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A Connection Between the Einstein and Yang-Mills Equations

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Abstract. It is our purpose here to show an unusual relationship between the Einstein equations and the Yang-Mills equations. We give a correspondence between solutions of the self-dual Einstein vacuum equations and the self-dual Yang-Mills equations with a special choice of gauge group. The extension of the argument to the full Yang-Mills equations yields Einstein's unifield equations. We try to incorporate the full Einstein vacuum equations, but the approach is incomplete. We first consider Yang-Mills theory for an arbitrary Lie-algebra with the condition that the connection 1-form and curvature are constant on Minkowski space. This leads to a set of algebraic equations on the connection components. We then specialize the Lie-algebra to be the (infinite dimensional) Lie-algebra of a group of diffeomorphisms of some manifold. The algebraic equations then become differential equations for four vector fields on the manifold on which the diffeomorphisms act. In the self-dual case, if we choose the connection components from the Lie-algebra of the volume preserving 4-dimensional diffeomorphism group, the resulting equations are the same as those obtained by Ashtekar, Jacobsen and Smolin, in their remarkable simplification of the self-dual Einstein vacuum equations. (An alternative derivation of the same equations begins with the self-dual Yang-Mills connection now depending only on the time, then choosing the Lie algebra as that of the volume preserving 3-dimensional diffeomorphisms.) When the reduced full Yang-Mills equations are used in the same context, we get Einstein's equations for his unified theory based on absolute parallelism. To incorporate the full Einstein *vacuum* equations we use as the Lie group the semi-direct product of the diffeomorphism group of a 4-dimensional manifold with the group of frame rotations of an $SO(1,3)$ bundle over the 4-manifold. This last approach, however, yields equations more general than the vacuum equations.

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1. Introduction

In this paper we provide a novel connection between Yang-Mills theory and general relativity. We give a correspondence between a class of solutions of the Yang-Mills equations on the one hand and solutions of Einstein's equations on the other.

This correspondence is most satisfactory when the fields are both self dual. When the Yang-Mills theory is not self-dual we obtain either Einstein's unified field equations or a set of equations which reduce to the Einstein vacuum equations when the torsion of a certain connection vanishes.

The group of gauge transformations for general relativity can be taken either as the semidirect product of the diffeomorphism group with the group of frame rotations or just the diffeomorphism group on its own. This is not isomorphic to an ordinary group of gauge transformations (maps from a 4-manifold into the *finite* dimensional gauge group [1]). From this it would seem unlikely that one would be able to make a correspondence between solutions of some version of the Yang-Mills equation and solutions of the vacuum equations.

We get around this by using infinite dimensional gauge groups; in particular, we choose for the gauge groups the group of (volume preserving) diffeomorphisms of some 4-manifold, $\mathscr{M}^4$, or the semi-direct product of this group with the group of frame rotations of an $SO(1,3)$ bundle on this space (this incorporates frame rotations). The first possibility is sufficient for the self-dual case, and the second possibility seems to be required to incorporate general vacuum fields. We then freeze the Minkowski space dependence of the gauge transformations by requiring that the vector potentials be constant so that the group of gauge transformations be reduced to the original gauge group. That is, we require that the connection be invariant under the translation subgroup of the Poincaré group.

Our basic procedure is thus to first consider the Yang-Mills equations for an arbitrary group and impose the symmetry condition that the connection be independent of the Minkowski space points, i.e. it is constant. This leads to an algebraic condition on the connection components. This algebraic condition becomes differential equations on the auxiliary $\mathscr{M}^4$ when the above infinite dimensional groups are used.

(There is also an alternative formulation in which we consider a Yang-Mills field which is independent of the spatial variables, but still depends on time. The appropriate gauge group is now the diffeomorphism group of some three manifold Σ^3.)

An interesting question is: what of interest can be obtained by using other infinite dimensional groups and other symmetry reductions for the Yang-Mills or self-dual Yang-Mills equations?

