

Quantum Field Theories of Vortices and Anyons

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Abstract. We develop the quantization of topological solitons (vortices) in three-dimensional quantum field theory, in terms of the Euclidean region functional integral. We analyze in some detail the vortices of the abelian Higgs model. If a Chern–Simons term is added to the action, the vortices turn out to be “anyons,” i.e. particles with arbitrary real spin and intermediate (Θ) statistics. Localization properties of the interpolating field, scattering theory and spin-statistics connection of anyons are discussed. Such analysis might be relevant in connection with the fractional quantum Hall effect and two-dimensional models of High T_c superconductors.

1. Introduction

In this paper we consider the quantum theory of vortices in three space-time dimensions. Following the strategy of [1–2], we construct Euclidean Green functions of local order fields and vortex fields in terms of Euclidean region functional integrals. In these correlation functions the basic Euclidean fields of the theory are distributional sections of some non-trivial bundle over punctured 3-d Euclidean space-time, with those space-time points deleted where a vortex field is inserted.

We discuss, in detail, the non-compact abelian Higgs model, but the main strategy applies to a large class of models with vortices. The quantum vortex fields of these models can be localized in bounded regions and satisfy the dual algebra with the Wilson loop operators. In the Higgs phase, vortex fields of unit vorticity couple the vacuum to a massive stable one particle state; [for a rigorous proof on the lattice see [1]]. If a Chern–Simons term is added to the action of the model, then vortex fields still exist, carry a fractional electric charge proportional to the coefficient of the Chern–Simons term and cannot be localized in bounded regions. According to a general analysis of Buchholz and Fredenhagen [3], the electrically charged vortex fields can be localized in space-like cones in 3-d Minkowski space-time. Those with unit vorticity couple the vacuum to a massive stable one-particle state; with an “extended particle” structure.

Such particles are called “anyons”, according to a terminology introduced by

Wilczek [4]: they can have “any” real spin depending on the coefficient of the Chern–Simons term. Using the transformation properties of anyon states under rotations, we can show that “anyons” obey anomalous statistics (the so-called Θ statistics) and establish a spin-statistics connection. The analysis of vortices and anyons performed in this paper is non-perturbative and does not rely on the semi-classical approximation.

From the point of view of Relativistic Quantum Field Theory, the discussion of vortices is essentially of academic interest, since it refers to $d = 2 + 1$ dimensions. However, it may have some interest in models of Solid State Physics which share properties similar to those of the models analyzed here. Besides the obvious relation between vortices of the Higgs model and the Abrikosov vortices in type II superconductors, [5] an analogy can be found between the anyons discussed in this paper and the lowest lying excitations appearing in the Fractional Quantum Hall Effect [6]. Actually the action discussed here has been proposed in [7] as a phenomenological action to describe the excitations of the F.Q.H.E. Furthermore, speculations on the role of anyons in high T_c superconductors appeared recently in the literature [8]

The organization of our paper is as follows: In Sect. 2, we briefly review the classical theory of vortices. In Sect. 3, we recall some basic facts about bundles and (de Rham) currents to make the paper self-contained. [The currents are needed in a “distributional” generalization of the gauge field involved in the functional integral representation of vortex-correlation functions.] In Sect. 4, we discuss the vortices of the non-compact abelian Higgs model in the formal continuum limit. At the end of the section we show how this construction is related to the rigorous one performed in the lattice approximation in [1] and comment on possible generalizations. In Sect. 5, we discuss the electrically charged vortices of the model with Chern–Simons term, in the formal continuum limit. In Sect. 6, we analyze the localization properties of anyons, construct asymptotic states describing free anyons and prove that anyons obey the anomalous Θ statistics. In Sect. 7, we analyze rigorously the lattice model with Chern–Simons term, we give a defect representation of anyon correlation functions and prove that anyon fields with unit vorticity couple the vacuum to a stable massive one-particle state.

