\frac{p_1(g_j)}{p_0(g_j)}(1+\beta_0)(1+(j-i)^{1/2})\\&\quad\cdot\exp(-\delta d(y,\Omega^c_{j+1}\backslash Z''_{j+1}))\varepsilon_i(L^{k-i}\eta)^2\end{aligned}$$

for $p\in B^i(y)$, where $y\in(Z''_{i+1}\backslash Z''_i)^{(i)}=\Gamma''_i$ for

$$i=j-1,j-2,\dots,\quad\text{and}\quad y\in(\Omega^c_{j+1}\backslash Z''_j)^{(j)}\quad\text{for}\quad i=j. \tag{1.46}$$

Let us remark that the scaled distance in the above estimates is defined in terms of M_1-cubes on corresponding scales. Thus, by the definition of the domains Z''_i, we obtain

$$\exp(-\delta d(y,\Omega^c_{j+1}\backslash Z''_{j+1}))\leqq\exp\left(-\delta\frac{M}{M_1}(j-i)\right)\quad\text{for}\quad y\in\Gamma''_i,\quad i<j. \tag{1.47}$$

We choose M large enough, so that $\delta(M/M_1)\geqq 2$. Then the product of $1+(j-i)^{1/2}$ and the exponential factor in (1.47) can be estimated by $\exp(-(j-i))$. The remaining numerical factors in front of the exponential can be estimated by $9B_3(A_1/A_0)(p_1(\gamma)/p_0(\gamma))(1+\beta_0)$, and can be made arbitrarily small by choosing γ, or A_1/A_0 sufficiently small. Notice, that the conditions on M, γ, A_0, and A_1 introduced above do not depend on any scale, i.e., on i,j,k, and so on. This applies also to other conditions introduced before. The numerical factor $8B_3L^{-i}\delta'_j<8B_3\gamma A_1p_1(\gamma)$ can be also chosen arbitrarily small for γ sufficiently small. Let us choose a bound for these two factors in the form $\alpha\beta$, where an absolute constant α will be chosen later. Then the estimate (1.46) and the above bounds imply

$$\begin{aligned}|U^{(n)}_{k,Z}(\partial p)-1|&\leqq|U^{(n+1)}_{k,Z}(\partial p)-1|(1+\alpha\beta 2^{-(j-i)})\\&\quad+\alpha\beta 2^{-(j-i)}\varepsilon_i(L^{k-i}\eta)^2\quad\text{for}\quad p\in B^i(y),\end{aligned} \tag{1.48}$$

where y is as in (1.46). Now consider the conditions in the definition (1.24) of $\chi^{(n)}_k$. The condition on the domain $Z''_{i+1}\backslash Z''_i$, for $i=h,h+1,\dots,j-1$, can be written in the form

$$|U^{(n)}_{k,Z}(\partial p)-1|<(1-\beta(1-2^{-(j-i+1)}))L_0^{2\max\{0,i-k_0-1\}}\varepsilon_i(L^{k-i}\eta)^2. \tag{1.49}$$

The corresponding condition for the configuration $U_{k,Z}^{(n+1)}$ is of the same form, only j is replaced by $j+1$. Introducing this bound into (1.48) we obtain

$$\begin{aligned} |U_{k,Z}^{(n)}(\partial p)-1| &< (1-\beta(1-2^{-(j-i+2)}))L_0^{2\max\{0,i-k_0-1\}}\varepsilon_i(L^{k-i}\eta)^2 \\ &\quad \cdot(1+\alpha\beta 2^{-(j-i)})+\alpha\beta 2^{-(j-i)}\varepsilon_i(L^{k-i}\eta)^2 \\ &< (1-\beta(1-2^{-(j-i+1)})-\beta 2^{-(j-i+2)} \\ &\quad +2\alpha\beta 2^{-(j-i)})L_0^{2\max\{0,i-k_0-1\}}\varepsilon_i(L^{k-i}\eta)^2, \end{aligned} \tag{1.50}$$

and the last inequality implies (1.49) if $8\alpha \leqq 1$. On the domain $\Omega_l \backslash \Omega_{l+1}$, for $l=k_0+1,\dots,j$, and on the domain $\Omega_{k_0+1}^c \backslash Z_j''$ for $l=k_0$, the condition on the configuration $U_{k,Z}^{(n)}$ is

$$|U_{k,Z}^{(n)}(\partial p)-1| < (1-\beta\tfrac{1}{2})\,L_0^{2\max\{0,j-l-1\}}\varepsilon_j(L^{k-j}\eta)^2. \tag{1.51}$$

The condition on the configuration $U_{k,Z}^{(n+1)}$ has the same form, only j is replaced by $j+1$. Introducing this bound into the inequality (1.48), which holds on those domains with $i=j$, we obtain

$$\begin{aligned} |U_{k,Z}^{(n)}(\partial p)-1| &< (1-\beta\tfrac{1}{2})L_0^{2\max\{0,j-l\}}\varepsilon_{j+1}(L^{k-j-1}\eta)^2(1+\alpha\beta) \\ &\quad +\alpha\beta\varepsilon_j(L^{k-j}\eta)^2 < \left((1+\beta_0)(1+\alpha\beta)\frac{L_0^2}{L^2}+\frac{2\alpha\beta}{2-\beta}\right) \\ &\quad \cdot(1-\beta\tfrac{1}{2})L_0^{2\max\{0,j-l-1\}}\varepsilon_j(L^{k-j}\eta)^2. \end{aligned} \tag{1.52}$$

The above inequality implies (1.51), under the usual restrictions on β_0, β, L_0 (i.e., $\beta_0 \leqq 1/2$, $\beta \leqq 1/2$, $L_0 < (1/2)L$), and $\alpha \leqq 1/4$. Thus taking $\alpha = 1/8$ in (1.48) we satisfy all the conditions. We have proved that the function $\chi_k^{(n)}$ in (1.29) is equal to 1. This completes the proof of the equality (1.29).

Now let us come back to the integral (1.25) with all the newly introduced functions. Using the equality (1.29) we get the integral of the same form multiplied by $\chi_k^{(n+1)}$, in which the only characteristic functions are the functions (1.9), (1.27) restricting the fluctuation field on the domains $\Omega_{j+1} \cap Z_{j+1}, \Omega_{j+1}^c \cap Z_{j+1}''^c$ correspondingly. There are also the gauge fixing terms (4.6) on the last domain. For this integral we perform all the operations discussed in Sect. 3 [III]. We introduce the fluctuation field V_j' on $(\Omega_{j+1}^c \backslash Z_j'')^{(j)}$,

$$V_j' = V_j(V^{(j)})^{-1}, \quad V^{(j)} = M^j(U_k^{(n+1)}), \quad B_j' = \frac{1}{i}\log V_j'. \tag{1.53}$$

This field is small, because

$$|V_j'-1| \leqq |V_j(V_Z^{(j)})^{-1}-1| + |V_Z^{(j)}(V^{(j)})^{-1}-1| < 3\delta_j', \tag{1.54}$$

by (1.27), and by the fact that the second expression on the right-hand side is much smaller than δ_j'. This we will show later. Thus B_j' is small, $|B_j'| < 6\delta_j'$, and we expand all the expressions in the integral with respect to B_j', we linearize the expressions in the δ-functions, we remove the δ-functions using the operator C, and finally we perform the scaling transformation $B_j' = g_j B_j$. All the formulas are the same as in Sect. 3, or in Sect. 2[I], only we have to replace the function $(g_k^{(n)}(\cdot))^{-2}$ in the Wilson term of the action by g_j^{-2}, the term with the difference $(g_k^{(n)}(\cdot))^{-2} - g_j^{-2}$ is put into the interaction. We obtain an equality with an expression analogous to the expression on the right-hand side of (3.15) [III]. The differences are obvious, mainly that the integration variable is denoted by B_j, and that there is the additional integration with respect to the variables A_j, the first integration in (1.25). In the obtained integral we introduce the decomposition of unity $1 = \chi'^{(j)} + (1 - \chi'^{(j)})$ in each component of Z, where

$$\chi'^{(j)} = \chi(\{|B_j(b)| < \delta'_j \text{ for } b \in (\Omega^c_{j+1} \setminus Z''_{j+1})^{(j)*}\}). \tag{1.55}$$

We exclude from Z the components with the large field function $1 - \chi'^{(j)}$. In these components we have to localize the dependence on the field V in the characteristic functions (1.27). Once more we write the representation

$$U_k^{(n+1)} = \left(\exp i\eta \mathbb{H}_{k,Z}^{(n+1)}\left(\frac{1}{i}\log\left[M^{\bullet}(U_k^{(n+1)})(M^{\bullet}(Q_k^{s*}V_k))^{-1}\right]\right)U_{k,Z}^{(n+1)}\right)^{(u_{k,Z}^{(n+1)})^{-1}} \tag{1.56}$$