In Sect. 2 we will discuss the reduced Yang-Mills equations, i.e. the algebraic equations obtained from the symmetry conditions and in Sect. 3 we will outline the version of the self-dual Einstein equations due to Ashtekar, Jacobsen and Smolin. In Sect. 4 we will show how the self-dual Einstein equations are equivalent

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[1] Here we refer to the gauge group as the group in whose Lie algebra the vector potentials take their values and the group of gauge transformations as the infinite group of frame rotations of the Yang-Mills vector bundle

to the reduced, self-dual Yang-Mills equations (identical to the Ashtekar, Jacobsen and Smolin version) when the gauge group is the volume preserving four-dimensional diffeomorphism group. Finally in Sect. 5 we discuss how the full Einstein equations (and various generalizations) arise from these considerations.

2. The Reduced Yang-Mills Equations

Consider a vector potential γ_a on Minkowski space, $\mathbb{M}$, where for each $a=0,\dots,3$ $\gamma_a(x)\in l$ for some Lie algebra l. The curvature is then given by:

$$F_{ab}=[\partial_a-\gamma_a,\partial_b-\gamma_b]=[D_a,D_b],$$

where $D_a=\partial_a-\gamma_a$.

The full Yang-Mills (YM) equations are:

$$D^aF_{ab}=0,$$

and the self-dual Yang-Mills (SDYM) equations are:

$$*F_{ab}=\tfrac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}=iF_{ab}$$

(or in spinors, setting $F_{ab}=\Phi_{AB}\varepsilon_{A'B'}+\Phi_{A'B'}\varepsilon_{AB}$ the SDYM become $\Phi_{AB}=0$).

Note that we have the Minkowski metric, η_{ab}, at our disposal to raise and lower the indices, $a,b,c,\dots$.

We first consider the case in which the vector potentials are independent of all the spacetime coordinates. Then we have that the only invariant gauge transformations are constant, and the γ_a are each constant elements of the Lie algebra l. The full YM equations then reduce to:

$$[\gamma^a,[\gamma_a,\gamma_b]_l]_l=0, \tag{1}$$

where the brackets, $[\ ,\ ]_l$ are the Lie algebra brackets for l. The SDYM equations reduce to:

$$[\gamma_a,\gamma_b]_l=-\frac{i}{2}\varepsilon_{ab}{}^{cd}[\gamma_c,\gamma_d]_l \tag{2}$$

(in spinors we have: $[\gamma_{A'(A},\gamma^{A'}_{B)}]_l=0$). (We retain Lorentzian conventions for the SD fields in order to avoid confusion when we discuss full Yang-Mills fields.)

The Jacobi identity, $\varepsilon^{abcd}[\gamma_b,[\gamma_c,\gamma_d]_l]_l=0$, implies that solutions of (2) also solve (1).

{For completeness we also consider the case in which the γ_a only depend on t. The remaining gauge transformations only depend on t, and can be used to eliminate γ_0 ($t=x^0$). We resolve the above equations parallel and perpendicular to the time-like direction. We find, the indices $i,j,\dots$ ranging from 1 to 3,

$$[\gamma^i,\dot{\gamma}_i]_l=0 \quad\text{and}\quad \ddot{\gamma}_i-[\gamma^j,[\gamma_j,\gamma_i]_l]_l=0 \tag{3}$$

for full YM and

$$\dot{\gamma}_i=-\frac{i}{2}\varepsilon_i{}^{jk}[\gamma_j,\gamma_k]_l \tag{4}$$

for SDYM (these are also known as Nahm's equations). Here ε_{ijk} is the projection of ε_{0abc}.}