2. Classical Vortices

In this section we briefly recall the classical vortex solutions of the non-compact abelian field theory in three dimensions. The Lagrangian density of the model is given by

$$\mathcal{L} = \frac{1}{2e^2}|dA|^2 + \frac{1}{2}|(d - iA)\phi|^2 - \lambda(|\phi|^2 - 1)^2, \quad (2.1)$$

where e , the electric charge, and λ are positive constants, ϕ is a complex scalar field and A is a real gauge field, i.e.

$$A = A_\mu(x)dx^\mu, \quad A_\mu(x) \in \mathbf{R}.$$

The energy of a configuration (A, ϕ) is given by

$$E(A, \phi) = \int d^2\vec{x} \Theta_{00}[A, \phi](x^0, \vec{x}),$$

where $\Theta_{\mu\nu}$ is the stress-energy tensor corresponding to \mathcal{L} . A configuration (A, ϕ) is called static iff it is time-independent, and $A_0 = 0$. For static configurations

$$E(A, \phi) = \int_{x_0=0} d^2\vec{x} (-\mathcal{L}). \tag{2.2}$$

Assuming that a static configuration satisfies the following boundary condition:

$$\begin{aligned} \lim_{R \rightarrow \infty} \sup_{|\vec{x}|=R} |1 - |\phi(\vec{x})|| &= 0, \\ |\vec{x}|^{1+\delta} |(d - iA)\phi|(\vec{x}) &\leq \text{const.}, \end{aligned} \tag{2.3}$$

for some positive δ , then the gauge orbit $[A, \phi]$ defines a homotopy class in $\pi_1(S^1) \cong \mathbf{Z}$. This class coincides with the homotopy class $[\phi]$ of the maps

$$\left. \begin{array}{l} \phi(\vec{x}) \\ |\phi(\vec{x})| \end{array} \right|_{|\vec{x}|=R} : S^1 \rightarrow S^1 \tag{2.4i}$$

for R sufficiently large [9]. The integer $[\phi] = q$ is called vortex number or vorticity.

Alternatively, the vorticity is also given by

$$q = \frac{1}{2\pi} \lim_{R \rightarrow \infty} \int_{|\vec{x}| \leq R} dA. \tag{2.4ii}$$

It is conjectured that the homotopy class $[A, \phi]$ can be defined under the simple assumption of finiteness of energy [9]. Hence, classically, static finite-energy configurations (A, ϕ) fall into disjoint homotopy classes labelled by the vorticity $q \in \mathbf{Z}$.

The configurations minimizing the energy in the class $q = +1, -1$ are called, respectively, vortex and antivortex. These configurations are solutions of the classical equations of motion, and their existence is discussed in [9]. An important feature of these solutions is that $|dA|, |1 - |\phi||, |(d - A)\phi|$ are large only in a compact region of space outside of which they converge to 0 exponentially fast. Therefore the energy density of vortices is negligibly small outside a compact region of space. For this reason one can think of vortices and anti-vortices as (extended) classical particles. In Sect. 4, we show how to construct their quantum counterparts, namely local quantum fields which couple the physical vacuum to one-vortex, or one-antivortex state.

The model (2.1) does not admit electrically charged vortex solutions [10]. However, if we add a so-called Chern–Simons term, $\mathcal{L}_{\text{c.s.}}$, to the Lagrangian density (2.1), where

$$\mathcal{L}_{\text{c.s.}} = \frac{\mu}{4\pi} A \wedge dA, \tag{2.5}$$

it has been shown [11] that time-independent, finite-energy solutions of the corresponding equations of motion exist for which, however, $A_0 \neq 0$.

From the equations of motion

$$\partial_i F^{i0} + \frac{\mu}{2\pi} \varepsilon^{0ij} F_{ij} = J^0, \quad (2.6)$$

it follows by integration that for such solutions the electric charge, q_e is related to the vorticity by

$$q_e = \mu q. \quad (2.7)$$

These charged classical vortex solutions, too, have a quantum counterpart. The quantum particle to which they correspond has been called “anyon” [4], since, as we shall see, it can have “any” spin [and “any” statistics parameter] depending on the value of μ .