The field in the argument of the function $\mathbb{H}_{k,Z}^{(n+1)}$ has a support in the boundary layer of the width $2M_1$ at the boundary of Z, and is bounded by $44d^2B_3\varepsilon_k$. Thus the function considered on the domain $\Omega^c_{j+1} \setminus Z''_{j+1}$ satisfies the bound

$$\begin{aligned}|\mathbb{H}_{k,Z}^{(n+1)}| &\leq (L^{j+1}\eta)^{-1}B_3\exp(-\delta 10M(R_{j+1} + \dots + R_{k-1}) - \delta MR_k)44d^2B_3\varepsilon_k \\ &< 44d^2B_3^2(1+\beta_0)\exp(-R_j)\varepsilon_j.\end{aligned} \tag{1.57}$$

For the averages $V^{(j)}$, $V_Z^{(j)}$ we have, as in (1.40)

$$V^{(j)} = \exp i\mathbb{H}_Z^{(j)} V_Z^{(j)}, \tag{1.58}$$

where $\mathbb{H}_Z^{(j)}$ is given by the formula corresponding to (1.41). It is an analytic function of the configuration $U_k^{(n+1)}$ restricted to Z, and on $(\Omega^c_{j+1} \setminus Z''_j)^{(j)}$ it is bounded by $O(1)\exp(-R_j)\varepsilon_j$. This bound is very small, smaller than any positive power of g_j. Now we make the change of variables (1.22) [III], with $\mathbb{H}_Z^{(j)}$ instead of $\mathbb{H}_{1,\mathrm{Ax}}$, for the bonds belonging to $(\Omega^c_{j+1} \setminus Z''_{j+1})^{(j)*}$. or rather to the components of this set with the large field function $1 - \chi'^{(j)}$. This change of variables transforms the function $\chi'_j(1 - \chi'^{(j)})$ into

$$\chi(\{|\exp ig_jB_j(b) - 1| < 2\delta'_j \text{ for } b \in (\Omega^c_{j+1} \setminus Z''_{j+1})^{(j)*}\})(1 - \chi'^{(j)}), \tag{1.59}$$

which has the required property, in fact it does not depend at all on the background fields. The change of variables gives also the usual new terms in the action.

Let us consider the components of Z with the small field function $\chi'^{(j)}$, or rather we assume that Z is a union of such components. The integral corresponding to (1.25) has the form of the fluctuation field integral in (3.25) [III], more precisely it has the form

$$\begin{aligned}&\chi_k^{(n+1)}\exp\left[A_k^{(n)}(U_k^{(n+1)}, A_j = 0) + E_0^{(j)}(\Omega^c_{j+1} \setminus Z''_{j+1})\right] \\ &\cdot \int dA_j \Gamma_{Z_{j+1} \cap \Omega_{j+1}} \chi^{(j)}(Z_{j+1} \cap \Omega_{j+1})\exp\left[-\tfrac{1}{2}\langle A_j, C^*\Delta^{(j)}_{\Lambda^c_{j+1}} CA_j\rangle\right] \\ &\cdot z^{(j)}\int dB_j \Gamma_{\Omega^c_{j+1} \setminus Z''_{j+1}} \chi'^{(j)}\exp\Bigg[-\tfrac{1}{2}\langle B_j, C^*\Delta^{(j)}CB_j\rangle + \mathbb{P}^{(j)}(g_j, A_j, B_j) \\ &+ \Bigg\{(\mathbb{E}_k + \mathbb{R}_k + \mathbb{B}_k + \mathbb{B}_k^{(n)})(U_k^{(n)}(\exp i[g_jCB_j - h\tilde{D}(g_jCB_j)]V^{(j)}))\end{aligned}$$

$$
\begin{aligned}
&-(\mathbb{E}_k+\mathbb{R}_k+\mathbb{B}_k+\mathbb{B}_k^{(n)})(U_k^{(n+1)},A_j=0)\\
&+A\left(\frac{1}{(g_k^{(n)}(\cdot))^2}-\frac{1}{g_j^2},\,U_k^{(n)}(\exp i[g_jCB_j-h\tilde{D}(g_jCB_j)]V^{(j)})\right)\\
&-A\left(\frac{1}{(g_k^{(n)}(\cdot))^2}-\frac{1}{g_j^2},\,U_k^{(n+1)}\right)\Bigg\}\Bigg].
\end{aligned}
\tag{1.60}
$$

The notation used here is the same as in Sect. 3 [III], in particular the constant $E_0^{(j)}(\Omega_{j+1}^c\backslash Z_{j+1}'')$ is equal to the constant in the first exponential in (3.15) [III], only for k replaced by j, and the sets $B(\Gamma_{k+1})^*$, Γ_{k+1} replaced by $(\Omega_{j+1}^c\backslash Z_{j+1}'')^{(j)*}$, $(\Omega_{j+1}^c\backslash Z_{j+1}'')^{(j+1)}$ correspondingly. The fundamental difference in comparison with the integral in Sect. 3 [III] is, that now the integration in (1.60) is localized in the large field region Z, therefore the integral contributes to the boundary terms only. Thus in the previous n integrations new boundary terms were created, and their sum is denoted by $\mathbb{B}_k^{(n)}$. We may specify the structure of this term a bit more writing it as the sum

$$
\mathbb{B}_k^{(n)}=\sum_{i=1}^{n}\mathbb{C}_k^{(i)},
\tag{1.61}
$$

where $\mathbb{C}_k^{(i)}$ is the one-step contribution, i.e., the contribution from the i^{th} integration. Now it is clear that the integral in (1.60) determines a main contribution to $\mathbb{C}_k^{(n+1)}$, another comes from the renormalization of the Wilson action, thus

$$
\begin{aligned}
&\mathbb{C}_k^{(n+1)}=\log(\text{the }\textstyle\int\text{ in (1.60)})+A\left(\frac{1}{(g_k^{(n+1)}(\cdot))^2}-\frac{1}{(g_k^{(n)}(\cdot))^2},\,U_k^{(n+1)}\right),\\
&\mathbb{B}_k^{(n+1)}=\mathbb{B}_k^{(n)}+\mathbb{C}_k^{(n+1)}.
\end{aligned}
\tag{1.62}
$$

The integral in (1.60) is of the same type as the integral studied in Sect. 3 [III], therefore we can apply the results of this section and write the term $\mathbb{C}_k^{(n+1)}$ as a sum of localized terms with the corresponding exponential decay and analyticity properties. There is one peculiar feature of this expansion, namely its localization domains are not from $\mathbb{D}_{j+1}$ only, but they are of a more general type, and can be described by the following properties: $X\cap\Omega_{j+2}^c\in\mathbb{D}_{j+1}$, $X\cap(\Omega_m\backslash\Omega_{m+1})\in\mathbb{D}_m$, $m=j+2,\dots,k$ (where $\Omega_{k+1}=\emptyset$), and $X\cap(Z_{j+1}\backslash Z_{j+1}'')\neq\emptyset$. For a given X there is a minimal index m such, that $X\subset\Omega_{m+1}^c$. We take the smallest domain from $\mathbb{D}_m$ containing X, and we resum over all X determining the same domain. As a result we obtain the representation

$$
\mathbb{C}_k^{(n+1)}=\sum_X\mathbb{C}_k^{(n+1)}(X,U_k^{(n+1)}),
\tag{1.63}
$$

where the summation is over the localization domains X satisfying the following properties: $X\subset\Omega_{m+1}^c$ for some $m\geqq j+1$, $X\cap\Omega_m\neq\emptyset$, $X\in\mathbb{D}_m$, and $X\cap(Z_{j+1}\backslash Z_{j+1}'')\neq\emptyset$. The terms in the sum satisfy an exponential decay bound, which will be formulated later. These results are standard, but there are some new aspects of the basic properties, like analyticity properties and bounds, which we have to discuss carefully.

Let us start with the analyticity properties. The aim of this whole preliminary step is to gain a larger domain of analyticity for the effective action, in order to perform the basic step in the $\mathbb{R}$-operation. This is achieved by two ways for the two parts of the action $A_k^{(n)}$. The regularity properties of the configuration $U_k^{(n)}$, and correspondingly of the complex field $\mathbb{U}$ which replaces it, are improved in comparison with U_k; therefore the old action A_k has effectively a larger analyticity domain with respect to the new field. For the new term $\mathbb{B}_k^{(n)}$ of the action we have to enlarge the analyticity domain by proper inductive assumptions, in agreement with the restrictions introduced by the characteristic function $\chi_k^{(n)}$. In order to describe the assumptions we have to define new spaces of complex gauge field configurations. They are modelled on the definition (1.24) of the characteristic functions $\chi_k^{(n)}$, and or the definition of the spaces $\tilde{U}_k^c(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$.

