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A One-Dimensional N Fermion Problem with Factorized S Matrix

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Abstract. A one dimensional N Fermion problem with attractive or repulsive δ function interaction is solved by Bethe's hypothesis. The S matrix factorizes and is explicitly given.

1. Introduction

We report a one dimensional N Fermion problem for which the S matrix is completely solved. The solution depends on extensive uses of the Yang-Baxter equation [1]. The corresponding problem for Boltzmann statistics was solved [2] in 1968, but specialization to Fermi statistics is algebraically involved and was never done. Here we approach the problem directly without going through the Boltzmann case.

The Hamiltonian for the problem is

$$H=-\sum_i \frac{\partial^2}{\partial x_i^2}+2c\sum_{i<j}\delta(x_i-x_j), \qquad (i,j=1,2,\ldots,N), \tag{1}$$

where $c=$real. Each particle has m "spin" states designated by $s_1, s_2 \ldots s_N$ where

$$1 \leqq s_i \leqq m. \tag{2}$$

The Schrödinger equation is $H\psi=E\psi$, where

$$\psi=m^N \times 1, \qquad \text{(column)}. \tag{3}$$

For the Fermion problem we are only interested in wave functions ψ that are antisymmetrical with respect to the interchange:

$$Q^{ij}:(x_i, s_i)\leftrightarrow(x_j, s_j). \tag{4}$$

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The Hamiltonian commutes with the full $m^N \times m^N$ unitary matrices $SU(m^N)$ that operate on ψ. But these matrices in general do not commute with Q^{ij}. A subgroup of $SU(m^N)$ consisting of identical $SU(m)$'s [operating on each subspace designated by s_i] does commute with Q^{ij}. Thus the Fermion problem for the Hamiltonian (1) has $SU(m)$ symmetry.

If we choose $m=4$ and identify the four states as

$$p\uparrow, p\downarrow, n\uparrow, n\downarrow \tag{5}$$

we have a one dimensional $SU(4)$ model of nucleons interacting through a δ-function interaction.

2. Bethe's Hypothesis

Consider the scattering of two particles with initial momenta k_1 and k_2 into states with final momenta k_1' and k_2'. Momentum and energy conservation give

$$\begin{aligned} k_1' + k_2' &= k_1 + k_2 , \\ k_1'^2 + k_2'^2 &= k_1^2 + k_2^2 , \end{aligned} \tag{6}$$

which has two and only two solutions

$$(k_1' k_2') = (k_1, k_2) \quad \text{or} \quad (k_2, k_1) .$$

The two solutions are *reflections* of each other in the (k_1, k_2) plane with respect to the mirror $k_1 = k_2$.

For the scattering of three particles, momentum and energy conservation still give two equations

$$\begin{aligned} k_1' + k_2' + k_3' &= k_1 + k_2 + k_3 , \\ k_1'^2 + k_2'^2 + k_3'^2 &= k_1^2 + k_2^2 + k_3^2 , \end{aligned} \tag{7}$$

which has obviously the following six special solutions:

$$(k_1', k_2', k_3') = (k_1, k_2, k_3) \quad \text{or 5 other permutations} . \tag{8}$$

The six solutions exhibited in (8) represent *reflections* of each other in (k_1, k_2, k_3) space. But there are many other solutions of (7) which represent *diffractions* in (k_1, k_2, k_3) space. The quantum mechanical three body scattering problem would in general yield outgoing states including both *diffracted* and *reflected* waves, and is therefore difficult to solve.

However, in some special cases, the outgoing waves consist of *only reflected waves*, a hypothesis first proposed by Bethe [3, 4]. If the hypothesis works, the solution of the Schrödinger equation becomes an *algebraic* problem, as we shall illustrate in the present work.

