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On the Construction of Monopoles for the Classical Groups*

Jacques Hurtubise[1] and Michael K. Murray[2]

[1] School of Mathematics, Institute for Advanced Study, Princeton, N.J. 08540, USA and Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke St. W., Montreal, Quebec, Canada H3A 2K6

[2] Department of Mathematics. R.S. Phys. S., The Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

Abstract. For G a classical group, an equivalence is exhibited between:

A) G monopoles over $\mathbb{R}^3$, with maximal symmetry breaking at infinity,

B) families of (rank(G)) algebraic curves in $T\mathbb{P}_1$, along with divisors on those curves, satisfying certain constraints,

C) solutions of Nahm's equations over (rank(G)) intervals, satisfying the appropriate boundary conditions.

A) and B) are linked by twistor techniques, B) and C) via the Krichever method for solving non-linear differential equations, and A) and C) via the ADHMN construction, providing a unified picture of techniques for solution. Amongst other things, an asymptotic formula for the Higgs field of the monopole is computed.

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Introduction

In recent years, monopoles have been studied quite extensively, from different points of view. One method is direct, involving analysis [JT, T1, T2]; another is complex-analytic, and employs twistor methods [W, Hi1]; yet another, due to Nahm [N] is an infinite dimensional version of the algebraic ADHM construction of instantons, and involves the solution of some non-linear ordinary differential equations, Nahm's equations.

From all of this, a fairly complete picture has emerged of the $SU(2)$-case. In particular, a beautiful paper of Hitchin [Hi2] gives the equivalence between

—an $SU(2)$-monopole
—an algebraic curve in $T\mathbb{P}_1(\mathbb{C})$ satisfying certain constraints
—a solution to Nahm's equations satisfying the appropriate boundary condition.

Using this equivalence, Donaldson [D] was able to give a description of the moduli space. Recently, the dynamics of monopoles have been studied in terms of geodesic motion on this space [AHi].

Our aim is to extend these results of Hitchin to the other classical groups, in the case of maximal symmetry breaking at infinity. As in [Hi2], we will prove an equivalence between three types of objects. These are defined as follows:

I) The Case of $SU(N)$. Our objects are:
$A(SU)$: $SU(N)$ Monopoles: Let H be a rank N complex vector bundle over $\mathbb{R}^3$. Let ∇ be an $SU(N)$ connection on H, and let Φ (the "Higgs field") be a section of the associated $su(N)$ adjoint bundle. (H, ∇, Φ) is an $SU(N)$ monopole if

A1) (∇, Φ) satisfies the Bogomoln'yi equation, $*F = \nabla\phi$, where F is the curvature of ∇ and $*$ is the Hodge duality operator.

A2) One has uniform asymptotic expansions, up to gauge transformation,

$$\Phi = i \operatorname{diag}(\mu_j - (k_j/2r)) + O(1/r^2),$$

$$|\nabla\Phi| = O(1/r^2), \quad \text{and}$$

$$\frac{\partial|\Phi|}{\partial\Omega} \overset{\mathrm{def}}{=} \left(\left(\frac{\partial|\Phi|}{\partial\theta}\right)^2 + \sin^2\theta\left(\frac{\partial|\Phi|}{\partial\rho}\right)^2\right)^{1/2} = O(1/r^2),$$

where (r, θ, ρ) are spherical coordinates in $\mathbb{R}^3$.

The μ_j and k_j are fixed, independent of direction and satisfy $\Sigma\mu_j = \Sigma k_j = 0$. The condition of maximal symmetry breaking is that the μ_j are distinct; we order them by

$$\mu_1 > \mu_2 > \cdots > \mu_N.$$

The k_j's are integers; we define the p^{th} magnetic charge m_p, $p = 1, \ldots, N-1$, by

$$m_p = k_1 + \cdots + k_p. \tag{0.1}$$

The second type of objects we are going to study is

$B(SU)$: *Nahm Data.* In the $SU(N)$ case, one has analytic hermitian vector bundles X_p of rank m_p on the intervals $[\mu_{p+1}, \mu_p]$, with, on the interior of each interval, an analytic hermitian connection ∇_t and three analytic skew-hermitian endomorphisms $T_i(t)$ satisfying:

B1) *Nahm's Equations*:

$$\nabla_t T_i = \tfrac{1}{2}\Sigma\varepsilon_{ijk}[T_j, T_k].$$

B2) *Boundary Conditions.* We adopt the convention $m_0 = m_N = 0$. At a boundary point μ_p, we distinguish three cases:

i) $m_p > m_{p-1}$:

In this case, there should be at μ_p an injection $X_{p-1} \to X_p$, compatible with the hermitian structure such that

—there exist well defined limits from above:

$$T_i^+ = \lim_{t \to \mu_p^+} T_i(t).$$

—for $t < \mu_p$, setting $z = t - \mu_p$, one has in a covariant constant basis, the expansion:

$$T_i(z) = \begin{array}{cc} \begin{array}{cc} k_p & m_{p-1} \end{array} & \\ \left(\begin{array}{c|c} r_i/z + O(1) & O(z^{(k-1)/2}) \\ \hline O(z^{(k-1)/2}) & T_i^+ + O(z) \end{array}\right) & \begin{array}{c} k_p \\ m_{p-1} \end{array} \end{array}.$$

The diagonal blocks are meromorphic; the off-diagonal blocks are $z^{((k-1)/2)} \times$ analytic. The residues $r_i, i = 1, 2, 3$ define an irreducible k_p-dimensional representation of $su(2)$.

ii) $m_p < m_{p-1}$:

One imposes the same boundary conditions, but with the roles of (μ_{p+1}, μ_p), (μ_p, μ_{p-1}) reversed.

iii) $m_p = m_{p-1}$:

We then have an identification at μ_p of X_p with X_{p-1}, such that if one sets

$$A(t, \zeta) = (T_1(t) + iT_2(t)) + 2iT_3(t)\zeta + (T_1(t) - iT_2(t))\zeta^2, \tag{0.2}$$

one asks that the one-sided limits $A^+(\zeta)$, $A^-(\zeta)$ of $A(t,\zeta)$ exist at μ_p, and that $A^+(\zeta) - A^-(\zeta)$ be at most of rank one, for all ζ. This is equivalent to asking that there be vectors $u_0, u_1, \in \mathbb{C}^m$ with

$$A^+(\zeta) - A^-(\zeta) = (u_0 + u_1\zeta)(\bar{u}_1 - \bar{u}_0\zeta)^T. \tag{0.3}$$

For both $SU(N)$ monopoles and solutions to Nahm's equations we can define spectral curves $S_i \subset T\mathbb{P}_1(\mathbb{C})$, $i = 1, \ldots, N-1$. For monopoles, this is outlined in Sect. 1; for the case of Nahm's equations, let ζ be an affine coordinate on $\mathbb{P}_1$, and let η be the associated fiber coordinate in $T\mathbb{P}_1$; the i^{th} spectral curve is defined by

$$\det(\eta \mathbb{1} - A(t,\zeta)) = 0$$

for $t \in (\mu_{i+1}, \mu_i)$. Nahm's equations are isospectral, so this is independent of the t chosen. Let $\mathcal{O}(k)$ denote the lift to $T\mathbb{P}_1$ of the line bundle $\mathcal{O}(k)$ on $\mathbb{P}_1$; in both cases the curves S_p belong to the linear system $|\mathcal{O}(2m_p)|$, and are compact. We will say that the monopole or the Nahm's data is *generic* if

$$S_p \cap S_{p-1} \textit{ consists of } 2m_p m_{p-1} \textit{ distinct points, for } p = 2, \ldots, N-1,$$

i.e. S_p and S_{p-1} intersect transversally

It is a non-trivial fact that generic monopoles and Nahm data exist. Let $L^\mu(k)$ denote the line bundle over $T\mathbb{P}_1$ with transition function $\exp(\mu\eta/\zeta)\zeta^k$ from $U_1 = \{\zeta \neq 0\}$ to $U_0 = \{\zeta \neq \infty\}$. Let $\tau\colon T\mathbb{P}_1 \to T\mathbb{P}_1$ denote the real structure $\tau(\eta,\zeta) = (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$. We will show that, from a generic monopole, or generic Nahm data, one can extract:

C(SU) Spectral Data. This consists of the compact, real (τ-invariant) curves $S_p \in |\mathcal{O}(2m_p)|$, $p = 1, \ldots, N-1$, in generic position, along with a splitting

$$S_p \cap S_{p-1} = S_{p,p-1} \cup S_{p-1,p} \qquad p = 2, \ldots, N-1$$

into disjoint subsets of points of equal cardinality, such that

C1) Over S_p

$$\mathcal{O} \cong L^{\mu_{p+1} - \mu_p}(m_{p-1} + m_{p+1})[-S_{p,p+1} - S_{p,p-1}].$$

C2) One has the vanishing theorem

$$H^0(S_p, L^{(\mu_p - z)}(m_p + m_{p-1} - 2)[-S_{p-1,p}]) = 0$$

for a) $\mu_{p+1} < z < \mu_p$
b) $z = \mu_p$ *if* $m_{p-1} \geqq m_p$
and c) $z = \mu_{p+1}$ if $m_p \leqq m_{p+1}$.

C3) The reality constraint

$$\tau(S_{p,p+1}) = S_{p+1,p}.$$

C4) The positivity constraint.

Let ψ_p be the section realizing the isomorphism in C1; set $\psi_p^* = \tau^*(\psi_p)$. Then $\psi_p \psi_p^*$ cuts out in S_p the divisor of $S_{p+1} \cap S_{p-1}$; the union of these curves can be given real equation

$$g_{p-1}g_{p+1} = \eta^{m_{p-1}+m_{p+1}} + (\text{lower order in } \eta).$$

One then has that $(\psi_p\psi_p^*)/(g_{p-1}g_{p+1})$ is a real constant e_p; one asks that

$$-(-1)^{m_p+m_{p-1}}e_p > 0.$$

II) The Cases of *SO(k), Sp(k)*. For the other classical groups, the definitions given above must be modified somewhat.

A(SO), A(Sp): G-Monopoles, G = SO(k), Sp(k). One adds an orthogonal or symplectic structure to the bundle H over R^3; ∇ is then compatible with this structure, and Φ is a section of the associated *so-*, *sp-*adjoint bundle.

We will treat the case of *SO-*, *Sp-*monopoles as SU-monopoles endowed with extra structure. One has the following table: (note [M] that a G-monopole has rank (G) magnetic charges):

(0.4)

A G-monopole for $G=$	with G-charges	embedded in $SU(N)$	As an $SU(N)$-monopole, its Higgs field is asymptotic to $\mathrm{diag}(\mu_i)$ with:	and it has $SU(N)$ charges m_i, with
$Sp(k)$	$r_1,\ldots,r_k$	$N=2k$	$\mu_i = -\mu_{2k+1-i}$ $i=1,\ldots,k$	$m_i = m_{2k-i} = r_i$ $i=1,\ldots,k.$
$SO(2k)$	$r_1,\ldots,r_{k-2}$, r_+,r_-	$N=2k$	$\mu_i = -\mu_{2k+1-i}$ $i=1,\ldots,k$	$m_i = m_{2k-i} = r_i$ $i=1,\ldots,k-2$ $m_{k-1} = m_{k+1} = r_+ + r_-$ $m_k = 2r_+.$
$SO(2k+1)$	$r_1,\ldots,r_k$	$N=2k+1$	$\mu_i = -\mu_{2k+2-i}$ $i=1,\ldots,k+1$	$m_i = m_{2k+1-i} = r_i$ $i=1,\ldots,k-1$ $m_k = m_{k+1} = 2r_k$

With this table in mind, one asks that the monopole conditions A-1 and A-2 again be satisfied.

B(SO), B(Sp): Nahm Data. In these cases, referring to (0.4), the Nahm data is the same as in the $SU(N)$-case, with the added condition:

(B-3) There are matrices C_j such that, for $z \in (\mu_{j+1}, \mu_j)$, $i = 1,2,3$,

$$T_i(-z)^T = C_j T_i(z) C_j^{-1}.$$

C_j and C_{j-1} are compatible in the obvious way at μ_j. Also,

$$C_{N-j+1} = -C_j^T \quad \text{for} \quad SO, C_j^T \quad \text{for} \quad Sp.$$

C(SO), C(Sp): Spectral Data:

C(Sp): For $Sp(k)$ monopoles, one can define [M] spectral curves $R_p \in |\mathcal{O}(2r_p)|$, $p=1,\ldots,k$. The genericity condition is that $R_p \cap R_{p+1}$ consist of $2r_pr_{p+1}$ distinct points, i.e. that the intersection of R_p and R_{p+1} be transversal. Under (0.4), the

SU-spectral curves are:

$$S_p = S_{2k-p} = R_p, \quad i = 1, \ldots, k.$$

The conditions $C(Sp)$ are then exactly those of $C(SU)$.

$C(SO(2k))$: Here, [M] yields spectral curves $R_1, \ldots, R_{k-2}, R_+, R_-$, $R_p \in |\mathcal{O}(2r_p)|$, $R_\pm \in |\mathcal{O}(2r_\pm)|$. The genericity conditions are that R_p and R_{p+1}, $p = 1, \ldots, k-3$, R_+ and R_-, R_{k-2} and $(R_+ \cup R_-)$ intersect transversally. Under (0.4), the associated SU-spectral curves are:

$$\begin{aligned} S_p = S_{2k-p} &= R_p, \quad p = 1, \ldots, k-2, \\ S_{k-1} = S_{k+1} &= R_+ \cup R_-, \\ S_k &= 2R_+ \text{ (i.e., with multiplicity two).} \end{aligned}$$

Also

$$\begin{aligned} S_{p,p+1} &= S_{2k-p,2k-p-1}, \\ S_{p+1,p} &= S_{2k-p-1,2k-p}. \end{aligned}$$

The conditions on $S_1, \ldots, S_{k-2}, S_{k+2}, \ldots, S_{2k-1}$ are then exactly the same as for SU. In addition, one has:

C-$1_\pm$) Over R_+,

$$\mathcal{O} \approx L^{\mu_{k-1}-\mu_k}(m_{k-2})[-S_{k-2,k-1}]$$

and over R_-,

$$\mathcal{O} \approx L^{\mu_{k-1}+\mu_k}(m_{k-2})[-S_{k-1,k-2}].$$

C-2_{k-1}) For S_{k-1}, the same vanishing theorem as for SU
C-2_k) The isomorphisms C-$1_\pm$ yield an identification

$$L^{-\mu_k}(m_{k-1}) \approx L^{\mu_k}(m_{k-1})$$

over $R_+ \cap R_-$. Define the bundle Q_k over R_+ by the exact sequence.

$$0 \to Q_k \to L^{\mu_k}(m_{k-1}) \oplus L^{-\mu_k}(m_{k-1}) \to L^{\mu_k}(m_{k-1})|_{R_+ \cap R_-} \to 0.$$

The vanishing theorem is then:

$$H^0(R_+, Q_k \otimes L^{-z}(-2)) = 0 \quad \text{for} \quad \begin{aligned} &\text{a) } -\mu_k \leqq z \leqq \mu_k \\ &\text{b) } z = \mu_k, \quad \text{if} \quad r_+ \leqq r_-. \end{aligned}$$

C-2_{k+1}) The vanishing theorem is:

$$H^0(S_{k+1}, L^{(\mu_{k+2}-z)}(m_{k+1} + m_{k+2} - 2)[-S_{k+1,k+2}]) = 0 \quad \text{for}$$
$$\begin{aligned} &\text{a) } \mu_{k+2} \leqq z \leqq \mu_{k+1} \\ &\text{b) } z = \mu_{k+1} \quad \text{if} \quad r_+ \geqq r_- \\ &\text{c) } z = \mu_{k+2} \quad \text{if} \quad m_{k+1} \leqq m_{k+2} \end{aligned}$$

C-$4_\pm$) The extra positivity constraints:

Let $\psi_\pm$ realize the isomorphisms in C-$1_\pm$; then $\psi_\pm \psi_\pm^* = e_\pm g_{k-2}$, with $g_{k-2} = \eta^{m_{k-2}} +$

(lower order in η), $e_\pm$ a real constant. One asks that:

$$-(-1)^{(r_{k-2}+r_\pm)}e_\pm > 0.$$

$C(SO(2k+1))$: In this case, one has spectral curves $R_p \in |\mathcal{O}(2r_p)|, p = 1,\ldots,k$, with the genericity condition that R_p and R_{p+1}, $p = 1,\ldots,k-1$ intersect transversally. One has:

$$\begin{aligned} S_p &= S_{2k+1-p} = R_p, \quad p = 1,\ldots,k-1,\\ S_k &= S_{k+1} = 2R_k. \end{aligned}$$

Also,

$$\begin{aligned} S_{p,p+1} &= S_{2k+1-p,\,2k-p},\\ S_{p+1,p} &= S_{2k-p,\,2k+1-p}. \end{aligned}$$

The conditions on $S_1,\ldots,S_{k-1},S_{k+2},\ldots,S_{2k}$ are as in the SU-case. In addition, one has:

C-1_k) Over R_k,

$$\mathcal{O} \approx L^{\mu_k}(m_{k-1})[-S_{k-1,k}].$$

C-2_k), C-2_{k+1}) The same vanishing theorems over S_k, S_{k+1} as in the SU-case.

C-4_k) The extra positivity constraint:

Let ψ_k realize the isomorphism C-1_k; then $\psi_k\psi_k^* = e_k g_{k-1}, g_{k-1} = \eta^{m_{k-1}} +$ (lower order in η), e_k a real constant. Then

$$-(-1)^{(m_{k-1}+r_k)}e_k > 0.$$

It is thus our intention to prove, for the cases $G = SU(N)$, $SO(k)$, $Sp(k)$ with maximal symmetry breaking:

Theorem 1. *There is a natural equivalence between*

A) *Generic monopoles,*
B) *Generic Nahm data,*
C) *Spectral data.*

The condition of genericity, using analyticity results of [JT], is a (real) Zariski-open one. Let M_m, $m = (m_1,\ldots,m_{\mathrm{rank}(G)})$ be the union of the connected components of the charge m G monopole moduli space which contain generic monopoles; elements of M_m are limits of generic monopoles. One can then show

Theorem 2. *There is a natural equivalence between*

A) *Monopoles in* M_m
B) *Nahm data.*

This result is used in [Hu3] to describe M_m in terms of rational maps of $\mathbb{P}_1$ into flag manifolds; amongst other things, M_m is connected. It is conjectured that the moduli space of charge m monopoles is connected, and so is equal to M_m; this is indeed the case for $SU(2)$ and $SU(3)$ [T2]. For an arbitrary monopole, one still obtains curves, and a generalization of the vanishing theorem. This in turn enables us to define a solution to Nahm's equations; the problem lies in showing that it satisfies the boundary conditions.

The paper is organized as follows. In Sects. 1 to 4, we concentrate on the case of $SU(N)$.

Section 1 is connected with the passage $A \Rightarrow C$. We recall from [M] how, from a monopole, we can obtain a holomorphic bundle E over $T\mathbb{P}_1$, along with two flags of subbundles E_i^+, E_i^-. The spectral curve S_p is (set-theoretically) the support of the sheaf $E/(E_p^+ + E_{N-p}^-)$.

We then prove the vanishing theorem and show how in the generic case, the conditions on the spectral data are satisfied. We also show how one can construct E from the spectral data, and derive an asymptotic formula for the Higgs field of the corresponding solution to the Bogomoln'yi equations in $\mathbb{R}^3$.

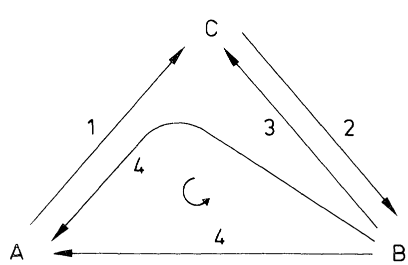
In Sect. 2, we study the correspondence $C \Rightarrow B$; from this we show how any monopole gives a solution to Nahm's equations, and prove that monopoles in M_m give solutions satisfying the boundary conditions.

Section 3 gives the inverse of the construction of Sect. 2; from a generic solution to B, we obtain the spectral data C.

Section 4 is concerned with the ADHMN construction of a monopole from a solution to Nahm's equations. This construction is described; we also show that, in the generic case, under the equivalence given in Sects. 2 and 3, it gives the same monopole as the twistor construction. This fact is then exploited: regularity is immediate from the ADHMN point of view, whereas using the twistor construction, one easily obtains from the asymptotic formulae of Sect. 1 that (∇, Φ) satisfies the boundary conditions.

In Sect. 5 we explain very briefly how these constructions must be modified for the cases of $SO(k)$, $Sp(k)$.

Section 6 provides a summary and conclusion, showing that the circle of ideas does indeed close.



1. From Monopoles to Spectral Data

1a) Bundles and Flag Structures. In [M], it was shown that, from a solution (H, ∇, Φ) to the $SU(N)$ Bogomolny equations over $\mathbb{R}^3$, one can obtain a rank $N(Sl(N, \mathbb{C}))$ holomorphic vector bundle E over $T\mathbb{P}_1(\mathbb{C})$. Recall [Hi1] that the space of oriented lines in $\mathbb{R}^3$ has a natural complex structure, and is holomorphically equivalent to $T\mathbb{P}_1(\mathbb{C})$. This correspondence can be given in coordinates as follows.

Let ζ be an affine coordinate on $\mathbb{P}_1(\mathbb{C})$, and let η be the corresponding fiber coordinate in $T\mathbb{P}_1(\mathbb{C})$ $(\eta \to \eta\, d/d\zeta)$. Note that $T\mathbb{P}_1(\mathbb{C})$ is covered by two coordinate patches $U_0(\zeta \neq \infty)$ and $U_1(\zeta \neq 0)$ with coordinates (η, ζ) and $(\eta', \zeta') = (\eta/\zeta^2, 1/\zeta)$ respectively. The correspondence between $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and (η, ζ) is then:

$$\eta = (x_1 + ix_2) - 2x_3\zeta + (-x_1 + ix_2)\zeta^2. \tag{1.1}$$

This can be viewed in two ways: fixing η, ζ, it defines a line $l(\eta, \zeta)$ in $\mathbb{R}^3$; fixing x, it defines the image C_x of a section $\mathbb{P}_1 \to T\mathbb{P}_1$. Also, $T\mathbb{P}_1(\mathbb{C})$ has a real structure $\tau: T\mathbb{P}_1 \to T\mathbb{P}_1$, given invariantly by reversal of orientation along a line, and in coordinates by $\tau(\eta, \zeta) = (-\bar{\eta}/\bar{\zeta}^2, \ -1/\bar{\zeta})$.

If l is a line in $\mathbb{R}^3$, define E_l = space of solutions s along l to $(\nabla_u - i\Phi)\, s = 0$, where u is the positive unit vector field on l. As (H, ∇, Φ) satisfies the Bogomoln'yi equations, E has an integrable holomorphic structure [Hi1].

The fact that ∇ is an $SU(N)$ connection, and that Φ is skew adjoint implies that if $(\nabla_u - i\Phi)s = 0$ along a line l, and if $(\nabla_u + i\Phi)t = 0$, then $\partial/\partial u \langle s, t\rangle = 0$. Thus the dual of E_l, via the isomorphism given by the metric on H, is the space of solutions to $(\nabla_u + i\Phi)t = 0$. This however, is the same as the solutions to $(\nabla_{-u} - i\Phi)t = 0$, i.e. $E_{\tau(l)}$. In short, there is an antilinear map $\sigma: E \to E^*$, lifting the map $\tau: T\mathbb{P}_1 \to T\mathbb{P}_1$.

If (H, ∇, Φ) satisfies the boundary conditions A-2, then it is shown in [M] that E possesses additional structure. Before recalling this, we again define some basic line bundles over $T\mathbb{P}_1$: first, one has the pull-back from $\mathbb{P}_1$ of the standard line bundles $\mathcal{O}(k)$, $k \in \mathbb{Z}$. These have ζ^k as a standard transition function from U_1 to U_0, i.e. a section of $\mathcal{O}(k)$ is described by functions f_i on U_i with $f_0 = \zeta^k f_1$ on the overlap. Next, define line bundles L^μ, $\mu \in \mathbb{R}$ by the transition function $e^{\mu\eta/\zeta}$ from U_1 to U_0. If F is any bundle, define $F(k)$ to be $F \otimes \mathcal{O}(k)$.

Lemma 1.2.

a) *In the standard trivialisations over* U_0, $H^0(T\mathbb{P}_1, \mathcal{O}(k))$ = *polynomials in* η, ζ *of degree* $\leqq k$, *where degree* $(\eta) = 2$, *degree* $(\zeta) = 1$. *Therefore*, $h^0(T\mathbb{P}_1, \mathcal{O}(2j)) = (j+1)^2$, *and* $h^0(T\mathbb{P}_1, \mathcal{O}(2j+1)) = (j+1)(j+2)$.

b) $H^0(T\mathbb{P}_1, L^\mu(k)) = 0$, *for all* $\mu \neq 0$, *for all* k.

c) $H^1(T\mathbb{P}_1, \mathcal{O}(k)) = \mathcal{O}(U_0 \cap U_1)/\mathcal{O}(U_0) \oplus \zeta^k \mathcal{O}(U_1)$. *Thus*, $H^1(T\mathbb{P}_1, \mathcal{O}(k))$ *is infinite dimensional. With respect to the covering by* U_0, U_1 *the cocycles* η^i/ζ^j, $i \geqq 0$, $j \in \{1, \ldots, 2i - k - 1\}$ *respect non-zero elements in* $H^1(T\mathbb{P}_1, \mathcal{O}(2k))$ *which are all linearly independent.*

Proof. a) is the result of explicit computation, using the transition function from U_1 to U_0.

b) is proven in [Hi2], p. 164.

c) follows from the fact that U_0, U_1 form a Leray cover of $T\mathbb{P}_1$.