The space $\tilde{U}_k^{(n)c}(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$, for one of the domains X in the representation (1.63), is the set of configurations $(\mathbb{U}, \mathbb{J})$ defined on X, such that $\mathbb{U} = U'U$, U has values in the group G, $U' = \exp i\eta A'$, A' and $\mathbb{J}$ have values in the complex algebra $\mathbf{g}^c$. For any cube $\square$ from the family of cubes described below there exists a gauge transformation u defined on $\square \cap X$, and such that $U^u = \exp i\eta A$, A has values in the algebra $\mathbf{g}$. The configurations $\mathbb{U}$, $U_{p,X}(M^{\cdot}(\mathbb{U})) = U(\mathbb{B}_p(X) \cup \{\Gamma_i^{(n)}\}_{i<p}, M^{\cdot}(\mathbb{U}))$, $\mathbb{J}, J_{p,X}(M^{\cdot}(\mathbb{U}))$, U, A', and A satisfy the following conditions:

(i) $|\partial \mathbb{U} - 1|, \quad |\partial U_{p,X}(M^{\cdot}(\mathbb{U})) - 1|, \quad |\mathbb{J}|, \quad |J_{p,X}(M^{\cdot}(\mathbb{U}))|,$
$|\partial U - 1|, \quad |A'|, \quad |\nabla_U^\eta A'|$

$$< \left(1 - \beta \sum_{i=h+1}^{j} 2^{-|m-i|} - \beta \sum_{q=m+1}^{k} 2^{-(q-m)}\right) L_0^{2\max\{0, m-k_0\}}$$
$$\cdot [\alpha_{0,m}\eta^2(L^m\eta)^{-2}, \alpha_{0,m}L^{-2p}(L^mL^{-p})^{-2}, \alpha_{0,m}(L^m\eta)^{-3},$$
$$\alpha_{0,m}L^{m-\min\{p,m\}}(L^mL^{-p})^{-3}, \alpha_{0,m}\eta^2(L^m\eta)^{-2}, \alpha_{1,m}(L^m\eta)^{-1}, \alpha_{1,m}(L^m\eta)^{-2}]$$

on $(\Omega_m'' \backslash \Omega_{m+1}'') \cap X$ for $m = 1, \dots, j-1$, and on $(\Omega_j'' \backslash \Omega_{k_0+1}) \cap X$ for $m = j$, $p = 1, \dots, k$.

$$L^m\eta|A|, (L^m\eta)^2|\nabla^\eta A| < L_0^{2\max\{0, m-k_0\}} BCM\alpha_{0,m} \tag{1.65}$$

for $\square \subset \Omega_m''$, $\square \cap \Omega_{m+1}''^c \neq \emptyset$ if $m = 1, \dots, j-1$, and for $\square \subset \Omega_j''$, $\square \cap \Omega_{k_0+1}^c \neq \emptyset$
if $m = j$, $\square$ is of the size $CML^m\eta$.

(ii) $|\partial \mathbb{U} - 1|, \quad |\partial U_{p,X}(M^{\cdot}(\mathbb{U})) - 1|, \quad |\mathbb{J}|, \quad |J_{p,X}(M^{\cdot}(\mathbb{U}))|, \quad |\partial U - 1|, \quad |A'|, \quad |\nabla_U^\eta A'|$

$$< \left(1 - \beta \sum_{i=h+1}^{j} 2^{-|j-i|} - \beta \sum_{q=j+1}^{k} 2^{-(q-j)}\right) L_0^{2(j-m)}$$
$$\cdot [\alpha_{0,j}\eta^2(L^j\eta)^{-2}, \alpha_{0,j}L^{-2p}(L^jL^{-p})^{-2}, \alpha_{0,j}(L^j\eta)^{-3}, \alpha_{0,j}L^{j-\min\{p,j\}}(L^jL^{-p})^{-3},$$
$$\alpha_{0,j}\eta^2(L^j\eta)^{-2}, \alpha_{1,j}(L^j\eta)^{-1}, \alpha_{1,j}(L^j\eta)^{-2}] \tag{1.66}$$

on $(\Omega_m \backslash \Omega_{m+1}) \cap X$ for $m = k_0 + 1, \dots, j-1, j$.

$$L^j\eta|A|, (L^j\eta)^2|\nabla^\eta A| < L_0^{2(j-m)} BCM\alpha_{0,j} \tag{1.67}$$

for $\square \subset \Omega_m$, $\square \cap \Omega_{m+1}^c \neq \emptyset$, $\square$ is of the size $CML^j\eta$, $m = k_0 + 1, \dots, j-1, j$.

(iii) $|\partial \mathbb{U}-1|, |\partial U_{p,X}(M^{\bullet}(\mathbb{U}))-1|, |\mathbb{J}|, |\mathbb{J}_{p,X}(M^{\bullet}(\mathbb{U}))|, |\partial U-1|, |A'|, |\nabla_U^{\eta} A'|$

$$< \left(1-\beta \sum_{i=h+1}^{j} 2^{-|m-i|}-\beta \sum_{q=m+1}^{k} 2^{-(q-m)}\right)$$

$$\cdot c[\alpha_{0,m}\eta^2(L^m\eta)^{-2}, \alpha_{0,m}L^{-2p}(L^m L^{-p})^{-2}, \alpha_{0,m}(L^m\eta)^{-3}, \alpha_{0,m}L^{m-\min\{p,m\}}(L^m L^{-p})^{-3},$$

$$\alpha_{0,m}\eta^2(L^m\eta)^{-2}, \alpha_{1,m}(L^m\eta)^{-1}, \alpha_{1,m}(L^m\eta)^{-2}] \tag{1.68}$$

on $(\Omega_m \backslash \Omega_{m+1}) \cap X$ for $m=j+1,\ldots,k-1$, and on $\Omega_k \cap X$ for $m=k$, where $c=1$ for $m<k$, and $c=3$ for $m=k$,

$$L^m\eta|A|, (L^m\eta)^2|\nabla^{\eta}A| < BCM\alpha_{0,m} \tag{1.69}$$

for $\square \subset \Omega_m$, $\square \cap \Omega_{m+1}^c \neq \emptyset$ if $m=j+1,\ldots,k-1$, and for $\square \subset \Omega_k$ if $m=k$, $\square$ is of the size $CML^m\eta$.

The constant C above is a positive integer. It is usually small, because of geometric constraints on the cubes, for example we can assume that $C \leqq 2$. The constant B is a fixed absolute constant, for example we can take the constant $B=B_3$ from Theorem 1 [16].

The above definition is so complicated because of the needs of the inductive procedure. It has been written for the case where $j>k_0$. For $j \leqq k_0$ it is simpler, there are no powers of L_0 in the point (i), and the point (ii) is empty. Let us discuss briefly basic properties of these spaces. They are invariant with respect to G-valued gauge transformations. For $n=j-h=0$ the above space coincides with the space $\tilde{U}_k^c(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$. They form a descending sequence for increasing n, if β and L_0 satisfy certain conditions, for example if $\beta \leqq 1/4$ and $L_0^2 \leqq (1/3)L$. Notice that we get the stronger restriction on L_0 because of the bounds for A'. We can improve the restriction to the previous one changing the bounds, but it does not matter. In fact we need not only the statement that the sequence is descending, but a stronger statement connected with the expressions we have to consider in the $n+1^{\text{st}}$ step. For example, we have to prove that if $\mathbb{U}$ is an element of the space $\tilde{U}_k^{(n+1)c}(X, \tilde{\alpha}_0\tilde{\alpha}_1)$, then $\exp i\eta \mathbb{H}_k^{(n)}(g_j CB_j - h\tilde{D}(g_j CB_j))\mathbb{U}$ is an element of the space $\tilde{U}_k^{(n)c}(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$, and the corresponding statement for $\mathbb{J}$. The proof is almost identical to the proof of a similar statement for the configurations $U_k^{(n)}$, given in (44)–(52), and we will not repeat it. Finally let us remark that if in a component of Z we finish the integrations for some n, because some new large field has been introduced, then we leave the new terms $\mathbb{B}_k^{(n)}$ as a part of all boundary terms, and it is easy to generalize the definition of the spaces in such a case for the next steps.

Now we can formulate bounds for terms in the representations (1.63). They are easily obtained estimating the integral (1.60), after the renormalization and the localization. All the expressions in the exponential in this integral can be uniformly bounded in the usual way, except possibly expressions connected with $\mathbb{B}_k^{(n)}$. Assuming uniform bounds for $\mathbb{C}_k^{(i)}$, $i \leqq n$, we still have a problem with a bound for the whole term, because the basic localization regions $Z_{h+i} \backslash Z''_{h+i}$ contain the common domain $\Omega_{k_0+1}^c \backslash \Omega''_k$ for $h+i>k_0$, hence the bounds from the corresponding terms cumulate. There are at most N_0 such terms, and to estimate their sum we use the fact that they are multiplied by g_j. More precisely, the bounds of the terms involve δ'_j, but $\delta'_j \leqq (1+\beta_0)^2(k-j)^{\beta_0}\delta'_k \leqq (1+\beta_0)^2 N_0^{\beta_0}\delta'_k$, so $N_0\delta'_j \leqq (1+\beta_0)^2 N_0^{1+\beta_0}\delta'_k < 4N_0^2\delta'_k$, and we assume that N_0^2 makes only a logarithmic contribution to δ'_k, e.g., $N_0 = O(\log g_k^{-2})$. In fact we will see that it is smaller. With such an assumption we get uniform bounds for the sum of the terms also. Let us formulate the bounds for $\mathbb{C}_k^{(n)}$. It has the representation (1.63), with n instead of $n+1$, and each term $\mathbb{C}_k^{(n)}(X, U_k^{(n)})$ can be extended to an analytic function defined on the space $\tilde{U}_k^{(n)c}(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$ and satisfying the bound

$$|\mathbb{C}_k^{(n)}(X, (\mathbb{U}, \mathbb{J}))| \leqq C_0 \exp(-(1+3\beta)\kappa d_m(X)) \tag{1.70}$$

for $X \in \mathbb{D}_m$ in the representation (1.63).