3. A Simple Form of the Self-Dual Vacuum Equations

Recently Ashtekar et al. (1988) have shown that the Self-Dual Einstein equations, when written out in $3+1$ form, are remarkably simple

$$\dot{V}_i = -\frac{i}{2}\varepsilon_i{}^{jk}[V_j, V_k] \tag{5}$$

and

$$\mathscr{L}_{V_i}(v_3) = 0\,. \tag{6}$$

Here $V_i, i=1,2,3$, are a triad of independent divergence free time dependent vector fields on a three manifold, Σ^3, the bracket, $[\,,]$, is now *the standard Lie bracket of vector fields (Lie derivative)* and $\varepsilon_{ijk}=\varepsilon_{[ijk]}$, $\varepsilon_{123}=1$, is the 3-dimensional alternating tensor and v_3 is a fixed volume form (say $v_3=\alpha dx^1\wedge dx^2\wedge dx^3$ with $x^1,\ldots,x^3$ coordinates on Σ^3 and α some nonvanishing function of x). The metric is then retrieved on $\Sigma^3\times\mathbb{R}$ as follows.

Let t be a coordinate on the $\mathbb{R}$ factor, then the SD vacuum metric on $\Sigma^3\times\mathbb{R}$ is determined by the requirement that the frame, $\sigma_a=\left(f^{-1}\frac{\partial}{\partial t}, f^{-1}V_i\right)$ $a=0,\ldots,3$, of the tangent bundle be orthonormal where f is determined by

$$f^2 = v_3(V_1, V_2, V_3)\,. \tag{7}$$

A proof of these results can be found in Ashtekar et al. (1987); however a direct proof of the above is also implicit in the following material.

4. The SDYM Equations and the Self Dual Vacuum Equations

It can now be seen that, by a special choice of gauge group, our Eqs. (4) become the Ashtekar Eqs. (5) and (6). Let $S\,\mathrm{Diff}(\Sigma^3)$ be the group of diffeomorphisms of Σ^3 which preserve the volume form v_3. Then the Lie algebra $LS\,\mathrm{Diff}(\Sigma^3)$ of $S\,\mathrm{Diff}(\Sigma^3)$ consists of vector fields V satisfying (6). *The commutator in the Lie algebra is just the Lie bracket (Lie derivative) of vector fields.* So we see that, identifying the V_i with the γ_i, Eq. (5) really is Eq. (4) where each γ_i, $i=1,\ldots,3$ take values in $LS\,\mathrm{Diff}(\Sigma^3)$.

Alternatively, we can see that (5) is a special case of Eq. (2). Let $\mathscr{M}^4=\Sigma^3\times\mathbb{R}$ and let $v_4=dt\wedge v_3$. Then consider the group of diffeomorphisms of $\mathscr{M}^4$ preserving v_4, $S\,\mathrm{Diff}(\mathscr{M}^4)$. If we write $V_a=(V_0,V_i)=\left(\frac{\partial}{\partial t}, V_i\right)$, $a=0,\ldots,3$, we have that the $V_a\in LS\,\mathrm{Diff}(\mathscr{M}^4)$ the Lie algebra of $S\,\mathrm{Diff}(\mathscr{M}^4)$. So, identifying the V_a with the γ_a, we have that Eq. (5) is now equivalent to Eq. (2).

We now wish to show (without relying on the observations of Ashtekar et al.) the following:

Proposition. *Let $V_a\in S\,\mathrm{Diff}(\mathscr{M}^4)$ for each $a=0,\ldots,3$, then if:*

$$\tfrac{1}{2}\varepsilon_{ab}{}^{cd}[V_c, V_d] = i[V_a, V_b]\,, \tag{8a}$$

$$\mathscr{L}_{V_a}v_4 = 0 \tag{8b}$$

with v_4 some non-vanishing 4-form, then the V_a are conformal to an orthonormal frame $\sigma_a = f^{-1} V_a$ for a self-dual vacuum space-time, where

$$f^2 = v_4(V_0, V_1, V_2, V_3). \tag{9}$$

Conversely, given a self-dual space-time, there will always exist an orthonormal frame, σ_a, and a nonvanishing function, f, such that $V_a = f\sigma_a$ preserve some volume form v_4 so that $V_a \in LS\,\mathrm{Diff}(\mathscr{M}^4)$ and the V_a satisfy Eq. (8).