We now can recall results from [M], specialised here to the case of $SU(N)$. One has, as $r \to \infty$, that $\Phi = i\,\mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N) - i\,\mathrm{diag}(k_1, \ldots, k_N)/2r + O(r^{-2})$; as $\mu_1 > \mu_2 > \cdots > \mu_N$, this implies that along each line, for each $p \in \{1, \ldots, N\}$, there is a p-dimensional subspace E_p^+ of solutions to $(\nabla_u - i\Phi)s = 0$ which are bounded by const. $\exp(-\mu_p r) r^{(k_p)}$ as $r \to \infty$. This defines a flag $0 \subset E_1^+ \subset \cdots \subset E_{N-1}^+ \subset E$, which varies holomorphically; or, in other terms, a reduction of the structure group of E from $SL(N,\mathbb{C})$ to a Borel (e.g. upper triangular) subgroup. Considering boundary behaviour as $r \to -\infty$ along each oriented line similarily gives a flag $0 \subset E_1^- \subset E_2^- \subset \cdots \subset E_{N-1}^- \subset E$. The real structure on E maps $E_p^{\pm}$ to the annihilator of $E_{N-p}^{\mp}$. Moreover, one has the identifications:

$$
\begin{aligned}
&E_1^+ \cong L^{\mu_1}(-k_1),\\
&0 \to E_1^+ \to E_2^+ \to L^{\mu_2}(-k_2) \to 0,\\
&0 \to E_2^+ \to E_3^+ \to L^{\mu_3}(-k_3) \to 0,\\
&\text{-------------------}\\
&0 \to E_{N-1}^+ \to E \to L^{\mu_N}(-k_N) \to 0,
\end{aligned} \tag{1.3}
$$

$$
\begin{aligned}
&E_1^- \cong L^{\mu_N}(k_N),\\
&0 \to E_1^- \to E_2^- \to L^{\mu_{N-1}}(k_{N-1}) \to 0,\\
&0 \to E_2^- \to E_3^- \to L^{\mu_{N-2}}(k_{N-2}) \to 0,\\
&\text{-------------------}\\
&0 \to E_{N-1}^- \to E \to L^{\mu_1}(k_1) \to 0.
\end{aligned} \tag{1.4}
$$

In [M], it is shown that for (ζ,η) generic, and in particular for (ζ,η) outside a compact set, these two flags are transversal, i.e. $E_p^+ \cap E_{N-p}^- = \{0\}$ for all p. It is the set where these are not transversal, however, which is important.

Definition 1.4. The p[th] spectral curve S_p $(p \in \{1 \cdots N-1\})$ of the monopole is defined by the vanishing of the map $\wedge^p E_p^+ \to \wedge^p(E/E_{N-p}^-)$.

Remarks. 1) From (1.3), $\wedge^p E_p^+$ is $L^{\mu_1+\cdots+\mu_p}(-k_1-\cdots-k_p)$, and $\wedge^p(E/E_{N-p}^-)$ is $L^{\mu_1+\cdots+\mu_p}(k_1+\cdots+k_p)$. S_p is therefore a curve in the linear system $|\mathcal{O}(2m_p)|$, where $m_p = k_1+\cdots+k_p$ is the p^{th} magnetic charge. As noted above, S_p is compact; it can thus be given the equation

$$g_p(\eta,\zeta) = \eta^{m_p} + a_{p,1}(\zeta)\eta^{m_p-1} + \cdots + a_{p,m_p}(\zeta) = 0, \tag{1.5}$$

where the $a_{p,i}$ are of degree $2i$.

2) An alternative definition of S_p is by the vanishing of $\wedge^{N-p}(E_{N-p}^-) \to \wedge^{N-p}(E/E_p^+)$. As a set, S_p is the locus where $E_p^+ \cap E_{N-p}^- \neq \{0\}$.
3) The real structure $\sigma: E \to E^*$ maps E_p^+ at (η,ζ) to the annihilator $(E_{N-p}^-)^{\perp}$ of E_{N-p}^- at $\tau(\eta,\zeta)$; E_{N-p}^- is mapped to $(E_p^+)^{\perp}$. However $E_p^+ \cap E_{N-p}^- \neq \{0\} \Leftrightarrow (E_p^+)^{\perp} \cap (E_{N-p}^-)^{\perp} \neq \{0\}$; therefore S_p is real (preserved by τ).

From (1.3) there is an exact sequence

$$0 \to \wedge^p E_p^+ \to \wedge^p E_{p+1}^+ \to \wedge^{p-1} E_p^+ \otimes L^{\mu_{p+1}}(-k_{p+1}) \to 0. \tag{1.6}$$

The map $\wedge^p E_{p+1}^+ \to \wedge^p(E/E_{N-p}^-)$ then passes to the quotient over S_p; there is over S_p a well defined map $\wedge^{p-1}E_p^+ \otimes L^{\mu_{p+1}}(-k_{p+1}) \to \wedge^p(E/E_{N-p}^-)$. Restrict this to $\wedge^{p-1}(E_{p-1}^+) \otimes L^{\mu_{p+1}}(-k_{p+1})$; using (1.3), the restricted map gives an element ρ_p of $H^0(S_p, L^{\mu_p-\mu_{p+1}}(m_{p-1}+m_{p+1}))$. In a similar vein, one obtains from the map $\wedge^{N-p-1}(E_{N-p-1}^-) \otimes L^{\mu_{p+1}}(k_{p+1}) \to \wedge^{N-p}(E/E_p^+)$ over S_p an element ξ_p of $H^0(S_p, L^{\mu_{p+1}-\mu_p}(m_{p-1}+m_{p+1}))$.

Another way of defining the spectral curves is as follows. Choose a trivialisation of E in which the positive flag E_i^+ is mapped to the standard flag in $\mathbb{C}^n$. The negative flag E_i^- then defines, locally, a map from $T\mathbb{P}_1$ into the flag manifold. The spectral curves are then the pull-backs of the closures of the codimension one Bruhat cells. In a similar fashion, the intersections of the spectral curves are the pull-backs of the closures of the codimension two cells; in particular, the intersection of the curves S_p and S_{p+1} corresponds to the closure of two codimension two cells; correspondingly, one can write $S_p \cap S_{p+1}$ as the union of two pieces:

$$\begin{aligned} S_{p,p+1}&:\dim(E_p^+ \cap E_{N-p-1}^-) \geqq 1,\\ S_{p+1,p}&:\dim(E_{p+1}^+ \cap E_{N-p}^-) \geqq 2. \end{aligned} \tag{1.7}$$

It is easy to check that they are interchanged by the real structure.

We now discuss genericity. If one refers to Taubes' construction of $SU(N)$ monopoles [T1], one finds that they are obtained there from approximate solutions which are the superposition of Σm_p well separated $SU(2)$-monopoles of charge 1. These have spectral curves which are real lines C_x in $T\mathbb{P}_1$. Furthermore, as in [AHi, Proposition 3.10; M], the spectral curves of the monopole one obtains approximate this union of lines; this approximation improves with the separation of the $SU(2)$ monopoles. As the C_x can be chosen with well separated distinct intersections, one obtains monopoles whose spectral curves intersect in distinct points.

Our generic monopoles are then those for which $S_p \cap S_{p+1}$ consists of $2m_p m_{p+1}$ distinct points, for $p = 1, \ldots, N-1$. When $S_p \cap S_{p+1}$ have no common components, we have the lemma:

Lemma 1.8[M]. *Over* S_p,

a) *The divisor* $S_{p-1,p} + S_{p,p-1}$ *is cut out by* S_{p-1}, *and therefore is in the linear system* $|\mathcal{O}(2m_{p-1})|$; *similarly*, $S_{p,p+1} + S_{p+1,p}$ *is cut out by* S_{p+1}, *and is in the linear system* $|\mathcal{O}(2m_{p+1})|$.
b) *The divisor of* ξ_p *in* $H^0(S_p, L^{\mu_{p+1}-\mu_p}(m_{p-1}+m_{p+1}))$ *is* $S_{p,p-1} + S_{p,p+1}$; *that of* ρ_p *is* $S_{p-1,p} + S_{p+1,p}$. ξ_p, ρ_p *are interchanged by the real structure.*

If [D] refers to the line bundle corresponding to a divisor D, then the fact that $L^{\mu_{p+1}-\mu_p}(m_{p-1}+m_{p+1}) \cong [S_{p,p-1} + S_{p,p+1}]$ over S_p imposes non-trivial constraints on the curve. This is condition C-1

From this lemma, we find by computing degrees that $S_{p,p+1}$ and $S_{p+1,p}$ both consist of $m_p m_{p+1}$ points, and so in the generic case:

$$S_{p,p+1} \text{ and } S_{p+1,p} \text{ are disjoint.} \tag{1.9}$$

This has several consequences:

$$\begin{aligned} \dim(E_p^+ \cap E_{N-p-1}^-) = 1 \quad &\text{on} \quad S_{p,p+1}, \quad 0 \text{ elsewhere},\\ \dim(E_p^+ \cap E_{N-p}^-) = 1 \quad &\text{on} \quad S_p^{\bullet}, \quad 0 \text{ elsewhere}. \end{aligned}$$

1b) A Meromorphic Reduction to a Torus. Both the positive and the negative flags define reductions of the structure group of E from $SL(N, \mathbb{C})$ to Borel subgroups. Away from the spectral curves, the two flags are transversal, and define holomorphic reductions to a (complex) torus (i.e. a Cartan subgroup): E can be thought of as a sum of line bundles $E/(E_p^+ + E_{N-p-1}^-)$. This reduction fails over the spectral curves; this is why we refer to the reduction as "meromorphic." This failure encodes the essential structure of E. Before examining this, we first study the structure of certain quotient sheaves of E, when E is generic.

i) $E/(E_p^+ + E_{N-p}^-)$: This sheaf is concentrated over S_p, Its structure can be obtained as follows. Using the genericity conditions (1.9), one sees that $\dim(E_p^+ \cap E_{N-p}^-) = 1$ over S_p; the same is then true of $E/(E_p^+ + E_{N-p}^-)$ and the sections of $E/(E_p^+ + E_{N-p}^-)$ form a locally free rank one sheaf over S_p. Next, from the exact sequence $0 \to E_p^+ \to E_{p+1}^+ \to L^{\mu_{p+1}}(-k_{p+1}) \to 0$, one sees that the natural map $E_{p+1}^+ \to E/(E_p^+ + E_{N-p}^-)$ factors through $L^{\mu_{p+1}}(-k_{p+1})$. Referring to (1.7), this is zero precisely at $S_{p+1,p}$. Therefore, over S_p, $E/(E_p^+ + E_{N-p}^-) \cong L^{\mu_{p+1}}(-k_{p+1})$ $[S_{p+1,p}]$. Referring to (1.8), this can be written as:

$$E/(E_p^+ + E_{N-p}^-) \cong L^{\mu_{p+1}}(m_p + m_{p+1})[-S_{p,p+1}]. \tag{1.10}$$

Similarily, one can get

$$E/(E_p^+ + E_{N-p}^-) \cong L^{\mu_p}(m_{p-1} + m_p)[-S_{p-1,p}]$$

ii) $E/(E_p^+ + E_{N-p-1}^-)$: One first notes that $E_p^+ \cap E_{N-p-1}^- = \{0\}$, except over $S_{p,p+1}$, where $\dim E_p^+ \cap E_{N-p-1}^- = 1$, by genericity. The quotient $Q = E/(E_p^+ + E_{N-p-1}^-)$ is then free (a line bundle) away from $S_{p,p+1}$. We first examine its global structure, then consider the local structure near $S_{p,p+1}$. From the exact sequence of sheaves

$$0 \to E_p^+ \oplus E_{N-p-1}^- \to E \to Q \to 0,$$

one has an injection of sheaves $\wedge^{N-1}(E_p^+ \oplus E_{N-p-1}^-) \otimes Q \to \wedge^N E$. Referring to (1.3), this yields an injection $Q \to L^{\mu_{p+1}}(m_p + m_{p+1})$, which is an isomorphism away from $S_{p,p+1}$. Locally, this can be thought of as an injection $Q \to \mathcal{O}$, the sheaf of functions of $T\mathbb{P}_1$; Q is an ideal sheaf. We now show that in fact Q is locally the ideal sheaf $\mathscr{I}(S_{p,p+1})$ of $S_{p,p+1}$. Let v_1^+ be a section of E_p^+, v_1^- a section of E_{N-p-1}^-, with $v_1^+ = v_1^-$ at a point x of $S_{p,p+1}$. We complete to local bases:

$$\begin{array}{ll} v_1^+, \ldots, v_p^+ \text{ of } E_p^+, & v_1^- \cdots v_{N-p-1}^- \text{ of } E_{N-p-1}^-, \\ v_1^+, \ldots, v_p^+, v_{p+1}^+ \text{ of } E_{p+1}^+, & v_1^- \cdots v_{N-p-1}^-, v_{N-p}^- \text{ of } E_{N-p}^-, \end{array}$$

by genericity, as $S_{p+1,p}$ is disjoint from $S_{p,p+1}$, E is the sum of E_{p+1}^+ and E_{N-p}^- near x. One has:

at x: v_{p+1}^+, v_{N-p}^- span $E/(E_p^+ + E_{N-p-1}^-)$

along $S_p \backslash \{x\}$: v_{p+1}^+ spans $E/(E_p^+ + E_{N-p-1}^-) \cong E/(E_p^+ + E_{N-p}^-)$, but v_{N-p}^- maps to zero.

along $S_{p+1} \backslash \{x\}$: v_{N-p}^- spans $E/(E_p^+ + E_{N-p-1}^-) \cong E/(E_{p+1}^+ + E_{N-p-1}^-)$, but v_{p+1}^+ maps to zero.

One can in fact choose local defining equations $a = 0$, $b = 0$ for S_p, S_{p+1}, so that locally one has the exact sequence

$$\begin{array}{rcl} 0 \to \mathcal{O} \to & \mathcal{O}^{\oplus 2} \to Q \to 0. & \\ f \mapsto & (af, bf) & \\ & (s,t) \mapsto [sv_{p+1}^+ + tv_{N-p}^-]. & \end{array}$$

This is, however, precisely the form of the Koszul resolution of $\mathscr{I}(S_{p,p+1})$:

$$\begin{array}{l} 0\to\mathcal{O}\to\quad \mathcal{O}^{\oplus 2}\to\mathscr{I}(S_{p,p+1})\to 0\\ \qquad f\mapsto (af,bf)\\ \qquad\qquad (s,t)\mapsto(sb-ta),\end{array}$$

and so, locally, $Q\cong\mathscr{I}(S_{p,p+1})$. Therefore, globally,

$$E/(E_p^+ + E_{N-p-1}^-)\cong L^{\mu_{p+1}}(m_p+m_{p+1})\otimes\mathscr{I}(S_{p,p+1}). \tag{1.11}$$

We now give a description of E valid for any monopole bundle.

Proposition 1.12. *One has the short exact sequence of sheaves:*

$$0\to E\xrightarrow{i}\begin{array}{c} E/E^-_{N-1}\\ \oplus\\ E/(E_1^+ + E_{N-2}^-)\\ \oplus \\ E/(E_2^+ + E_{N-3}^-)\\ \oplus\\ \cdots\cdots\\ E/(E^+_{N-2}+E^-_1)\\ \oplus\\ E/E^+_{N-1}\end{array} \xrightarrow{\pi} \begin{array}{c} E/(E_1^+ + E_{N-1}^-)\\ \oplus\\ E/(E_2^+ + E_{N-2}^-)\\ \oplus\\ \cdots\cdots\\ E/(E^+_{N-1}+E^-_1)\end{array}\to 0,$$

where the map between the second and third terms is of the form $(a_1\cdots a_N)\mapsto(a_1-a_2, a_2-a_3,\dots,a_{N-1}-a_N)$.

Proof. The only non-trivial part is showing that $\ker\pi\subset\operatorname{Im} i$, i.e. that $\pi=0$ imposes sufficient constraints on a section of the middle term for it to come from E. Let $(x_i+(E_i^+ + E_{N-i}^-))$ $i=0,\dots,N-1$ represent a local section in $\ker\pi$. One wants a section y of E such that

$$y+(E_i^+ + E_{N-i-1}^-)=x_i+(E_i^+ + E_{N-i-1}^-) \text{ for all } i, \text{ and so } y\in\bigcap_{i=0}^{N-1} x_i+(E_i^+ + E_{N-1}^-).$$

Now if A,B are subsheaves of E, one has the sequence

$$0\to E/(A\cap B)\to E/A\oplus E/B\to E/(A+B)\to 0. \tag{1.13}$$

As x_0,x_1 map to the same element in $E/(E_1^+ + E_{N-1}^-)$, there is a $y_1+(E_{N-1}^-\cap(E_1^+ + E_{N-2}^-))$ mapping to both $x_0+E_{N-1}^-$ and $x_1+(E_1^+ + E_{N-2}^-)$. The problem is now to find $y\in\left(\bigcap_{i=2}^{N-1} x_i+(E_i^+ + E_{N-i-1}^-)\right)\cap(y_1+(E_{N-1}^-\cap (E_1^+ + E_{N-2}^-)))$. As $E_{N-2}^-\subset E_{N-1}^-\cap(E_1^+ + E_{N-2}^-)$, it suffices to find $y\in \left(\bigcap_{i=2}^{N-1} x_i+(E_i^+ + E_{N-i-1}^-)\right)\cap(y_1+E_{N-2}^-)$. Now, by hypothesis, y_1 and x_2 map to the same element in $E/(E_2^+ + E_{N-2}^-)$; proceeding as above, there is a $y_2+(E_{N-2}^-\cap(E_2^+ + E_{N-3}^-))$ mapping to both $y_1+E_{N-2}^-$ and $x_2+(E_2^+ + E_{N-3}^-)$, and the problem then reduces to find $y \in \left(\bigcap_{i=3}^{N-1} x_i + (E_i^+ + E_{N-i-1}^-) \right) \cap (y_2 + E_{N-3}^-)$. Iterating this procedure, one obtains $y_3, y_4, \ldots,$ and $y_{N-1} = y$.

Using the identifications of the quotients given above, we therefore see that in the generic case E fits into an exact sequence:

$$0 \to E \to \begin{array}{c} L^{\mu_1}(m_1) \\ \oplus \\ L^{\mu_2}(m_1 + m_2) \otimes \mathscr{I}(S_{1,2}) \\ \oplus \\ L^{\mu_3}(m_2 + m_3) \otimes \mathscr{I}(S_{2,3}) \\ \cdots\cdots\cdots\cdots \\ \oplus \\ L^{\mu_{N-1}}(m_{N-1} + m_{N-2}) \otimes \mathscr{I}(S_{N-2,N-1}) \\ \oplus \\ L^{\mu_N}(m_{N-1}) \end{array} \to \begin{array}{c} L^{\mu_1}(m_1)|_{S_1} \\ L^{\mu_2}(m_1 + m_2)[-S_{1,2}]|_{S_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ L^{\mu_{N-1}}(m_{N-1} + m_{N-2})[-S_{N-2,N-1}]|_{S_{N-1}} \end{array} \to 0. \tag{1.14}$$

Let r_p denote restriction to the p^{th} spectral curve, and, referring to (1.8) and (1.5), let $f_p = \rho_p g_{p+1}^{-1}$ be a meromorphic section over S_p of $L^{\mu_p - \mu_{p+1}}(m_{p-1} - m_{p+1})$, which has poles at $S_{p,p+1}$ and zeroes at $S_{p-1,p}$. The map between the second and third terms above is:

$$\begin{aligned} \Pi(a_1, \ldots, a_N) = (r_1(a_1) - f_1 r_1(a_2), r_2(a_2) - f_2 r_2(a_3), \ldots, r_{N-1}(a_{N-1}) \\ - f_{N-1} r_{N-1}(a_N)). \end{aligned}$$

We call the curves S_p and the splitting $S_p \cap S_{p+1} = S_{p,p+1} \cup S_{p+1,p}$ the *spectral data* of the monopole. As the divisors $S_{p,p+1}, S_{p-1,p}$ determine the sections f_p, we have from (1.14) a result from [M]:

Proposition 1.15. *A generic monopole is determined by its spectral data.*

Remark 1.16. Note that one can also interpret the flag structure of E in terms of (1.12). Local sections of E_p^+ are elements in the kernel of Π of the form $(a_1, \ldots, a_p, 0, \ldots, 0)$. This forces a_p not only to vanish on $S_{p-1,p}$, but on the whole of S_p; a_p is then a local section of $L^{\mu_p}(m_{p-1} - m_p) = L^{\mu_p}(-k_p)$. From this, one can reobtain the extensions

$$0 \to E_{p-1}^+ \to E_p^+ \to L^{\mu_p}(-k_p) \to 0.$$

Similarly, local sections of E_{N-p}^- are of the form $(0, \ldots, 0, a_{N-p+1}, \ldots, a_N)$. One also notes that one has exact sequences, for any monopole

$$0 \to L^{\mu_p}(-k_p) \to E/(E_{p-1}^+ + E_{N-p}^-) \to E/(E_p^+ + E_{N-p}^-) \to 0,$$
$$0 \to L^{\mu_p}(k_p) \to E/(E_{p-1}^+ + E_{N-p}^-) \to E/(E_{p-1}^+ + E_{N-p+1}^-) \to 0.$$

1C) A Vanishing Theorem.

Theorem 1.17. *Let S_p be the p^{th} spectral curve of a generic monopole. Then $W_z = H^0(S_p, L^{-z}(-2) \otimes E/(E_p^+ + E_{N-p}^-)) = 0$. (Generically,*

$$W_z = H^0(S_p, L^{\mu_p - z}(m_p + m_{p-1} - 2)[-S_{p-1,p}]).)$$

a) $\forall z \in (\mu_{p+1}, \mu_p)$.
b) *For* $z = \mu_p$, *if* $m_p \leqq m_{p-1}$. $(1 < p)$.
c) *For* $z = \mu_{p+1}$, *if* $m_p \leqq m_{p+1}$. $(p < N)$.

Proof. The idea is to use (1.12). By a coboundary map δ, the space W_z above maps into $H^1(T\mathbb{P}_1, EL^{-z}(-2))$. This, in turn, by the twistor transform, corresponds to solutions to a Laplace type equation over $\mathbb{R}^3$. By computing boundary behaviour of solutions in $\delta(W_z)$, we show that they must vanish.

We start by showing that δ is an injection. An element s of W_z can be thought of as a section $(0, \ldots, 0, s, 0 \cdots 0)$ of the right-hand term of (1.12) (twisted by $L^{-z}(-2)$). If it maps to zero in $H^1(T\mathbb{P}_1, EL^{-z}(-2))$, then it is the image of a section of the middle term of (1.12). Let us first consider the generic E. For these, in case a), the middle term has only the zero section, as by (1.2), $H^0(T\mathbb{P}_1, L^t(k)) = 0 \ \forall k, \forall t \neq 0$, and so $s = 0$. In case b), consider the following portion of $(1.14) \otimes L^{\mu_p}(-2)$:

$$\begin{array}{ll}
D = L^{\mu_{p-1}-\mu_p}(m_{p-2} + m_{p-1} - 2) \otimes \mathscr{I}(S_{p-2,p-1}) & \\
\qquad\qquad\qquad\qquad\qquad\qquad \searrow & \\
 & B = L^{\mu_{p-1}-\mu_p}(m_{p-2} + m_{p-1} - 2)[S_{p-2,p-1}]|_{S_{p-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad \nearrow & \\
C = \mathcal{O}(m_p + m_{p-1} - 2) \otimes \mathscr{I}(S_{p-1,p}) & \\
\qquad\qquad\qquad\qquad\qquad\qquad \searrow & \\
 & A = \mathcal{O}(m_p + m_{p-1} - 2)[-S_{p-1,p}]|_{S_p} \\
\qquad\qquad\qquad\qquad\qquad\qquad \nearrow & \\
E = L^{\mu_{p+1}-\mu_p}(m_{p+1} + m_p - 2) \otimes \mathscr{I}(S_{p,p+1}). &
\end{array}$$

For the section $(0, s)$ of (B, A) that we are considering to be in the image of the left-hand side, one must have sections (d, c, e) of (D, C, E) with $(0, s) = (r_{p-1}(d) - f_{p-1} r_{p-1}(c), r_p(c) - f_p r_p(e))$. However, by (1.2), $d = e = 0$. Therefore e vanishes on S_{p-1}, and so can be thought of as a section of $\mathcal{O}(m_p - m_{p-1} - 2)$. By hypothesis, this has negative degree; therefore $c = 0$, and so $s = 0$. Injectivity for the third case of the theorem is proven in a similar fashion.

When E is non-generic, the injectivity is proven in essentially the same fashion but is notationally more complicated: one uses the sequences of (1.16) to express the lifts to the middle portion of $((1.12) \otimes L^{-z}(-2)$, of sections of the right-hand side of the form $(0, 0, \ldots, 0, s, 0, \ldots, 0)$ in terms of sections of $\oplus_i L^{\mu_i - z}(k_i - 2)$, which must vanish.