We close this section by reviewing the vortex solutions of the simplest non-abelian model, so $SO(3)$ Higgs model. A Lagrangian which admits vortex solutions to the equation of motion is obtained by coupling an $SU(2)$ gauge field $A = \sum_{a=1}^3 \tau^a A^a$ to two Higgs field ϕ_1, ϕ_2 , in the adjoint representation of $SU(2)$. One may choose

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \|F^2\| + \frac{1}{2} \|\nabla_A \phi_1\|^2 + \frac{1}{2} \|\nabla_A \phi_2\|^2 \\ & - (\lambda_1 (\|\phi_1\|^2 - \eta_1^2)^2 + \lambda_2 (\|\phi_2\|^2 - \eta_2^2)^2 + \lambda' (\phi_1^a \cdot \phi_2^a)^2), \end{aligned} \quad (2.8)$$

where $\|(\cdot)\|$ is the norm induced by the inner product $(\cdot, \cdot) = \text{Tr}(\cdot \cdot)$,

$$F = (dA + A \wedge A), \quad \nabla_A = (d + iA).$$

Vortex solutions correspond [12] to non-trivial element of

$$\pi_1(SO(3)) \cong \pi_1(SU(2)/\mathbf{Z}_2) \cong \mathbf{Z}_2.$$

[In this model, too, the addition of a Chern–Simons term,

$$\mathcal{L}_{\text{C.S.}} = \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (2.9)$$

permits the existence of charged vortices [13].] Generalizations to $SU(N)/\mathbf{Z}_N$ are, of course, possible, at both, the classical and the quantum level.

3. Mathematical Preliminaries

In a geometrical framework, classical matter fields are interpreted as sections of a fibre bundle, and gauge fields as connections on a principal fibre bundle [14]. In our construction of quantum kinks in two dimensions [2], we have seen that the quantum field theory of solitons involves the dual of the space of C^∞ sections, i.e. a space of section distributions.

To follow closely the analogy with quantum kinks, we require, in our construction of quantum vortices, a distribution generalization of connections, A , and matter fields, ϕ . The proper mathematical framework for such a generalization is given by the theory of (de Rham) currents.

In this section, we thus collect some basic definitions and properties of fibre bundles and currents, which are needed later on. The following definitions and

results can be found in standard textbooks (e.g. [15]) and are summarized here merely for the reader's convenience.

3.1. Fibre Bundles and Connections. Let M be a topological space, and let G be a group acting continuously on a topological space, F . Then a fibre bundle, W , with structure group G and fibre F over M can be constructed by choosing an open covering of M , $\mathcal{U} = \{U_i\}_{i \in I}$, and continuous transition functions:

$$g_{ij}: U_i \cap U_j \rightarrow G, \quad (3.1)$$

satisfying $g_{ij}g_{jk}g_{ki}(x) = 1_G$, for

$$x \in U_i \cap U_j \cap U_k. \quad (3.2)$$

One takes the disjoint union, $\bigcup_i U_i \times F$, and identifies $(x, f) \in U_i \times F$ with $(x, g_{ij}(x)f) \in U_j \times F$ for $x \in U_i \cap U_j$. The identification space is the fibre bundle W .

Assume, for simplicity, that the covering \mathcal{U} is such that every multiple intersection $U_{i_1} \cap \dots \cap U_{i_n}$ of open sets in \mathcal{U} is contractible. Then two fibre bundles, constructed as indicated above, are called *isomorphic* iff, for every U_i , there exists a continuous function

$$g_i: U_i \rightarrow G \quad (3.3)$$

such that the transition functions of the two fibre bundles, $\{g_{ij}\}, \{g'_{ij}\}$, are related by

$$g'_{ij} = g_i^{-1} g_{ij} g_j. \quad (3.4)$$

A fibre bundle, P , in which $F = G$, with G acting on itself by left translation, is called a *principal bundle*. A fibre bundle in the isomorphism class of P is called a bundle associated to P . An isomorphism class of fibre bundles with structure group G is called a G -bundle.

The following classification theorem is useful.

Theorem 3.1.