3. $N=2$

For two particles the Schrödinger equation describes two free particles except on the line $x_1 = x_2$ in Fig. 1. Bethe's hypothesis states that

in region I $(x_1 < x_2)$:

$$\psi = \alpha_{12} e^{i(k_1 x_1 + k_2 x_2)} + \alpha_{21} e^{i(k_2 x_1 + k_1 x_2)} , \tag{9}$$

where α_{12} and α_{21} are $m^2 \times 1$ column matrices. The antisymmetrization requirement for ψ says that

in region II $(x_2 < x_1)$:

$$\psi = -(P^{12}\alpha_{12})e^{i(k_1 x_2 + k_2 x_1)} - (P^{12}\alpha_{21})e^{i(k_2 x_2 + k_1 x_1)}, \tag{10}$$

where

$$P^{12} = \text{operator on the } m^2 \times 1 \text{ column that interchanges } s_1 \leftrightarrow s_2 . \tag{11}$$

Now we transform to the center of mass coordinate X, and the relative coordinate y:

$$X = \tfrac{1}{2}(x_1 + x_2), \qquad y = x_2 - x_1 . \tag{12}$$

The Schrödinger equation becomes

$$H\psi = \left[-\frac{1}{2}\frac{\partial^2}{\partial X^2} - 2\frac{\partial^2}{\partial y^2} + 2c\delta(y) \right] \psi = E\psi .$$

H commutes with $i\dfrac{\partial}{\partial X}$. Hence we can put $\dfrac{\partial}{\partial X} = 0$, obtaining

$$\left[-2\frac{\partial^2}{\partial y^2} + 2c\delta(y) \right] \psi = E\psi . \tag{13}$$

Thus ψ is continuous at $y = 0$:

$$\alpha_{12} + \alpha_{21} = -P^{12}(\alpha_{12} + \alpha_{21}), \tag{14}$$

and $\dfrac{\partial^2}{\partial y^2}\psi$ has a δ-function singularity at $y = 0$. Integrating (13) from $y = 0-$ to $y = 0+$ we obtain

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=0+} - \left.\frac{\partial\psi}{\partial y}\right|_{y=0-} = c\psi|_{y=0}$$

Substituting (9) and (10) into this equation results in

$$\frac{i}{2}(1 - P^{12})(k_1 - k_2)(-\alpha_{12} + \alpha_{21}) = c(\alpha_{12} + \alpha_{21}) . \tag{15}$$

Eliminating the term $P^{12}\alpha_{21}$ from (14) and (15) leads to

$$[i(k_1 - k_2) - c]\alpha_{21} = [-i(k_1 - k_2)P^{12} + c]\alpha_{12} . \tag{16}$$

Thus

$$\alpha_{21} = Y\alpha_{12}, \tag{17}$$

where

$$Y = \frac{-i(k_1 - k_2)P^{12} + c}{i(k_1 - k_2) - c} \tag{18}$$

if the denominator is nonvanishing. It is easy to check that if (17) is satisfied, then (14) and (15) are satisfied.

Thus if $k_1 \neq k_2$ and are real, (9) and (10) give indeed a solution of the Schrödinger equation satisfying Bethe's hypothesis.

4. $N \geqq 3$

For $N \geqq 3$, Bethe's hypothesis states that in one of the regions, where $x_1 < x_2 < \ldots < x_N$ we have a generalization of (9):

$$\psi = \alpha_{12\ldots N} e^{i(k_1x_1 + k_2x_2 + \ldots + k_Nx_N)} + \alpha_{21\ldots N} e^{i(k_2x_1 + k_1x_2 + \ldots + k_Nx_N)} + (N! - 2) \text{ other terms}. \tag{19}$$

The columns α have the dimension $m^N \times 1$. In other regions the wave function ψ is determined from (19) by the requirement of antisymmetrization. The energy is given by

$$E = k_1^2 + k_2^2 + \ldots + k_N^2. \tag{20}$$

Now we examine the Schrödinger equation along the plane $x_3 = x_4$. The $N!$ terms in (19) form pairs, with the terms containing $\alpha_{1234\ldots N}$ and $\alpha_{1243\ldots N}$ forming one pair, $\alpha_{37154\ldots 2}$ and $\alpha_{37514\ldots 2}$ forming another pair, etc. In each pair only the third and fourth subscripts are different. Exactly the same procedures apply to the two terms in each pair as to the two terms in (9) of the last section. Thus we obtain in the same way that we obtained (17) and (18):

$$\alpha_{..lj\ldots} = Y_{jl}^{34} \alpha_{..jl\ldots}, \tag{21}$$

where

$$Y_{jl}^{34} = \frac{-i(k_j - k_l)P^{34} + c}{i(k_j - k_l) - c}, \tag{22}$$

provided the denominator does not vanish.