Having shown that δ is injective, we compute Cech and Dolbeault representatives for $\delta(s)$. Let V_i, $i = 0, \ldots, n$ be a sufficiently fine covering of $T\mathbb{P}_1$ with V_i, $i = 1, \ldots, n$ covering the spectral curves, $\bigcup_{i=1}^{n} V_i$ lying inside a compact set, and V_0 not intersecting the spectral curves. Over each V_i, one can pull back the section $(0, 0, \ldots, 0, s, 0 \cdots 0)$ to the middle term of $(1.12) \otimes L^{-z}(-2)$, in two particularly convenient ways:

1) as a local section f_1^+ of E_p^+, of the form $(f_{i,1}^+, \ldots, f_{i,p}^+, 0, \ldots, 0)$,
(2) as a local section f_i^- of E_{N-p}^-, of the form $(0, \ldots, 0, f_{i,p+1}^-, \ldots, f_{i,N}^-)$.

Setting $f_0^\pm = 0$, $\delta(s)$ then has two representative cocycles $f_\pm$ defined by $f_{ij}^\pm = f_i^\pm - f_j^\pm$ over $V_i \cap V_j$, differing of course by a coboundary; f_+ represents an element of $H^1(T\mathbb{P}_1, E_p^+ L^{-z}(-2))$, f_-, an element of $H^1(T\mathbb{P}_1, E_{N-p}^- L^{-Z}(-2))$.

Dolbeault representatives are obtained in a standard way. If σ_i is a partition of unity subordinate to V_i, set $\theta_j^\pm = \bar\partial(\sum_i \sigma_i f_{ji}^\pm)$; $\theta_j^\pm = \theta_k^\pm$ on overlaps, and so one gets globally defined forms $\theta^+ \in \Omega^{0,1}(E_p^+ L^{-z}(-2))$, $\theta^- \in \Omega^{0,1}(E_{N-p}^- L^{-z}(-2))$, both representing $\delta(s)$. Furthermore, if $\gamma = \sum_i \sigma_i (f_i^+ - f_i^-)$, $\theta^+ - \theta^- = \bar\partial\gamma$. Note that as $\mathcal{O}(-2) \simeq \Pi^*(K(\mathbb{P}_1))$ $(\Pi: T\mathbb{P}_1 \to \mathbb{P}_1)$, $\theta^\pm$ can be considered as $(1,1)$ forms with values in EL^{-z}; these forms have terms in $d\zeta \wedge d\bar\eta, d\zeta \wedge d\bar\zeta$ and none in $d\eta \wedge d\bar\zeta$, $d\eta \wedge d\bar\eta$; furthermore, θ^+, θ^- and γ are all compactly supported in some disk bundle D inside $T\mathbb{P}_1$.

The proof is now a slightly refined version of that found in Hitchin, [Hi2, p. 162] (see also [HiM]). Recall the twistor transform over $\mathbb{R}^3$. First, the bundle H over $\mathbb{R}^3$ is reobtained by $H_x = H^0(C_x, EL^{-z})$. Secondly, as EL^{-z} trivial over C_x, $H^1(C_x, EL^{-z}(-2)) \simeq H^0(C_x, EL^{-z}) \otimes H^1(C_x, \mathcal{O}(-2))$. By Serre duality (integrating a representative $(1,1)$ form), $H^1(C_x, \mathcal{O}(-2)) \simeq \mathbb{C}$. Thus, restricting our element $\delta(s)$ to C_x, we get an element $F(x)$ of H_x. F is in fact the solution to a Laplace type equation over $\mathbb{R}^3$; the map from $H^1(T\mathbb{P}_1, EL^{-z}(-2))$ to the space of such solutions is bijective.

We examine the behaviour of our solution at a point x of $\mathbb{R}^3$ as $x \to \infty$. For concreteness, take x to be the point $(0, 0, -b/2)$; then C_x is defined by $\eta = b\zeta$. The intersection of C_x with D, for $|x|$ large, is the disjoint union of two open sets A_x^+, A_x^- centred around $\zeta = 0$, $\zeta = \infty$ respectively on C_x. Note that their radii (in ζ) tend to zero linearly in b^{-1} as $b \to \infty$.

Let $\theta^\pm$ be written locally as $\alpha^\pm d\zeta \wedge d\bar\eta + \beta^\pm d\zeta \wedge d\bar\zeta$; let their restrictions to C_x be $\sum_i s_i(\alpha_i^\pm d\zeta \wedge d\bar\eta + \beta_i^\pm d\zeta \wedge d\bar\zeta) = \sum_i s_i(b\alpha_i^\pm + \beta_i^\pm) d\zeta \wedge d\bar\zeta$, where the s_i are an orthonormal basis of the sections of E over C_x. Similarly, set γ over C_x to be $\Sigma s_i(\rho_i d\zeta)$. One has, over C_x, $[(b\alpha_i^+ + \beta_i^+) - (b\alpha_i^- + \beta_i^-)] d\zeta \wedge d\bar\zeta = \bar\partial(\rho_i d\zeta) = d(\rho_i d\zeta)$. The coefficient of s_i in $F(x)$ is:

$$\int_{C_x} (b\alpha_i^+ + \beta_i^+) d\zeta \wedge d\bar\zeta = \int_{A_x^+} (b\alpha_i^+ + \beta_i^+) d\zeta \wedge d\bar\zeta + \int_{A_x^-} (b\alpha_i^- + \beta_i^-) d\zeta \wedge d\bar\zeta,$$

using Stokes' theorem.

We now use the fact that $\theta^+ \in \Omega^{1,1}(E_p^+ L^{-z})$. Elements of $E_p^+ L^{-z}$ at a point of $T\mathbb{P}_1$ correspond to solutions of $(\nabla_u - i\Phi)s = 0$ which are bounded by const. $[\exp(-(\mu_p - z)b) b^{-(k_p)}]$ as $b \to \infty$. Following Hitchin [Hi2, p. 163], this means that the coefficients α_i^+, β_i^+ are bounded over A_x^+ by const. $[\exp(-(\mu_p - z)b) b^{-(k_p)}]$. In the cases a, b, c which interest us, the integrand is then always bounded by const. $[b]$; however the area of A_+ is bounded by const. $[b^{-2}]$, and so the integral over A_x^+ is bounded by const. $[b^{-1}]$. The same argument, applied to $\theta^- \in \Omega^{1,1}(E_{N-p} - L^{-z})$ bounds the integral over A_x^- by const. $[b^{-1}]$, and so $|F(x)|$ is bounded by const. $[b^{-1}]$. As in [Hi2, p. 164], keeping track of the derivatives gives $|\nabla F(x)|$ bounded by const. $[b^{-2}]$, and so the argument given there (essentially the maximum principle) applies, forcing $F = 0$. Therefore $s = 0$, and the vanishing theorem is proved.

1d) The Asymptotic Higgs Field. To begin, note that the spectral curves all lie within some compact disc bundle D over $\mathbb{P}_1$. Then, for each $x \in \mathbb{R}^3$ outside a compact set K, $C_x \cap S_p$ is a set of $2m_p$ points which partitions naturally into two clusters $C^0_{p,x}, C^\infty_{p,x}$ of m_p points: the points in $C^0_{p,x}$ are lines in $\mathbb{R}^3$ through x which point (approximately) away from the origin, and the points in $C^\infty_{p,x}$ are the same lines, but with orientation reversed, and so pointing towards the origin; $\tau(C^0_{p,x}) = C^\infty_{p,x}$.

Let $U \subset V$ be open sets in $\mathbb{P}_1$ representing a "cone" of directions in $\mathbb{R}^3$. Set $A = (T\mathbb{P}_1 \backslash D) \cup \pi^{-1}(V) \cup \tau(\pi^{-1}(V))$, where $\pi: T\mathbb{P}_1 \to \mathbb{P}_1$ is the projection. If $|x|$ is large enough, and the line Ox has direction in U, then $C_x \subset A$.

Let $g_p = 0$ be the equation defining S_p; one can define an element of $H^1(A, \mathcal{O}(-2))$ by the cocycle -$(\partial g_p/\partial\eta)/(2\pi i g_p)$, relative to the covering of A by $T\mathbb{P}_1 \backslash \pi^{-1}(V)$, $T\mathbb{P}_1 \backslash \tau(\pi^{-1}(V))$. This in turn, by the twistor transform [Hi3], corresponds to a solution of the (ordinary) Laplace equation, defined for the x's in $\mathbb{R}^3$ such that $C_x \subset A$. Varying U, these solutions patch together to give a global solution ψ_p which is defined outside a compact set of $\mathbb{R}^3$. ψ_p has the following alternate formulations, proven in [Hu1, p. 386–387]:

Lemma (1.18).

a) *Along the line in $\mathbb{R}^3$ corresponding to sections $\eta = b\zeta$, $b \to -\infty$,*

$$\psi_p = \partial_b \log\left(\prod_{(b\zeta_i, \zeta_i) \in C^0_{x,p}} \zeta_i \right) = -\partial_b \log\left(\prod_{(b\zeta_i, \zeta_i) \in C^\infty_{x,p}} \zeta_i \right).$$

b) *Along C_x, $T(T\mathbb{P}_1)$ splits into the sum of two canonically isomorphic bundles: TF, the tangents to the fibers of $T\mathbb{P}_1 \to \mathbb{P}_1$. and TC_x. Assuming that C_x intersects S_p transversally, at smooth points of S_p; then a tangent vector v to S_p at $q \in S_p \cap C_x$ then decomposes at (v_f, v_{c_x}), and the "slope" $s(q) = (v_{c_x})/(v_f)$ of S_p at q is well defined. Then*

$$\psi_p(x) = \sum_{q \in C^0_{x,p}} s(q).$$

Theorem (1.19). *Let E be a vector bundle E over $T\mathbb{P}_1$ defined by the sequence (1.14), where the S_p's are real. Then,*

a) *For X outside a compact set K in $\mathbb{R}^3$, E is trivial when restricted to C_x.*

b) *E defines a Higgs field Φ and a connection ∇ over $\mathbb{R}^3 \backslash K$ such that asymptotically, the eigenvalues Φ_j of Φ are approximated by*

$$(\Phi_j)_{\text{as}} = i\mu_j + i(\psi_j - \psi_{j-1}),$$

where ψ_0, ψ_N are defined to be zero. This approximation is valid up to exponentially decreasing terms with exponentially decreasing derivatives.

c) *(∇, Φ) satisfy the boundary conditions of a monopole.*

Proof. We will compute the Higgs field for $x = (0, 0, -b/2)$, as $b \to -\infty$: this corresponds to lines C_x: $\eta = b\zeta$. We will suppose that $(\eta, \zeta) = (0, 0) \notin S_p$, and so $\tau(0, 0) \notin S_p$. This can be done without loss of generality, as for a given small cone of directions in $\mathbb{R}^3$, one can shift the origin so that this is true. For convenience, we will suppose that the intersection of C_x with the curves S_p consists of $2m_p$ distinct points; the presence of multiple points does not change the proof, but, as we are using Lagrange interpolation, it does change the formulae.

Over the line $\eta = b\zeta$, a section of $L^{\mu}(k)$ is represented over U_0 by polynomial s_0 in ζ of degree $\leqq k$, and over U_1 by a polynomial s_1 in ζ^{-1} of degree $\leqq k$, such that on the overlap:

$$s_0 = e^{\mu b}\zeta^k s_1. \tag{1.20}$$

Let C_x intersect S_p in $2m_p$ points with ζ-coordinates $\zeta_{p,i} = \zeta_{p,i}(b)$, $i = 1, \ldots, 2m_p$. As C_x and S_p are real, one can order the points so that $\zeta_{p,i} \in C^0_{x,p}$ (therefore $\zeta_{p,i} \to 0$ as $b \to -\infty$) for $i = 1, \ldots, m_p$, and so that $\zeta_{p,m_p+i} = -1/\bar{\zeta}_{p,i}$, $i = 1, \ldots, m_p$. In the exact sequence (1.14), meromorphic sections f_p of $L^{\mu_p - \mu_{p+1}}(m_{p-1} - m_{p+1})$ over S_p are involved. Represent these by functions ${}_0f_p$ over U_0, ${}_1f_p$ over U_1, with ${}_0f_p = \exp((\mu_p - \mu_{p+1})\eta/\zeta)\zeta^{(m_{p-1} - m_{p+1})}{}_1f_p$ on the overlap. Set $f_{p,j}(b)$ to be the value of ${}_0f_p$ at $\zeta_{p,j}(b)$. Then, referring to (1.14), a section of E over C_x can be represented in the U_0 trivialisation by polynomials s_p of degree $m_p + m_{p-1}$ $(m_0 = m_N = 0)$ with the constraints

$$s_p(\zeta_{p,j}) = f_{p,j}s_{p+1}(\zeta_{p,j}), \quad j = 1 \cdots 2m_p, \quad p = 1 \cdots N-1. \tag{1.21}$$

As b tends to infinity, the points $\zeta_{p,j}$, $j = 1, \ldots, m_p$ tend to zero; the ζ_{p,m_p+j} tend to infinity, linearly in b. As ${}_1f_p$ is bounded near $\zeta = \infty$, then, as $\eta = b\zeta$ on C_x, the f_{p,m_p+j} converge to zero, exponentially in b. The equations (1.21) for a section are then, up to exponentially decreasing terms:

$$\begin{aligned} s_p(\zeta_{p,j}) &= f_{p,j}s_{p+1}(\zeta_{p,j}), \\ s_p(\zeta_{p,m_p+j}) &= 0, \quad j = 1, \ldots, m_p. \end{aligned} \tag{1.22}$$

As $b \to -\infty$, (1.22) does not quite tend to a finite, well defined limit. However, note that $b\zeta_{p,j} \to c_{p,j}$, where $c_{p,j}$ are the η-coordinates of the points of intersection of $\{\zeta = 0\}$ with S_p. We therefore set $t_p(b\zeta) = s_p(\zeta)$; if $g_{p,j}$ are the values of ${}_0f_p$ at $\{\zeta = 0\} \cap S_p$, then (1.22) becomes in the limit:

$$\begin{aligned} &t_p(c_{p,j}) = g_{p,j}t_{p+1}(c_{p,j}), \text{ and } t_p \text{ has an } m_p\text{-fold zero at } \zeta = \infty \\ &\cdot(\text{i.e. } t_p \text{ is of degree } m_{p-1}) \end{aligned} \tag{1.23}$$

For E to be trivial over a line C_x, it suffices to find a basis of solutions ${}^is = ({}^is_p)$, $i = 1, \ldots, N$, to (1.22), spanning the fiber of E at a point. Taking this point to be $\zeta = 0$. one adds to conditions (1.22) the extra constraint

$${}^is_p(0) = \delta_{p,i}. \tag{1.24}$$

Adding this condition into the limit equations (1.23), the linear system one obtains is non-degenerate, and in fact solvable by Lagrange interpolation. Therefore, for $|b|$ large, (1.22) and (1.24) are also solvable and E is trivial over $\eta = b\zeta$. By varying our coordinate system (η, ζ) (i.e. rotating in $\mathbb{R}^3$), one obtains part a).

We next compute the Higgs field, again along $\eta = b\zeta$. We begin with a remark: all the functions involved in this computation are of the form (in b) $\exp(-kb) \times$ (meromorphic in b^{-1}). Thus, in general terms if we obtain an exponential approximation q' of a quantity q, the derivatives of q' will also approximate the derivatives of q exponentially.

With this in mind, we construct an exponential approximation ${}^is_p$ to a basis of sections of E over $\eta = b\zeta$; this amounts to finding ${}^is_p$ solving (1.22) and (1.24), which is simply a matter of Lagrange interpolation. For i fixed, one gets:

$$-{}^{i}s_p \equiv 0 \quad \text{for} \quad p < i,$$

$$-{}^{p}s_p = \prod_{j=1}^{m_{p-1}} \frac{(\zeta - \zeta_{p-1,j})}{(-\zeta_{p-1,j})} \prod_{j=1}^{m_p} \frac{(\zeta - \zeta_{p,m_p+j})}{(-\zeta_{p,m_p+j})}. \tag{1.25}$$

The ${}^{i}s_p$ for $i < p$ can be computed similarily. Thus the matrix $S = ({}^{i}s_p)$ is lower triangular.

We then construct the asymptotic Higgs field in this basis. The family of sections C_x all intersect at $(0,0)$ and $\tau(0,0)$. There are two natural connections on the bundle H ($H_x = H^0(C_x, E)$) over the line $(0,0,-b/2)$ in $\mathbb{R}^3$: one, ∇_0, has flat sections defined by fixing values of elements of H_x at $(0,0)$; the other ∇_∞, is similarily defined at $\tau(0,0)$. The Higgs field is then [Hi1]

$$\Phi db = i(\nabla_0 - \nabla_\infty).$$

The basis S above is ∇_0-constant by (1.24), and ∇_0 then has zero matrix. To obtain the matrix of ∇_∞, one must first evaluate the basis at $\zeta = \infty$; taking the change of trivialisations (1.20) into account, one evaluates (in our approximation) $[\operatorname{diag}\{\exp(-\mu_i b)\zeta^{-(m_i + m_{i-1})}\} \cdot S]$ at ∞, obtaining a matrix M. Write M as $\operatorname{diag}(\exp(-\mu_i b)).\ T$; T is lower triangular, with diagonal elements

$$T_{p,p} = \prod_{j=1}^{m_{p-1}} (-\zeta_{p-1,j})^{-1} \prod_{j=1}^{m_p} (-\zeta_{p,m_p+j})^{-1}.$$

The matrix of ∇_∞ is $\mathrm{M}^{-1}(\partial_b M)db$; that of Φ_{as} is then

$$\Phi_{\mathrm{as}} = -iM^{-1}\partial_b M = -i(T^{-1}\operatorname{diag}(-\mu_i)T + T^{-1}\partial_b T).$$

Φ_{as} is lower triangular; its diagonal entries $(\Phi_{\mathrm{as}})_p$ (eigenvalues) can be computed from those of T, to obtain

$$(\Phi_{\mathrm{as}})_p = i\mu_p + i\partial_b \log\left(\prod_{j=1}^{m_{p-1}} \zeta_{p-1,j} \prod_{j=1}^{m_p} \zeta_{p,m_p+j}\right),$$

referring to Lemma (1.18)

$$(\Phi_{\mathrm{as}})_p = i\mu_p + i(\psi_p - \psi_{p-1}).$$

To prove c), one notes that

$$\prod_{j=1}^{m_{p-1}} (\zeta_{p-1,j}) \prod_{j=1}^{m_p} (\zeta_{p,m_p+j})$$

has leading term $b^{m_p - m_{p-1}} = b^{k_p}$ as $b \to \infty$; this gives $\Phi_p = i(\mu_p - k_p/2r) + O(r^{-2})$. The other estimates follow by taking derivatives as in [Hu1]. One uses $|\nabla\Phi|^2 = \Delta|\Phi|^2/2$.

1e) Reality Conditions. Our monopole bundles E are equipped with an anti-holomorphic map $\sigma: E \to E^*$ lifting τ. We examine how this is encoded in the spectral data; we begin by showing how to define σ for a bundle E derived from the spectral data via (1.14). Note that the two real structures σ, τ define an operation $*$ from sections of E to sections of E^* by $f^*(p) = \sigma f(\tau p)$ for p in $T\mathbb{P}_1$, and if f is holomorphic so also is f^*. If f and g are sections of E over a real section C_x, we can define a pairing by $\langle f, g\rangle = f(g^*)$ which on a real section is a holomorphic function, and so a constant complex number. This defines a pairing on the bundles H on $\mathbb{R}^3$ ($H_x = H^0(C_x, E)$).

We shall show that this pairing is hermitian and positive definite, and also, that if one performs the twistor construction of the (∇, Φ) associated to E, the pairing is compatible with the connection and Higgs field. This means the monopole we have constructed is an $SU(n)$ monopole.

First let us make some normalization conventions. Every bundle $L^\mu(k)$ has a real structure σ mapping $L^\mu(k)$ to $L^{-\mu}(k)$ and covering τ. This can be chosen so that when bundles are tensored the real structures are tensored, and we shall denote all these different real structures by the same symbol. The operator * on sections of $L^\mu(k)$ then satisfies $** = (-1)^k$. If g_p is our normalised section of $\mathcal{O}(2m_p)$ defining S_p, then we have

$$g_p^* = g_p \cdot (-1)^{m_p}.$$

Similarly the meromorphic sections f_p in (1.14) are normalized to satisfy

$$f_p f_p^* = e_p \frac{g_{p-1}}{g_{p+1}}, \quad \text{where} \quad e_p = \pm 1,$$

and

$$-(-1)^{m_{p-1}+m_p} e_p > 0$$

for all p from the conditions on the spectral data. We shall see below that it is this condition on the sign of e_p that makes the hermitian form positive definite.

We start by defining a pairing between the fibres $E(\gamma)$ and $E(\tau(\gamma))$ for any γ in minitwistor space. Consider an open set U about γ. Then a local section of E over U, from diagram (1.14), can be regarded as a collection $s = (s_1, \ldots, s_N)$, where

$$s_p \in H^0(U, L^{\mu_p}(m_p + m_{p-1}) \otimes \mathcal{I}(S_{p-1,p})),$$

and

$$\begin{aligned} s_1 &= f_1 s_2 \quad &\text{on} \quad S_1, \\ s_2 &= f_2 s_3 \quad &\text{on} \quad S_2, \\ &\cdots\cdots\cdots\cdots && \\ s_{N-1} &= f_{N-1} s_N \quad &\text{on} \quad S_{N-1}. \end{aligned}$$

Let s be a local section of E over U and t a local section of E over $\tau(U)$. Then t^* is a local section over U and we can form the interesting expression

$$\rho(s, t) = \sum_{p=1}^{N} (-1)^p e_1 e_2 \cdots e_{p-1} s_p t_p^* g_1 \cdots g_{p-2} g_{p+1} \cdots g_{N-1}.$$

By using the above formulae it is easy to check that this section of $\mathcal{O}(2m_1 + \cdots + 2m_{N-1})$ vanishes on all of the spectral curves so we can define

$$\langle s(\gamma), t(\gamma)\rangle = \frac{\rho(s, t)}{g_1 \cdots g_{N-1}}(\gamma). \tag{1.26}$$

Using the normalization for the g_p, the result on * squared and the fact that σ on functions is conjugation, it follows that the induced inner product on H is hermitian. We shall show next that this inner product is compatible with the connection and Higgs field.

Consider any line γ in $\mathbb{R}^3$ not on a spectral curve and the corresponding family C_x, $x\in\gamma$ of real sections in $T\mathbb{P}_1$. If s and t are sections of E over any of these real sections, then the inner product $\langle s,t\rangle$ is a holomorphic function and therefore constant, so it can be determined by evaluating at any point, for example γ or $\tau(\gamma)$ the two intersection points of the family of real sections. Choose two sections s and t of the bundle H over the line in $\mathbb{R}^3$ so that as induced sections of E over the family of real sections s is constant at γ and t is constant at $\tau(\gamma)$. Then $\langle s,t\rangle$ evaluated at γ is independent of where we are on the line in $\mathbb{R}^3$. Using the definition of the connection and Higgs field this means that if $(\nabla_\gamma - i\phi)(s)=0$ and $(\nabla_\gamma + i\phi)(t)=0$ then their inner product is constant. It is easy to deduce then that

$$\frac{d}{dz}\langle s,t\rangle = \langle(\nabla_z - i\Phi)s,t\rangle + \langle s,(\nabla_z + i\Phi)t\rangle$$

for z a parameter along the line, and if we expand this out and equate the pieces which are symmetric and conjugate-symmetric in s and t we obtain the desired result. We have, in fact, only proved the result for lines not on the spectral curves, but these are dense in the space of all lines so the complete result follows by continuity.

Because the inner product is invariant under the connection, it is enough to show that it is definite somewhere to deduce that is definite everywhere or instead to show that it is asymptotically definite, which is the approach we shall take.

Consider again the family of real sections parametrised by the points of the line $\gamma=(0,0,-b/2)$ in $\mathbb{R}^3$. Assume that we are in the generic situation of Sect. 1d); we have an asymptotic basis of sections ${}^is$ satisfying (1.22), (1.24), and so (1.25). It suffices to evaluate (1.26) at the point $(\eta,\zeta)=(0,0)\in C_b$ for $b\to\infty$, and show that it is positive. It is easy to check that this asymptotic basis is orthogonal; to check definiteness, we need to show that the sign of

$$\frac{\langle {}^ps, {}^ps\rangle}{\langle {}^{p+1}s, {}^{p+1}s\rangle}(\gamma)$$

is positive; at $(0,0)$, this equals

$$-e_p\frac{{}^ps_p\cdot{}^ps_p^*}{{}^{p+1}s_{p+1}\cdot{}^{p+1}s_{p+1}^*}\frac{g_{p+1}}{g_{p-1}}(\gamma). \tag{1.27}$$

Evaluating, we find

$$\frac{-e_p\Pi(-\bar{\zeta}_{p-1,i})^{-1}\cdot\Pi(-\bar{\zeta}_{p,m_p+i})^{-1}g_{p+1}(0)}{\Pi(-\bar{\zeta}_{p,i})^{-1}\Pi(-\bar{\zeta}_{p+1,m_{p+1}+i})^{-1}g_{p-1}(0)}.$$

Using $\zeta_{p,m_p+i}=-1/\bar{\zeta}_{p,i}$, this is

$$\frac{-e_p(-1)^{m_p+m_{p-1}}\Pi(\zeta_{p,i}\bar{\zeta}_{p,i})\Pi(\bar{\zeta}_{p+1,i})g_{p+1}(0)}{\Pi(\zeta_{p+1,i}\bar{\zeta}_{p+1,i})\Pi(\bar{\zeta}_{p-1,i})g_{p-1}(0)}$$

from the fact that $g_p^*=(-1)^{m_p}g_p$, and that $b\zeta_{p,i}\to c_{p,i}$, one can easily show that

$\left(\prod_{i=1}^{m_p} \bar{\zeta}_{p,i}\right) g_p(0) > 0$ for all p. The expression above is then indeed positive, by the condition on the e_p.