There is one additional remark we have to make in connection with the definitions of the new background fields and the new spaces. In these definitions there are powers of L_0^2, the highest power appearing is $k - k_0 = N_0$. This number is defined by the equality $L^{-N_0+1}R_{k-N_0+1} = 1$, from which we get $L^{N_0-1} = R_{k-N_0+1} \leqq (L+1)(N_0+1)^{\beta_0}R_k$. The number L_0 satisfies the restriction $L_0^2 \leqq (1/3)L$, hence $L_0^{2N_0} \leqq (1/3)^{N_0}L(L+1)(N_0+1)^{\beta_0}R_k < L(L+1)N_0^{-1}R_k$. Thus the powers of L_0 make only unessential logarithmic contributions to the constants $\alpha_{0,j}, \alpha_{1,j}, \varepsilon_j$ in the definitions.

Let us write now the expressions we obtain after the N integrations, omiting the $\mathbb{T}$-operations for the large field regions not satisfying the conditions (i), (ii). These expressions have he form

$$\chi_k(\Omega_k^{\sim 4}) \prod_{j=k-1}^{h} \mathbb{T}''^{(j)}(Z''_{j+1})\chi_h(\Omega_h \cap Z_h)\mathbb{T}_h(Z_h)\chi''_k \exp A''_k, \tag{1.71}$$

where the superscript (N) in symbols of the characteristic function and the action has been replaced by the double prime, and the operations $\mathbb{T}''^{(j)}$ are defined by

$$\mathbb{T}''^{(j)}(Z''_{j+1}) = \int dV_j [_{Z''_{j+1}} \delta(\bar{V}_j V_{j+1}^{-1}). \tag{1.72}$$

The new action depends on the background field $U''_k = U_k^{(N)}$, and it has the form $A''_k = A_k + \mathbb{B}''_k$, where $\mathbb{B}''_k$ is the sum of the new boundary terms described above.

The new large field region Z''_k is given by (1.10), i.e., $Z''_k = (\Omega_{k_0+1}^{\sim 5})^c \cap Z$, hence it is a union of M-cubes, by the definition of the index k_0, and it is a rectangular parallelepiped. Define

$$\Lambda = (\Omega_{k_0+1}^{\sim 4})^c \cap Z. \tag{1.73}$$

This domain, obtained by adding one layer of M-cubes to Z''_k, is also a union of M-cubes, and by the condition (i) it is a rectangular parallelepiped contained in a cube of the size $100M$. The domain Λ plays a fundamental role in the definition of the $\mathbb{R}$-operation, it is the domain on which we integrate out all field variables. Now we want to restrict further the variables V_k outside Λ. Take the function $U_{k,Z}(V_k)$ given by

$$U_{k,Z} = U_{k,Z}(V_k) = U(\mathbb{B}_k(Z), M^{\cdot}(Q_k^{s*}V_k)). \tag{1.74}$$

This function is also important for subsequent constructions. It is defined in each component of Z separately. In the components introduce the decompositions of unity $1=\chi_{k,\Lambda}+(1-\chi_{k,\Lambda})$, where

$$\chi_{k,\Lambda}=\chi\left(\left\{\inf_{V_k\lceil_\Lambda}\sup_{p\in\Omega_k^c}|U_{k,Z}(\partial p)-1|<2\varepsilon_k\eta^2\right\}\right). \tag{1.75}$$

The restriction introduced by this function is on the field $V_k\lceil_{Z\cap\Lambda^c}$ only. It means that this field has an extension on the whole domain Z, such that the extended field satisfies the regularity condition in (1.75). Let us elaborate a bit this point, because we want to understand also a meaning of the restriction established by the function $1-\chi_{k,\Lambda}$. Take a configuration V_k defined on $Z\cap\Lambda^c$ and satisfying the regularity condition $|V_k(\partial p')-1|<\varepsilon$ for $p'\subset Z\cap\Lambda^c$, $\varepsilon>0$ is sufficiently small. There are many ways of extending it to the domain Λ. The way we describe here depends only on V restricted to $\partial^+\Lambda=\{y\in\Lambda^c$: there exists a nearest neighbor point $y'\in\Lambda$, or $\langle y,y'\rangle\in\Lambda\}$. Introduce a generalized axial gauge on the surface $\partial^+\Lambda$. The configuration $V_k\lceil_{\partial^+\Lambda}$ is transformed into a small configuration V_k', $|V_k'(b')-1|<O(1)M^2\varepsilon$ for $b'\subset\partial^+\Lambda$. We extend it putting $V_k'(b')=1$ for $b'\in\Lambda$. The extended configuration satisfies the regularity condition $|V_k'(\partial p')-1|<O(1)M^2\varepsilon$ for $p'\subset\Lambda\cup\partial^+\Lambda$. Now we apply the inverse gauge transformation on $\partial^+\Lambda$, and we get a configuration V_k defined on the whole domain, equal to the given one on $Z\cap\Lambda^c$, and satisfying the regularity condition $|\partial V_k-1|<O(1)M^2\varepsilon$. The corresponding configuration $U_{k,Z}$ satisfies the condition $|\partial U_{k,Z}-1|<O(1)B_3M^2\varepsilon\eta^2$. We have a couple of conclusions from the above reasoning. At first, recall that the functions $\chi_k(\Omega_k^{\sim 4})$, χ_k'' restrict the field variables V_k on Ω_k'', e.g., $|\partial V_k-1|<2L_0^{2N_0}\varepsilon_k<2L(L+1)N_0^{-1}R_k\varepsilon_k$. The above reasoning implies that V_k has an extension to Λ, satisfying the regularity condition $|\partial V_k-1|<O(1)M^2L^2R_k\varepsilon_k$ on Z, hence the function $U_{k,Z}$ is defined for all those fields, and the condition in (1.75) has a meaning. Now, take a field V_k in the domain determined by the function $1-\chi_{k,\Lambda}$, and denote $\varepsilon=\sup_{p'\subset Z\cap\Lambda^c}|V_k(\partial p')-1|$. Of course $\varepsilon<L^2R_k\varepsilon_k$. By the above reasoning the field $V_k\lceil_{Z\cap\Lambda^c}$ has an extension, for which $|\partial U_{k,Z}-1|<O(1)B_3M^2\varepsilon\eta^2$. On the other hand, for any extension we have $|\partial U_{k,Z}-1|\geqq 2\varepsilon_k\eta^2$, hence $2\varepsilon_k<O(1)B_3M^2\varepsilon$, and $|V_k(\partial p')-1|>(O(1)B_3M^2)^{-1}\varepsilon_k$ for some $p'\subset Z\cap\Lambda^c$. This condition is enough to get the exponential small factor, estimating in the usual way the Wilson action. Thus $1-\chi_{k,\Lambda}$ is a large field function, and we exclude from Z the components with this function. The union of the remaining components is denoted by Z again, and the corresponding expressions are given by the formula (1.71) multiplied by $\chi_{k,\Lambda}$.

All the above transformations preserve the k^{th} density ρ_k, they change only the representation of this density. It is represented in the same general form (2.18) [III], but with different operations $\mathbb{T}_k$, and different effective actions. Now we define the next operation, which changes the density. The equality sign is replaced by the equivalence sign, the equivalence means that both sides have equal integrals over the space of fields V_k. To define the operation we formulate briefly the result of the previous operations. We have represented the density ρ_k as a sum of terms of the general form (2.18) [III]. Each term determines uniquely a large field region, and in particular the components of this region, satisfying the conditions (i), (ii). Such a term is represented as a $\mathbb{T}_k$-operation for the remaining components, acting on the corresponding expression of the form (1.71) multiplied by the function $\chi_{k,\Lambda}$. The next operation is an integration of this term with respect to the field variables

V_k over the domain Λ. We obtain a new expression, which we consider as a function of all field variables V_k, but independent of $V_k\restriction_\Lambda$. This operation changes essentially the sum of these terms, and we can write only that the density is equivalent to the sum of the new, integrated terms. We remark also that regions of integration are usually different for terms in the sum. Let us write the effect of this operation for each term. Omitting the $\mathbb{T}_k$-operation for the remaining components, as in (1.71), and using the form (1.72) of the operations $\mathbb{T}''^{(j)}$, we obtain the expressions

$$\chi_k(\Omega_k^{\sim 4})\chi_{k,\Lambda}\int dV_h\restriction_{Z_h}\chi_h(\Omega_h \cap Z_h)\mathbb{T}_h(Z_h)\int dV''\restriction_{\mathbb{B}_k''\cap(\Lambda\cap Z_h^c)}\chi_k'' \exp A_k''. \tag{1.76}$$

The variables V'' are determined by, and defined on the set $\mathbb{B}_k''$, i.e., $V''=V_j$ on Γ_j'', $j=0,1,\ldots,k-1,k$. The two integrals above are over the disjoint regions of integrations, hence we can consider them independently. In fact the first integral, over the region Z_h, is left unchanged in the following discussion.

