Commun. Math. Phys. 122, 171–173 (1989)

Communications in
Mathematical
Physics


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# Redundancy of Conditions for a Virasoro Algebra*

Jack L. Uretsky

High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439, USA
Department of Natural Sciences, College of DuPage, Glen Ellyn, IL 60137, USA

**Abstract.** I show that the Fairlie, Nuyts, Zachos construction of Virasoro algebra contains redundant conditions.

Fairlie et al. [1] construct a Virasoro algebra from two starting generators and eight conditions on the commutators. I show that the eight conditions are not independent.

The authors of [1] start with two generators called $L_3$ and $L_{-2}$ and the following definitions:

$$\text{D1} \quad 5L_1 = [L_3, L_{-2}],$$

$$\text{D1} \quad 3L_{-1} = [L_1, L_{-2}],$$

$$\text{D1} \quad 2L_0 = [L_1, L_{-1}],$$

$$\text{D1} \quad 4L_2 = [L_2, L_{-1}],$$

$$\text{D1} \quad (n-1)L_{n+1} = [L_n, L_1] \qquad n \geqq 3,$$

$$\text{D1} \quad (n+1)L_{n-1} = [L_n, L_{-1}] \qquad n \leqq -2.$$

The authors then impose 8 conditions. I shall limit my discussion to positive values of the index $n$. The conditions that will be of interest are, then,

$$\text{C1} \quad [L_3, L_0] = 3L_3 \qquad \text{(Cond. 1 of ref. [1])},$$

$$\text{C2} \quad [L_0, L_{-2}] = 2L_{-2} \qquad \text{(Cond. 2 of ref. [1])},$$

$$\text{C3} \quad [L_2, L_{-2}] = 4L_0 + 6c \qquad \text{(Cond. 4 of ref. [1])},$$

$$\text{C4} \quad [L_2, L_1] = L_3 \qquad \text{(Cond. 3 of ref. [1])},$$

$$\text{C5} \quad [L_3, L_2] = L_5 \qquad \text{(Cond. 5 of ref. [1])},$$

$$\text{C6} \quad [L_5, L_2] = L_5 \qquad \text{(Cond. 5 of ref. [1])}.$$

* Work supported in part by the U.S. Department of Energy, Division of High Energy Physics, Contract W-31-109-ENG-38

In C3, $c$ is an arbitrary constant.

The definitions, conditions C1 and C2 and the Jacobi identities imply Lemmas 1 and 2 of ref. 1, namely,

**Lemma FNZ 1.** $[[L_m, L_n], L_0] = (m+n)[L_m, L_n]$.

**Lemma FNZ 2.** $[L_m, L_0] = mL_m$.

I shall show that C4 is implied by the definitions and the other three conditions.

Define, without prejudice, an operator $K_3$ according to

$$[L_2, L_1] = K_3\,. \tag{1}$$

Using the definition of $L_1$, the Jacobi identities, and the conditions other than C4 gives

$$5K_3 = 12L_3 - [L_5, L_{-2}]\,. \tag{2}$$

The following statements, which I list as lemmas, are consequences of the definitions and the two previously lemmas.

**Lemma 1.** $[L_4, L_{-1}] = \frac{1}{2}[[L_3, L_1], L_{-1}] = 2K_3 + 3L_3$.

**Lemma 2.** $[L_4, L_{-2}] = \frac{1}{2}[[L_3, L_1], L_{-2}] = \frac{1}{2}[L_3, [L_1, L_{-2}]] = 6L_2$.

Lemmas 1 and 2 and the definition of $L_5$ lead to.

**Lemma 3.** $[L_5, L_{-2}] = \frac{1}{3}[[L_4, L_1], L_{-2}] = 24K_3 + 3L_3$.

Substitution of the last relation into Eq. 2 gives the result that

$$K_3 = L_3\,. \tag{3}$$

I have shown that C1 through C3 and C5 imply C4. Alternatively, I show that C1 through C4 and C6 imply C5. The steps of the proof may be applied symmetrically to the negative index conditions to likewise show the redundancy of condition 7 of [1]. I use the following lemma which is readily proved by induction:

**Lemma 4.** $[L_n, L_{-1}] = (n+1)L_{n-1}$, $n \geqq 0$.

Then, using Lemma 1, the definitions and the Jacobi identities,

$$[L_3\, L_2] = (\tfrac{1}{5})\,[[L_4, L_{-1}], L_2] = (\tfrac{1}{5})([[L_4, L_2], L_{-1}] - 9L_5)\,, \tag{4}$$

$$[L_4, L_2] = (\tfrac{1}{6})\,[[L_5, L_{-1}], L_2] = 2L_6\,, \tag{5}$$

from which condition C5 follows. Condition C6 was used in obtaining Eq. (5).

If conditions C4 and C5 are now both abandoned, then Lemmas 1 and 3 are specific cases of two general relationships that are provable by induction, namely

**Lemma 5.** $[L_n, L_{-1}] = \left(n+1-\dfrac{4}{n-2}\right)L_{n-1} + \left(\dfrac{4}{n-2}\right)K_{n-1}$, $n \geqq 4$, *and*

**Lemma 6.** $[L_n, L_{-2}] = \left(n+2-\dfrac{12}{n-2}\right)L_{n-2} + \left(\dfrac{12}{n-2}\right)K_{n-2}$, $n \geqq 5$. *Here I have defined*

$$(n-1)K_{n+1} = [K_n, L_1] \quad \textit{for} \quad n \geqq 3\,. \tag{6}$$

The relationships in Lemmas 5 and 6 impose consistency requirements upon any conditions that might replace C4 or C5, that are placed upon the commutators of the algebra. These consistency requirements arise by virtue of the fact that the $L_n$'s and $K_n$'s for $n$ greater than 2 may be approached from below – because of the defining relation $D+$ (and Eq. 6) – and from above – because of Lemmas 5 and 6. It appears to be an open question whether the remaining conditions, and their negative integer counterparts, are sufficient to define a commutator algebra.

*Acknowledgements.* I am greatly indebted to Dr. C. K. Zachos for many stimulating discussions and helpful suggestions, and to the High Energy Physics Division of Argonne National Laboratory for its hospitality.

## References

1. Fairlie, D.B., Nuyts, J., Zachos, C.K.: A presentation for the Virasoro and Super-Virasoro algebras. Commun. Math. Phys. **117**, 595 (1988)

Communicated by L. Alvarez-Gaumé

Received September 1, 1988