To finish this section we show that when the spectral data is obtained from a monopole that the inner product on the monopole has this form and therefore the spectral data of a monopole satisfies the condition C-4. By continuity it suffices to work over a real section which doesn't intersect the intersections of the spectral curves.

Consider the part of the dual of figure (1.14) over a real section which is

$$\begin{array}{c} E_{N-1}^{-\perp} \\ \oplus \\ E_1^{+\perp} \cap E_{N-2}^{-\perp} \xrightarrow{i^*} E^*. \\ \vdots \\ E_{N-1}^{+\perp} \end{array}$$

It is straightforward to check that the real structure on E induces a map

$$E/E_p^+ \cap E_{N-p-1}^- \to E_p^{+\perp} \cap E_{N-p-1}^{-\perp},$$

which covers the real structure on $T\mathbb{P}_1$ and is a multiple of σ composed with multiplication by $g_{p-1}g_p$. It follows that the inner product on holomorphic sections of E over a real section obtained by using the real structure on E must take the form

$$\langle s, t \rangle = \sum_{p=1}^{N} a_p \frac{s_p t_p^*}{g_{p-1} g_p}.$$

We can now use the fact that this is a holomorphic function over the real section and defines a positive definite, hermitian form to reverse the arguments above and discover that if we set $a_1 = -1$, then we have

$$a_p = (-1)^p e_1 \cdots e_{p-1}$$

and

$$-(-1)^{m_{p-1}+m_p} e_p > 0.$$

To obtain the first condition one just applies the inner product to a section vanishing at a point of $S_p \cap C_x$, and to obtain the second one uses an asymptotic orthonormal basis as above.

2. From Spectral Data to Nahm's Equations

2a) Introduction and Notation. The purpose of this section is to obtain from the spectral data a solution $T_i(t)$ to Nahm's equations which satisfies the conditions B of the introduction. Let us first combine the matrices $T_i(t)$ into a polynomial; define

$$A(t, \zeta) = (T_1 + iT_2)(t) - 2iT_3(t)\zeta + (T_1 - iT_2)(t)\zeta^2 = A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2;$$

set $A_\#(t, \zeta) = \frac{1}{2} A_1(t) + A_2(t)\zeta$; Nahm's equations are then equivalent to

$$\dot{A} + [A_\#, A] = 0. \tag{2.1}$$

To solve this over each interval (μ_{p+1}, μ_p), we first define a vector bundle X over (μ_{p+1}, μ_p); then we define a polynomial $\underline{A}(t, \zeta)$ with coefficients in Γ (End(X)). From there, to get our matrices $A(t, \zeta)$, it suffices to choose a correct trivialisation of X; we will do this by specifying a connection. The main problem will be to show that the solution one obtains satisfies the boundary conditions B-2.

As all our computations will be on the fixed interval (μ_{p+1}, μ_p), to reduce the number of indices we will define some notation particular to this section:

- t: flow parameter for solutions to Nahm's equations.
- z: $z = \mu_p - t$.
- m: $m = m_p$.
- n: $n = m_{p-1}$.
- g: $g(\eta, \zeta) = \eta^m + a_1(\zeta)\eta^{m-1} + \cdots + a_m(\zeta) = g_p(\eta, \zeta)$. $g = 0$ is a local equation of S_p.
- h: $h(\eta, \zeta) = \eta^n + b_1(\zeta)\eta^{n-1} + \cdots b_n(\zeta) = g_{p-1}(\eta, \zeta)$. $h = 0$ is a local equation for S_{p-1}.
- D: $D = S_{p-1,p}$; $\tau(D) = S_{p,p-1}$.
- M: $M = \mathbb{C} \times S_p$; we will denote line bundles and divisors on S_p and their lifts to M by the same symbol.
- $\mathscr{L}$: $\mathscr{L}$ is the line bundle over M defined by $\mathscr{L}|_{\{z\} \times S_p} = L^z$.
- X: $X =$ the direct image sheaf $P_*(\mathscr{L}(m+n-1)[-D])$ ($P: M \to \mathbb{C}$ is the projection).
- Γ: $\Gamma = P_*(\mathscr{L}(m+n+1)[-D])$.
- Ξ: $\Xi = P_*(\mathscr{L}(m+n-1))$.
- N: $N = \mathbb{C} \times S_{p-1}$.
- Y: Defining $\mathscr{L}, P$ as above, but with respect to N instead of M, $Y = P_*(\mathscr{L}(m+n-1)[-D])$.
- $\tilde{g}$: $\tilde{g}(z, \eta, \zeta) = z^m g(\eta/z, \zeta) = \eta^m + z a_1(\zeta)\eta^{m-1} + \cdots + z^m a_m(\zeta)$.
- $\tilde{h}$: $\tilde{h}(z, \eta, \zeta) = z^n h(\eta/z, \zeta)$.
- zS_p: zS_p is S_p "shrunk" by a factor of z in the η-direction; it is defined (fixing z) by $\tilde{g}(z, \eta, \zeta) = 0$.
- $F^{(k)}$: The k^{th} formal neighbourhood of the zero section in $T\mathbb{P}_1$, defined by $\eta^{k+1} = 0$.
- $\tilde{M}$: Surface in $\mathbb{C} \times T\mathbb{P}_1$ defined by $\tilde{g}(z, \eta, \zeta) = 0$. Let $\tilde{P}: \tilde{M} \to C$ be the projection; then $\tilde{P}^{-1}(0) = F^{(m-1)}$, and for $z \neq 0$, $\tilde{P}^{-1}(z) = zS_p$. Again, bundles on $T\mathbb{P}_1$, their lifts to $\mathbb{C} \times T\mathbb{P}_1$, and their restriction to $\tilde{M}$ will be denoted by the same letter.
- $\tilde{\Xi}$: $\tilde{\Xi} = \tilde{P}_*(L(m+n-1))$.

2b) Solution Over (μ_{p+1}, μ_p). In this section, we will recall the version of the "Krichever construction" of the solution to Nahm's equations which is due to Hitchin [Hi2]. Let V_t denote the sheaf $L^{-t}(-1) \otimes (E/E_p^+ + E_{N-p}^-))$; for generic monopoles, $V_t = L^{\mu_p - t}(m+n-1)[-D]$. If T_a denotes the fiber of $T\mathbb{P}_1 \to \mathbb{P}_1$ at $a \in \mathbb{P}_1$, then one has the exact sequence over S_p:

$$0 \to V_t(-1) \to V_t \to V_t|_{S_p \cap T_a} \to 0. \tag{2.2}$$

For $t \in (\mu_{p+1}, \mu_p)$, $H^0(S_p, V_t(-1)) = 0$, by the vanishing theorem (1.17). As the genus of S_p is $(m-1)^2$, and the degree of $L^{\mu_p - t}(m+n-2)$ is $(m-1)^2 - 1$, Riemann Roch then implies that $H^1(S_p, V_t(-1)) = 0$, and referring to (2.2),

$$\bar{X}_t = H^0(S_p, V_t) \simeq H^0(S_p \cap T_a, V_t) \simeq \mathbb{C}^m. \tag{2.3}$$

That $h^0(S_p \cap T_a, V_t) = m$ is easy to see in the generic case, as $S_p \cap T_a$ consists of m points, counted with multiplicity and V_t is a line bundle. In general, if one writes out a local basis for E, E_p^+, E_{n-p}^- along T_a, it is straightforward to show that the number of independent sections of V_t along T_a is exactly the multiplicity of S_p. Next, as the dimension of $\bar{X}_t$ is constant in t (2.3), the space $\bar{X}_t$ fit together nicely to form a vector bundle; in sheaf theoretic terms, $\bar{X}_t$ is the fiber X_t at t of the direct image sheaf X.

In a similar vein, one obtains

$$\Gamma_t \simeq H^0(S_p, V_t(2)) \simeq \mathbb{C}^{3m}.$$

We next define the endomorphisms $\underline{A}(t, \zeta)$. One way to do this [AHH] is to use (2.3). If $\zeta = \zeta_0$ corresponds to the point $a \in \mathbb{P}_1$, then one defines $\underline{A}(t, \zeta_0)$ by the commuting diagram:

$$\begin{tikzcd} X_t \arrow[r, "\text{restr}", "\sim"'] \arrow[d, "\underline{A}(t,\zeta_0)"'] & H^0(S_p \cap T_a, V_t) \arrow[d, "\times\eta"] \\ X_t \arrow[r, "\text{restr}", "\sim"'] & H^0(S_p \cap T_a, V_t) \end{tikzcd} \tag{2.4}$$

A section at a point (ζ_0, η_{i0}) of multiplicity k of $S_p \cap T_a$ is a truncated power series $\Sigma \alpha_j (\eta - \eta_{i0})^j$; the map $\times \eta$ multiplies this series by η. Note that this means that the spectrum of $\underline{A}(t, \zeta_0)$ is the set of η_{i0}; also, in the generic case when V_t is a line bundle, that $\dim\ker(\eta \mathbb{1} - \underline{A}(t, \zeta)) \leq 1$, for all ζ, η.

An equivalent way of defining $\underline{A}(t, \zeta_0)$ is that of [Hi2]: the map

$$H^0(S_p, \mathcal{O}(2)) \otimes X_t \to \Gamma_t \tag{2.5}$$

has an m dimensional kernel; taking a basis η, 1, ζ, ζ^2 of $H^0(S_p, \mathcal{O}(2))$ (in the standard trivialisation), it is shown that there are endomorphisms $\underline{A}_0$, $\underline{A}_1$, $\underline{A}_2$ of X_t, such that the map $s \mapsto \eta \otimes s - 1 \otimes \underline{A}_0(s) - \zeta \otimes \underline{A}_1(s) - \zeta^2 \otimes \underline{A}_2(s)$ is an isomorphism of X_t onto the kernel of (2.5). One thus sees that $\underline{A}(t, \zeta)$ is a polynomial of degree 2 in ζ.

One must then give the trivialisation of X. As this is a bundle over a one-dimensional base, then, up to an irrelevant overall change of basis this is equivalent to giving a connection on X. One fixes trivialisations of $L^\mu(k)$ over $T\mathbb{P}_1$ in which the transition functions from U_1 to U_0 are $\exp(\mu\eta/\zeta)\zeta^k$; this determines an isomorphism $e: H^0(S_p \cap U_0, V_t) \to H^0(S_p \cap U_0, V_{t_0})$ for a fixed t_0. One then shows that there is a well defined connection ∇ acting on sections of X over $S_p \times \mathbb{C}$ defined by

$$\nabla_t(s) = e^{-1}(\partial_t e + e\underline{A}_\#)s. \tag{2.6}$$

It then follows [Hi2] that writing $\underline{A}$ in a ∇-flat basis gives a solution $A(t, \zeta)$ to Nahm's equation (2.1).

2c) Behaviour of X at μ_p. One thus obtains flows over the intervals (μ_{p+1}, μ_p), $(\mu_p, \mu_{p-1}), \ldots$ etc. The problem then arises of studying boundary behaviour, at μ_p, say. The particular boundary conditions we are studying are preserved under limits; therefore, it suffices to consider the generic case. As in the definition of the solution over (μ_{p+1}, μ_p), we will analyse this in three steps, studying

i) the fiber X_{μ_p} of X at μ_p,
ii) the endomorphism $\underline{A}(t, \zeta)$ at $t = \mu_p$,
iii) the matrices $A(t, \zeta)$ at $t = \mu_p$.

The next three sections are devoted to this analysis. As everything we do concerns behaviour at $t = \mu_p$, it is more convenient to change our parameter, and set $z = (\mu_p - t)$. We incorporate this change into our definitions of X, $\underline{A}$, etc., and so we will refer to X_{μ_p} as X_0, $\underline{A}(\mu_p, \zeta)$ as $\underline{A}(0, \zeta)$, etc.

The first thing to do is to analyse X at $z = 0$. Approaching 0 from below, one is studying $L^z(m+n-1)[-D] \simeq L^{\mu_p - t}(m_p + m_{p-1} - 1)[-S_{p-1}, p]$ over S_p as $t \to \mu_p$; approaching μ_p from above, one is interested in $L^{\mu_{p-1} - t}(m_{p-1} + m_{p-2} - 1)[-S_{p-2,p-1}]$ over S_{p-1} as $t \to \mu_p$; however, using the section of (1.8), this last bundle is isomorphic over S_{p-1} to $L^z(m+n-1)[-D]$. Thus, in both cases one is looking at the same bundle, as $z \to 0$. Symmetry then allows us to consider only one case, say that of S_p.

By general theory, as X is torsion free over a one dimensional base, X is locally free. However, at $z = 0$, at least for $m > n$, one does not necessary have the isomorphism (2.4). The evaluation map on the fiber X_0 of X at $z = 0$:

$$\mathrm{ev}_0 : X_0 \to H^0(S_p, \mathcal{O}(m+n-1)[-D]) \tag{2.7}$$

is not necessarily surjective: the dimension of the space $H^0(S_p, L^z(m+n-1)[-D])$ may jump (upward) at $z = 0$. ev_0 is, however, injective; any section of X mapping to zero in (2.7) corresponds to a local section of $\mathscr{L}(m+n-1)[-D]$ vanishing at $z = 0$; this is then divisible by z, and so cannot be a generator of X. The problem is then to determine what the image is in (2.7).

To do this we will consider the sheaves Ξ, $\tilde{\Xi}$ (see 2.a) over $\mathbb{C}$, which in a similar fashion are locally free; the evaluation maps:

$$\begin{aligned} &\mathrm{ev}_z : \Xi_z \to H^0(S_p, L^z(m+n-1)), \\ &\tilde{\mathrm{ev}}_z : \tilde{\Xi}_z \to H^0(\tilde{P}^{-1}(z), L(m+n-1)), \end{aligned} \tag{2.8}$$

are again injections, and are bijective for generic z. *Sections of X, Ξ (and of $\tilde{\Xi}$) over $V \subset \mathbb{C}$ are to be thought of in terms of the corresponding sections of the bundles over $P^{-1}(V)(\tilde{P}^{-1}(V))$; using the U_0 trivialisation, they will be thought of as functions $f(z, \eta, \zeta)$ over $P^{-1}(V)(\tilde{P}^{-1}(V))$ satisfying certain constraints.* X is a subsheaf of Ξ, consisting of those sections which vanish along $\mathbb{C} \times D$.

There is a very useful link between Ξ and $\tilde{\Xi}$:

Lemma (2.9). *There is a map of sheaves*

$$\rho : L(k)|_{\tilde{M}} \to \mathscr{L}(k)|_M$$

given locally as follows. If in the standard trivialisation over U_i a section s of $L(k)$ over $\tilde{M}$ is represented by $\tilde{f}_i(z, \eta, \zeta)$ then $\rho(s)$ is represented over M by $f_i(z, \eta, \zeta) = \tilde{f}_i(z, \eta z, \zeta)$. Away from $z = 0$, this is an isomorphism.

Proof. It suffices to verify that over M, $f_0 = \exp(z\eta/\zeta) f_1$: this is immediate.

Corollary (2.10). *Taking direct images, there is a map of sheaves over* $\mathbb{C}$

$$\rho: \tilde{\Xi} \to \Xi,$$

which is an isomorphism away from $z = 0$.

We now study the sheaf $\tilde{\Xi}$; we find that the map $\tilde{\mathrm{ev}}$ of (2.8) is always a bijection near $z = 0$ by showing that the right-hand side has constant dimension. First, let us consider $z = 0$; $\tilde{P}^{-1}(0)$ is $F^{(m-1)}$. Note that $SL(2, \mathbb{C})$ acts on $F^{(m-1)}$; this action lifts to $L(k)$ [Hi2].

Lemma (2.11). *One has the decomposition*

$$\begin{aligned} H^0(F^{(m-1)}, L(m+n-1)) \simeq\ & H^0(F^{(m-1+n)}, L(m-1+n)) \\ & \oplus \eta H^0(F^{(m-1+n-2)}, L(m-1+n-2)) \\ & \ldots\ldots \\ & \oplus \eta^s H^0(F^{(m-1+n-2s)}, L(m-1+n-2s)), \end{aligned}$$

where $s = \min(m-1, n)$. *The* η^i *are to be thought of as sections of* $\mathcal{O}(2i)$, *and there are implicit restrictions to* $F^{(m-1)}$ *on the right-hand side. The decomposition is* $SL(2, \mathbb{C})$ *invariant.*

Proof. We use Proposition 5.4 of [Hi2] repeatedly, which states:

Under restriction, one has an $SL(2, \mathbb{C})$ invariant isomorphism

$$H^0(F^{(j)}, L(j)) \simeq H^0(F^{(0)}, L(j) \simeq H^0(\mathbb{P}_1, \mathcal{O}(j)) \tag{2.12}$$

and

$$H^0(F^{(j)}, L(j-1)) = 0. \tag{2.13}$$

Let σ be a section of $L(m+n-1)$ over $F^{(m-1)}$; then (2.12) implies that there is a section σ_0 of $L(m+n-1)$ over $F^{(m+n-1)}$ such that the restriction of σ, σ_0 to $F^{(0)}$ are the same: then $\sigma - \sigma_0 = \eta\sigma_1$, say, over $F^{(m-1)}$, where as $\eta\sigma_1$ is defined over $F^{(m-1)}$, σ_1 is defined over $F^{(m-2)}$; as η is a section of $\mathcal{O}(2)$, σ_1 is a section of $L(m+n-1-2)$. One can then reapply (2.12) to σ_1; iterating this procedure, one obtains

$$\sigma - (\sigma_0 + \eta\sigma_1 + \cdots + \eta^s\sigma_s) = \eta^{s+1}\sigma_{s+1},$$

where the σ_i are restrictions to $F^{(m-1-i)}$ of sections of $L(m+n-1-2i)$ over $F^{(m+n-1-2i)}$, and $\sigma_{s+1} \in H^0(F^{(m-1-(s+1))}), L(m-1-2(s+1))$. This procedure truncates either when $s = m-1$ as then $\eta^{s+1} = 0$, or when $s = n$, as then (2.13) implies that $\sigma_{s+1} = 0$. It is easy to see that this decomposition is an isomorphism; $SL(2, \mathbb{C})$ invariance follows from the naturality of the construction.

Corollary (2.14).

1) $h^0(F^{(m-1)}, L(m+n-1)) = m(n+1)$.

2) *If* $k > j$, *the restriction map* $H^0(F^{(k)}, L(k+j)) \to H^0(F^{(j)}, L(k+j))$ *is an isomorphism.*

One next looks at the case $z \neq 0$:

Lemma (2.15) *There is an interval I containing 0 such that for $z \neq 0$, $z \in I$,*

$$h^0(zS_p, L(m+n-1)) = m(n+1).$$

Proof. By Lemma (2.9), $h^0(zS_p, L(m+n-1)) \simeq h^0(S_p, L^z(m+n-1))$. By the vanishing theorem (1.17), and Riemann Roch, $h^1(S_p, L^z(m+n-2)[-D]) = 0$; a fortiori, $h^1(S_p, L^z(m+n-1)) = 0$. Riemann Roch then gives $h^0(S_p, L^z(m+n-1)) = m(n+1)$.

As (2.8) is always an isomorphism, from (2.11) one therefore has a clear picture of what the fiber of $\tilde{\Xi}$ is at $z=0$; the next step is to exploit this to study Ξ at $z=0$, using the map ρ of (2.9). We start with a lemma describing $H^0(S_p, \mathcal{O}(m+n-1))$.

Lemma (2.16). *The restriction map:*

$$H^0(T\mathbb{P}_1, \mathcal{O}(j)) \to H^0(S_p, \mathcal{O}(j))$$

is surjective. Its kernel is the set of sections of the form $g(\eta, \zeta) \cdot f(\eta, \zeta)$, with $f(\eta, \zeta) \in H^0(T\mathbb{P}_1, \mathcal{O}(j\text{-}2m))$, and so is zero for $j < 2m$.

Proof. We use the exact sequence $0 \to \mathcal{O}(j\text{-}2m) \to \mathcal{O}(j) \to \mathcal{O}(j)|_{S_p} \to 0$. The only non-trivial statement is surjectivity; to prove this, one shows that the map (multiplication by g) from $H^1(T\mathbb{P}_1, \mathcal{O}(j\text{-}2m)) \to H^1(T\mathbb{P}_1, \mathcal{O}(j))$ is injective. This is easy to see if one uses the explicit description of these spaces given in (1.2).

Combining (2.16) and (1.2), one sees that $H^0(S_p, \mathcal{O}(m+n-1))$ is composed of sections given in the U_0 trivialisation by $\sum_{i=0}^{m-1} \eta^i f_i(\zeta)$, where the f_i are polynomials of degree $m+n-1-2i$ ($f_i = 0$ if $m+n-1-2i < 0$).

Proposition (2.17). *The fiber Ξ_0 of Ξ at $z=0$ is mapped under the evaluation map*

$$\mathrm{ev}_0 \colon \Xi_0 \to H^0(S_p, \mathcal{O}(m+n-1))$$

to the subspace of sections of the form (in the U_0 trivialisation):

$$\sum_{i=0}^{s} \eta^i f_i(\zeta), \quad \textit{where the } f_i \textit{ are polynomials of degree } (m+n-1-2i), \tag{2.18}$$

where $s = \min(m-1, n)$. Thus, ev_0 is a bijection $\Leftrightarrow m-2 \leq n$.

Proof. As the subspace of $H^0(S_p, \mathcal{O}(m+n-1))$ represented by (2.18) is in all cases of dimension $m(n+1)$, i.e. the dimension of Ξ_0, it suffices to show that all sections of the form (2.18) are in the image of ev_0.

Let $\tilde{\xi}$ be a local section of $\tilde{\Xi}$ near $z=0$; $\tilde{\xi}$ can be written $\tilde{\xi} = \sum_{i=0}^{s} \eta^i \tilde{f}_i(\eta, \zeta) + O(z)$, where $\tilde{f}_i$ represents a section of $L(m+n-1-2i)$ over $F^{(m-1-i)}$; then $\xi = \rho(\tilde{\xi})$ is $\sum_{i=0}^{s} (\eta z)^i \tilde{f}_i(\eta z, \zeta) + O(z) = \tilde{f}_0(0, \zeta) + O(z)$. As $\tilde{f}_0(0, \zeta)$ can be an arbitrary polynomial of degree $(m+n-1)$, this shows that the $i=0$ portion of (2.18) is realised; to get the rest, one has to be a bit more careful.

Let $0 \leq r \leq s$. Suppose one has a section $\tilde{\xi} = \Sigma \tilde{\xi}_{ijk} \eta^i \zeta^j z^k$ of $\tilde{\Xi}$, such that $\tilde{\xi}_{ijk} = 0$ for $i+k < r$; $\xi = z^{-r} \rho(\tilde{\xi})$ is then a section of Ξ. We will show inductively in r that one obtains in this way, at $z=0$, all sections in (2.18) of the form $\sum_{i=0}^{r} \eta^i f_i(\zeta)$.

Over $\tilde{M}$, one has the exact sequence of formal neighbourhoods of $\{z=0\}$,

$$0 \to \mathcal{O}_{F^{(m-1)}} \to \mathcal{O}_{\{z^{k+1}=0\}} \to \mathcal{O}_{\{z^k=0\}} \to 0. \tag{2.19}$$

If one tensors this sequence with $L(m+n-1)$, and recalls from Hitchin ([Hi2, p. 174]) that $H^1(F^{(m-1)}, L(m+n-1))=0$ for $n \geqq -1$, then it is clear that any section of $L(m+n-1)$ over $\{z^{k-1}=0\}$ extends to $\{z^k=0\}$. By Grothendieck's theorem (see, e.g. [Ha]), this means that any section over a formal neighbourhood of $z=0$ extends to an actual neighbourhood; it is thus sufficient to work over formal neighbourhoods.

Over $\{z=0\}=F^{(m-1)}$, consider a section $\tilde{\xi}$, given over U_0 by $\eta^r \tilde{f}_0(\eta,\zeta)$, and over U_1 by $(\eta/\zeta^2)^r \tilde{f}_1(\eta,\zeta)$, where the $\tilde{f}_i$ represent any element of $H^0(F^{(m-1+n-2r)}, L(m+1+n-2r))$. One therefore has over $\tilde{M} \cap \{z=0\}$,

$$\eta^r \tilde{f}_0(\eta,\zeta) = \exp(\eta/\zeta)\zeta^{m+n-1}((\eta/\zeta^2)^r \tilde{f}_1(\eta,\zeta)),$$

and so, over $T\mathbb{P}_1$,

$$\eta^r \tilde{f}_0 + \eta^m s = \exp(\eta/\zeta)\zeta^{m+n-1}((\eta/\zeta^2)^r \tilde{f}_1)$$

where s is some function. However this implies that modulo z^{r+1},

$$(\tilde{g}/\eta^{m-r})\tilde{f}_0 + \tilde{g}s = \exp(\eta/\zeta)\zeta^{m+n-1}\cdot(\tilde{g}/\eta^{m-r}\zeta^{2r})\cdot \tilde{f}_1),$$

and so $(\tilde{g}/\eta^{m-r})\tilde{f}_0, (\tilde{g}/\eta^{m-r}\zeta^{2r})\tilde{f}_1$ define a section of L^{m+n-1} over $\{z^{r+1}=0\}$; it is easy to check that this section satisfies $\xi_{ijk}=0$ for $i+k<r$, and that $\xi = z^{-r}\rho(\tilde{\xi})$ is at $z=0$ of the form:

$$\tilde{f}_0(0,\zeta)\{\eta^r + a_1(\xi)\eta^{r-1} + \cdots + a_r(\xi)\}.$$

One sees from this that one can obtain all terms in (2.18) of order r in η; inductively, one already had all terms of order $\leqq r-1$, and so one now has all terms of order $\leqq r$.