The above integral should be analyzed now in the same way as small field integrals are analyzed. i.e., we should find a minimum of the Wilson action $A(U_k'')$ on the domain of integration, we should expand the whole action around the minimum, introduced a fluctuation field together with a gauge fixing, restrict the fluctuation field and analyze the fluctuation field integral. Instead, we will do these steps on the domain $\Lambda\cap Z_h$ only, except the first one, the solution of the variational problem, which we will consider on the whole domain of integration. Let us start with this problem. The integration in (1.76) does not involve any δ-functions, they are all integrated out, therefore we look for a minimum of the function $A(U_k'')$ for $V''\restriction_\Lambda$ restricted by regularity conditions only. This problem can be divided into two steps, at first we look for a minimum of $A(U_k'')$ with the averages $M^k(U_k'')$ fixed, and then we look for a minimum removing the restrictions on the averages on the domain Λ. A solution of the last problem should give a solution of the original problem. These remarks serve only as a justification of the following construction. Consider the function

$$V_k\restriction_\Lambda \to A(U_{k,Z}(V_k)). \tag{1.77}$$

It is defined on configurations V_k satisfying mild regularity conditions, e.g., $|\partial V_k - 1| < a_1$ on Z. The function is invariant with respect to the group of all gauge transformations defined on Λ, hence it is natural to consider it on orbits of this group. We look for a minimal orbit. We will prove later the following theorem.

Proposition 1. *For a configuration $V_k\restriction_{Z\cap\Lambda^c}$, satisfying the regularity condition $|\partial V_k - 1| < \varepsilon$ on the domain $Z\cap\Lambda^c$, for $\varepsilon > 0$ sufficiently small, there exists exactly one critical orbit of the function (1.77). An element of the orbit is a minimum of the function, and is denoted by $V_\Lambda = V_\Lambda(V_k\restriction_{Z\cap\Lambda^c})$. It satisfies the regularity condition*

$$|V_\Lambda(\partial p') - 1| < B_5 M^5 \varepsilon \quad \text{for} \quad p'\in\Lambda. \tag{1.78}$$

The orbit-valued function $V_\Lambda(V_k\restriction_{Z\cap\Lambda^c})$ has an analytic extension for G^c-valued configurations $\mathbb{V}_k = V_k' V_k = \exp iB' V_k$ satisfying the same regularity condition as V_k, with $B'\in\mathbf{g}^c$ and small, e.g., $|B'| < \varepsilon$.

The constant B_5 is determined by the geometry of the problem, more precisely by the condition (i). The power M^5 is not the optimal one, in fact we can take just M, but we will not prove it.

Extend the function V_Λ on the whole domain Z putting $V_\Lambda = V_k$ on $Z \cap \Lambda^c$, and define

$$U_0 = U_{k,Z}(V_\Lambda). \tag{1.79}$$

This is a fundamental background field for the constructions of this and the next sections. In particular we expand the action A_k'' in (1.76) around this configuration on $\Lambda \cap Z_h^c$; therefore it is important to understand its regularity properties. To simplify the description we take into account the characteristic function $\chi_{k,\Lambda}$. The restrictions introduced by this function, and the method of construction of the function V_Λ, imply the estimate

$$|U_0(\partial p) - 1| < 2\varepsilon_k \eta^2 + O(1) B_3 B_5 M^5 \exp(-\delta \operatorname{dist}(p, \Lambda)) \varepsilon_k \eta^2 \tag{1.80}$$

for $p \in \Omega_k$. We will discuss it together with the proof of Proposition 1, because then it will be immediate, but it follows also quite generally from the definition (1.79) and the bound (1.78). From the estimate (1.80) we can see easily that the configuration U_0 belongs to the integration domain determined by the characteristic function χ_k'' in (1.76), and to the analyticity domains of terms in the effective action A_k''. In fact, it is the reason why we have done the preliminary integrations, and why we have introduced the function χ_k'', and the spaces $\tilde{U}_k^{(n)c}(X, \tilde{\alpha}_0, \tilde{\alpha}_1)$. We do not show the above statement now, because we will need a stronger statement in the future.

Now we define the fluctuation field V' on the set $\mathbb{B}_k'' \cap \Lambda \cap \Omega_{h+1}''^{\sim 2}$. Let us recall that $\Omega_{h+1}'' = \Omega_{h+1}^{\sim 5} = Z_h'^{\sim 3}$, hence $\Omega_{h+1}''^{\sim 2} = \Omega_{h+1}^{\sim 7} = (Z_h'^{\sim})^c$, and denote $\Lambda_0 = \Lambda \cap \Omega_{h+1}''^{\sim 2}$. We put

$$V'' = V' V_0 \quad \text{on} \quad \Lambda_0, \quad V_0 = M_{\mathbb{B}_k''}(U_0). \tag{1.81}$$

More precisely the equality is on the set $\mathbb{B}_0 = \mathbb{B}_k'' \cap \Lambda_0$. Let us recall that according to our convention, bonds intersecting $\partial\Lambda$ belong to $\mathbb{B}_0$, and bonds intersecting $\partial\Omega_{h+1}''^{\sim 2}$ do not belong to $\mathbb{B}_0$.

The measure and the underintegral expressions in the second integral in (1.76) are gauge invariant; more precisely they are invariant with respect to the gauge transformations defined on $\mathbb{B}_0$. We remove this invariance by fixing a gauge for the variables V'. We fix a gauge which is a modification and a generalization of the axial gauge in cubes used in the previous papers. At this point we make an essential use of the fact that the domains $\Lambda \supset Z_k'' \supset \cdots \supset Z_{h+1}'' \supset (\Omega_{h+1}''^{\sim 2})^c$ are rectangular parallelepipeds. This allows us to give a simple description of the gauge fixing. Consider two successive rectangular parallelepipeds in the sequence, for example

$$P_1 = [a_1, b_1] \times \cdots \times [a_d, b_d] \supset P_2 = [a_1', b_1'] \times \cdots \times [a_d', b_d'],$$

hence $[a_\mu', b_\mu'] \subset [a_\mu, b_\mu]$. We have to fix a gauge in $P_1 \backslash P_2$, more precisely in the intersection with the corresponding lattice. We may assume, rescaling properly, that it is the unit lattice. The gauge is fixed by a tree graph built of bonds contained in $P_1 \backslash P_2$. We fix the initial point $y=(y_1,\ldots,y_d)=(a_1+1/2,\ldots,a_d+1/2)$, and a number $a\in(a_1', b_1')$. For a given point x of the lattice in $P_1\backslash P_2$ we choose a contour connecting x to y. If $x_1 \leqq a$, then we take the usual contour

$$\Gamma_{y,x}=[y,(y_1,\ldots,y_{d-1},x_d)]\cup\cdots\cup[(y_1,x_2,\ldots,x_d),x].$$

If $x_1 > a$, then we take the contour

$$\begin{aligned}\Gamma_{y,x}=&[y,(y_1,\ldots,y_{d-1},x_d)]\cup\cdots\cup[(y_1,y_2,y_3,x_4,\ldots,x_d),(y_1,y_2,x_3,x_4,\ldots,x_d)]\\ &\cup[(y_1,y_2,x_3,\ldots,x_d),(b_1-1/2,y_2,x_3,\ldots,x_d)]\\ &\cup[(b_1-1/2,y_2,x_3,\ldots,x_d),(b_1-1/2,x_2,x_3,\ldots,x_d)]\\ &\cup[(b_1-1/2,x_2,x_3,\ldots,x_d),x].\end{aligned}$$