Proof. We first prove the second part of the theorem. Assume that we have a space-time with self-dual curvature. This implies that the curvature of the unprimed spin connection is zero so that we can find a basis of covariantly constant unprimed spinors, Penrose (1976). This implies that we can choose a frame, σ_a, so that the corresponding unprimed spin frame is covariantly constant. Define the rotation coefficients, $\Gamma_c{}^a{}_b$, as follows:

$$\nabla_{\widehat{\sigma_c}} \sigma_b = \Gamma_c{}^a{}_b \sigma_a. \tag{10}$$

Since σ_a is orthonormal we have $\Gamma_{cab} = \Gamma_{c[ab]}$ and, as a consequence of the constancy of the corresponding unprimed spin frame we have: $\frac{1}{2}\Gamma_{cab}\varepsilon^{ab}{}_{de} = i\Gamma_{cde}$, i.e. Γ_{cab} is self-dual on the ab index pair.

Define the structure functions, $C_{ab}{}^c$ associated to σ_a as follows:

$$[\sigma_a, \sigma_b] = C_{ab}{}^c \sigma_c. \tag{11}$$

Then they are related to the rotation coefficients by

$$C_{abc} = \Gamma_{acb} - \Gamma_{bca}. \tag{12}$$

We wish to express the self-duality condition on Γ_{abc} as a condition on C_{abc}. To that end we can write

$$C_{abc} = -\Gamma_{abc} - \Gamma_{bca} = -3\Gamma_{[abc]} + \Gamma_{cab}.$$

Define $\Gamma_a = \Gamma_c{}^c{}_a$. Then the trace of the self-duality equation for Γ_{abc} implies that

$$3\Gamma_{[abc]} = -i\varepsilon_{abc}{}^d \Gamma_d.$$

So the self duality of Γ_{cab} on the ab index pair implies the self duality of $C_{abc} - i\varepsilon_{abc}{}^d\Gamma_d$. However the combination $i\varepsilon_{abc}{}^d\Gamma_d + 2\Gamma_{[a}\eta_{b]c}$ is automatically self dual on the ab index pair (where η_{ab} is $\mathrm{diag}\{1, -1, -1, -1\}$). So we see that the combination $C_{abc} + 2\Gamma_{[a}\eta_{b]c}$ is also self dual,

$$i\{C_{abc} + 2\Gamma_{[a}\eta_{b]c}\} = \tfrac{1}{2}\varepsilon_{ab}{}^{ef}\{C_{efc} + 2\Gamma_{[e}\eta_{f]c}\}. \tag{13}$$

Let us assume for the moment that Γ_a is a gradient

$$\Gamma_a = \sigma_a(\log f), \tag{14}$$

where $V(f)$ denotes the derivative of the function f along the vector field V.

Define $V_a = f\sigma_a$. Then we have $[V_a, V_b] = f\{C_{ab}{}^c + 2\Gamma_{[a}\delta_{b]}{}^c\} V_c$. So (13) implies Eq. (8):

$$\tfrac{1}{2}\varepsilon_{ab}{}^{cd}[V_c, V_d] = i[V_a, V_b].$$

Equations (9) can be obtained as follows. Let v_g be the metric volume form so that

$$v_g(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = 1. \tag{15}$$

Then define $v_4 = f^{-2} v_g$ in order to ensure Eq. (8b). We wish to see that Eq. (9), $\mathcal{L}_{V_a} v_4 = 0$, follows. Differentiation of Eq. (15) along σ_a yields the equation[2] $\mathcal{L}\sigma_a v_g = \Gamma_a v_g = \sigma_a(\log f) v_g$. From (14) and the various definitions, this is equivalent to

$$0 = \mathcal{L}\sigma_a f^{-1} v_g = d\{f^{-1}\sigma_a \lrcorner v_g\} = d\{f\sigma_a \lrcorner f^{-2} v_g\} = d\{V_a \lrcorner v_4\} = \mathcal{L}_{V_a} v_4,$$

where d denotes the exterior derivative on forms. Equation (9) now follows from Eq. (15) and the definitions of V_a and v_4.