5. Yang-Baxter Equation

The subscripts in Eq. (21) can be chosen in $N!$ different ways. The superscript 34 can be replaced by $12, 23, \ldots, (N-1)N$. Thus Eq. (21) actually is representative of $(N-1)(N!)$ different linear equations between the $N!$ columns $\alpha\ldots$. Are these equations mutually consistent? For example, for $N=3$,

$$\alpha_{123} = Y_{21}^{12}\alpha_{213} = Y_{21}^{12}Y_{31}^{23}\alpha_{231} = Y_{21}^{12}Y_{31}^{23}Y_{32}^{12}\alpha_{321},$$

but also

$$\alpha_{123} = Y_{32}^{23}\alpha_{132} = Y_{32}^{23}Y_{31}^{12}\alpha_{312} = Y_{32}^{23}Y_{31}^{12}Y_{21}^{23}\alpha_{321}.$$

Thus for Bethe's hypothesis to be consistent, we require

$$Y_{23}^{12}Y_{31}^{23}Y_{32}^{12} = Y_{32}^{23}Y_{31}^{12}Y_{21}^{23}.$$

More generally

$$Y_{ij}^{a,a+1}Y_{kj}^{a+1,a+2}Y_{ki}^{a,a+1} = Y_{ki}^{a+1,a+2}Y_{kj}^{a,a+1}Y_{ij}^{a+1,a+2}.$$

Now if a, b, c are unequal,

$$P^{ac} Y_{ij}^{ab} P^{ac} = Y_{ij}^{cb} = Y_{ij}^{bc} .$$

Thus

$$Y_{ij}^{ab} Y_{kj}^{bc} Y_{ki}^{ab} = Y_{ki}^{bc} Y_{kj}^{ab} Y_{ij}^{bc} , \tag{23}$$

$$Y_{ij}^{ab} Y_{ji}^{ab} = 1 ,$$

and

$$Y_{ij}^{ab} Y_{kl}^{cd} = Y_{kl}^{cd} Y_{ij}^{ab} , \quad \text{if } a, b, c, d \text{ are all unequal}.$$

These equations are now called the Yang-Baxter equations. In the present problem, they represent consistency conditions for Bethe's hypothesis.

Using definition (22) of the operators Y, we find that indeed Eqs. (23) are satisfied. Thus for real values of $k_1, \ldots, k_N$, all different from each other, the $(N-1)(N!)$ Eqs. (21) are consistent and define uniquely all the α's once one of them is given. From these α's we can then construct a wave function ψ.

Equation (23) can be cast [1] into a different form by defining

$$X_{ij}^{ab} = -Y_{ij}^{ab} P^{ab} = X_{ij}^{ba} . \tag{24}$$

One finds

$$X_{ij}^{ab} X_{ij}^{ba} = 1 ,$$

$$X_{ij}^{ab} X_{kj}^{cb} X_{ki}^{ca} = X_{ki}^{ca} X_{kj}^{cb} X_{ij}^{ab} , \tag{25}$$

$$X_{ij}^{ab} X_{kl}^{cd} = X_{kl}^{cd} X_{ij}^{ab} \quad \text{if } a, b, c, d \text{ are all unequal}.$$

6. *S*-Matrix

Consider N real k's ordered in the following way:

$$k_1 < k_2 < \ldots < k_N . \tag{26}$$

In each coordinate region, such as in

$$x_1 < x_2 < \ldots < x_N \tag{27}$$

we have one term in the wave function ψ in (19), namely,

$$\alpha_{12\ldots N} e^{i(k_1 x_1 + k_2 x_2 + \ldots + k_N x_N)} \tag{28}$$

which designates an *outgoing* wave. This is an outgoing wave because we can make a wave packet by superposing wave functions slightly different from (28). Such a wave packet centered at x_1, $x_2, \ldots,$ for the N particles at $t=0$ would move with velocities $2k_1, 2k_2, \ldots, 2k_N$. Thus x_1 moves slowest and x_N moves fastest, because of (26). Therefore in all future times the particles would be separated by larger and larger distances and would not collide with each other. Hence (28) is an *outgoing* wave.

Similarly in region (27) the *incoming* wave in (19) is

$$\alpha_{N(N-1)\ldots 1} e^{i(k_N x_1 + k_{N-1} x_2 + \ldots + k_1 x_N)} \tag{29}$$

because if one constructs a wave packet out of terms (29) with centers for the particles at $x_1, x_2, \ldots, x_N$ at $t=0$, the particles would move respectively with velocities $2k_N > 2k_{N-1} > \ldots > 2k_1$. Hence as we go *back* into negative time, they would never collide with each other.