- a) $\{U(1)\text{-bundles over } M\} \cong H^2(M, \mathbf{Z})$.
- b) Let M be a three-dimensional manifold, \tilde{G} a simply connected compact Lie group, and \mathbf{Z}_G a subgroup contained in the centre of \tilde{G} . Let $G = \tilde{G}/\mathbf{Z}_G$. Then $\{G\text{-bundles over } M\} \cong H^2(M, \mathbf{Z}_G)$.

Here $H^2(M, \mathbf{Z})$, $(H^2(M, \mathbf{Z}_G))$ denotes the group of closed 2-cochains with coefficient in $\mathbf{Z}(\mathbf{Z}_G)$, modulo exact 2-cochains.

If M is simply connected, then $H^2(M, \mathbf{Z})$ is isomorphic to the subgroups of $H^2_{\text{deRham}}(M)$ given by the group of closed two forms with integral periods, modulo exact forms. We recall that a k -form, α , is said to have integral periods if, for every k -cycle, c_k , in M ,

$$\int_{c_k} \alpha \in \mathbf{Z}.$$

The map between $H^2(M, \mathbf{Z})$ and $H^2_{\text{deRham}}(M)$ is obtained as follows. If c^2 denotes a closed 2-cochain, then we associate a 2-form α with c^2 ,

$$H^2(M, \mathbf{Z}) \ni [c^2] \rightarrow [\alpha] \in H^2_{\text{deRham}}(M), \quad (3.5)$$

such that, for all α -cycles c_2 ,

$$\int_{c_2} \alpha = c^2(c_2).$$

If, moreover, M is an orientable Riemannian manifold, then the cohomology class $[\alpha]$ has a canonical representative, the *harmonic* form.

Let d denote the exterior differential on M , and $*$ the Hodge dual. We set $\delta \equiv \pm *d*$, and define the Laplacian by $\Delta \equiv d\delta + \delta d$. A form α_h is called harmonic if $\Delta\alpha_h = 0$, which, for compact M , is equivalent to $\delta\alpha_h = d\alpha_h = 0$. Therefore, if M is simply connected, a $U(1)$ -bundle over M is completely specified by giving a harmonic 2-form with integral periods. A generalization to \mathbf{Z}_G (or to non-simply connected M) requires the *Allendoerfer–Eells* forms [16].

Next, we introduce connections. A connection \tilde{A} on a principal G -bundle is a collection of 1-forms $\{A_i\}$ on \mathcal{U} with values in the Lie algebra of G , $\text{Lie } G$, such that for $x \in U_i \cap U_j$,

$$A_j(x) = [g_{ij}^{-1}dg_{ij} + g_{ij}^{-1}A_i g_{ij}](x). \tag{3.6}$$

Given a connection, \tilde{A} , we define its curvature $F = F(\tilde{A})$ as the collection of 2-forms,

$$F_i = dA_i + A_i \wedge A_i \tag{3.7}$$

on \mathcal{U} . Notice that if G is abelian

$$F_i(x) = F_j(x), \quad \text{for all } x \in U_i \cap U_j,$$

i.e. $\{F_i\}$ defines a closed 2-form F .

For $G = U(1)$, the map between the set of $U(1)$ -bundles and $H_{\text{deRham}}^2(M)$ can be realized, explicitly, as follows: Let P denote a $U(1)$ -principal bundle and \tilde{A} a connection on P . Then the elements in $H_{\text{deRham}}^2(M)$ corresponding to P are given by $(i/2\pi)F(\tilde{A})$.

The space of smooth connections of a bundle P , $\mathcal{A} \equiv \mathcal{A}(P)$, is an affine space modelled on the vector space $\Gamma^1(\text{ad } P)$ whose elements are collections of Lie G -valued 1-forms $\{A_i\}$ with the patching property,

$$A_j(x) = g_{ij}^{-1}(x)A_i(x)g_{ij}(x), \quad \text{for } x \in U_i \cap U_j.$$

Therefore, given any connection, A_0 , on P , any other connection \tilde{A} can be written as

$$\tilde{A} = A_0 + A, \tag{3.8}$$

where $A \in \Gamma^1(\text{ad } P)$. Notice that if G is abelian, then A is a *globally defined 1-form*.