The last step is to reinsert the constraint of vanishing at D, that is to consider X instead of Ξ. In this, it is useful to distinguish two cases:

i) $m \leqq n$.
In this case, the preceding discussion is not even necessary. One has, at $z=0$, the same vanishing theorem (1.17) one had for $z<0$; this implies $h^0(S_p, \mathcal{O}(m+n-1)[-D])=m$, which in turn gives us:

Proposition (2.20). *If $m \leqq n$, the map ev_0 in (2.7) is an isomorphism.*

ii) $m>n$:

Proposition (2.21). *If $m>n$, $\mathrm{ev}_0(X_0)$ in (2.7) splits into two summands $Z_0' \oplus Z_0''$.*

Z_0' is of dimension $(m-n)$ and consists of sections vanishing on $S_p \cap S_{p-1}$; in the U_0 trivialisation, these are of the form $h(\eta,\zeta)\cdot p(\zeta)$, $p(\zeta)$ a polynomial of degree $\leqq m-n-1$.

Z_0'' is of dimension n, and consists of all the sections of the form $\sum_{s=0}^{n-1} \eta^i t_i(\zeta)$ which vanish on D.

Proof. Let $W = H^0(S_p, \mathcal{O}(m+n-1)[-D])$; we will show that $\mathrm{ev}_0(X_0) = W \cap \mathrm{ev}_0(\Xi_0)$; as $\mathrm{ev}_0(X_0) \subset W \cap \mathrm{ev}_0(\Xi_0)$, it suffices then to show that $W \cap \mathrm{ev}_0(\Xi_0)$ is of dimension m.

By Proposition (2.17), the space Z'_0 sits inside $W \cap \mathrm{ev}_0(\Xi_0)$, and has dimension $(m-n)$; furthermore, if s is any element of $W \cap \mathrm{ev}_0(\Xi_0)$ then there exists an element s' of Z'_0 (determined by the η^n component of s) such that $s - s' = s'' \in Z''_0$. The only thing to be shown is that Z''_0 has dimension n. As $\mathrm{ev}_0(X_0) \subset Z'_0 \oplus Z''_0$, its dimension is at least n. Let $s'' \in Z''_0$. By (2.16), it is the restriction to S_p of a section t'' over $T\mathbb{P}_1$ which vanishes on D; as t'' is of the form $\sum_{i=0}^{n-1} \eta^i p_i(\zeta)$, t'' does not vanish when restricted to S_{p-1}. One then has an injective map from Z''_0 to $H^0(S_{p-1}, \mathcal{O}(m+n-1)[-D])$; by case i), this latter space is of dimension n, and so $\dim(Z''_0) = n$.

The proof of (2.21) also tells us how the bundles X, Y should be glued at $z=0$. One uses the diagram:

$$\begin{array}{ccc} & H^0(T\mathbb{P}_1, \mathcal{O}(m+n-1) \otimes \mathscr{I}(D)) & \\ \swarrow & & \searrow \\ H^0(S_p, \mathcal{O}(m+n-1)[-D]) & & H^0(S_{p-1}, \mathcal{O}(m+n-1)[-D]). \end{array} \tag{2.22}$$

When $m > n$, Y_0 is mapped isomorphically into the summand Z''_0 of X_0. Note that when $m = n$, the maps in (2.22) are both isomorphisms.

2d) The Endomorphism $\underline{A}(z, \zeta)$ at μ_p. Having now established what the fiber of X is at $z=0$, we turn our attention to the behaviour of the endomorphisms $\underline{A}(z,\zeta)$ defined by (2.4) as $z \to 0$. As $z \to 0$ from above, $\underline{A}(z,\zeta)$ is defined using the curve S_{p-1}; as $z \to 0$ from below, $\underline{A}(z,\zeta)$ is defined using the curve S_p. As the two situations are symmetric, we confine our study to the latter. As before, it will be useful to distinguish the cases: $m < n$, $m > n$, $m = n$.

i) $m < n$.
Referring to the exact sequence (2.2), and case b) of the vanishing theorem, (1.17), one can apply the same arguments one had for $t \in (\mu_{p+1}, \mu_p)$ to the case $t = \mu_p$ $(z=0)$, showing that the restriction map in (2.3) is an isomorphism at $z=0$. This enables one to show that there is a well defined finite limit $\underline{A}(0,\zeta)$. of $\underline{A}(z,\zeta)$.

ii) $m > n$.
In this case, the restriction map in (2.3) is no longer an isomorphism; as we shall see, $\underline{A}(z,\zeta)$ has a pole at $z=0$. As before, when analysing this case, it is more convenient to work with $\tilde{M}$ than with M. Let

$$\mathscr{I}_p = \text{ideal of functions } \Sigma f_{ijk} \eta^i \zeta^j z^k \text{ with } f_{ijk} = 0 \textit{ for } i + k < p. \tag{2.23}$$

The proof of (2.17) gave an isomorphism

$$z^{-n}\rho : \mathscr{I}_n \tilde{\Xi} \to \Xi.$$

Let $\tilde{X} = (z^{-n}\rho)^{-1}(X)$. In a similar fashion, define $\tilde{\Gamma} = (z^{-n}\rho)^{-1}(\Gamma)$. The following lemma explains how the maps η, $\underline{A}(z,\zeta)$ involved in (2.4) transport from M to $\tilde{M}$:

Lemma (2.24). *The diagram*

$$\begin{tikzcd} X \arrow[r, "F"] & \Gamma \\ \tilde{X} \arrow[u, "z^{-n}\rho"] \arrow[r, "\tilde{F}"] & \tilde{\Gamma} \arrow[u, "z^{-n}\rho"'] \end{tikzcd}$$

commutes, where either i) $F = z\eta, \tilde{F} = \eta$
ii) $F = A(z,\zeta) = \tilde{F}$.

The proof is straightforward; as a consequence, if one sets $\underline{B}(z,\zeta) = z\underline{A}(z,\zeta)$, $(\eta - \underline{A}(z,\zeta))X = 0 \Leftrightarrow (\eta - \underline{B}(z,\zeta))\tilde{X} = 0$. We will compute $\underline{B}$ instead of $\underline{A}$.

As a first step, we define near $z = 0$ two subbundles Z', $Z''(\zeta_0)$ of X of rank $(m-n)$, n respectively, which extend the summands Z'_0, Z''_0 of X_0 defined in (2.21), in such a way that:

—local sections of Z' correspond over M to sections which are divisible by h, to order $(m-n)$ (inclusively) in z.

—local sections of $Z''(\zeta_0)$, considered as sections over M, restricted to $\zeta = \zeta_0$ and written, using Lagrange interpolation in η, as polynomials in η of degree $< m$, are of the form $\sum_{i=0}^{n-1} a_i(z)\eta^i$; all such functions can be obtained.

The subbundle Z' is obtained by constructing the appropriate subbundle $\tilde{Z}'$ of $\tilde{X}$. Apply the division algorithm in η to obtain:

$$g(\eta,\zeta) = k(\eta,\zeta)h(\eta,\zeta) + r(\eta,\zeta), \tag{2.25}$$

where k is of degree $(m-n)$ in η, r of degree less than n in η. One has

$$z^m g(\eta/z;\zeta) = (z^{m-n}k(\eta/z,\zeta))(z^n h(\eta/z,\zeta)) + z^m r(\eta/z,\zeta);$$

rewrite this as

$$\tilde{g}(z,\eta,\zeta) = \tilde{k}(z,\eta,\zeta)\cdot\tilde{h}(z,\eta,\zeta) + z^m r(\eta/z,\zeta). \tag{2.26}$$

Thus, modulo the ideal $\mathscr{I}_m$ (2.23), $\tilde{g} = \tilde{k}\tilde{h}$. Now consider the surface $\tilde{Q} \subset \mathbb{C} \times T\mathbb{P}_1$ defined by $\tilde{k} = 0$; let $\tilde{R}: \tilde{Q} \to \mathbb{C}$ be the projection. Define

$$\tilde{\Sigma} = \tilde{R}_*(L(m-n-1)) \tag{2.27}$$

then $\tilde{\Sigma}_0$, by (2.12), is

$$H^0(F^{(m-n-1)}, L(m-n-1)) \simeq H^0(F^{(0)}, L(m-n-1)). \tag{2.28}$$

Let $\tilde{C}_i, i = 1,\dots,m-n$ be the elements of a local basis of $\tilde{\Sigma}$, which over $F^{(0)}$ correspond to the section ζ^{i-1}; $\tilde{C}_i$ can be represented by functions $\tilde{f}_{ji}$ over U_j with $\tilde{f}_{0i} = \exp(\eta/\zeta)\zeta^{m-n-1}\tilde{f}_{1i} + \tilde{k}s_i$, where s_i is some function. Therefore,

$$\tilde{h}\tilde{f}_{0i} = \exp(\eta/\zeta)\zeta^{m+n-1}((\tilde{h}/\zeta^{2n})\tilde{f}_{1i}) + \tilde{k}\tilde{h}s. \tag{2.29}$$

i.e., modulo $\mathscr{I}_m$, $\tilde{h}\tilde{f}_{0i} = \exp(\eta/\zeta)\zeta^{m+n-1}((\tilde{h}/\zeta^{2n})\tilde{f}_{1i}) + \tilde{g}s$; modulo $\mathscr{I}_m$ one has local sections of $L(m+n-1)$ over $\tilde{M}$ which are divisible by $\tilde{h}$. One checks, order by order in z, similarily to the proof of Lemma (2.17), that such sections extend to a neighbourhood of $z=0$. Applying $z^{-n}\rho$ to the $\tilde{e}_i$, one obtains sections $e_1 \cdots e_{m-n}$, divisible by h to order $(m-n)$ in z; Z' is the subbundle spanned by these sections.

The definition of $Z''(\zeta_0)$ is more straightforward. The elements of X_z whose Lagrange interpolations over $\zeta=\zeta_0$ are of the form $\sum_{i=0}^{n-1} a_i\eta^i$, form a rank n subbundle of X_z; at $z=0$, this is a consequence of (2.17), and away from $z=0$, of the isomophism (2.3). One has a natural basis $e_{n-m+1} \cdots e_m$ of $Z''(\zeta_0)$ restricting over $\zeta=\zeta_0$ to $\eta^{n-1}, \ldots, 1$.

One would like to study $\underline{B}(z,\zeta_0)$ at $z=0$; as for $\underline{A}$, one has two definitions of $\underline{B}$. The first uses a diagram similar to (2.4).

$$\begin{array}{ccc} \tilde{X}_z & \xrightarrow{\mathrm{restr}_{z,a}} & H^0(zS_p \cap T_a, L(m+n-1)) \simeq \mathbb{C}^m \\ \underline{B}(z,\zeta) \Big\downarrow & & \Big\downarrow \times\eta \\ \tilde{X}_z & \xrightarrow{\mathrm{restr}_{z,a}} & H^0(zS_p \cap T_a, L(m+n-1)) \simeq \mathbb{C}^m \end{array} \tag{2.30}$$

Again, the spaces on the right-hand side are to be thought of as the space V of polynomials in η of degree less than m, using Lagrange interpolation. The diagram (2.30) defines $\underline{B}(z,\zeta)$, as long as $\mathrm{restr}_{z,a}$ is an isomorphism; the problem is that this is not the case at $z=0$. Let us consider the action of $\mathrm{restr}_{z,a}$ on the subbundles $\tilde{Z}', \tilde{Z}''(\zeta_0)$ of $\tilde{X}$ corresponding under $z^{-n}\rho$ to $Z', Z''(\zeta_0)$. Remember $e_i = z^{-n}\rho(\tilde{e}_i)$. For e_i, $i=1,\ldots,m-n$ (i.e. the local basis of $\tilde{Z}'$) $\mathrm{restr}_{0,a}(\tilde{e}_i)=\eta^n$ (polynomial $p_i(\eta)$), as $\tilde{h}=\eta^n$ over $z=0$. Referring to (2.11), (2.28), the $\tilde{e}_i$ at zero form a basis of $\eta^n H^0(F^{(m-n-1)}, L(m-n-1))$; as $H^0(F^{(m-n-1)}, L(m-n-2))=0$, the $\mathrm{restr}_{0,a}(\tilde{e}_i)$ form a basis of the subspace V_n of V consisting of polynomials of the form $\sum_{i=n}^{m} a_i\eta^i$.

On the other hand, for $i=m-n+1,\ldots,m$, $\mathrm{restr}_{z,a}(\tilde{e}_i)$ has leading term $\eta^{m-i}z^{i-n-m}$, and so $\mathrm{restr}_{0,a}(\tilde{e}_i)=0$; $Z''(\zeta_0)$ maps to zero.

We modify the map restr_a to obtain an isomorphism at $z=0$. Set for $z=0$,

$$\begin{aligned} m\,\mathrm{restr}_{z,a}(\tilde{e}_i) &= \mathrm{restr}_{z,a}(\tilde{e}_i), \quad i=1,\ldots,m-n, \\ &= z^{m-n-i}\,\mathrm{restr}_{z,a}(\tilde{e}_i) \quad i=m-n+1,\ldots,m. \end{aligned} \tag{2.31}$$

Then $m\,\mathrm{restr}_{z,a}$ is an isomorphism for all $z\neq 0$, and the limit $m\,\mathrm{restr}_{0,z}$ is an isomorphism also. Define $\underline{B}'(z,\zeta_0)$ by

$$\begin{array}{ccc} \tilde{X}_z & \xrightarrow{m\,\mathrm{restr}_{z,a}} & V \\ \underline{B}'(z,\zeta_0) \Big\downarrow & & \Big\downarrow \times\eta. \\ \tilde{X}_z & \xrightarrow{m\,\mathrm{restr}_{z,a}} & V \end{array}$$

$\underline{B}'$ is continuous, well defined at $z=0$; in fact, changing the basis $\tilde{e}_1 \cdots \tilde{e}_{m-n}$ of $\tilde{Z}'$ so that $\mathrm{restr}_{0,a}(\tilde{e}_i)=\eta^{m-i+1}$,

$$\begin{aligned} \underline{B}'(0,\zeta_0)(e_i) &= 0 \quad &\text{if} \quad i=1 \\ &= e_{i-1} \quad &\text{if} \quad i=2,\dots,m. \end{aligned}$$

Proposition (2.32).

i) *$\underline{B}(z,\zeta)$ is continuous at $z=0$.*

ii) *With respect to the decomposition $\tilde{X}=\tilde{Z}'\oplus Z''(\zeta)$, $\underline{B}(z,\zeta)$ is of the form:*

$$\begin{bmatrix} \alpha(z,\zeta) & O(z) \\ O(z^{m-n}), & O(z) \end{bmatrix}.$$

iii) *Writing $\alpha(z,\zeta)=\alpha_0(z)+\alpha_1(z)\zeta+\alpha_2(z)\zeta^2$ the $\alpha_i(0)=z_i$, $i=0,1,2$ are the generators of an irreducible representation of $sl(2,\mathbb{C})$.*

Proof. i) One uses the expression above for $\underline{B}'(z,\zeta)(e_i)$; one gets, e.g., for $\zeta=\zeta_0$:

$$\begin{aligned} \underline{B}(z,\zeta_0)(e_i) = O(z) \quad & i=1, \\ e_{i-1}+O(z), \quad & i=2,\dots,m-n, \\ ze_{i-1}, \quad & i=m-n+1,\dots,m. \end{aligned} \tag{2.33}$$

iii) Equation (2.33) shows that $\underline{B}(0,\zeta)(Z'_0)\subset Z'_0$; to see that one gets the representation, it is more convenient to use the second definition of $\underline{B}$ analogous to that of $\underline{A}$ given after (2.5): one has the map

$$H^0(T\mathbb{P}_1,\mathcal{O}(2))\otimes\tilde{X}\to\tilde{\Gamma}.$$

$\underline{B}$ is defined by $(\eta-\underline{B}(z,\zeta))(\tilde{X})=0$. At $z=0$, restricted to Z'_0, the map $\times\,\eta$ is:

$$\times\,\eta\colon \eta^n H^0(F^{(m-n-1)},L(m-n-1))\to\eta^n H^0(F^{(m-n-1)},L(m-n+1)).$$

Using (2.11), this becomes:

$$\times\,\eta\colon \eta^n H^0(F^{(m-n-1)},L(m-n-1))\to\eta^n\begin{bmatrix} H^0(F^{(m-n+1)},L(m-n+1)) \\ \oplus\,\eta H^0(F^{(m-n-1)},L(m-n-1)) \\ \oplus\,\eta^2 H^0(F^{(m-n-3)},L(m-n-3)) \end{bmatrix}$$

$\times\,\eta$ is just the $Sl(2,\mathbb{C})$ invariant isomorphism into the second summand. Referring to Hitchin [Hi2, p. 178], if one sets in the basis $\tilde{e}_i$ which, at $z=0$, over $\eta^{n+1}=0$, corresponds to the monomial $\eta^n\zeta^{i-1}$:

$$\begin{aligned} a_0(\tilde{e}_j) &= (j-(m-n))\tilde{e}_{j+1}, \\ a_1(\tilde{e}_j) &= (-2j+(m-n)+1)\tilde{e}_j, \\ a_2(\tilde{e}_j) &= (j-1)\tilde{e}_{j-1}, \end{aligned} \tag{2.34}$$

then the a's give an irreducible representation of $sl(2,\mathbb{C})$, and $(\eta+\Sigma a_i\zeta^i)(\tilde{Z}'_0)=0$.

ii) One must show that the fact that $\underline{B}(z,\zeta)(\tilde{Z}'_0)\subset\tilde{Z}'_0$ at $z=0$ extends to order $(m-n)$ in z. For this, we go back to the construction of the subbundle $\tilde{Z}'$; referring to (2.25), (2.27) and setting S' to be the curve defined by $(k=0)$, then $\tilde{R}^{-1}(z)=zS'$, for $z\neq 0$, and $\tilde{R}^{-1}(0)=F^{(m-n-1)}$. One can show that $h^0(\tilde{R}^{-1}(z), L(m-n+1))=(m-n)$ for z near 0, and the restriction maps:

$$\tilde{\Sigma}_z = H^0(\tilde{R}^{-1}(z), L(m-n-1)) \to H^0(\tilde{R}^{-1}(z) \cap T_a, L(m-n-1)) \simeq \mathbb{C}^{m-n}$$

are isomorphisms for all z near 0. This implies that, as before, there are $\hat{\alpha}_i(z)$ with $(\eta - \hat{\alpha}_0(z) - \hat{\alpha}_1(z)\zeta - \hat{\alpha}_2(z)\zeta^2)\tilde{\Sigma} = 0$.

Again, let a local basis of sections of $\tilde{\Sigma}$ be represented by functions $\tilde{f}_{ji}$ on U_j; then

$$(\tilde{f}_{01}, \dots, \tilde{f}_{0,m-n})(\eta - \hat{\alpha}_0(z) - \hat{\alpha}_1(z)\zeta - \hat{\alpha}_2(z)\zeta^2) = (\tilde{v}_1, \dots, \tilde{v}_{m-n})\tilde{k},$$

where the $\tilde{v}_i$ are functions; multiply this by $\tilde{h}$:

$$(\tilde{h}\tilde{f}_{01}, \dots, \tilde{h}\tilde{f}_{0,m-n})(\eta - \hat{\alpha}_0(z) - \hat{\alpha}_1(z)\zeta - \hat{\alpha}_2(z)\zeta^2) = (\tilde{v}_1, \dots, \tilde{v}_{m-n})\tilde{g},$$

to order $(m-n)$ in z. Note that the $\tilde{h}\tilde{f}_{oi}$ are precisely what defined the basis of sections $\tilde{e}_i$ of $\tilde{Z}'$. One has then has, to order $(m-n)$ in z, that $\hat{\alpha}_i(z) = \alpha_i(z)$, and that $(\eta - \Sigma\alpha_i(z)\zeta^i)\tilde{Z}' = 0$.

This finishes the proof that the endomorphisms $\underset{\sim}{A}$ have the correct boundary behaviour on both sides of μ_p for $m \neq n$; one then must check that they patch together. Referring to (2.24), we have that in a basis of X whose first $(m-n)$ elements generate Z' and whose last n elements at $z=0$ generate Z_0'', $\underset{\sim}{A}(z, \zeta)$ is of the form, near $z = 0$:

$$\begin{bmatrix} a(\zeta)/z + a'(\zeta) + O(z), & b(\zeta) + O(z) \\ O(z^{m-n-1}), & c(\zeta) + O(z) \end{bmatrix} \tag{2.35}$$

with $a(z)$ corresponding to an irreducible representation of $SU(2)$. This is computed using S_p, for $z \in (0, \mu_p - \mu_{p+1})$; similarily, one has, for $z \in (\mu_p - \mu_{p-1}, 0)$, a solution $\underset{\sim}{A}(z, \zeta)$, computed using S_{p-1}; $\underset{\sim}{A}(z, \zeta)$ is there an endomorphism of the rank n bundle Y, and we saw that there is a well defined limit $\underset{\sim}{A}^-(\zeta)$ at $z = 0$. In fact, under the identification of Y_0 with Z_0'', we have that $\underset{\sim}{A}^- = c$, as follows.

Remember from (2.16), that, for $m > n$, $H^0(S_p, \mathcal{O}(m+n+1)[-D]) \simeq H^0(T\mathbb{P}_1, \mathcal{O}(m+n+1) \otimes \mathcal{I}(D))$. Thus, at $z = 0$, if $(e_1 \cdots e_m)(\eta - \underset{\sim}{A}(z, \zeta)) = 0$ over S_p, the same is true over $T\mathbb{P}_1$; in particular, at $z = 0$,

$$0 = \eta e_{m-n+i} + \sum_{j=1}^{m-n} e_j b_{j,i}(\zeta) + \sum_{j=1}^{n} e_{m-n+j} c_{j,i}(\zeta)$$

over $T\mathbb{P}_1$. However, at $z = 0$, over $S_{p-1}, e_1, \dots, e_{m-n}$ vanish, and so, over S_{p-1},

$$0 = \eta e_{n-m+i} + \sum_{j=1}^{n} e_{m-n+j} c_{ji}(\zeta).$$

Under the identification (2.22) this forces $c(\zeta) = \underset{\sim}{A}^-(\zeta)$.

iii) *The Case* $m = n$.

This case is rather similar to that of $m < n$; again, the vanishing theorem (1.17) applies in the limit at $z = 0$ and so one has well defined, finite limits $\underset{\sim}{A}^-(\zeta)$, $\underset{\sim}{A}^+(\zeta)$, as z tends to zero from below or above.

$\underset{\sim}{A}^+(\zeta)$ and $\underset{\sim}{A}^-(\zeta)$, are not the same, however. Whereas X_0 and Y_0 are well identified by (2.22), the spaces Γ_0 and $\Delta_0 = H^0(S_{p-1}, \mathcal{O}(m+n+1)[-D])$ are not, via the analogous diagram:

$$\begin{tikzcd} & W = H^0(T\mathbb{P}_1, \mathcal{O}(m+n+1)\otimes\mathcal{I}(D)) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \Gamma_0 & & \Delta_0 \end{tikzcd} \tag{2.36}$$

The map p has a two dimensional kernel, consisting of sections of the form $(x+y\zeta)g, x, y\in\mathbb{C}$; similarly, the map q has a kernel of sections of the form $(x+y\zeta)h$. Referring to (2.5), this allows $\underline{A}^+$ and $\underline{A}^-$ to be different.

Represent a basis of sections of $X_0 \cong Y_0 \cong H^0(T\mathbb{P}_1, \mathcal{O}(m+n-1)\otimes\mathcal{I}(D))$ by polynomials v_i in ζ, η of degree less than m in η; $\underline{A}^+, \underline{A}^-$ are defined by

$$(v_1, \dots, v_m)(\eta - \underline{A}^-(\zeta)) = (s_{0i} + \zeta s_{11}, \dots, s_{0m} + \zeta s_{1m})g,$$
$$(v_1, \dots, v_m)(\eta - \underline{A}^+(\zeta)) = (t_{0i} + \zeta t_{11}, \dots, t_{0m} + \zeta t_{1m})h,$$

$s_{ji}, t_{ji}\in\mathbb{C}$. $(s_{0i} + s_{1i}\zeta)$ is the (η^{m-1}) term of v_i, as is $(t_{0i}+t_{1i}\zeta)$; therefore $s_{0i} = t_{0i}$, $s_{1i} = t_{1i}$. Writing $s_j = (s_{j1}, \dots, s_{jm})$, we get

$$(v_1 \cdots v_m)(\underline{A}^+(\zeta) - \underline{A}^-(\zeta)) = (h-g)(s_0 + \zeta s_1). \tag{2.37}$$

$(h-g)$ represents a section of $\mathcal{O}(2m)$, which vanishes on $S_p \cap S_{p-1}$; as $H^0(S_p, \mathcal{O}(2m)[-D]) \cong H^0(S_p, \mathcal{O}(2m-1)[-D]) \otimes H^0(S_p, \mathcal{O}(1)), (h-g)$ decomposes as $\sum_{i=1}^{m} v_i(u_{0i} + \zeta u_{1i})$, with $u_{ji}\in\mathbb{C}$. Writing $u_j = (u_{j1}, \dots, u_{jm})$, and substituting into (2.36) yields

$$(v_1, \dots, v_m)(\underline{A}^+(\zeta) - \underline{A}^-(\zeta)) = (v_1, \dots, v_m)(u_0 + \zeta u_1)^T(s_0 + \zeta s_1),$$

and so:

Proposition (2.38).

$$\underline{A}^+(\zeta) - \underline{A}^-(\zeta) = (u_0 + \zeta u_1)^T(s_0 + \zeta s_1),$$

i.e., $\underline{A}^+(\zeta) - \underline{A}^-(\zeta)$ *is of rank one.*

2e) The Matrices $A(z, \zeta)$ at $z = 0$. One remembers that before one could obtain a solution to Nahm's equations, one had to trivialise the bundle X so that the endomorphisms $\underline{A}(z, \zeta)$ could be written as matrices $A(z, \zeta)$; this trivialisation is chosen to be flat with respect to a certain connection ∇.