It remains to prove that we can choose our tetrad, σ_a, so that (14) holds for some f. Indeed we will see that there exists at least a class of such frames depending on a free function of three variables.

We choose a $3+1$ dimensional decomposition of the space-time with the $\sigma_a = (n, \sigma_i)$, $i = 1, 2, 3$, so that $n = \sigma_0$ is hypersurface orthogonal. The condition that the associated unprimed spin frame be covariantly constant restricts the choice of the σ_i's up to a rigid rotation. So we have that the dual one form to n (also denoted n) satisfies

$$n = f dT, \tag{16}$$

and we chose the function T to satisfy the wave equation, $\Box T = 0$, so that $f^{-1} n$ is covariantly divergence free.

For the acceleration of n we have

$$a^i \sigma_i = \nabla_n n = \Gamma_0{}^i{}_0 \sigma_i = -\sigma^i(\log f)\sigma_i, \tag{17}$$

and for the divergence we have

$$\nabla \cdot n = \Gamma_i{}^i{}_0 = \nabla_n(\log f). \tag{18}$$

Decomposing Γ_a we have

$$\Gamma_0 = \Gamma_i{}^i{}_0 \quad \text{and} \quad \Gamma_i = \Gamma_a{}^a{}_i = \Gamma_0{}^0{}_i + \Gamma_j{}^j{}_i. \tag{19}$$

If the last term in (19), namely $\Gamma_j{}^j{}_i$, vanishes then from (17) and (18) we have that $\Gamma_a = \sigma_a(\log f)$ as required. That $\Gamma_j{}^j{}_i = 0$ can be seen from the hypersurface orthogonality of n and the self-duality of Γ_{abc} by the following argument: We have that $n_\wedge dn = 0$. This implies that

$$\Gamma_{[ij]0} = 0,$$

and the self-duality of Γ_{abc} implies that $\Gamma_{ijk} = \frac{1}{2}\varepsilon_{jk}{}^l \Gamma_{il0}$. So tracing over the ij index pair gives the result.

In order to prove the first part of the theorem we must run through the reverse of the above argument. The Lie derivative of Eq. (9) along $\sigma_a = \frac{1}{f} V_a$ yields the relation $C_{ab}{}^b = -\sigma_a(\log f)$. The rest of the proof is straightforward. $\Box$

Remark. It is worth noting that, if we fix a self-dual backround $\mathcal{M}^4$, Eqs. (8) also propagate the frame deterministically.

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[2] The covariant divergence $\nabla \cdot V$ of a vector field, V, can be obtained from the formula:

$$\mathcal{L}_V v_g = (\nabla \cdot V) v_g$$

clearly from the regular definition of divergence, $\nabla \cdot \sigma_a = \Gamma_a$

5. The Full Yang-Mills Equations and Einstein's Teleparallel Equations

An interesting question arises as to whether some analogue of the above procedure using the reduced *full* YM equations will yield the Einstein vacuum equations or the conformal Einstein vacuum equations.

We have not succeeded in obtaining the full Einstein vacuum equations using only the 4-dimensional diffeomorphism group. Instead, when we proceed by analogy with Sect. 4, we obtain Einstein's equations for his unified field theory based on "absolute parallelism". In this theory one is given not only a metric, but also a global orthonormal frame, and one can work with the connection with zero curvature but non-zero torsion obtained by defining the covariant derivative of a tensor to be the ordinary derivatives of the components of the tensor in the given orthonormal frame.