This slightly awkward definition describes a simplest family of contours connecting points of $P_1\backslash P_2$ with the point y. The union of all the contours is a tree graph T on $P_1\backslash P_2$, hence $T=\bigcup_{x\in P_1\backslash P_2}\Gamma_{y,x}$. This definition has the important property that if we enlarge P_2, i.e., we replace it by a rectangular parallelepiped P_2', $P_2\subset P_2'\subset P_1$, and we remove the contours corresponding to points of $P_2'\backslash P_2$, then the remaining contours build the corresponding graph T' on $P_1\backslash P_2'$. We fix the gauge putting the bond variables equal to 1 for bonds belonging to the tree graph. We use all gauge degrees of freedom connected with points of $P_1\backslash P_2$, except one, so we add an external bond to the graph. If is one of the bonds intersecting the boundary ∂P_1, but they are bonds of the larger scale (they are L-bonds for the unit scale in P_1), except when $P_1=\Lambda$, so it is simpler to describe the corresponding bond for P_2. We assume that P_2 is not the last domain in the sequence, i.e., $P_2\neq(\Omega''^{\sim 2}_{h+1})^c$, because no gauge is fixed in this domain, and no external bonds intersecting ∂P_2 are added to the graph. For the remaining domains P_2 we choose the bond

$$[(a_1'-1/2,a_2'+1/2,\ldots,a_d'+1/2),(a_1'+1/2,a_2'+1/2,\ldots,a_d'+1/2)],$$

and add it to the graph. For $P_1=\Lambda$ we add also the additional bond $[(y_1-1,y_2,\ldots,y_d),y]$. The union of the above described tree graphs and bonds is denoted by T_0. It is a tree graph in $\mathbb{B}_0$, fixing completely a gauge in this set. Using the Faddeev–Popov procedure we introduce the gauge fixing δ-function $\delta_{T_0}(V')$ in the integral (1.76).

The regularity conditions for V'', V_0, and the gauge fixing for V' introduce restrictions on this field. We can prove that it satisfies $|V'-1|<O(1)M^2NR_k^4\varepsilon_k$ on $\mathbb{B}_0$. We introduce stronger restrictions on the fluctuation field by the decomposition of unity $1=\chi'+(1-\chi')$, where

$$\chi'=\chi(\{|B'(b)|<\delta_k' \quad \text{for} \quad b\in\mathbb{B}_0\}), \tag{1.82}$$

and $V'=\exp iB'$. The decomposition of unity is introduced in each component of Z, and we exclude from Z the components with the large field function $1-\chi'$. In the remaining components, denoted by Z, the functions $\chi_{k,\Lambda}$, χ' should allow us to remove the function χ_k'', except that they do not give restrictions on the variables $V''=V_h$ on $Z_h'^{\sim}\backslash Z_h$. We shall introduce such additional restrictions, but at first we fix a gauge for the configuration U_0. We would like to represent it locally, on a subdomain of Λ, as $\exp i\eta\mathbb{A}_0$, with $\mathbb{A}_0$ sufficiently small. Such a representation can be obtained in two steps. The configuration $M^k(U_0) = V_\Lambda$ satisfies the regularity condition (1.78). We fix for it the axial gauge in $\Lambda^{(k)}$, hence the field V_Λ is small inside $\Lambda^{(k)}$. More precisely, it satisfies the bound

$$|V_\Lambda(b) - 1| < O(1)B_5M^6\varepsilon_k \quad \text{for} \quad b \subset \Lambda^{(k)}. \tag{1.83}$$

Consider the configuration U_0 inside the domain Λ. We take it in the axial gauge in k-blocks, hence $|M^j(U_0) - Q^{s*}_{k-j}V_\Lambda| < 11d^2O(1)B_3B_5M^5\varepsilon_k$ inside Λ, by the estimate (1.80), and (1.65) [14]. This and (1.83) imply

$$|M^j(U_0) - 1| < O(1)B_3B_5M^6\varepsilon_k \quad \text{inside} \quad \Lambda. \tag{1.84}$$

The field U_0 can be represented inside Λ as follows:

$$U_0 = U(\mathbb{B}_k(\Lambda), M^{\boldsymbol{\cdot}}(U_0)). \tag{1.85}$$

The field in the argument of the function on the right-hand side of the above equation is small, hence we can expand the function around the configuration identically equal to 1. We transform the function to the Landau gauge constructed around this configuration, and we get

$$U_0 = \left(\exp i\eta\mathbb{H}_{k,\Lambda}\left(\frac{1}{i}\log M^{\boldsymbol{\cdot}}(U_0)\right)\right)^{u_0^{-1}}. \tag{1.86}$$

Thus $U_0^{u_0}$ is in the required gauge, and we have

$$U_0^{u_0} = U_0^{(AL)} = \exp i\eta\mathbb{A}_0 \quad \text{inside} \quad \Lambda,$$

where

$$|\mathbb{A}_0|, |\nabla^\eta\mathbb{A}_0|, |\partial^{\eta*}\partial^\eta\mathbb{A}_0| < O(1)B_3^2B_5M^6\varepsilon_k \quad \text{on} \quad Z''_k. \tag{1.87}$$

The function $\mathbb{A}_0 = \mathbb{H}_{k,\Lambda}((1/i)\log M^{\boldsymbol{\cdot}}(U_0))$ is also an analytic function of $(1/i)\log M^{\boldsymbol{\cdot}}(U_0)$, on a much larger domain than the one determined by the bounds (1.84). The superscript (AL) in (1.87) indicates the mixture of the axial and the Landau gauges. Changing the gauge of U_0 in the integral (1.76) requires the corresponding compensating gauge transformations of the variables V_h, the gauge transformation restricted to $\partial^+(Z_h'^{\sim})^{(h)}$, and of the variables V', the gauge transformation in the adjoint representation. The expressions in the integral (1.76) are invariant with respect to these transformations. We can also choose from the beginning the configuration U_0 in the AL-gauge. Finally, we introduce the last decompositions of unity on components of $(\Omega''^{\sim}_{h+1})^c \cap \Omega_h$:

$$\begin{aligned}1 &= \sum_P \prod_{\square \subset P^c} \chi\left(\left\{\sup_{p \subset \square^\sim} |U_{h,\square}((1, V_h), \partial p) - 1| < \tfrac{1}{2}\varepsilon_h(L^{k-h}\eta)^2\right\}\right) \\ &\quad \cdot \prod_{\square \subset P} \chi\left(\left\{\sup_{p \subset \square^\sim} |U_{h,\square}((1, V_h). \partial p) - 1| \geqq \tfrac{1}{2}\varepsilon_h(L^{k-h}\eta)^2\right\}\right) \\ &= \sum_P \chi_{h,1/2}(P^c)\chi^c_{h,1/2}(P).\end{aligned} \tag{1.88}$$

The summation above is over subsets P of a component of the domain $(\Omega''^{\sim}_{h+1})^c \cap \Omega_h$, which are unions of LM_2R_h-cubes (for the L^{-h}-scale). The symbol $(1, V_h)$ means the configuration $(1\lceil_{\Omega''^{\sim 2}_{h+1}}, V_h\lceil_{(\Omega''^{\sim 2}_{h+1})^c})$. We have new large fields in components with nonempty sets P, so we exclude these components from Z. In the remaining components, which form a new domain Z, we have the function $\chi_{h,1/2}((\Omega''^{\sim}_{h+1})^c \cap \Omega_h)$. Notice, that if a cube $\square$ is contained in $\Omega''^{\sim 2}_{h+1}$, and is not touching the boundary of this domain, then $U_{h,\square}(1, V_h) = U_{h,\square}(1) = 1$, and the condition defining the corresponding characteristic function is always satisfied, i.e., the function is identically equal to 1. Therefore the above characteristic function is defined by the product over cubes $\square$ intersecting the domain $(\Omega''^{\sim 2}_{h+1})^c$, and contained in Ω_h.

Now we prove that, with the new functions introduced in the integral, we can drop the function χ''_k, i.e., we have the equality

$$\chi_k(\Omega_k^{\sim 4})\chi_{k,\Lambda}\chi_{h,1/2}((\Omega''^{\sim}_{h+1})^c \cap \Omega_h)\chi'\chi_h(\Omega_h \cap Z_h)\chi''_k = \chi_k(\Omega_k^{\sim 4})\chi_{k,\Lambda}\chi_{h,1/2}((\Omega''^{\sim}_{h+1})^c \cap \Omega_h)\chi'. \tag{1.89}$$

We have assumed that $\beta \leqq 1/4$. The equality $\chi_h(\Omega_h \cap Z_h) = 1$ is immediate, so we have to prove that $\chi''_k = 1$. We have already done several reasonings of this type, so now we will sketch only main points. Take a cube $\square \subset (\Omega''^{\sim}_{h+1})^c \cap \Omega_h$, and represent the functions $U''_{k,Z}$ on $\square^{\sim}$ in the usual way:

$$\begin{aligned} U''_{k,Z} &= U(\mathbb{B}_h(\square^{\sim 4}), [M^{\bullet}(U''_{k,Z})(M^{\bullet}(Q_h^{s*}V''))^{-1}]M^{\bullet}(Q_h^{s*}V'')) \\ &= \left(\exp iL^{k-h}\eta \mathbb{H}_{h,\square}\left(\frac{1}{i}\log[M^{\bullet}(U''_{k,Z})(M^{\bullet}(Q_h^{s*}V''))^{-1}]\right)U_{h,\square}(V'')\right)^{u_{h,\square}^{-1}}. \end{aligned} \tag{1.90}$$