The definition (2.6) of the connection involved the maps $\underline{A}(z, \zeta)$. When these have finite, well defined limits at $z = 0$, there is no problem obtaining a smooth ∇-flat trivialisation at $z = 0$; the boundary behaviour of A is then, in essence, that of $\underline{A}$, that is:

—for $m < n$, there is a finite well defined limit $A(0, \zeta) = \lim_{z\to 0^+} A(z, \zeta)$.

—for $m = n$, there are well defined limits $A^-(\zeta), A^+(\zeta)$ with

$$A^+(\zeta) - A^-(\zeta) = (\mu_0 + \mu_1\zeta)^T(a_0 + a_1\zeta).$$

It is the case $m > n$ which poses the problem, as $\underline{A}(z, \zeta)$ has a pole at $z = 0$. Let $e_1, \dots, e_m$ be a local basis of X; set $C(z, \zeta) = C_0(z) + C_1(z)\zeta + C_2(z)\zeta^2$ to be the expression of $\underline{A}(z, \zeta)$ in this trivialisation of X. If s is a section of X, let $(s)_0$

denote the function representing s in the U_0 trivialisation. The connection is defined by:

$$(\nabla_z e_i)_0 = \frac{\partial (e_i)_0}{\partial z} + \sum_j \left(e_j \left(\frac{C_1}{2} + \zeta C_2 \right)_{ji} \right)_0 .$$

Note that this implies that there is a matrix

$$D_{ji}(z,\zeta) \text{ such that } \frac{\partial (e_i)_0}{\partial z} = \sum_j (e_j D_{ji}(z,\zeta))_0 .$$

Now assume that the e_i's are chosen so that $e_1, \dots, e_{m-n}$ form a basis of Z'; then the $(e_i)_0$, $i = 1, \dots, m-n$, are divisible by h, to order $(m-n)$ in z. The same is then true of $\partial(e_i)_0/\partial z$, to order $(m-n-1)$; D thus has the block decomposition with respect to $X = Z' \oplus Z''$:

$$\begin{bmatrix} D' & D'' \\ O(z^{m-n-1}) & D''' \end{bmatrix}.$$

Referring to (2.35), similar decompositions hold for C_1, C_2, and so for the connection matrix $\Gamma = D + \frac{1}{2}C_1 + \zeta C_2$. Finally, as in [Hi2, p. 178], the polar part of Γ has the block diagonal form

$$\frac{(m-n-1)}{2z} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

These facts then imply that one can find a change of basis matrix S of the form

$$\begin{bmatrix} z^{-(m-n-1)/2} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \cdot [\mathbb{1} + O(z)]$$

from this basis to a ∇-flat basis, with the $O(z)$ term having the appropriate block form. Expressing $\underline{A}(z,\zeta)$ with respect to the ∇-flat basis, one obtains (referring to (2.35)

$$A(z,\zeta) = \begin{bmatrix} \alpha(\zeta)/z + \alpha'(\zeta) + O(z), & O(z^{(m-n-1)/2}) \\ O(z^{(m-n-1)/2}), & \gamma(\zeta) + O(z) \end{bmatrix}, \tag{2.39}$$

which completes the proof that the boundary behaviour of our solutions to Nahm's equations is that given in condition B-2.

2f) Real Structure. We now have a solution $T_i(z)$ to Nahm's equations, satisfying the right boundary conditions; the last step is to show that the solution can be made skew adjoint. As before, this is done in several steps:

i) One defines a positive definite hermitian form on X.
ii) One shows that the endomorphisms $\underline{T}_i$ are skew adjoint with respect to this form.
iii) One shows that the connection preserves the form, and so the matrices $T_i(z)$ can be made skew adjoint.
iv) One must show that the boundary conditions and the hermitian structure are compatible, i.e. that the block decomposition of $T_i(z)$ at $z = 0$ can be obtained in an orthonormal basis.

Step i), the definition of the form, is an adaptation of that of Hitchin [Hi2, p. 179]. Let $s,\ t\in X_z \approx H^0(S_p, L^z(m+n-1)[-D])$; the real structure pulls back $L^z(m+n-1)[-D]$ to $L^{-z}(m+n-1)[-\tau(D)]$; there is an antilinear map

$$\sigma\cdot H^0(S_p, L^z(m+n-1)[-D]) \to H^0(S_p, L^{-z}(m+n-1)[-\tau(D)]). \tag{2.40}$$

$s\cdot\sigma(t)$ is a section of $H^0(S_p, \mathcal{O}(2m+2n-2)[-D-\tau(D)])$; remember that $D\cup\tau(D) = S_p\cap S_{p-1}$; thinking of $s\cdot\sigma(t)$ as a section of $\mathcal{O}(2m+2n-2)$ over $T\mathbb{P}_1$ (2.11), this means that there exist $a\in H^0(T\mathbb{P}_1, \mathcal{O}(2n-2))$, $b\in H^0(T\mathbb{P}_1, \mathcal{O}(2m-2)$, with:

$$s\cdot\sigma(t) = ag + bh. \tag{2.41}$$

Therefore, over S_p:

$$s\cdot\sigma(t) = bh$$

and over S_{p-1}:

$$s\cdot\sigma(t) = ag.$$

By (2.16), and (1.2), b decomposes uniquely as $b_0\eta^{m-1} + b_1(\zeta)\eta^{m-2} + \cdots + b_{m-1}(\zeta)$, with b_i of degree $2i$. One then sets

$$\langle s, t\rangle_z = b_0. \tag{2.42}$$

Once $\langle,\rangle$ has been defined, steps ii) and iii) are proven exactly as in Hitchin [Hi2], and so will not be repeated here.

We now consider step iv). In the case $m > n$, what must be shown is that the decomposition $X = Z'\oplus Z''(\xi_0)$ is orthogonal at $z = 0$, to order $(m-n)/2$ in z. Let s be a local ∇-flat section of Z'; then $s' = z^{(m-n)/2}\, s$ is a smooth section of X, and $s' = hp'$ over $T\mathbb{P}_1$ to order $(m-n)$ in z, for some p'. Let t be a local section of $Z''(\xi_0)$. Using the fact that $s' = z^{-n}\rho(\tilde{s})$, $t = z^{-n}\rho(\tilde{t})$, it is fairly easy to see that $s'\sigma(t)$ has no terms of order $\geqq (m+n-1)$ in η, to order $(m-n)$ in z; therefore, to order $(m-n)$ in z, one has

$$s'\sigma(t) = hp'\sigma(t),$$

and so, to order $(m-n)$ in z, the (order $(m-1)$ term in η of $b = p'\sigma(t)$) is the (order$(m+n-1)$ term in η of $s'\sigma(t)$, which is zero.

Finally, we check compatibility at the boundary, and positivity. We note that one has

$$\begin{aligned} s, t &\in H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1} - 1)[-S_{p,p-1}]) \\ &\approx H^0(S_p, L^{\mu_p - t}(m_p + m_{p+1} - 1)[-S_{p,p+1}]) \end{aligned}$$

using the section f_p of (1.8). One then sees that

$$\begin{aligned} s\cdot\sigma(t) &\in H^0(S_p, \mathcal{O}(2m_p + 2m_{p-1} - 2)[-S_p\cap S_{p-1}]) \\ &= H^0(S_p, \mathcal{O}(2m_p + 2m_{p+1} - 2)[-S_p\cap S_{p+1}]). \end{aligned}$$

The technique used above therefore gives two possible forms $\langle,\rangle_z$, $\{,\}_z$; it is easy to check that $\langle,\rangle_z = e_p\{,\}_z$, where e_p is the constant of C-4 in the introduction normalised so that $|e_p| = 1$; one then has $e_p = -(-1)^{m_p + m_{p-1}}$.

We now check that the definition of $\langle,\rangle$ is compatible with the glueing of Y_0 and X_0 at $z = 0$. Without loss of generality, set $m \geqq n$. Let elements s, t of $Y_0 \cong H^0(S_{p-1}, \mathcal{O}(m+n-1)[-D])$ be represented, using (2.16), by polynomials of degree less than n in η:

$$\begin{aligned}\langle s,t\rangle_{Y_0} &= \text{degree } (n-1) \text{ term (in } \eta) \text{ of } a\\ &= -\text{degree } (m-1) \text{ term (in } \eta) \text{ of } b, \text{ by (2.41)}\\ &= -\{s,t\}_{X_0}\\ &= (-1)^{m_p+m_{p-1}}\langle s,t\rangle_{X_0}.\end{aligned} \tag{2.43}$$

Now let us show that the form is definite. We begin on the interval (μ_2, μ_1). The form is preserved by the connection, and is compatible with the T_i; in particular, at μ_1, the residues of T_i define an irreducible representation of $su(2)$; these residues are skew hermitian with respect to $\langle,\rangle$, which forces the form to be definite. In fact, one can compute its sign to be $(-1)^{m_1}$, from the basis of X_{μ_1}, using the description of (2.21). Transporting to μ_2, one has the same form there on Y_{μ_2}. If $m_2 \leqq m_1$, this form transfers to a definite form on $X_{\mu_2} \subset Y_{\mu_2}$, of sign $(-1)^{m_2}$, by (2.43). If $m_2 > m_1$, then $X_{\mu_2} = Z'_{\mu_2} \oplus Z''_{\mu_2}$ as in (2.21); on Z''_{μ_2}. the sign of the form is $(-1)^{m_2}$, by "transfer" from Y_{μ_2}, above; on Z'_{μ_2}, explicit computation from (2.21) gives that the sign is also $(-1)^{m_2}$. Continuing in this way, one checks positivity on each interval in turn.

3. From Nahm's Equations to Spectral Data

In this section, we show that a generic solution $A(t,\zeta)$ to Nahm's equations, satisfying conditions B, yields back the spectral data. From $A(t,\zeta)$, we must obtain:

i) real curves S_p, of degree $2m_p$, for $p=1,\dots,N-1$,
ii) the partition of $S_p \cap S_{p-1}$ into $S_{p,p-1}$ and $S_{p-1,p}$,
iii) the section ξ_p in $H^0(S_p, L^{\mu_{p+1}-\mu_p}(m_{p-1}+m_{p+1})[-S_{p,p+1}-S_{p,p-1}])$,
iv) the fact that S_p satisfies the vanishing theorem (1.17).

Part i) of this data is easy to obtain: let S_p be the curve in $T\mathbb{P}_1$ defined by

$$\det(\eta\mathbb{1} - A(t,\zeta)) = 0 \tag{3.1}$$

for any $t \in (\mu_{p+1}, \mu_p)$; S_p is thus the spectrum of $A(t,\zeta)$ (recall that Nahm's equations are isospectral). As the T_i's are skew adjoint, $\zeta^2\overline{A(-\bar{\zeta}^{-1})} = -A(\zeta)$, and so S_p is real.

To obtain parts ii), iii) and iv) of the data, we begin by inverting the procedure used in Chap. 2 to obtain $A(t,\zeta)$; here, therefore, from the flow $A(t,\zeta)$, we obtain the appropriate flow K_t of line bundles over the curves. Recall that in Chap. 2, $A(t,\zeta) = A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2$, for $t \in (\mu_{p+1}, \mu_p)$, was derived from the exact sequence

$$\begin{aligned}&0 \to H^0(S_p, L^{-t}(-1)\otimes E/(E_p^+ + E_{N-p}^-))\otimes \mathcal{O}(-2) \xrightarrow{\eta - A(t,\zeta)}\\ &H^0(S_p, L^{-t}(-1)\otimes E/(E_p^+ + E_{N-p}^-))\otimes \mathcal{O} \xrightarrow{\text{ev}} L^{-t}(-1)\otimes E/(E_p^+ + E_{N-p}^-) \to 0,\end{aligned} \tag{3.2}$$

where ev is the evaluation map. More succinctly:

$$0 \to \mathcal{O}(-2)^{\oplus m_p} \to \mathcal{O}^{\oplus m_p} \to L^{-t}(-1)\otimes E/(E_p^+ + E_{N-p}^-) \to 0.$$

Generically, of course, $E/(E_p^+ + E_{N-p}^-)$ is the restriction to S_p of $L^{\mu_p}(m_p + m_{p-1})[-S_{p-1,p}]$. We now invert this procedure. Let η also denote $\eta\mathbb{1}$. Define

$$K_t = \operatorname{coker}(\eta - A(t,\zeta)) : \mathcal{O}(-2)^{\oplus m_p} \to \mathcal{O}^{\oplus m_p}.$$

Note that K_t is not necessarily locally free over S_p; it can have torsion at eigenvalues of higher multiplicity. One way to smooth it out is to take a dual, which is torsion free; one has, over S_p,

$$K_t^* \simeq \ker(\eta - A(t,\zeta))^T : \mathcal{O}^{\oplus m_p} \to \mathcal{O}(2)^{\oplus m_p}. \tag{3.5}$$

Let the suffix adj denote the classical adjoint:

$$(\eta - A(t,\zeta))^T(\eta - A(t,\zeta))^T_{\mathrm{adj}} = \det(\eta - A(t,\zeta))\mathbb{1}. \tag{3.6}$$

One has, over S_p:

$$\operatorname{Im}(\eta - A(t,\zeta))^T_{\mathrm{adj}} \subset K_t^*. \tag{3.7}$$

When $(\eta - A(t,\zeta))$ has corank one $(\eta - A(t,\zeta))^T_{\mathrm{adj}}$ is of rank one, and one has equality in (3.7); K_t is then a line bundle. When the corank of $(\eta - A(t,\zeta))$ is greater than one, the classical adjoint vanishes.

To begin, we show that the flow of bundles is in the right direction. Let s be a (meromorphic) section of K_t^*, for $t = t_0$; let s be represented over $U_0 = \{\zeta \neq \infty\}$ by u, and over $U_1 = \{\zeta \neq 0\}$ by v; let $g(t_0)$ be a transition function for $K_{t_0}^*$; then $u = g(t_0)v$. One has over U_0,

$$(\eta - A(t_0,\zeta))^T u = 0,$$

and over U_1

$$\zeta^{-2}(\eta - A(t_0,\zeta))^T v = 0.$$

Let $A_\#(t,\zeta) = A_1(t)/2 + A_2(t)\zeta$; varying t, we ask that, as t varies:

$$\frac{\partial u}{\partial t} = A_\#^T u, \tag{3.8}$$

then, using Nahm's equations,

$$\frac{\partial}{\partial t}(\eta - A)^T u = A_\#^T(\eta - A)^T u$$

and so, if u is a solution to (3.8), $(\eta - A)^T u$ is also; as the initial condition for this linear equation is $(\eta - A)^T u = 0$, then $(\eta - A)^T u = 0$ for all t.

Similarly, one can ask that

$$\frac{dv}{dt} = -(A/\zeta - A_\#)^T v,$$

ensuring that $\zeta^{-2}(\eta - A)^T v = 0$ for all t; then,

$$A_\#^T u = \frac{\partial u}{\partial t} = \frac{\partial g}{\partial t} v - g(A/\zeta - A_\#)^T v,$$

and so

$$\frac{\eta}{\zeta}u = \frac{A}{\zeta}u = \frac{\partial g}{\partial t}g^{-1}u.$$

Integrating,

$$g(t,\eta,\zeta) = e^{t\eta/\zeta}g(t_0,\eta,\zeta)$$

therefore,

$$K_t^* = K_{t_0}^* \otimes L^{(t-t_0)}. \tag{3.9}$$

The flow K_t^* is, at least, in the right direction; to identify K_t, one must examine the boundary behaviour of $A(t,\zeta)$ at the points μ_p. In the study of boundary behaviour, we readopt the notation of Sect. 2. We begin with the case $m > n$.

Set $k = m - n$. Then, near $z = 0$, one has, for $z > 0$, the block decomposition:

$$A^T(z,\zeta) = \begin{pmatrix} \alpha^T(z,\zeta)z^{-1}, & \delta^T(z,\zeta)z^{(k-1)/2} \\ \beta^T(z,\zeta)z^{(k-1)/2}, & \gamma^T(z,\zeta) \end{pmatrix}. \tag{3.10}$$

A^T is $m \times m$, α^T is $k \times k$; $\alpha, \beta, \gamma, \delta$ are quadratic in ζ, and are written $\alpha(z,\zeta) = \alpha_0(z) + \alpha_1(z)\zeta + \alpha_2(z)\zeta^2$, etc. Let $\alpha^T(0,\zeta) = a(\zeta)$, $\beta^T(0,\zeta) = b(\zeta)$, $\gamma^T(0,\zeta) = c(\zeta)$, $\delta^T(0,\zeta) = d(\zeta)$. One also had, for $z < 0$, an $n \times n$ solution $A^T(z,\zeta)$, with $A^T(0,\zeta) = c(\zeta)$. Conjugating (3.10) by $\operatorname{diag}(z^{(k-1)/2}, \mathbb{1})$, one gets

$$\begin{pmatrix} \alpha^T(z,\zeta)z^{-1} & \delta^T(z,\zeta)z^{k-1} \\ \beta^T(z,\zeta) & \gamma^T(z,\zeta) \end{pmatrix}. \tag{3.11}$$

Recall that from $a(\zeta) = a_0 + a_1\zeta + a_2\zeta^2$, one defined an irreducible k-dimensional representation of $sl(2,\mathbb{C})$. For each ζ, $a(\zeta)$ has a one dimensional kernel; one can choose a basis e_i so that $\ker a(\zeta) = (\zeta^{k-1}, \zeta^{k-2}, \ldots, 1)^T$; in this basis, for example, $a(0)e_i = e_{i+1}$. We compute a section of $\ker(\eta - A)^T$, or equivalently, (as we shall see) a section of $\operatorname{Im}(\eta - A)^T_{\mathrm{adj}}$. We do this first at $\zeta = 0$; it suffices to compute the first column of $(\eta - A)^T_{\mathrm{adj}}$, i.e., the minors along the top row. We find:

$$\begin{array}{c} \textit{row:}\ 1 \\ \vdots \\ k \\ \hline k+1 \\ \vdots \\ m \end{array} \left[(-1)^k z^{-k+1} \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ -\det(\eta - c(0)) \\ \hline (\eta - c(0))_{\mathrm{adj}} \cdot b(0) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \end{array} \right] + O(z^{-k+2}) \right] \tag{3.12}$$

Multiplying through by $(-1)^k z^{-k+1}$, one obtains a non-zero limit vector, at $z = 0$. Similarly, when ζ is arbitrary, one has the limit section of $\ker(\eta - A)^T$,

$$\begin{pmatrix} f_1(\eta,\zeta) \\ \hline f_2(\eta,\zeta) \end{pmatrix} = \begin{pmatrix} \det(\eta - c(\zeta)) \cdot \mathbb{1}_{k \times k} \\ \hline (\eta - c(\zeta))_{\mathrm{adj}} \cdot b(\zeta) \end{pmatrix} \begin{pmatrix} \zeta^{k-1} \\ \zeta^{k-2} \\ \vdots \\ 1 \end{pmatrix}. \tag{3.13}$$

Note that, at $\zeta=0$, if one computes the $(k+i)^{th}$ column of $(\eta-A)^T_{\mathrm{adj}}$, one obtains for the first k entries, the finite limit at $z=0$,

$$e_1^i(\eta,0)^T=(0,\ldots,0,[(1,0,\ldots,0)d(0)(\eta-c(0))_{\mathrm{adj}}]_i)^T \tag{3.14}$$

similarly, as ζ varies, one has a finite limit $e_1^i(\eta,\zeta)$. As rank $(\eta-A(z,\zeta))^T_{\mathrm{adj}}\leqq 1$ over S_p, the $(k+i)^{th}$ column and the first column of A are proportional; at the limit $z=0$, the factor of proportionality normalised by $-(-z)^{k+1}$ is e_1^i/f_1. As f_2 is finite, the limit vector e_2^i of the last n entries in the $(k+i)^{th}$ column is finite, when $\det(\eta-c(\zeta))\neq 0$. Hartogs' theorem (in the variables ζ, z over $S_p\times\mathbb{C}$) then forces e_2^i to be finite even when $\det(\eta-c(\zeta))=0$. We write the limit of the $(k+i)^{th}$ column at $\zeta=0$ as:

$$(e_1^i,e_2^i)^T=(0,\ldots,0,((1,0,\ldots,0)d(0)(\eta-c(0))_{\mathrm{adj}})_i,e_{21}^i,\ldots,e_{2n}^i)^T. \tag{3.15}$$

Furthermore, one has, for some $p(\eta)$,

$$\det(\eta-A(0))=\det(\eta-c(0))\cdot p(\eta)-(1,0\cdots 0)d(0)(\eta-c(0))_{\mathrm{adj}}b(0)(0\cdots 0,1)^T, \tag{3.16}$$

which relates the equation of $S_p(\det(\eta-A(0))=0)$ and that of $S_{p-1}(\det(\eta-c(0))=0)$ over $\zeta=0$. Now suppose that we are in the generic case. Genericity precludes intersections of the spectral curves at multiple points; at $S_p\cap S_{p-1}$, therefore, $(\eta-A)^T_{\mathrm{adj}}$ is of rank one. On the other hand, away from the intersections, over S_p, where $\det(\eta-c(\zeta))\neq 0$, (3.13) shows us that $(\eta-A)^T_{\mathrm{adj}}$ is again non-zero, and so $(\eta-A)$ has everywhere corank one on S_p. Similarly, changing ζ-coordinates if necessary, from (3.16) one sees that $(\eta-c(\zeta))_{\mathrm{adj}}$ is non-zero on S_{p-1} away from S_p, and so $(\eta-c(\zeta))$ has everywhere corank one. Therefore, *the sheaves K_0, on S_p and S_{p-1} are line bundles.*

We now identify the bundles K_z at $z=0$.

Proposition (3.17). *Let $m>n$. Suppose that the Nahm data is generic. There is a partition of $S_p\cap S_{p-1}$ into divisors D, $\tau(D)$ such that, over S_p and over S_{p-1}, $K_0\simeq\mathcal{O}(m+n-1)[-D]$.*

Proof i). *Over S_p.* The column vector (3.13) defining a section of $K_0^*\subset\mathcal{O}^{\oplus m}$, has, as entries, polynomials of degree $\leqq(m+n-1)$; (3.13) can be thought of as a map $\mathcal{O}(-m-n+1)\to K_0^*$, or, dually, as a map

$$K_0\to\mathcal{O}(m+n-1). \tag{3.18}$$

We must show that this map vanishes on an appropriate D, and only on D. Let E be the divisor cut out on S_p by S_{p-1}; one wants $D+\tau(D)=E$. The f_1 portion of (3.13) vanishes on E, as S_{p-1} is defined by $\det(\eta-c(\zeta))=0$; f_1 will vanish on any $D<E$. Set $f_2^*(\eta,\zeta)=\zeta^{m+n-1}\overline{f_2(\tau(\eta,\zeta))}=\zeta^{m+n-1}\overline{f_2(-\bar{\eta}/\bar{\zeta}^2,-1/\bar{\zeta})}$; consider the $n\times n$ rank 1 matrix $f_2f_2^{*T}$.

Lemma (3.19). *There is a positive divisor D such that f_2 vanishes on D and $D+\tau(D)=E$ if and only if $f_2f_2^{*T}$ vanishes on E.*

Proof. The proof in one direction is obvious; in the other, let $f_2f_2^{*T}$ vanish on E; then $f_{2,i}f_{2,j}^*$ vanishes on $E,\forall_{i,j}$. The fact that E is real means that it can be written as $E=\Sigma m_k(p_k+\tau(p_k))$, where p_k are points of S. Let $f_{2,i}$ vanish at p_k with multiplicity $g_{k,i}$, and at $\tau(p_k)$ with multiplicity $h_{k,i}$. As $f_{2,i}f^*_{2,j}$ is zero on E, $g_{k,i}+h_{k,j}\geqq m_k$ for all i,j. Set $g_k=\min(g_{k,i})$, $h_k=m_k-g_k$; then $f_{2,i}$ vanishes at $D=\Sigma g_k p_k+h_k\tau(p_k)$ for all i, and $D+\tau(D)=E$.

Returning to (3.17), to show that $f_2 f^{*T}_2$ is zero over E, let $p\in S_p\cap S_{p-1}$. Changing coordinates on $T\mathbb{P}_1$ if necessary, let p be located at $\zeta=0$. One has from (3.13):

$$f^*_2(\eta,\zeta)=(-\eta-\zeta^2\overline{c(-\bar{\zeta}^{-1})})_{\mathrm{adj}}\zeta^2\overline{b(-\bar{\zeta}^{-1})}\begin{pmatrix}(-\zeta)^{-k+1}\\ \cdots\\ \cdots\\ \cdots\\ 1\end{pmatrix}\zeta^{k-1},$$

as the $T_i(t)$ are skew adjoint, $\zeta^2\overline{c(-\bar{\zeta}^{-1})}=-c(\zeta)^T$, $\zeta^2\overline{b(-\bar{\zeta}^{-1})}=-d(\zeta)^T$; therefore at $\zeta=0$, up to a sign,

$$f_2f^{*T}_2=(\eta-c(0))_{\mathrm{adj}}b(0)\begin{pmatrix}0\\ \cdots\\ \cdots\\ 1\end{pmatrix}(1,\dots,0)d(0)(\eta-c(0))_{\mathrm{adj}}.$$

One now uses the fact that (3.12) and (3.15) are proportional over S_p; as we are at a point where $\det(\eta-c(0))=0$, this forces either $(\eta-c(0))_{\mathrm{adj}}b(0)(0,\dots,1)^T=0$ or $(1,0,\dots,0)d(0)(\eta-c(0))_{\mathrm{adj}}=0$, by (3.16).