To recapitulate, in region (27), (28) is the outgoing wave and (29) the incoming wave. Similarly in every other region, there is one outgoing and one incoming wave.

The incoming wave that has the same exponential as (28) is in the region

$$x_N < x_{N-1} \ldots < x_1 , \tag{30}$$

and is given by

$$[(-P^{1N})(-P^{2(N-1)}) \ldots] \alpha_{N(N-1)\ldots 1} e^{i(k_N x_N + \ldots + k_1 x_1)} \tag{31}$$

which we shall write as

$$\alpha_{\mathrm{inc}} e^{i(\Sigma k_j x_j)} . \tag{32}$$

We shall also write

$$\alpha_{\mathrm{out}} = \alpha_{12\ldots N} ,$$

so that (28) becomes

$$\alpha_{\mathrm{out}} e^{i(\Sigma k_j x_j)} . \tag{33}$$

The S matrix is defined by

$$\alpha_{\mathrm{out}} = S\alpha_{in}$$

and can be calculated from the relationship between $\alpha_{12\ldots N}$ and $\alpha_{N(N-1)\ldots 1}$ which is, according to (21),

$$\begin{aligned} \alpha_{12\ldots N} &= [Y_{21}^{12} Y_{31}^{23} \ldots Y_{N1}^{(N-1)N}] \alpha_{2\ldots N1} \\ &= [\ldots][Y_{32}^{12} Y_{42}^{23} \ldots Y_{N2}^{(N-2)(N-1)}] \alpha_{3\ldots N21} \\ &= \ldots \end{aligned}$$

Write

$$S' = [Y_{21}^{12} Y_{31}^{23} \ldots Y_{N1}^{(N-1)N}][Y_{32}^{12} Y_{42}^{23} \ldots Y_{N2}^{(N-2)(N-1)}] \ldots [Y_{N(N-1)}^{12}] .$$

Then

$$\alpha_{\mathrm{out}} = S' \alpha_{N(N-1)\ldots 1} = S' R [P^{N1} P^{(N-1)2} \ldots] \alpha_{\mathrm{inc}} ,$$

where

$$R = \text{parity of permutation } N(N-1)\ldots 1 .$$

Hence

$$\begin{aligned} S &= S' R [P^{N1} P^{(N-1)2} \ldots] \\ &= RS' [P^{N1} P^{(N-1)2} \ldots] \\ &= RS' [P^{12}][P^{23} P^{12}][P^{34} P^{23} P^{12}] \ldots [P^{(N-1)N} \ldots P^{12}] . \end{aligned}$$

Inserting the explicit form of S' into the right-hand side of this last equation we obtain a product of many Y's followed by an equal number of P's. The first factor in this product is Y_{21}^{12} and the last P^{12}. We now permute this last P^{12} forward through all the P's and Y's until it reaches just behind Y_{21}^{12}, forming with it, according to (24), $[-X_{21}^{21}]$. The new product now has as a last factor P^{13} which we now permute forward through all the P's and Y's until it reaches just behind the new first Y, forming with it, according to (24), $[-X_{31}^{31}]$. Continuing this way we obtain

$$S=[X_{21}X_{31}\dots X_{N1}][X_{32}X_{42}\dots X_{N2}]\dots[X_{N(N-1)}], \tag{34}$$

where

$$X_{ij}=X_{ij}^{ij}=\frac{i(k_i-k_j)-cP^{ij}}{i(k_i-k_j)-c}, \tag{35}$$

which is a special case of the X_{ij}^{ab} defined in (24). The matrix elements of (34) have the following meaning:

$$\langle s_1' s_2' \dots s_N'|S|s_1 s_2 \dots s_N\rangle = S\text{-matrix matrix element for the process}$$

$$[\text{state: } (k_1 s_1)(k_2 s_2)\dots(k_N s_N)]\to[\text{state: } (k_1 s_1')(k_2 s_1')\dots(k_N s_N')]. \tag{36}$$