The function $\mathbb{H}_{h,\square}$ and it derivatives can be bounded on $\square^{\sim}$ by $B_3\exp(-\delta 2LM_2R_h)11d^2\varepsilon_h < 11d^2B_3\exp(-R_h)\varepsilon_h < \alpha\varepsilon_h$, where α is a small, absolute constant, which will be fixed later. Estimating as in (1.46) we get

$$\begin{aligned} |U''_{k,Z}(\partial p) - 1| &< |U_{h,\square}(V'', \partial p) - 1|(1 + L^{-h}\alpha\varepsilon_h) \\ &\quad + (\alpha + 8\alpha^2\varepsilon_h)\varepsilon_h(L^{k-h}\eta)^2 \quad \text{for} \quad p \subset \square^{\sim}. \end{aligned} \tag{1.91}$$

The field V'' is equal to V_h on $(\Omega''^{\sim 2}_{h+1})^c \cap \square^{\sim 4}$, and on $\Omega''^{\sim 2}_{h+1} \cap \square^{\sim 4}$ it is equal to

$$V'' = V'M^h(U_0^{(AL)}) = \exp iB' \exp i\tilde{Q}_h(\eta \mathbb{A}_0). \tag{1.92}$$

Expanding $U_{h,\square}(V'')$ with respect to the above field, we get

$$U_{h,\square}(V'') = \left(\exp iL^{k-h}\eta\mathbb{H}_{h,\square}\left(\frac{1}{i}\log V''\lceil_{\Omega''^{\sim 2}_{h+1}}\right)U_{h,\square}(1, V_h)\right)^{u'^{-1}_{h,\square}}. \tag{1.93}$$

The function $\mathbb{H}_{h,\square}$ above, and its derivatives, can be bounded on $\square^{\sim}$ by

$$\begin{aligned} B_3\delta'_k + L^{-(k-h)}O(1)B_3^3B_5M^6\varepsilon_k &\leqq B_3(1+\beta_0)N^{1/2}\frac{A_1}{A_0}\frac{p_1(g_h)}{p_0(g_h)}\varepsilon_h \\ &\quad + L^{-N}N^{1/2}O(1)B_3^3B_5M^6\varepsilon_h < \alpha\varepsilon_h. \end{aligned} \tag{1.94}$$

The last inequality holds under two restrictions on N. At first, we assume that $N \leqq O(1)(\log g_k^{-2})^v \leqq O(1)(1+\beta_0)(\log g_h^{-2})^v$ with a positive integer v satisfying $(1/2)v \leqq p_0 - p_1 - 1$. The second is that N has to be sufficiently large, so that the constant in the second term above can be bounded by $(1/2)\alpha$. These two conditions can be satisfied by N to the positive power of $\log g_k^{-2}$. From the representation

(1.93) and the above bounds we get

$$|U_{h,\square}(V'', \partial p) - 1| < |U_{h,\square}((1, V_h), \partial p) - 1|(1 + L^{-h}\alpha\varepsilon_h) + (\alpha + 8\alpha^2 2\varepsilon_h)\varepsilon_h(L^{k-h}\eta)^2$$
$$< \tfrac{1}{2}(1 + L^{-h}\alpha\varepsilon_h)\varepsilon_h(L^{k-h}\eta)^2 + (\alpha + 8\alpha^2\varepsilon_h)\varepsilon_h(L^{k-h}\eta)^2. \tag{1.95}$$

The estimates (1.91), (1.95) yield

$$|U''_{k,Z}(\partial p) - 1| < \tfrac{3}{4}\varepsilon_h(L^{k-h}\eta)^2 < (1 - \beta(1 - 2^{-(k-h+1)}))\varepsilon_h(L^{k-h}\eta)^2 \tag{1.96}$$

for $p \subset \square^{\sim}$, hence on the whole domain $(\Omega''^{\sim}_{h+1})^c \cap \Omega_h$. We have chosen $\alpha = 1/12$ in (1.91), (1.94), (1.95).

Consider now the function $U''_{k,Z}$ on the domain $\Omega''^{\sim}_{h+1} \cap \Omega^c_k$. We localize it in the domain $Z \cap \Omega''^{\sim 2}_{h+1}$, introducing the usual boundary conditions at the boundary $\partial\Omega''^{\sim 2}_{h+1}$ through the determining set $\mathbb{B}_h(\Omega''^{\sim 2}_{h+1})$. We write the identity corresponding to (1.90), with $\mathbb{B}_h(\square^{\sim 4})$ replaced by $\mathbb{B}''_{k,Z} \cup \mathbb{B}_h(\Omega''^{\sim 2}_{h+1})$, and with the argument

$$M^{\boldsymbol{\cdot}}(U''_{k,Z}) = [M^{\boldsymbol{\cdot}}(U''_{k,Z})(M^{\boldsymbol{\cdot}}(U_0^{(AL)}))^{-1}]M^{\boldsymbol{\cdot}}(U_0^{(AL)}).$$

The above field is equal to V'' outside the layer of thickness $2M_1$ (in L^{-h}-scale) at the boundary $\partial\Omega''^{\sim 2}_{h+1}$. Denote this layer by $\sum$. We expand the function with respect to $(1/i)\log[\cdots]\restriction_\Sigma$ by the usual formula. This field can be bounded by

$$22d^2 \max\{1, L^{-2N}N^{1/2}O(1)B_3B_5M^5(1 + \beta_0)\}\varepsilon_h + 2\delta'_k$$
$$\leqq 22d^2\varepsilon_h + 2(1 + \beta_0)N^{1/2}\frac{A_1}{A_0}\frac{p_1(g_h)}{p_0(g_h)}\varepsilon_h < 23d^2\varepsilon_h.$$

The $\mathbb{H}$-function in the expansion can be bounded on the domain $Z''_{j+1}\backslash Z''_j$ for $h < j < k$, $Z''_{h+1}\backslash(\Omega''^{\sim}_{h+1})^c$ for $j = h$, and $\Omega^c_k\backslash Z''_k$ for $j = k$, by

$$B_3\exp\left(-\delta\frac{M}{M_1}(j-h)\right)\exp(-\tfrac{1}{2}\delta MR_h)23d^2\varepsilon_h$$
$$\leqq 23d^2B_3(1 + \beta_0)^2(1 + (j-h)^{\beta_0})\exp(-(j-h))\exp(-R_h)\varepsilon_j < \alpha\varepsilon_j.$$

This yields the bound (1.91) with h replaced by j, and the configuration $U_{h,\square}(V'')$ replaced by

$$U(\mathbb{B}''_{k,Z} \cup \mathbb{B}_h(\Omega''^{\sim 2}_{h+1}), (\exp iB'M^{\boldsymbol{\cdot}}(U_0^{(AL)})\restriction_{\Sigma^c}, M^{\boldsymbol{\cdot}}(U_0^{(AL)})\restriction_\Sigma)). \tag{1.97}$$

We expand the above function with respect to the variables B'. The corresponding $\mathbb{H}$-function can be bounded on the same domains as above by

$$B_3\delta'_k \leqq B_3(1 + \beta_0)(1 + (k-j)^{1/2})\frac{A_1}{A_0}\frac{p_1(g_j)}{p_0(g_j)}\varepsilon_j < \alpha\varepsilon_j.$$

The plaquette variables for the configuration (1.97) are bounded again as in (1.91), with h replaced by j, and with $U_{h,\square}(V'')$ replaced by (1.97) for $B' = 0$. This configuration is equal to $U_0^{\bar{u}_0}$, where $\bar{u}_0$ is a gauge transformation constant in blocks of the determining set of (1.97), the constant equal to the value of u_0 at centers of the blocks. The plaquette variables of $U_0^{\bar{u}_0}$ satisfy the estimate (1.80). Combining the above estimates, and using the inequality

$$\varepsilon_k\eta^2 \leqq (1 + \beta_0)(k-j)^{1/2}L^{-2(k-j)}\varepsilon_j(L^{k-j}\eta)^2 \leqq L^{-(k-j)}\varepsilon_j(L^{k-j}\eta)^2$$

for $j<k$, we obtain

$$\begin{aligned}|U''_{k,Z}(\partial p)-1| &< (2\varepsilon_k\eta^2+O(1)B_3B_5M^5\exp(-\delta\operatorname{dist}(p,\Lambda))\varepsilon_k\eta^2)\\ &\quad\cdot(1+L^{-j}\alpha\varepsilon_j)^2+(2+L^{-j}\alpha\varepsilon_j)(\alpha+8\alpha^2\varepsilon_j)\varepsilon_j(L^{k-j}\eta)^2\\ &\leqq (2L^{-(k-j)}+4\alpha+O(1)B_3B_5M^5\\ &\quad\cdot\exp(-\delta\operatorname{dist}(p,\Lambda))L^{-(k-j)})\varepsilon_j(L^{k-j}\eta)^2,\end{aligned} \tag{1.98}$$