In this approach we take the $\sigma_a \in LS\,\mathrm{Diff}(\mathscr{M}^4)$. We do not need to use the conformal rescaling of Sect. 4, and so we proceed by directly identifying the σ_a with the γ_a of Eq. (1). This yields the equation:

$$\sigma^a(C_{ab}{}^c) + C_{ab}{}^d C^a{}_d{}^c = 0 \tag{5.1a}$$

for the structure functions as defined in Eq. (11). (Here, again, $V(C_{ab}{}^c)$ denotes the derivative of the functions $C_{ab}{}^c$ along V.) The volume preserving condition on σ_a is equivalent to the further field equations

$$\sigma_a(C_{bc}{}^a) = 0\,. \tag{5.1b}$$

This can be seen as follows. The trace of the Jacobi identity $[\sigma_{[a}, [\sigma_b, \sigma_{c]}]] = 0$ yields

$$\sigma_a(C_{bc}{}^a) = -\sigma_{[b}C_{c]} + C_{bc}{}^d C_d\,,$$

where $C_a = C_{ab}{}^b$. The vanishing of the right-hand side of this equation is the integrability condition for the existence of a function f such that $C_a = \sigma_a(\log f)$. If we define the volume form, v by the condition that

$$v(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = f\,, \tag{5.2}$$

then we have that $\mathscr{L}_{\hat{\sigma}_a} v = 0$. Conversely, if the σ_a's all preserve the volume form v, then $C_a = \sigma_a(\log f)$ follows from taking the Lie derivative of Eq. (5.2) along σ_a. This in turn implies Eq. (5.1b).

Equations (5.1a and b) are the equations originally put forward by Einstein for his unified field theory based on absolute parallelism, see for example Cartan and Einstein (1979) and references therein.

These equations do not give equations on the space-time metric independently of the choice of frame σ_a. These equations therefore describe a "teleparallel" theory in which the frame, σ_a, is thought of as a dynamical physical field which reacts back on the space-time geometry. This back reaction of the frame on the geometry makes it impossible to forget about the choice of frame in our procedure as was possible in the self-dual case (it is a familiar fact, from other field theories, that self dual fields have zero energy momentum tensor, and thus do not react back on the space-time).

The frame, σ_a, provides a parallelism of the space-time. Such theories were considered by Einstein and Cartan as candidates for a unified field theory in which the frame plays the role of the electromagnetic and matter fields. There are many possibilities for such a set of field equations (there are various places in which one can insert "coupling constants"). It is remarkable, therefore, that we obtain precisely the field equations preferred, and originally put forward, by Einstein.

6. The Full Vacuum Equations

We will now attempt to change the formalism in order to incorporate vacuum fields.

In order to make our considerations independent of the choice of frame for our space-time we can extend the diffeomorphism group of $\mathscr{M}^4$ to the group of automorphisms of a principal $SO(1,3)$ bundle $\mathscr{B}\to\mathscr{M}^4$. The automorphisms of $\mathscr{B}$ are diffeomorphisms of $\mathscr{B}$ which preserve the principal bundle structure. As will be seen, this incorporates frame rotations into the group of diffeomorphisms and thus has the effect of making our considerations independent of the choice of a frame.

Let $\mathscr{B}\to\mathscr{M}^4$ be a principal $SO(1,3)$ bundle over $\mathscr{M}^4$. On $\mathscr{B}$ we have an $SO(1,3)$ action which has the effect of rotating the fibres. The action is determined infinitesimally by the six "vertical" vector fields $D_{ab}=D_{[ab]}$, $a=0,\dots,3$, which satisfy the $SO(1,3)$ Lie algebra relations:

$$[D_{ab}, D_{cd}]=\eta_{a[d}D_{c]b}-\eta_{b[d}D_{c]a}\,,$$

where η_{ab} is the Lorentz metric diag$(1,-1-1-1)$.

The gauge group for our theory will be Aut$(\mathscr{B})$, the infinite group of maps from $\mathscr{B}$ to itself which preserve the $SO(1,3)$ action. The Lie algebra of Aut$(\mathscr{B})$, LAut$(\mathscr{B})$, consists of vector fields, V, which preserve the $SO(1,3)$ action so that

$$[D_{ab}, V]=0\,. \tag{6.1}$$

Such vector fields have a well defined projection down to $\mathscr{M}^4$ and so determine an infinitesimal diffeomorphism of $\mathscr{M}^4$. They also have a (non-canonical) vertical part which can be thought of as providing an infinitesimal frame rotation.