For example, for $N=2$,

$$S=X_{21}=\frac{i(k_2-k_1)-cP^{12}}{i(k_2-k_1)-c}. \tag{37}$$

For $m=2$, this gives the following matrix elements:

$$AA\to AA, \quad BB\to BB, \quad \langle|S|\rangle=1, \tag{38}$$

$$AB\to AB, \quad BA\to BA, \quad \langle|S|\rangle=i|k_B-k_A|[i|k_B-k_A|-c]^{-1}, \tag{39}$$

$$AB\to BA, \quad BA\to AB, \quad \langle|S|\rangle=-c[i|k_B-k_A|-c]^{-1}. \tag{40}$$

Equation (38) is in agreement with the simple argument that for the reaction $AA\to AA$, the "spin" part of the wave function is symmetrical, so the space part must be antisymmetrical. Thus $\psi=0$ at $x_1=x_2$ rendering the potential energy $\delta(x_1-x_2)$ inoperative. Thus for $AA\to AA$, there is no interaction and S must be $=1$.

7. Wave Packet Interpretation of Yang-Baxter Equation

For 3 particles, the S matrix is, by (34),

$$S=X_{32}X_{31}X_{21}. \tag{41}$$

It is also equal to, because of the Yang-Baxter equation,

$$S=X_{21}X_{31}X_{32}. \tag{42}$$

One could try to interpret (41) and (42) by the particle density diagram in Fig. 1(a) and Fig. 2(b). In the wave packet construction in Fig. 2(a), the three body collision occurs in three steps of two body collisions, suggesting

$$S=X_{32}(k_3-k_2)X_{31}(k_3-k_1)X_{21}(k_2-k_1),$$

which is (41). Similarly for (42).

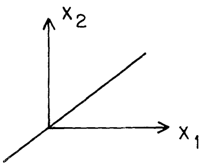

Fig. 1. Coordinate space for two particles. The interaction $\delta(x_1-x_2)$ is nonvanishing only along the line $x_1=x_2$

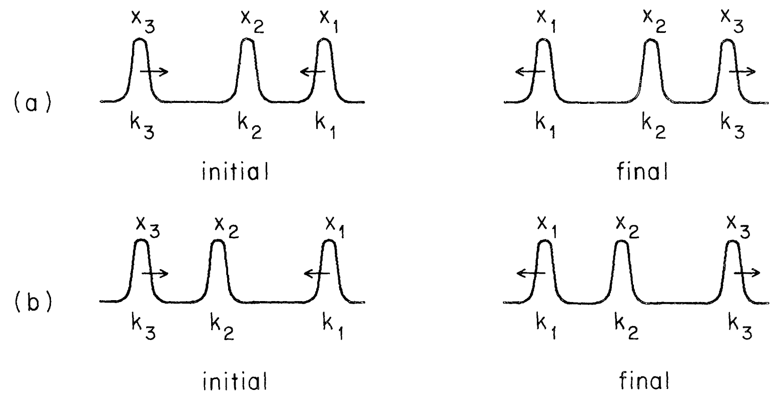

Fig. 2a and b. Particle density in coordinate space in collision of three particles, with momenta $k_1<k_2=0<k_3$. **a** Particles 2 and 1 collide first, then particles 3 and 1, then particles 3 and 2. **b** Particles 3 and 2 collide first, then particles 3 and 1, then particles 2 and 1

The above argument must be understood with great care: We had seen in Sect. 2 above that for two particles in one dimension, there is never diffraction because of energy and momentum conservation. An improper understanding of Fig. 2(a) may lead to the conclusion that for three particles there is also no diffraction, which is in general erroneous.

8. Bound States for $c<0$

For two particles, ignoring the "spin" index and Fermi statistics, the Schrödinger Eq. (13) in the relative coordinate $y=x_2-x_1$ has a bound state for the case $c<0$:

$$\psi=\exp\left[\tfrac{1}{2}c|x_2-x_1|\right]$$

for which

$$E=-\tfrac{1}{2}c^2 .$$

By inspection one obtains similarly for N particles a bound state:

$$\psi=\exp\left[\tfrac{1}{2}c\sum_{i>j}|x_i-x_j|\right] \tag{43}$$

with energy

$$E=-\frac{c^2}{12}N(N^2-1). \tag{44}$$

This bound state was first discovered by McGuire [3] who also noted that it is the only bound state for the N body problem with Boltzmann statistics.