The final step is to show that (3.18) vanishes only at D. This is done by remarking that $\deg(K_0)=\deg(\mathcal{O}(m+n-1)[-D])=m(m-1)$.

ii) *Over* S_{p-1}: K^*_0 is defined over S_{p-1} as the kernel of $(\eta-c(0))$; referring to (3.13), f_2 is a section of K^*_0. The proof that $K_0\sim\mathcal{O}(m+n-1)[-D]$ over S_{p-1} is then just the repetition of that given over S_p. ■

We now analyse the case $m=n$. At $z=0$, one has limits $A^+(\zeta), A^-(\zeta)$ with $\det(\eta-A^+(\zeta))=0$, $\det(\eta-A^-(\zeta))=0$ defining S_{p-1}, S_p respectively. Furthermore,

$$A^+(\zeta)-A^-(\zeta)=(s_0+\zeta s_1)(\bar{s}_1-\zeta\bar{s}_0)^T=s(\zeta)s^*(\zeta)^T, \tag{3.20}$$

where s_0, s_1 are column vectors.

Again we define a section of $\ker(\eta-A^+(\zeta))^T$ over S_{p-1} and of $\ker(\eta-A^-(\zeta))^T$ over S_p. Recall the "Weinstein–Aronzajn" relation:

$$\det(\eta-A^+)^T=\det(\eta-A^{-T}-s^*s^T)=\det(\eta-A^{-T})-s^T(\eta-A^-)^T_{\mathrm{adj}}s^*. \tag{3.21}$$

Then, over S_p,

$$(\eta-A^-)^T(\eta-A^-)^T_{\mathrm{adj}}s^*=\det(\eta-A^-)^Ts^*=0$$

and, over S_{p-1},

$$(\eta-A^+)^T(\eta-A^-)^T_{\mathrm{adj}}s^*=(\eta-A^-)(\eta-A^-)^T_{\mathrm{adj}}s^*-s^*s^T(\eta-A^-)^T_{\mathrm{adj}}s^*=0,$$
by (3.21).

$(\eta-A^-(\zeta))^T_{\mathrm{adj}}s^*$ thus defines a section of K^*_0, both over S_p and S_{p-1}. Formula (3.21) shows us that, away from $S_p\cap S_{p-1}$, $(\eta-A^-)^T_{\mathrm{adj}}$ is non-zero on S_p; symmetrically, $(\eta - A^+)^T_{\mathrm{adj}}$ is non-zero on S_{p-1}. The genericity condition then tells us that these are also both non-zero at the intersection. K_0 is then a line bundle, both on S_p and on S_{p-1}.

Proposition (3.22). *Let $m = n$. Suppose that the Nahm data is generic. There is a partition of $S_p \cap S_{p-1}$ into divisors D, $\tau(D)$ such that, over S_p and over S_{p-1}, $K_0 \simeq \mathcal{O}(m+n-1)[-D]$.*

Proof. As $(\eta - A^-(\zeta))^T_{\mathrm{adj}} s^*$ is of degree $(2m-1)$, as before, we have a map.

$$K_0 \to \mathcal{O}(2m-1).$$

To show that this vanishes on the appropriate D, we use Lemma (3.19). It suffices to show that $(\eta - A^-(\zeta))^T_{\mathrm{adj}} s^* s^T (\eta - A^-(\zeta))^T_{\mathrm{adj}}$ vanishes on $S_p \cap S_{p-1}$. From (3.21), over $S_p \cap S_{p-1}$, $s^T(\eta - A^-(\zeta))^T_{\mathrm{adj}} s^* = 0$. As $(\eta - A^-(\zeta))^T_{\mathrm{adj}}$ is of rank one over S_{p-1}, it can be written as a product uw^T, where u, w are column vectors. Therefore, $s^T u w^T s^* = 0$, and so either $s^T u = 0$, or $w^T s^* = 0$. Then, $(\eta - A^-)^T_{\mathrm{adj}} s^* = 0$ or $s^T(\eta - A^-)^T = 0$, which yields the result. As above, one shows that it only vanishes on D.

One now has the necessary material to obtain parts ii) and iii) of the spectral data. $S_{p-1,p}$, of course, is just the divisor D of the above propositions. As for part iii) consider the flow K_t of bundles over S_p for $t \in [\mu_{p+1}, \mu_p]$

-at $t = \mu_{p+1}$, one has $K_{\mu_{p+1}} \simeq \mathcal{O}(m_p + m_{p+1} - 1)[-S_{p,p+1}]$
--at $t = \mu_p$, $K_{\mu_p} = \mathcal{O}(m_p + m_{p-1} - 1)[-S_{p-1,p}]$.

One knows however, that $K_{\mu_p} \sim L^{\mu_{p+1} - \mu_p} K_{\mu_{p+1}}$; this can be written as an isomorphism

$$\begin{aligned} \mathcal{O} &\sim L^{\mu_{p+1} - \mu_p}(m_{p+1} - m_{p-1})[-S_{p,p+1} + S_{p-1,p}] \\ &\sim L^{\mu_{p+1} - \mu_p}(m_{p+1} + m_{p-1})[-S_{p,p+1} - S_{p,p-1}] \end{aligned}$$

as $\mathcal{O}(2m_{p-1}) \sim [+S_{p-1,p} + S_{p,p-1}]$. This isomorphism is the section ξ_p of iii).

For part iv), one must show that $H^0(S_p, K_t(-1)) = 0$, whenever $A(t,\zeta)$ is finite. By the definition (3.4) of K_t as a sheaf over $T\mathbb{P}_1$, one has the exact sequence,

$$\cdots \to H^0(T\mathbb{P}_1, \mathcal{O}(-1)^{\oplus m_p}) \to H^0(S_p, K_t(-1)) \to$$

$$H^1(T\mathbb{P}_1, \mathcal{O}(-3)^{\oplus m_p}) \xrightarrow{F = \eta - A(t,\zeta)} H^1(T\mathbb{P}_1, \mathcal{O}(-1)^{\oplus m_p}),$$

$H^0(T\mathbb{P}_1, \mathcal{O}(-1)^{\oplus m_p}) = 0$; referring to the explicit form of $H^1(T\mathbb{P}_1, \mathcal{O}(-j))$ given in Lemma (1.2), the map F is injective. Therefore, $H^0(S_p, K_t(-1)) = 0$.

To prove the positivity condition C-4, one considers both ends of the interval (μ_{p+1}, μ_p); one has, at μ_{p+1}, the section $f(\mu_{p+1})$ of $\mathcal{O}(m_p + m_{p+1})[-S_{p,p+1}]$, and, at μ_p, the section $\tilde{f}(\mu_p)$ of $\mathcal{O}(m_{p-1} + m_p)[-S_{p-1,p}]$, given by (3.13) or (3.21). One then "propagates" f from μ_{p+1} to μ_p, using Eq. (3.8), obtaining $f(\mu_p)$. Setting $v_p = f(\mu_p)/\tilde{f}(\mu_p)$, one then must compute

$$e_p = v_p^* v_p (g_{p-1}/g_{p+1}) = \frac{f(\mu_p)^{*T} f(\mu_p) g_{p-1}}{\tilde{f}(\mu_p)^{*T} \tilde{f}(\mu_p) g_{p+1}}.$$

However, from (3.8), $\partial_t(f^{*T}f)=0$, and so:

$$e_p = \frac{f(\mu_{p+1})^{*T}f(\mu_{p+1})}{g_{p+1}} \frac{g_{p-1}}{\tilde{f}(\mu_p)^{*T}\tilde{f}(\mu_p)},$$

referring to the definitions, and keeping careful track of signs, one obtains $-(-1)^{m_p+m_{p-1}}e_p>0$.

4. Reconstructing the Monopole

4a) Introduction. In this section we explain the ADHMN construction and show how it associates a monopole to Nahm data. Starting with a solution (T_1, T_2, T_3) to Nahm's equations, we show how to construct a rank N bundle C over $\mathbb{R}^3$ with connection and Higgs field (∇, Φ) which are regular and satisfy the Bogomolny equations. We then must show that it is a monopole, that is that (∇, Φ) satisfy the BPS boundary conditions.

We do this first in the generic case, when the Nahm data has associated spectral data to it and a holomorphic bundle E. From this, via twistor methods, one can construct a bundle H on a dense open subset U of $\mathbb{R}^3$, along with a connection and Higgs Field $(\tilde{\nabla}, \tilde{\Phi})$ which satisfy the Bogomolny equation. U consists of the $x\in\mathbb{R}^3$ for which $E|_{C_x}$ is holomorphically trivial (once we are done, we will see that $U=\mathbb{R}^3$). In Sect. 1.D, it was shown that $\mathbb{R}^3-U$ is compact, and that $(\tilde{\nabla}, \tilde{\Phi})$ satisfy the BPS coundary conditions.

We achieve our aim by defining an isomorphism $C\simeq H$ on a smaller open dense subset $\tilde{U}$ and by showing that $(\tilde{\nabla}, \tilde{\Phi})\simeq(\nabla, \Phi)$ over U. One then has a global regular solution over all of $\mathbb{R}^3$, which satisfies the boundary conditions.

4b) The ADHMN Construction. For each interval $[\mu_{p+1}, \mu_p]$ let $\tilde{\mathscr{H}}_p$ be the Sobolev space of L^2 sections of $Y_p=X_p\otimes\mathbb{C}^2$ which have L^2 derivative. (The subscript p denotes the interval.) By the Sobolev lemmas such sections are continuous. Similarly let $\mathscr{L}_p$ be the space of L^2 sections of Y_p.

At a boundary point μ_p of $[\mu_{p+1}, \mu_p]$, if $m_p \geqq m_{p-1}$, $Y_p(\mu_p)= Y_{p-1}(\mu_p)\oplus Y_{p-1}(\mu_p)^{\perp}$, and we adopt the terminology *continuing* for vectors in the first space and *terminating* for those in the second. If $m_p\leqq m_{p-1}$ all vectors of Y_p are continuing; thus, continuing vectors on both sides are identified. Define similar terminology for the other end.

The Sobolev space $\mathscr{H}_p$ is defined to be the subspace of $\tilde{\mathscr{H}}_p$ consisting of sections whose terminating components at each end are zero. Define the operator $D_p(x)$, for $x\in\mathbb{R}^3$, by

$$D_p(x): \mathscr{H}_p \to \mathscr{L}_p, \quad D_p(x)=i\nabla_t-(\underline{T}^p+i\underline{x}), \tag{4.1}$$

where $\underline{T}^p=\Sigma T_i\otimes e_i$ and $\underline{x}=\Sigma x_i(\mathbb{1}\otimes e_i)$ for (e_1, e_2, e_3) the unit imaginary quaternions. *Because the components of the section acted on by the singular part of* $\underline{T}^p$ are zero this operator is well defined and has image in $\mathscr{L}_p$.

If $m_p=m_{p-1}$ the boundary condition for the solution of Nahm's equation implies that for some $x\in X_p(\mu_p)$ and some $\alpha\in\mathbb{C}^2$,

$$\underline{T}^p(\mu_p)-\underline{T}^{p-1}(\mu_p)=x\otimes x^*\otimes(\alpha\otimes\alpha^*-\langle\alpha,\alpha\rangle\mathbb{1}/2)\in\mathrm{End}\,(X_p(\mu_p))\otimes sl(2,\mathbb{C}).$$

In such a case, let W_p be the span of $x\otimes\alpha$ and let $\pi_p\colon Y_p(\mu_p)\to W_p$ be the orthogonal projection.

Define $\mathscr{H}\subset\bigoplus_{p=1}^{N-1}\mathscr{H}_p$ to be the space of all sections $f=(f_1,\dots,f_{N-1})$ such that $f_p(\mu_p)=f_{p-1}(\mu_p)$. Nahm's operator is defined to be

$$\mathscr{D}(x)\colon\mathscr{H}\to\mathscr{L}=\left(\bigoplus_{p=1}^{N-1}\mathscr{P}_p\right)\oplus\left(\bigoplus_{\substack{q\\ m_q=m_{q-1}}}W_q\right)$$

$$f\colon\mapsto\{[(D_1(x)f_1,\dots,D_{N-1}(x)f_{N-1})],[(\pi_{q_1}f_{q_1}(\mu_{q_1}),\dots,\pi_{q_r}f_{q_r}(\mu_{q_r})]\},$$

where $q_1,\dots,q_r$ are all the indices for which the jump m_q-m_{q-1} is zero.

Note that the kernel of Nahm's operator is all $(f_1,\dots,f_{N-1})$ such that

—$D_pf_p=0$
—each f_p is L^2
—the terminating components are zero (4.2)
—the continuing components are continuous
—at zero jumps, $f_q(\mu_q)$ is in $W_q^{\perp}$.

Define

$$D_p^*(x)=i\nabla_t+(\underline{T}^p+i\underline{x}).$$

Then integrating by parts it is easy to deduce that the cokernel of $\mathscr{D}(x)$ is all $\{[g_1,\dots,g_{N-1}],[w_{q_1},\dots,w_{q_r}]\}$ such that:

—$D_p^*(x)g_p=0$, (4.3)
—g_p is in L^2
—the continuing components are continuous except at zero jumps, where

$$g_q(\mu_q)-g_{q-1}(\mu_q)=w_q\in W_q.$$

Notice that the terminating components of the g_p are not constrained except by the L^2 requirement.

Let us call these boundary conditions for the kernel of $\mathscr{D}(x)$ and the cokernel of $\mathscr{D}(x)$ the Nahm and co-Nahm boundary conditions.

We define the "bundle" $C(x)$ by

$$C(x)=\operatorname{coker}\mathscr{D}(x)\subset\mathscr{L}.$$

We shall show, by calculating the index of $\mathscr{D}(x)$ and proving that $\dim\ker\mathscr{D}(x)=0$, that rank $C(x)=N$, and so C is in fact a bundle.

To do this we investigate the behaviour of solutions to $D_p^*(x)\varphi=0$ on an interval (μ_{p+1},μ_p). If $z=t-\mu_p$ is a parameter near μ_p and $k=m_p-m_{p-1}>0$, then the theory of singular, regular, ordinary differential equations (as used in Hitchin [Hi2]) tells us that the $2m_p$ dimensional space of solutions to $D_p^*(x)\varphi=0$ decomposes into a direct sum of three pieces:

1) *A* $k-1$ dimensional space which are $O(z^{-(k-1)/2})$ near μ_p;
2) *A* $k+1$ dimensional space which are $O(z^{(k-1)/2})$ near μ_p; and
3) *A* $2m_{p-1}$ dimensional space of solutions which $O(z^0)$ at μ_p.

The first two of these are terminating and the third is continuing.

For each interval $[\mu_{p+1}, \mu_p]$, $p = 1, \ldots, N-1$, let V_p be the $2m_p$ dimensional kernel of D_p^*; let $V_0 = V_N = \langle 0 \rangle$. Define $U_p \subset V_p \times V_{p-1}$ for $p = 1, \ldots, N$ to be the space of pairs satisfying the co-Nahm boundary conditions at μ_p.

The analysis of the boundary behaviour shows that $(m_0 = m_N = 0)$

$$\dim U_p = m_p + m_{p-1} + 1.$$

Define a linear map

$$\chi\colon \bigoplus_{p=1}^{N} U_p \to \bigoplus_{p=1}^{N-1} V_p$$

$$\chi\colon ((0, \hat{u}_1), (u_1, \hat{u}_2), (u_2, \hat{u}_3), \ldots, (u_N, 0)) \mapsto (\hat{u}_1 - u_1, \hat{u}_2 - u_2, \ldots, \hat{u}_N - u_N),$$

whose kernel is clearly the cokernel of $\mathscr{D}(x)$. We want to show that the cokernel of χ is dual to the kernel of $\mathscr{D}$. It will then follow that the index of $\mathscr{D}$ is $\sum_{p=1}^{N-1} 2m_p - \sum_{p=1}^{N} (m_{p-1} + m_p + 1) = -N$.

Notice that because of the Liebniz rule if $D_p f = 0$ and $D_p^* g = 0$ then $\langle f, g \rangle$ is a constant in t. Hence V_p^* can be identified with the space of all solutions to $D_p f = 0$. For this equation we can repeat the analysis of boundary behaviour above and find a similar result, except that the dimensions $k-1$ and $k+1$ are interchanged.

Assume now that $(f_1, \ldots, f_{N-1})$ belongs $\bigoplus_{p=1}^{N-1} V_p^*$ and annihilates the image of χ. At μ_1 if we apply χ to $((0, u_1), (0,0) \cdots (0,0))$, then $\langle f_1, u_1 \rangle = 0$ for all u_1 in the $m_1 + 1$ dimensional space of decaying solutions. Hence f_1 is in the $m_1 - 1$ dimensional space of decaying solutions to $D_p f_1 = 0$. At a typical point μ_p with say, $k = m_p - m_{p-1} > 0$, if we take a pair $(u_{p-1}, \hat{u}_p)$ with $u_{p-1}(\mu_p) = \hat{u}_p(\mu_p)$ under the glueing then $0 = \langle \chi((0,0) \cdots (u_{p-1}, \hat{u}_p), \ldots, (0,0)), (f_1, \ldots, f_{N-1}) \rangle = \langle u_{p-1}, f_{p-1} \rangle - \langle \hat{u}_p, f_p \rangle = \langle u_{p-1}(\mu_p), f_{p-1}(\mu_p) - f_p(\mu_p) \rangle$, so the continuing components of f_{p-1}, f_p match.

Next, if we take $u_{p-1} = 0$ and $\hat{u}_p$ with zero continuing component and decaying terminating component, it follows that f_p has decaying terminating component. If $k = 0$ we can, in addition, choose $\hat{u}_p(\mu_p) - u_{p-1}(\mu_p) \in W_p$, so that $f_p(\mu_p) = f_{p-1}(\mu_p)$ and

$$0 = \langle \hat{u}_p(\mu_p) - u_{p-1}(\mu_p), f_p(\mu_p) \rangle.$$

In all cases, f satisfies the Nahm boundary conditions

Clearly, the converse is also true, if $(f_1, \ldots, f_{N-1})$ is in the kernel of $\mathscr{D}$ it annihilates the cokernel of χ. So we have proved:

Proposition 4.4. *The index of Nahm's operator is* $-N$.

It remains to prove a vanishing theorem for the kernel at $\mathscr{D}$. Using Nahm's equations, it is straightforward to calculate the Weitzenbock type formula

$$D_p^*(x) D_p(x) = -((d/(dz))(d/(dz)) + (\underline{T}^p + i\underline{x})^*(\underline{T}^p + i\underline{x}).$$

Hence if $D_p(x) f_p = 0$, we have

$$0=\int\left\langle\left(-\frac{d}{dz}\frac{d}{dz}f_p\right),f_p\right\rangle+\langle(\underline{T}^p+i\underline{x})f_p,(\underline{T}^p+i\underline{x})f_p\rangle$$
$$=\left\|\frac{df_p}{dz}\right\|^2+\|(\underline{T}^p+i\underline{x})f_p\|^2-\left\langle\frac{df_p}{dz},f_p\right\rangle(\mu_{p+1})-\left\langle\frac{df_p}{dz},f_p\right\rangle(\mu_p)$$
$$=\left\|\frac{df_p}{dz}\right\|^2+\|(\underline{T}^p+i\underline{x})f_p\|^2-\langle(\underline{T}^p+i\underline{x})f_p,f_p\rangle(\mu_{p+1})+\langle(\underline{T}^p+i\underline{x})f_p,f_p\rangle(\mu_p).$$

If $(f_1,\dots,f_{N-1})$ is in the kernel of $\mathscr{D}(x)$ and there are no zero jumps, then the continuity of f_p and $\underline{T}^p$ gives

$$0=\sum_{p=1}^{N-1}\left\|\frac{df_p}{dz}\right\|^2,$$

and hence $f_p=0$ for all p.

If there are zero jumps, this expression has additional terms of the form

$$-\langle(\underline{T}^p-\underline{T}^{p-1})f_p,f_p\rangle(\mu_p),$$

but inspection of the boundary behaviour of $\underline{T}^p-\underline{T}^{p-1}$ shows that for $f_p(\mu_p)\in W_p^\perp$ this whole term is non-negative.

Therefore $\ker\mathscr{D}(x)=0$ and we have that $C(x)$ is a rank N bundle on $\mathbb{R}^3$.

The connection and Higgs field for C are defined by composing differentiation and multiplication by iz with the orthogonal projection $\pi:\mathscr{L}\to C$:

$$\nabla_i=\pi\circ(\partial/\partial x^i),\quad \Phi=\pi\circ iz. \tag{4.5}$$

The same proof as that of [Hi2] shows that this defines a smooth solution to the Bogomoln'yi equations; in the next section we shall relate these constructions to the twistor approach for generic Nahm data.

4c) Link to the Twistor Approach. In the previous discussion we realized the cokernel of Nahm's operator $\mathscr{D}(x)$ as the kernel of an exact sequence

$$0\to C(x)\to\begin{matrix}U_1\\\oplus\\\vdots\\\oplus\\U_N\end{matrix}\xrightarrow{\ \ell\ }\begin{matrix}V_1\\\oplus\\\vdots\\\oplus\\V_{N-1}\end{matrix}\to 0. \tag{4.6}$$

This should be a familiar sight to the reader by now! If we take a real section C_x not intersecting any of the $S_{p,p+1}$ or contained in any spectral curve, then restricting 1.14 to this real section gives an exact sequence

$$0\to H^0(C_x,E)\to\begin{matrix}H^0(C_x,L^{\mu_1}(m_1))\\\oplus\\H^0(C_x,L^{\mu_2}(m_2+m_1))\\\oplus\\\vdots\\\oplus\\H^0(C_x,L^{\mu_N}(m_{N-1}))\end{matrix}\to\begin{matrix}H^0(S_1\cap C_x,L^{\mu_1}(m_1))\\\oplus\\H^0(S_2\cap C_x,L^{\mu_2}(m_1+m_2))\\\vdots\\\oplus\\H^0(S_{N-1}\cap C_x,L^{\mu_{N-1}}(m_{n-1}+m_{N-2})).\end{matrix}\to 0 \tag{4.6a}$$

Our purpose is to identify this term by term with (4.6). This will give us $H^0(C_x, E) = \operatorname{coker} \mathscr{D}$ and identify the bundle obtained over $\mathbb{R}^3$ by this construction with that obtained by the twistor construction.

Notice first that because L is trivial on C_x we have

$$\dim H^0((C_x, L^{\mu_p}(m_p + m_{p-1})) = m_p + m_{p-1} + 1,$$

and because $S_p \cap C_x$ is $2m_p$ points

$$\dim H^0(S_p \cap C_x, L^{\mu_p}(m_p + m_{p-1})) = 2m_p.$$

Recall the vanishing theorem 1.17 which can easily be extended to

$$\begin{aligned} H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1} + r)[-S_{p,p-1}]) &= 0 \quad \text{for} \quad r \leqq -2, \\ H^1(S_p, L^{\mu_p - t}(m_p + m_{p-1} + r)[-S_{p,p-1}]) &= 0 \quad \text{for} \quad r \geqq -2 \end{aligned}$$

for $\mu_{p+1} < t < \mu_p$ with the appropriate results also at the boundary points. The Riemann–Roch theorem gives

$$\begin{aligned} &h^0(S_p, L^{\mu_p - t}(m_p + m_{p-1} + r)[-S_{p,p-1}]) - h^1(S_p, L^{\mu_p - t}(m_p + m_{p-1} + r)[-S_{p,p-1}]) \\ &\quad = (2 + r)m_p. \end{aligned}$$

From the exact sequence on $\mathbb{P}_1$

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \to 0$$

obtained by evaluating sections on $\mathbb{C}^2 \simeq H^0(\mathbb{P}_1, \mathcal{O}(1))$, and the vanishing theorem we have

$$\begin{aligned} H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1})[-S_{p,p-1}]) &= H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1} - 1)[S_{p,p-1}]) \otimes \mathbb{C}^2 \\ &= X_p(t) \otimes \mathbb{C}^2. \end{aligned}$$

This space is then naturally identified with $Y_p(t)$. Also, from the vanishing theorem and the sequence $0 \to \mathcal{O}(-2) \to \mathcal{O} \to \mathcal{O}_{C_x} \to 0$,

$$H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1})[-S_{p,p-1}]) = H^0(S_p \cap C_x, L^{\mu_p - t}(m_p + m_{p-1})[-S_{p,p-1}]). \tag{4.7}$$

Over C_x we can fix a "C_x-trivialization" of L^t which is defined relative to the standard trivialisations over U_i by the functions ϕ_i,

$$\begin{aligned} \phi_0(t, \zeta) &= \exp(t((x_1 - ix_2)\zeta + x_3)), \\ \phi_1(t, \zeta) &= \exp(t((-x_3 + (x_1 + x_2)/\zeta). \end{aligned}$$

Evaluating with respect to this C_x trivialization at these $2m_p$ points fixes an isomorphism $H^0(S_p \cap C_x, L^{\mu_p - t}(m_p + m_{p-1})[-S_{p,p-1}])$ into a fixed $2m_p$ dimensional space; composing with (4.7), one then has a $2m_p$ dimensional space $\underline{V}_p$ of sections of Y_p defined by asking that the image be constant under this map.