As the "γ_a" of Sect. 2 we will consider a collection of 4 such vector fields D_a. However, since the index a should be thought of as a Lorentz index, we shall require that D_a transform as a covector under the Lorentz group so that we shall require

$$[D_{ab}, D_c]=\eta_{c[a}D_{b]}$$

instead of the trivial transformation law (6.1). (If we were not to require this, we would have two distinct Lorentz groups in the theory, one acting on the index of D_a and one acting on $\mathscr{B}$.)

We can now use the D_a to identify $\mathscr{B}$ with the bundle of orthonormal frames of $\mathscr{M}^4$, where $\mathscr{M}^4$ is endowed with the metric determined by the push down from $\mathscr{B}$ of $\eta^{ab}D_a\otimes D_b$. The D_a determine the horizontal subspaces on $\mathscr{B}$ of some connection (possibly with torsion) compatible with the metric. The curvature, R_{abcd}, and torsion, $T_{ab}{}^c$, of the connection are defined by

$$[D_a, D_b]=T_{ab}{}^cD_c+R_{ab}{}^{cd}D_{cd}\,.$$

With the identification of D_a with the γ_a of Sect. 2 we have the field equations:

$$[D^a, [D_a, D_b]] = 0 .$$

This yields after a straightforward calculation

$$R_{ab} = -\nabla_c T^c{}_{ab} - T_{ca}{}^d T^c{}_{db} , \tag{6.2a}$$

$$\nabla_d R^d{}_{abc} = T^d{}_{ae} R^e{}_{dbc} , \tag{6.2b}$$

with $R_{ab} = R^d{}_{abd}$, the Ricci tensor and ∇_c the covariant derivative.

If the connection is torsion free, then (6.2) become the standard vacuum equations, $R_{ab} = 0$. Unfortunately we have not, as yet, been able to articulate this condition in the spirit of the Yang-Mills theory which motivated these considerations. However it seems likely that these are a decent set of equations in the sense that there are enough equations to propagate both the metric and the torsion deterministically.

To our knowledge, the field Eqs. (6.2) have not been previously discussed in the literature; they do not appear to be the Einstein Cartan equations or their subsequent generalizations.

7. Conclusions

We have shown how, starting from the reduced YM equations on Minkowski (or Euclidean) space with infinite dimensional gauge groups, one can obtain non-linear field equations, for example, as demonstrated previously, the "teleparallel" Eqs. (5.1) or the generalized Einstein equations, (6.2). There are many other equations one can obtain in this fashion. In particular if one extends the group $\mathrm{Aut}(\mathscr{B})$ by incorporating also the group of frame rotations of some vector bundle on $\mathscr{M}^4$, then one obtains the minimally coupled Yang-Mills equation on the backround satisfying Eq. (6.2). Unfortunately the Yang-Mills field does not back react on the geometry at all.

Though the meaning of our "derivation" of these equations is slightly obscure, they nevertheless appear to have a certain naturality, the Bianchi identities $\varepsilon^{abcd}[D_a, [D_b, D_c]] = 0$ can be written as

$$[D^a, [D_a, D_b]^*] = 0 ,$$

the $*$ indicating Hodge dual on the ab index pair, and thus our equations

$$[D^a, [D_a, D_b]] = 0$$

are in some sense dual to the Bianchi identities, a known feature of the Yang-Mills equations.

An alternative method of generalizing the ideas presented here would be to avoid using the full reduction of the YM equations as we did in Eqs. (1) and (2), i.e. leave in some or all of the Minkowski space dependence but still work with these infinite dimensional groups. One could also investigate, for example, reduced SD or full Yang-Mills equations based on Kac-Moody or Virasoro Lie algebras.

There appear to be many interesting questions that can be asked of the source free Yang-Mills equations.

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