The wave function (43) is symmetrical with respect to the exchange of any two coordinates $x_i \leftrightarrow x_j$. Since the Hamiltonian (1) is "spin" independent it follows that for N Fermions, one has a bound state by multiplying (43) with a spin wave function which is antisymmetrical:

$$\psi = \alpha_0 \exp\left[\tfrac{1}{2} c \sum_{i>j} |x_i - x_j|\right], \tag{45}$$

where

$$P^{ij}\alpha_0 = -\alpha_0, \quad (\text{any } i \neq j). \tag{46}$$

Furthermore this is the only bound state for the N body Fermion problem.

It is obvious that if $m < n$ there exists no "spin" function that satisfies (46), while if $m \geqq N$, there exist C_N^m linearly independent solutions of (46).

Does the wave function (45) satisfy Bethe's hypothesis? The answer is yes, because in each region, e.g. $x_1 x_2 < \ldots < x_N$, (45) is of the form (19) with imaginary valves of $k_1, k_2 \ldots k_N$. It is obvious that

$$k_1 = \frac{i}{2} c(N-1), \quad k_2 = k_1 - ic, \quad k_3 = k_2 - ic, \ldots, k_N = -k_1, \tag{47}$$

which is plotted in Fig. 3.

If we take as a model $m = 4$ and consider the bound states for $N = 1, 2, 3, 4, 5$, we obtain the states listed in column 2 of Table 1. The model has $SU(4)$ symmetry. In

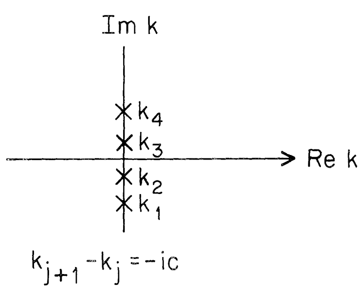

Fig. 3. The "momenta" for the N particle bound state

Table 1. Bound states in model with $SU(4)$ symmetry compared with ground states of light nuclei. In the model we take $c = -2$. The binding energies of the model are listed in column 4. Columns 5 and 6 list ground states (in *italics*) of light nuclei. I = isotopic spin, J = total spin

N	States	$SU(4)$ Represen-tation	Binding Energy	Nuclear States	Nuclear Binding Energy (MeV)
1	A, B, C, D	$1+0+0+0$	0	p, p, n, n	0
2	AB, AC, AD	$1+1+0+0$	2	d: $(I=0, J=1)$	2
	BC, BD, CD			(pp): $(I=1, J=0)$	0
3	$ABC, ABD,$	$1+1+1+0$	8	He^3	8
	ACD, BCD			H^3	
4	$ABCD$	$1+1+1+1$	20	He^4	28
5	(no bound states)		(no bound states)		

columns 5 and 6 are listed the ground states for real light nuclei. The number of bound states for each N and their binding energies are quite similar to those of the model. That light nuclei observe approximate $SU(4)$ symmetry was first discussed by Wigner [5].

Why are the binding energies of the model so similar to that of real three dimensional nuclei? The model is very different from real nuclei in many respects. Two of these are especially important: (1) The model is in one dimension while real nuclei are in three dimensions. The lower dimensionality enhances the "Fermi-Dirac" repulsion for the model. (2) The model does not have a hard core repulsion which is present between nucleons. These two differences have opposite signs and apparently cancel each other rather effectively.

9. Scattering of Bound States ($c<0$)

The wave function (45) does satisfy Bethe's hypothesis and is of the form (19). Section 4 and 5 above apply to this wave function. But the α's are all zero except for $\alpha_{12\ldots N}$. Equation (21) is still valid if we multiply both sides by the denominator $i(k_j-k_l)-c$ (which is sometimes zero.). We have

$$\begin{gathered}\alpha_{12\ldots N}=\alpha_0\neq 0\,, \qquad \text{all other } \alpha=0\,,\\ P^{12}\alpha_0=P^{23}\alpha_0=\ldots=P^{(N-1)N}\alpha_0=-\alpha_0\,.\end{gathered} \tag{48}$$

This bound state has total momentum zero. To give it a positive (negative) momentum, one simply displaces to the right (left) the k's of Fig. 3 by equal amounts, so that they remain on a vertical line. Condition (48) remains, and the wave function remains of the form (19).