On the space, \mathcal{A} , of connections the *gauge group*, \mathcal{G} , acts whose elements are collections of G -valued functions, $\{g_i\}$, with the patching property

$$g_j(x) = g_{ij}(x)^{-1}g_i(x)g_{ij}(x), \quad \text{for } x \in U_i \cap U_j. \tag{3.9}$$

Let G be as in Theorem 3.1, and denote by \mathcal{G}_0 the subgroup of \mathcal{G} satisfying

$$g(x_0) = 1_G, \tag{3.10}$$

for some fixed point, x_0 , in M ; (if M is non-compact we can interpret (3.10) as a

boundary condition at infinity, i.e., formally, $x_0 = \infty$). Then \mathcal{A} is a principal fibre bundle over the orbit space $\mathcal{M} = \mathcal{A}/\mathcal{G}_0$ with structure group \mathcal{G}_0 [17]. The base space \mathcal{M} is usually viewed as the configuration space of classical (pure) gauge theories with structure group G .

If, in addition, matter fields (= sections in $\Gamma(W)$) are involved, where W is an associated bundle whose fibre carries a representation, U , of G , then the classical configuration space can be taken as $(\mathcal{A} \times \Gamma(W))/\mathcal{G}_0$, where \mathcal{G}_0 acts on $(A, \phi) \in \mathcal{A} \times \Gamma(W)$ by

$$g \in \mathcal{G}: (A_i, \phi_i) \rightarrow (g_i^{-1} A_i g_i + g_i^{-1} dg_i, U(g_i^{-1}) \phi_i).$$

In the abelian models, with $G = U(1)$, the vector space, $\Gamma^1(\text{ad } P)$, on which \mathcal{A} is modelled, is simply the space of (real) 1-forms, $\Lambda^1(M)$. If M is connected and simply connected, then $\mathcal{G}_0 \cong$ space of exact 1-forms, $d\Lambda^0(M)$. In fact, using connectedness of M ,

$$\mathcal{G}/U(1) \cong \mathcal{G}_0,$$

where the isomorphism is obtained by considering the map

$$g(\cdot) \in \mathcal{G} \rightarrow g(\cdot)/g(x_0) \in \mathcal{G}_0.$$

One also has

$$\mathcal{G}/U(1) \cong d\Lambda^0(M),$$

where the isomorphism is obtained by considering

$$\mathcal{G} \ni g \rightarrow g^{-1} dg \in d\Lambda^0(M),$$

where $g^{-1} dg$ which is always closed is, in fact, exact, because $H_{\text{deRham}}^1(M) = 0$, by the assumption that M is simply connected. Moreover,

$$\frac{\Lambda^1(M)}{d\Lambda^0(M)} \cong d\Lambda^1(M),$$

since

$$d: \alpha \in \Lambda^1(M) \rightarrow d\alpha \in d\Lambda^1(M),$$

and injectivity follows, because $H_{\text{deRham}}^1(M) = 0$. Therefore, symbolically,

$$\mathcal{M} = \frac{\mathcal{A}}{\mathcal{G}_0} = \frac{\tilde{\mathcal{A}}_0 + \Lambda^1(M)}{d\Lambda^0(M)} \sim F_0 + d\Lambda^1(M),$$

and one observes that \mathcal{M} is still an affine space.

3.2 Currents. In order to define a quantum measure in scalar models, like $\lambda\phi^4$, one needs to enlarge the space of configurations from smooth functions to distributions. Similarly, in gauge theory one must enlarge the space of connections, \mathcal{A} , and the quotient space, \mathcal{M} , to a “distributional completion.” For abelian models, with $G = U(1)$, both spaces, \mathcal{A} and \mathcal{M} , are modelled on a space of forms, as we have seen. A natural distributional completion of the space of k -forms is well known to mathematicians and is called the space of (deRham) k -currents.