Proposition 4.8. *$\underline{V}_p$ is the kernel of $D_p^*(x)$*, i.e., $\underline{V}_p = V_p$.

Proof. Start with a section s of

$$H^0(S_p, L^{\mu_p - t}(m_p + m_{p-1})[-S_{p,p-1}]),$$

which is some

$$s(t,\eta,\zeta)=p_0(\zeta)f_0(t,\eta,\zeta)=e^{(\mu_p-t)\eta/\zeta}\zeta^{m_p+m_{p-1}}p_1(\zeta)f_1(t,\eta,\zeta)q(\eta,\zeta),$$

where q is the transition function for $[-S_{p-1,p}]$, the f_i represent sections in $X_p(t)$ over U_i and the $p_i(\zeta)$ represent sections of $\mathcal{O}(1)$.

If we restrict to $S_p \cap C_x = \{(\eta_i,\zeta_i),\ i=1,\dots,2m_p\}$ we find that the coefficients of the section with respect to the C_x trivialisation are

$$p_0(\zeta_i)f_0(t,\eta_i,\zeta_i)e^{(\mu_p-t)(x_3+(x_1-ix_2)\zeta_i)},$$

and these are required to be constant.

From the definition of the connection (2.6)

$$\frac{d}{dz}=\nabla_z-iT_3+\zeta(T_1-iT_2),$$

and moreover as we are on $S_p\cap C_x$ we have that

$$\eta_i=x_1+ix_2-2x_3\zeta_i+(\zeta_i)^2(x_1-ix_2)$$

and

$$\eta_i=T_1+iT_2+2iT_3\zeta_i+(\zeta_i)^2(-T_1+iT_2)$$

from the definition of the T^i in Sect. 2.

Combining all these and choosing a basis of $H^0(T\mathbb{P}_1,\mathcal{O}(1))$ gives us

$$[(\nabla_t+\underline{T}^p+i\underline{x})s](\eta_i,\zeta_i,t)=0.$$

But the isomorphism (4.7) then implies that

$$(\nabla_t+\underline{T}^p+i\underline{x})s=0$$

as required.

We now have an isomorphism $V_p\simeq H^0(S_p\cap C_x,\ L^{\mu_p}(m_p+m_{p-1}))$ and an embedding

$$\begin{aligned}&H^0(C_x,L^{\mu_p}(m_p+m_{p-1}))\\&\quad\to H^0(S_p\cap C_x,L^{\mu_p}(m_p+m_{p-1}))\oplus H^0(S_{p-1}\cap C_x,L^{\mu_p}(m_{p-1}+m_{p-2})=V_p\oplus V_{p-1},\end{aligned}$$

the next step is to show that the image is U_p.

We start with the case of $m_p>m_{p-1}$. Recall the analysis above of the solutions about μ_p. To make statements about the decay of sections we have to choose a trivialization of the bundle Y. We shall consider two different kinds. The first is a covariantly constant (using the connection on X_p defined in 2*e* and $Y_p=X_p\otimes\mathbb{C}^2$) trivialization, which we shall call a Nahm trivialization, and the second is a trivialization using sections of

$$H^0(S_p,L^{\mu_p-t}(m_p+m_{p-1})[-S_{p,p-1}])$$

which are holomorphic in t and defined at μ_p. This we shall call a holomorphic trivialization.

We saw above that, in a Nahm trivialisation, letting $k=m_p-m_{p-1}$ and setting $z=t-\mu_p$ the $2m_p$ dimensional space of all solutions is a direct sum of three pieces:

A) *A* $k-1$ dimensional space of solutions blowing up like $z^{-(k-1)/2}$,

B) *A* $k+1$ *dimensional space of solutions decaying like* $z^{(k-1)/2}$, (4.9)
C) *A* $2m_{p-1}$ *dimensional space of solutions which are of order* z^0 *at* μ_p.

If, as is more natural in the twistor picture, we use a holomorphic trivialization, the results of section 2*e* imply that these spaces become:

A) *A* $k-1$ *dimensional space of sections blowing up like* $z^{-(k-1)}$,
B) *A* $k+1$ *dimensional space of sections which are of order* z^0 *at* μ_p, (4.10)
C) *A* $2m_{p-1}$ *dimensional space of sections which are also of order* z^0 *at* μ_p.

The upshot of this is that to satisfy the co-Nahm boundary conditions a section in the holomorphic trivialization has to live in the sum of the B) and C) components on the "larger side" and also be continuous in the C) components.

First, the decay behaviour. From the results of Sect. 2*c*),

$$Y_p(\mu_p) \subset H^0(S_p, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}]) \tag{4.11}$$

breaks into three pieces:

Y_A: *A* $k-1$ dimensional space of sections of $\mathcal{O}(m_p+m_{p-1})$ of the form $\rho_x g_{p-1} s$, where $(\rho_x=0)$ defines C_x and s is pulled back from $\mathbb{P}_1$.

Y_B: *A* $k+1$ dimensional space of sections of $\mathcal{O}(m_p+m_{p-1})$ of the form $g_{p-1}s$ where s is pulled back from $\mathbb{P}_1$.

Y_C: *A* $2m_{p-1}$ dimensional space of sections of $\mathcal{O}(m_p+m_{p-1})$ which are of degree $m_{p-1}-1$ in η and vanish on $S_{p-1,p}$. (4.12)

Consider now the restriction maps

$$\begin{tikzcd} Y_p(\mu_p) \arrow[r, "\sigma"] & H^0(C_x \cap S_p, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}]) \\ & H^0(C_x, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}]). \arrow[u, "\rho"] \end{tikzcd} \tag{4.13}$$

For $t<\mu_p$ the map σ in (4.13) is an isomorphism and for $t=\mu_p$ it clearly kills the space Y_A. Notice that the map ρ is an inclusion because $H^0(\mathbb{P}_1, \mathcal{O}(-m_p+m_{p-1}))=0$ when $m_p>m_{p-1}$. In addition we have

Proposition 4.14. *The map* σ *in* (4.13) *maps* $Y_B \oplus Y_C$ *isomorphically onto the image of the map* ρ.

Proof. As, from 2c), $H^0(T\mathbb{P}_1, \mathcal{O}(m_p+m_{p-1}) \otimes \mathcal{I}(-S_{p,p-1})) = H^0(S_p, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}])$ then extending to $T\mathbb{P}_1$ everything in $\sigma(Y_p(\mu_p))$ is in the image of the vertical map.

Let s in $Y_B \oplus Y_C$ vanish on $C_x \cap S_p$. Divide s by $\rho_x \in \Gamma(\mathcal{O}(2))$, lift to $T\mathbb{P}_1$, then restrict the result to S_{p-1}; the vanishing theorem tells us that the result is zero. s is then divisible by ρ_x and g_{p-1}, which contradicts the fact that its degree in η is less than m_{p-1}.

The map is thus an injection and so, counting dimensions, an isomorphism.

Thus, at μ_p, the isomorphism (4.7) fails. The values in $H^0(C_x \cap S_p, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}])$ that do correspond to solutions in V_p are those in $\sigma(Y_p(\mu_p)$; by Proposition (4.14), this is $H^0(C_x, \mathcal{O}(m_p+m_{p-1})[-S_{p,p-1}])$.

As there are no constraints on decay behaviour on the "smaller" side, $H^0(C_x, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}])$ therefore corresponds on both sides of μ_p to solutions with the right decay behaviour; to see that it also matches them up correctly we note that at the endpoints the glueing of the spaces is accomplished by the diagram

$$\begin{tikzcd}[column sep=tiny] & H^0(T\mathbb{P}_1, \mathcal{O}(m_p + m_{p-1}).\mathcal{I}(S_{p,p-1})) \arrow[dl] \arrow[dr] & \\ H^0(S_p, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}]) & & H^0(S_{p-1}, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}]) \\ Y_p(\mu_p) \arrow[u, phantom, "\cup" description] & & Y_{p-1}(\mu_p), \arrow[u, phantom, "\|" description] \end{tikzcd} \tag{4.15}$$

where the left diagonal map is an isomorphism. Lifting back and pushing down defines an isomorphism from Y_C to $Y_{p-1}(\mu_p)$.

If we restrict to C_x and use the proposition we have

$$\begin{tikzcd}[column sep=tiny] & H^0(C_x, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}]) \arrow[dl, "\simeq"'] \arrow[dr] & \\ Y_B \oplus Y_C \subset Y_p(\mu_p) & & Y_{p-1}(\mu_p), \end{tikzcd} \tag{4.16}$$

so a bounded solution is continuous if its component in Y_C and its component in $Y_{p-1}(\mu_p)$ are related by this map. The patching condition follows tautologically and so $H^0(C_x, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}])$ really does correspond to U_p.

Consider now the case of $m_p - m_{p-1} = k_p = 0$. There is no problem with decay behaviour; as for patching, the identification of $V_p(\mu_p)$ and $V_{p-1}(\mu_p)$ induces an identification of $Y_p(\mu_p)$ and $Y_{p-1}(\mu_p)$. If we start with a section in $H^0(C_x, \mathcal{O}(m_p + m_{p-1})$ $[-S_{p,p-1}])$ its images in $Y_p(\mu_p)$ and $Y_{p-1}(\mu_p)$ are related by the pulling back and pushing down described above. As $k_p = 0$, $m_p + m_{p-1} = 2m_p = 2m_{p-1}$, and there is an ambiguity in lifting back a section from $C_x \cap S_p$ to C_x, namely all the multiples of g_p; similarly, that in lifting from $C_x \cap S_{p-1} C_x$ is g_{p-1}. The net effect of this is that a section in the kernel of Nahm's operator arising from a section of E over C_x may have a discontinuity at μ_p which is a multiple of $(g_p - g_{p-1})$. Comparing with (2.27) we see that this means that the discontinuity is in the image of $\underline{A}^+(\zeta) - \underline{A}^-(\zeta)$, which is precisely the result required. Again U_p is identified with $H^0(C_x, \mathcal{O}(m_p + m_{p-1})[-S_{p,p-1}])$.

4d) The Equivalence of the Connections and of the Higgs Fields. The isomorphism of (4.6) and (4.6a) now yields an isomorphism of two bundles over R^3,

$$C(x) = H(x) = H^0(C_x, E), \tag{4.17}$$

each of which is equipped with a solution to the Bogomoln'yi equations. We complete the discussion by showing that the connections and Higgs fields are equivalent.

Recall from [Hu1] that fixing a direction $(0,0,x_3)$ in $\mathbb{R}^3$ means looking at a family of real sections in $T\mathbb{P}_1$ all intersecting on $\mathbb{P}_1$ in the same two points, 0 and ∞. We can trivialize the bundle E over any of these real sections by evaluation at either of these points and define two "evaluation" connections ∇_0 and ∇_∞ in H along $(0,0,x_3)$. Then these relate to the connection and Higgs field by

$$\Phi dx^3 = \frac{i}{2}(\nabla_0 - \nabla_\infty), \quad \nabla_{x^3} = \tfrac{1}{2}(\nabla_0 + \nabla_\infty). \tag{4.18}$$

Notice that if we know these two quantities for every choice of line, then the connection and Higgs field are completely determined, and vice versa. It is enough then to show that under the identification of C with H, both constructions give rise to the same ∇_0 and ∇_∞ or, because of the symmetry, the same ∇_0.

From Nahm's point of view the operator ∇_0 is defined by $\pi(\partial/\partial x_3 + t)$, where π is the orthogonal projection onto the cokernel of $\mathscr{D}$. To establish the equivalence we will show that a section constant for Nahm's ∇_0 when interpreted on $T\mathbb{P}_1$ via the isomorphism (4.17) is constant for the twistor ∇_0, that is it takes a constant value in the fibre of E over 0.

Suppose for simplicity that none of the k_p are zero. Let $f(t, x_3) = f = (f_1, \dots, f_{N-1})$ lie in $\operatorname{coker}(\mathscr{D}(x))$ and suppose that $\pi(\partial/\partial x_3 + t)f = 0$. There then exists a $g = (g_1, \dots, g_{N-1}) \in \mathscr{H}$ with $(\partial/\partial x_3 + t)f_p = D_p(x)g_p$ for all p. Expanding the 2×2 matrices in D_p^*, we find

$$\left[D_p^*(x_3), \frac{\partial}{\partial x_3} + t \right] = \begin{bmatrix} 0, & 0 \\ 0, & 2 \end{bmatrix}.$$

Writing $f_p = (f'_p, f''_p)^T$, we have that $\mathscr{D}^*\mathscr{D}g = (0, 2f'')^T$. From the positivity and reality of $\mathscr{D}^*\mathscr{D}$ it follows that $g = (0, g'')$ and therefore

$$\left(\frac{\partial}{\partial x^3} + z \right) f_p = - \begin{bmatrix} (T_1 + iT_2)g''_p \\ h_p \end{bmatrix} \tag{4.19}$$

for some h_p.

If we think of these as sections over $T\mathbb{P}_1$ we can use 1 and ζ as a basis for $\mathbb{C}^2 = H^0(T\mathbb{P}_1, \mathcal{O}(1))$ and obtain

$$\left(\frac{\partial}{\partial x^3} + t \right) f_p = -(T_1 + iT_2)g''_p + \zeta h_p = -A_0 g''_p + \zeta h_p = -(\eta - \zeta A_1 - \zeta^2 A_2)g''_p + \zeta h_p, \tag{4.20}$$

where $f_p = f'_p + \zeta f''_p$. To obtain a section over C_x we have to change the trivialization to $F_p = \exp(x_3 t) f_p$, and therefore using this and Eq. (4.20) we obtain

$$\frac{\partial F_p}{\partial x_3} = -e^{tx_3}(\eta - \zeta A_1 - \zeta^2 A_2)g''_p + e^{tx_3}\zeta h_p. \tag{4.21}$$

Evaluating at $\zeta = \eta = 0$ we see that F_p is constant in the x_3 direction as required. The case when some of the k_p are zero is proven similarly.

5. Modifications for the Cases $SO(k)$, $Sp(k)$

In this section, we briefly summarize the modifications necessary for treating the cases of $SO(k)$, $Sp(k)$.

5a) From Monopoles to Spectral Data. We now suppose that the bundle H over

$\mathbb{R}^3$ is equipped with a (symmetric or skew) bilinear form, compatible with the unitary structure, preserved by the connection, with respect to which the Higgs field is skew adjoint. The asymptotics of the Higgs field satisfy $\mu_i = -\mu_{N-i+1}$, $k_i = -k_{N-i+1}$. Along a line, the form applied to a pair of solutions s, s' of $(\nabla - i\Phi)s = 0$ is constant; the bilinear form thus passes over to a bilinear form $(\ ,\ )$ defined on the bundle E over TP_1. Alternately, one has an antilinear map

$$J: E \to E \tag{5.1}$$

lifting the map τ. Composing with the map σ of 1e), one has a holomorphic bundle map $\sigma J: E \to E^*$. The form is then given by $(a,b) = \sigma J(a)\,(b)$. In the orthogonal case, $J^2 = 1$; in the symplectic case $J^2 = -1$.

As the flags are defined by decay rates at $\pm\infty$ of solutions to $(\nabla - i\Phi)s = 0$, evaluating the bilinear form near $\pm\infty$ gives us:

$$(E_p^{\pm})^{\perp} = (E_{N-p}^{\pm}),$$

i.e. the flags are "isotropic-coisotropic." As a consequence, $((E_p^+ \cap E_{N-p}^-) = 0) \Leftrightarrow ((E_{n-p}^+ \cap E_p^-) = 0)$, i.e.

$$S_p = S_{N-p}.$$

Similarly,

$$S_{p,p+1} = S_{N-p,N-p-1}$$
$$S_{p+1,p} = S_{N-p-1,N-p}.$$

In [M], spectral curves R_p are defined for G-monopoles, G any compact Lie group. Some Lie theory then shows that in our case, the curves R_p and S_q are linked by the relations given in the introduction.

The existence of monopoles with the R_q in general position is proven in the same way as in Sect. 1. In the case of $Sp(k)(N = 2k)$, it then follows that the curves S_p are in general position, and the whole of Sect. 1 goes through verbatim.

In the orthogonal case, one must recompute some of the quotients in (1.12). For $SO(2k)$, the isomorphisms C-1 (proven in [M]), give us, over $R_+ \cap R_-$, an isomorphism $L^{-\mu_k}(m_{k-1}) = L^{\mu_k}(m_{k-1})$. Using this, one has:

$$0 \to E/(E_k^+ + E_k^-) \to \{L^{\mu_k}(m_{k-1}) \oplus L^{-\mu_k}(m_{k-1})\}|_{R_+} \to L^{\mu_k}(m_{k-1})|_{R_+ \cap R_-} \to 0,$$

$$0 \to E/(E_{k-1}^+ + E_k^-) \to L^{\mu_k}(m_{k-1}) \oplus \{L^{-\mu_k}(m_{k-1})\}|_{R_+} \to L^{\mu_k}(m_{k-1})|_{R_+ \cap R_-} \to 0,$$

$$0 \to E/(E_k^+ + E_{k-1}^-) \to \{L^{\mu_k}(m_{k-1})|_{R_+}\} \oplus L^{-\mu_k}(m_{k-1})\} \to L^{\mu_k}(m_{k-1})|_{R_+ \cap R_-} \to 0.$$

The other quotients are as in the unitary case. For $SO(2k+1)$, one has from [M] the isomorphism

$$s: \mathcal{O} \approx L^{\mu_k}(m_{k-1})[-S_{k-1,k}].$$

Consider now the exact sequence:

$$0 \to \mathcal{O}(-m_k)|_{R_k} \to \mathcal{O}|_{2R_k} \to \mathcal{O}|_{R_k} \to 0,$$

and tensor it by $L^{\mu_k}(m_{k-1})[-S_{k-1,k}]$. The coboundary $\delta(s)$ in

$$H^1(R_k, L^{\mu_k}(m_{k-1} - m_k)[-S_{k-1,k}]) \approx H^1(R_k, \mathcal{O})$$

defines an extension P_k over $T\mathbb{P}_1$:

$$0 \to \mathcal{O}|_{R_k} \to P_k \to \mathcal{O}(m_k) \to 0,$$

one has $P_k \approx E/(E_k^+ + E_k^-)$. All the other quotients are as in (1.12).

From this point, the vanishing theorems, as well as the asymptotic estimates on the Higgs field, follow more or less as in the unitary case.

One must also show that one can construct an appropriate form on a bundle built starting from the spectral data, as in (1.14). It is easiest, in fact, to build the map J. Some thought shows that J should descend to the sum of the quotients $P_i = E/(E_i^+ + E_{N-i-1}^-)$, interchanging P_i and P_{N-i+1}. In the $Sp(k)$ case, for example, this amounts to finding a $J: L^{\mu_i}(m_{i-1} + m_{i+1}) \otimes \mathcal{I}(S_{i-1,i}) \to L^{-\mu_i}(m_{i-1} + m_{i+1}) \otimes \mathcal{I}(S_{i,i-1})$ lifting τ, which certainly is possible.

5b) From Spectral Data to Nahm Data. Given the spectral data, the next step is to construct bundles X over the intervals (μ_{i+1}, μ_i), as well as a solution to Nahm's equations over these intervals. This is done in essentially the same way as in Sect. 2, the proof that the boundary conditions are satisfied being modified slightly to take the different structure of the quotients $Q_p = E/(E_p^+ + E_{N-p}^-)$ into account.

One must also construct the matrices C_j of condition B-3. Invariantly, this is equivalent to giving a pairing of X_z with X_{-z}, covariant constant with respect to the connection on X, and such that $T_i(z)$ and $T_i(-z)$ are adjoints of one another. As above, this form is most easily defined by giving an antilinear map $\tilde{J}: X_z \to X_{-z}$, and then using the unitary structure. Note that for $z \in (\mu_{i+1}, \mu_i)$, $X_z \approx H^0(S_i, Q_i \otimes L^{-z}(-1))$. To define $\tilde{J}$, one uses the map J given above; J can be "pushed down" to a map $J': Q_i \to Q_{N-i}$; one also has a map $J'': L^z(-1) \to L^{-z}(-1)$, with $(J'')^2 = -1$. $\tilde{J}$ is then the map induced on sections by $J' \otimes J''$; one has $(\tilde{J}^2 = \pm 1)$ $(J^2 = \mp 1)$, and so $\tilde{J}^2 = 1$ for Sp, -1 for SO. An alternative definition of the form is given in [Hu3].

6. Summary and Conclusion

We have now built up all the ingredients of our theorem; it is perhaps appropriate to sum up by showing how they all fit together to give the desired result.

First, the generic case. In Sect. 1, we showed how a generic monopole gave one spectral data; we also showed that the map was injective; the inverse (twistor) construction gives back the original monopole. In Sect. 2 and 3, we proved that there is an equivalence between spectral data and generic Nahm data. In Sect. 4, we showed that any spectral data yielded back a monopole. This is done by building a bundle E on $T\mathbb{P}_1$, and applying the twistor transform. To see that applying the construction of Sect. 1 gives back the same spectral data, one notes that the bundle one obtains from the monopole by the inverse twistor transform must be E[Hi1]; to see that the spectral data is the same, it is sufficient to show that the flags E_i^+, E_i^- are the same. Referring to Sect. 1d on the asymptotic Higgs field, one sees that this is indeed the case, as the sum of the asymptotic eigenspaces corresponding to $\mu_1 \cdots \mu_i$ is indeed E_i^+; and similarly, for E_i^-. Alternately, one can apply the result of [HiM], showing that the spectral data is the same. Thus, theorem 1 is proven.

In the non-generic case, Sect. 2 showed that a monopole gives a solution to Nahm's equations; if the monopole is a limit of generic monopoles, then the solution satisfies the boundary conditions. Conversely, given Nahm data, we showed in Sect. 4 how to construct a solution to the Bogomolny equations over $\mathbb{R}^3$. The set of Nahm data is connected [Hu3] and so our Nahm data is the limit of generic Nahm data with spectral curves of the same degree. If one examines the construction, it is easy to see that the solution to the Bogomolny equations is a limit of monopoles. The monopole version of the Uhlenbeck compactness theorem [AHi, Proposition 3.9] implies that this limit is a monopole of possibly lower charge. However, if the charge is lower, this implies that the spectral curves do not stay bounded as we approach our limit, which is precluded in this case. Alternately, one could use an improved formula for the asymptotic Higgs field, with explicit bounds on the exponential error term; note that this formula remains defined in the limit, as it involves only the geometry of the spectral curves.

The circle therefore closes, giving one various points of view for attacking the problem. Each construction highlights certain aspects: the twistor viewpoint emphasizes the role of algebraic curves, and gives us asymptotic behaviour quite neatly; the regularity, however, is easiest to see from the Nahm viewpoint. This latter is also the most convenient for computing moduli [Hu3].

Several problems remain: one is showing that this construction yields all monopoles. This is equivalent to showing that the monopole moduli space for fixed charge is connected; it seems quite likely that this is the case [T2]. Another problem is extending these ideas to arbitrary groups, and to non-maximal symmetry breaking (μ_i not distinct.)

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References

[AHH] Adams, M., Harnad, J., Hurtubise, J.: Isospectral Hamiltonian flows in finite and infinite dimensions: II. Integration of flows. Preprint

[AHi] Atiyah, M. F., Hitchin, N. J.: The geometry and dynamics of magnetic monopoles. Princeton: P.U.P. 1988

[D] Donaldson, S. K.: Nahm's equations and the classification of monopoles. Commun. Math. Phys. **96**, 387–408 (1984)

[JT] Jaffe, A., Taubes, C. H.: Vortices and monopoles. PPh2, Boston: Birkhaüser 1980

[Ha] Hartshorne, R.: Algebraic geometry. GTM vol. **52**. Berlin Heidelberg, New York: Springer 1977

[Hi1] Hitchin, N. J.: Monopoles and geodesics. Commun. Math. Phys. **83**, 579–602 (1982)

[Hi2] Hitchin, N. J.: On the construction of monopoles. Commun. Math. Phys. **89**, 145–190 (1983)

[Hi3] Hitchin, N. J.: Linear field equations on self dual spaces. Proc. R. Soc. Lond. **A370**, 173–191 (1980)

[HiM] Hitchin, N. J., Murray, M. K., Spectral curves and the ADHM construction. Commun. Math. Phys. **118**, 463–474 (1988)

[Hu1] Hurtubise, J. C.: The asymptotic Higgs field of a monopole. Commun. Math. Phys. **97**, 381–389 (1985)

[Hu2] Hurtubise, J. C: Monopoles and rational maps: A note on a theorem of Donaldson. Commun. Math. Phys. **100**, 191–196 (1985)

[Hu3] Hurtubise, J. C.: The classification of monopoles for the classical groups. Commun. Math. Phys. **120**, 613–641 (1989)

[M] Murray, M. K : Non-Abelian magnetic monopoles. Commun. Math. Phys. **96**, 539–565 (1984)
[N] Nahm, W.: The construction of all self-dual multi-monopoles bys the ADHM method, in Monopoles in quantum field theory. Craigie et al. (ed.). Singapore: World Scientific 1982
[T1] Taubes, C. H.: The existence of multi-monopole solutions to the non-abelian Yang-Mills Higgs equations for arbitrary simple gauge groups. Commun. Math. Phys. **80**, 343 (1981)
[T2] Taubes, C. H.: Min-max theory for the Yang-Mills-Higgs equations. Commun. Math. Phys. **97**, 473–540 (1985)
[W] Ward, R. S.: Ansätze for self-dual Yang-Mills fields. Commun. Math. Phys. **80**, 563–574 (1981)

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