on the j^{th} domain described above. Now we have to analyze this bound on all the domains in the definition (1.24). Consider $\Omega_m\backslash\Omega_{m+1}$ for $k_0<m<k$. We have $j=k$, and the exponential can be bounded by

$$\begin{aligned}&\exp(-\delta(L^{-(k-m)}8MR_m+\dots+L^{-(k-k_0-2)}8MR_{k_0+2}+L^{-(k-k_0-1)}4MR_{k_0+1}))\\ &\quad\leqq\exp(-4\delta(m-k_0)M),\end{aligned}$$

hence the last term in the bound (1.98) can be made arbitrarily small for M large enough. Making it smaller than α, and taking $\alpha\leqq 1/8$, we estimate the right-hand side of (1.98) by $(21/8)\varepsilon_k\eta^2\leqq 3(1-\beta(1/2))\varepsilon_k\eta^2$ for $m=k-1$, and $(21/8)\varepsilon_k\eta^2\leqq(1-\beta(1/2))L_0^{2(k-m-1)}\varepsilon_k\eta^2$ for the remaining m. Next, on the domain $\Omega^c_{k_0+1}\cap\Omega''^{\sim}_{h+1}$ the right-hand side of (1.98) can be bounded by $(4\alpha+O(1)B_3B_5M^5L_0^{-2(k-j)})\varepsilon_j(L^{k-j}\eta)^2$, with an increased $O(1)$. On the domain $Z''_{j+1}\backslash Z''_j$ for $k_0<j<k$, and on $\Omega^c_{k_0+1}\backslash Z''_k$ for $j=k$, we have

$$(4\alpha+O(1)B_3B_5M^5L_0^{-2(k-j)})\varepsilon_j(L^{k-j}\eta)^2\leqq(1-\beta)L_0^{2(j-k_0-1)}\varepsilon_j(L^{k-j}\eta)^2$$

if $O(1)B_3B_5M^5L_0^{-2(N_0-1)}\leqq 1/4$. From the definition of the number N_0 (recall that it is defined by the equation $L^{-N_0+1}R_{k-N_0+1}=1$) we get

$$O(1)B_3B_5M^5L_0^{-2(N_0-1)}\leqq O(1)B_3B_5M^5L_0^2R_k^{-\alpha_0},$$

where $\alpha_0=(\log L_0^2)/(\log L)$. The number on the right-hand side is $\leqq 1/4$ for $R_k^{-\alpha_0}$, or γ sufficiently small. Finally, on the domain $Z''_{j+1}\backslash Z''_j$ for $h<j\leqq k_0$, and on $Z''_{h+1}\cap\Omega''^{\sim}_{h+1}$ for $j=h$, we have

$$(4\alpha+O(1)B_3B_5M^5L_0^{-2(k-j)})\varepsilon_j(L^{k-j}\eta)^2\leqq(1-\beta)\varepsilon_j(L^{k-j}\eta)^2,$$

because $k-j\geqq N_0$ and we use the above bound again. Thus we have proved that the configuration $U''_{k,Z}$ satisfies all the conditions in the definition (1.24), hence $\chi''_k=1$, and the equality (1.89) is proved.

Let us now summarize the result of all the transformations we have done on the density ρ_k. We have obtained a density, which may not be equal to ρ_k, but is equivalent to it, in the sense that they have equal integrals. This density is written as a sum over large field regions. The components of the regions are divided into two classes: the components satisfying the conditions (i), (ii), for which the corresponding integral operations are given by the integrals in (1.76), and the remaining components, for which the integral operations have many forms, varying from the old $\mathbb{T}$-operations to the integrals as in (1.76), but with some large field characteristic functions, through the intermediate operations described above. These operations are denoted by $\mathbb{T}''_k$. Thus we have obtained the equivalence

$$\rho_k(V_k)\equiv\sum_{\{\Omega_j,\Lambda_j\}}\chi_k(\Omega_k^{\sim 4})\chi_{k,\Lambda}\mathbb{T}''_k(Z_k\backslash Z)\int dV_h\big[_{(\Omega''^{\sim 2}_{h+1})^c\cap Z}\chi_{h,1/2}$$

$$\cdot((\Omega''^{\sim}_{h+1})^c \cap \Omega_h \cap Z)\mathbb{T}_h(Z_h \cap Z)$$
$$\cdot \int dV' \ulcorner_{B_0} \delta_{T_0}(V')\chi' \exp A''_k. \tag{1.99}$$

The operation $\mathbb{T}''_k$ above includes a summation over all possible forms of this operation in various components of $Z_k \backslash Z$. Now we rearrange the sum above. We denote $Z_k \backslash Z$ by Z_k, we write Z explicitly as a union of components, $Z = X_1 \cup \cdots \cup X_n$, and we separate the summation over the admissible Z_k, n, $X_1, \ldots, X_n$, from the remaining summations, which are factorized in those domains. In such a representation it is natural to replace the summation over $\{\Omega_j, \Lambda_j\}$ by summations over the corresponding sequences $\{\Omega^c_j, Z_j\}$ localized in the components. Denote the sequence localized in X_i by $\{\Omega^c_{i,j}, Z_{i,j}\}$, and $\mathbb{B}_0$, T_0, Λ localized in X_i by $\mathbb{B}_i$, T_i, Λ_i. The integrations in (1.99) are also factorized in those components. This way we write the expression in (1.99) in a form similar to a polymer expansion, suggesting explicitly localization operations and an exponentiation. Finally, we are ready to define an operation, which is not a complete $\mathbb{R}$-operation yet, but is a basic part of it:

$$(\mathbb{R}'\rho_k)(V_k) = \sum_{Z_k} \Bigg(\sum_{\{\Omega^c_j, Z_j\}} \mathbb{T}''_k(Z_k) \Bigg) \Bigg\{ \sum_{n \geq 0} \sum_{\{X_1, \ldots, X_n\}} \chi_k(\Omega_k^{\sim 4}) \prod_{i=1}^n \Bigg[\frac{1}{N_i} \sum_{\{\Omega^c_{i,j}, Z_{i,j}\}, G_i, T_i} \chi_{k,\Lambda_i}$$
$$\cdot \frac{\delta_{G_i}(V'_k)\chi(\Lambda_i) \exp\left[-\frac{1}{g_k^2} A(\zeta_i, U_{k,X_i}(V'_k V_{\Lambda_i})) \right]}{\int dV' \ulcorner_{\Lambda_i} \delta_{G_i}(V')\chi(\Lambda_i) \exp\left[-\frac{1}{g_k^2} A(\zeta_i, U_{k,X_i}(V' V_{\Lambda_i})) \right]}$$
$$\cdot \int dV_h \ulcorner_{(\Omega''^{\sim 2}_{i,h+1})^c} \chi_{h,1/2}((\Omega''^{\sim}_{i,h+1})^c \cap \Omega_{i,h}) \mathbb{T}_h(Z_{i,h}) \int dV' \ulcorner_{B_i} \delta_{T_i}(V')\chi'_i \Bigg] \exp A''_k \Bigg\}, \tag{1.100}$$

where G_i is a graph in Λ_i fixing the axial gauge, $\zeta_i \in C_0^\infty(X_i)$, ζ_i changes from 0 to 1 in a neighborhood of $\partial X_i^{\sim -2}$, and $\chi(\Lambda_i)$ is given by

$$\chi(\Lambda_i) = \chi\left(\left\{ \left| \frac{1}{i} \log V'(b) \right| < M_0 \varepsilon_k \quad \text{for} \quad b \in \Lambda_i \right\} \right). \tag{1.101}$$

Because of the gauge fixing terms the expressions in (1.100) are not Euclidean invariant, and we have introduced the averaging over families of graphs, such that the averaged expressions are invariant. With our prescription of building the graphs, dependent on the chosen coordinate system, these averages can be replaced by averages over Euclidean rotations leaving the lattice invariant, or even simpler, by averages over $d!$ permutations of coordinates, and 2^d reflections in subsets of coordinates. Then $N_i = 1/(2^d d!)$. We can also choose other ways of fixing a gauge, generalizing the Landau gauge, which are Euclidean invariant, but they are analytically much more complicated.

The above operation has the fundamental normalization property

$$\int dV_k (\mathbb{R}'\rho_k)(V_k) = \int dV_k \rho_k(V_k). \tag{1.102}$$

In fact, the $\mathbb{R}'$-operation changes essentially the initial density ρ_k only in a neighborhood of the large field region; the changes are decaying exponentially fast with the distance to the region. It is now clear what the next operations are. At first, we have to extract from the expression in the curly bracket $\{\cdots\}$ the density $\exp A_k$, where A_k is the effective action determined by the assumption that Z_k is the only large field region. The remaining expression should be localized around the components $X_1, \ldots, X_n$, represented as a polymer expansion, and finally exponentiated. All these operations will be described in the next paper.

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