We first give the formal definition of a k -current and then we exhibit its relation with distributions, forms and chains.

Definition 3.2. On a d -dimensional smooth manifold, M , a k -current (or current of degree k , or of rank k), is a functional, $T(\alpha)$, defined on the space of all smooth $(d - k)$ -forms, α , with compact support, which is linear and which is continuous in the sense of distributions.

Currents of rank 0 are ordinary distributions. If $T_{i_1, \dots, i_k} (i_1 < i_2 < \dots < i_k)$ are $\binom{d}{k}$ currents of rank 0 defined in the domain, D , of a coordinate system, $\{x^1 \dots x^d\}$, then, according to the definition,

$$T = \sum_{i_1 < i_2 < \dots < i_k} T_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

is a k -current defined in D , and any k -current in D can be represented by such an expression.

Therefore every current can be represented by a differential form whose coefficients are distributions.

The notion of current generalizes the notions of form and of chain.

In fact a k -form, A , defines a k -current by setting

$$A(\alpha) = \int_M \alpha \wedge A, \tag{3.11}$$

where α is a $(d - k)$ -form, and a k -chain, c_k , defines a $(d - k)$ -current, c_k , by setting

$$c_k(\alpha) = \int_{c_k} \alpha \tag{3.12}$$

for every $\alpha \in \Lambda^k(M)$.

Given a k -current, T , we define a $k + 1$ current by

$$dT(\alpha) \equiv (-1)^{k+1} T(d\alpha). \tag{3.13}$$

Note that if T is given by a form, A , then $dT = dA$, and if T is equal to a $(d - k)$ -chain c_{d-k} , then

$$dc_{d-k} = (-1)^{k+1} \partial c_{d-k}. \tag{3.14}$$

If M is a Riemannian manifold, then one can generalize the Hodge decomposition to currents. We denote by $*$ the Hodge dual, defined for currents in an obvious way, set $\delta = \pm * d *$, with d defined as in (3.13), and define the Laplacian $\Delta = \delta d + d \delta$.

Any current T can be decomposed as

$$T = \delta dGT + d\delta GT + HT, \tag{3.15}$$

where HT is called the harmonic part of T and satisfies

$$\Delta HT = 0.$$

and G is the Green function for Δ . If M is compact this is equivalent to

$$dHT = \delta HT = 0.$$

The following theorem ensures that all information on cohomology is already contained in the subspace of currents given by smooth forms.

Theorem 3.3 [15b]

- a) Each closed current is cohomologous to a form.
- b) If, given a form α , there exists a current T such that $\alpha = dT$, then there exists a form β such that $\alpha \doteq d\beta$.
- c) A harmonic current is equal to a C^∞ form, the harmonic form.

4. Quantum Vortices

We now turn to the Quantum Field Theory of the non-compact abelian Higgs model. In order to associate a Euclidean quantum field theory with the classical action functional

$$S(A, \phi) = \int_{\mathbf{R}^3} d^3x \left\{ \frac{1}{2e^2} (dA)^2(x) + \frac{1}{2} |(d - iA)\phi|^2(x) + \lambda |\phi|^4(x) - \frac{1}{2} m^2 |\phi|^2(x) + \text{counterterms} \right\}. \tag{4.1}$$

the formal prescription is to construct a “probability measure”

$$d\mu(A, \phi) = \frac{1}{Z} e^{-S(A, \phi)} D\phi DA, \tag{4.2}$$

where $D\phi, DA$ are formal Lebesgue measures on $\prod_{x \in \mathbf{R}^3} \mathbf{C}, \prod_{x \in \mathbf{R}^3} \mathbf{R}^3$, respectively.

Actually, since $S(A, \phi)$ is gauge-invariant, one should interpret the measure (4.2) to be defined on gauge-equivalence classes of configurations (A, ϕ) , i.e. on \mathcal{M} , or add some gauge fixing term to the action. Formula (4.2) is the Euclidean Gell–Mann–Low formula. [Henceforth we omit the explicit reference to *counterterms* in the action.]