Similarly we can write down the wave function that corresponds to the k distribution of Fig. 4(a), which represents a scattering of a bound state of two particles on a state of three particles, the former having a momentum of k_1+k_2, and the latter having a momentum of $k_3+k_4+k_5$. The wave function still satisfies Bethe's ansatz, but many of the columns are zero. In fact

$$\alpha_P=0\,, \quad \text{unless } P \text{ is of type } A\,, \tag{49}$$

where type A means in $P=[P1\ P2\ P3\ P4\ P5]$ 1 is to the left of 2, 3 to the left of 4, and 4 to the left of 5. Equation (21) remains valid if both sets of subscripts in the

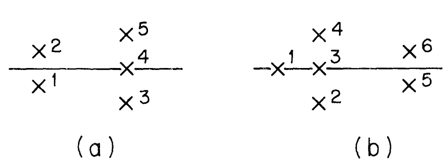

Fig. 4a and b. $k_1k_2\ldots k_N$ positions for scattering states. **a** A two particle bound state with momentum k_1+k_2 scattering on a three particle bound state with momentum $k_3+k_4+k_5$. **b** A particle of momentum k_1 and a three particle bound state of momentum $k_2+k_3+k_4$ scattering on a three particle bound state with momentum k_5+k_6. The difference between two successive k's in any vertical column is always $-ic$

equation are of type A, in which case definition (22) is meaningful. There is in addition the condition

$$-\alpha_{12345}=P^{12}\alpha_{12345}=P^{34}\alpha_{12345}=P^{45}\alpha_{12345}. \tag{50}$$

To analyse the incoming and outgoing parts of the wave function we follow the same procedure as Sect. 6 above, except that the incoming wave is not given by α_{54321} which is zero because of (49). Instead it is α_{34512}, (i.e. the block 345 to the left of the block 12, but within each block type A must obtain.) The S matrix is now

$$S=[X_{32}X_{42}X_{52}][X_{31}X_{41}X_{51}]. \tag{51}$$

This S matrix is understood to operate between states ϕ that satisfy the condition

$$-\phi=P^{12}\phi=P^{34}\phi=P^{45}\phi. \tag{52}$$

ϕ of course is a $m^N\times 1$ column matrix. It can be proved that if ϕ satisfies (52), then $S\phi$ also satisfies (52).

The above considerations can be generalized to the collision of many bound or unbound particles. For example for Fig. 4(b), we have a collision of an unbound particle with momentum k_1 and a bound particle of momentum $k_2+k_3+k_4$ with a bound particle of momentum k_5+k_6. We have in place of (51) and (52),

$$S=(X_{21}X_{31}X_{41}X_{51}X_{61})(X_{54}X_{64})(X_{53}X_{63})(X_{52}X_{62}), \tag{51$'$}$$

$$-\phi=P^{23}\phi=P^{34}\phi=P^{56}\phi. \tag{52$'$}$$

From (51) we can read off such matrix elements as

$$\begin{aligned}(AB)+(ACD)&\to(AB)+(ACD)\\&\to(AC)+(ABD)\\&\to(AD)+(ABC)\\(AB)+(CDE)&\to(AB)+(CDE)\\&\to(CD)+(ABE)\\&\quad\text{etc.}\end{aligned}$$

10. Properties of the S Matrix

The S matrix should be unitary between allowed states. Furthermore since the Hamiltonian is time reversal invariant, S should be symmetrical between allowed states. Both statements can be proved by repeated use of the Yang-Baxter equation

$$X_{ij}X_{kj}X_{ki}=X_{ki}X_{kj}X_{ij}.$$

The procedure is exactly as in [2] and will not be repeated here.

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References

1. Yang, C.N.: Phys. Rev. Lett. **19**, 1312 (1967)
2. Yang, C.N.: Phys. Rev. **168**, 1920 (1968)
3. Bethe, H.A.: Zeit. Phys. **71**, 205 (1931); L. Hulthen Arkiv. f. Matematik, Astr. Och Fysik **26**A, No. 11 (1938). The first application of Bethe's hypothesis to the Hamiltonian (1) for Bosons was in E. Lieb and W. Liniger. Phys. Rev. **130**, 1605 (1963) and McQuire, J. B.: J.Math. Phys. **5**, 622 (1964)
4. The name Bethe's hypothesis or Bethe's ansatz was coined by C. N. Yang and C. P. Yang in 1966. See Chen Ning Yang, Selected Papers with Commentary, p. 63 (Freeman, 1983)
5. Wigner, E.: Phys. Rev. **51**, 106 (1937)

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