In a series of papers [18], a rigorous meaning has been given to the measure (4.2), with a mass term for A and a gauge fixing term, by means of a lattice approximation. The renormaliation theory for (4.1), (4.2), i.e. a choice of counterterms is also contained in [18].

The lattice measures are shown to converge to the continuum limit, using a sequence of block-spin renormalization transformations. A similar technique is combined in [19] with cluster expansions, to construct the continuum limit of the lattice measure with a compact action for A . Such methods appear to also suffice to rigorously establish the so-called *Higgs mechanism*. That is, for λ and e small, all gauge-invariant correlation functions exhibit exponential clustering. Heuristically, this is understood by a polar decomposition of ϕ :

$$\phi(x) = \rho(x) e^{i\Theta(x)}; \quad \rho(x) \in \mathbf{R}_+; \quad \Theta(x) \in (-\pi, \pi), \tag{4.3}$$

and the choice of the unitary gauge:

$$\Theta(x) = 0. \tag{4.4}$$

In this gauge the fields A and ϕ acquire masses

$$m_A \sim \frac{me}{\sqrt{8\lambda}}, m_\phi \sim m. \quad (4.5)$$

The apparently massless rotational degrees of freedom described by Θ are suppressed by the gauge condition (4.4). Therefore one obtains exponential clustering, at least heuristically.

Remark 4.1. On the unit lattice one can prove [20] that the model with non-compact action (4.1) has also a Coulomb phase with massless photons for λ and/or e large enough.

Even if successful in exhibiting clustering (one of the main inputs in our construction of vortices), the lattice approximation involved in the strategy of [18, 19] somewhat obscures the geometry of the gauge fields (see however [21]), which plays a key role in our definition of vortex correlation functions.

Our approach is then as follows: We describe the construction of neutral [and charged = anyons] quantum vortices in the formal continuum limit, using the measure (4.1), (4.2) to exhibit the geometrical structure of our construction. The construction of vortex sectors indicated for the continuum Higgs model in this paper can be justified, rigorously, for the lattice Higgs model. At the end of the section, we make some comments about the lattice model and the requirements needed to completely justify our discussion in the continuum.

Euclidean Green functions of the non-compact Abelian Higgs model can be defined as expectation values in the measure corresponding to the formal expression (4.2) of monomials of gauge invariant observables such as $|\phi^2|; (z)$ and (renormalized) Wilson loops (see e.g. [12]):

$$W_\alpha(\mathcal{L}) = \lim_{\varepsilon \downarrow 0} e^{C(\varepsilon, \alpha)|\mathcal{L}|} \exp(i\alpha \oint_{\mathcal{L}} A_\varepsilon). \quad (4.6)$$

Here $::$ denotes the normal ordering with respect to the free complex gaussian measure of mass $\mathbf{1}$, \mathcal{L} is a loop, ε a lattice cutoff, α a real number and $C(\varepsilon, \alpha)$ a constant $\propto \alpha^2$ which diverges when $\varepsilon \searrow 0$.

A relativistic quantum field theory can be reconstructed from these Euclidean Green functions, assuming they satisfy the Osterwalder–Schrader axioms in the form given in [23–24]. The Hilbert space of states, \mathcal{H}_0 , obtained via O.S. reconstruction turns out to be the vacuum sector of the model.

In the Higgs phase, however, the physical state space is much larger than \mathcal{H}_0 . One can in fact define a vorticity operator Q , the quantum counterpart of the vorticity number (2.4ii), and one can prove (see Eq. (4.29)) that

$$Q\mathcal{H}_0 = 0, \quad (4.7)$$

i.e. all states in \mathcal{H}_0 have total vorticity zero.

As suggested by the classical vortex solutions, however, there exist physical states $|v\rangle$ of finite total energy which are eigenvectors of the vorticity operator Q corresponding to integer eigenvalues. By Eq. (4.7) they cannot belong to \mathcal{H}_0 .

Our purpose is now to construct a local field $v_q(x)$ the quantum vortex field

operator, carrying a vorticity of $q \neq 0$, out of which states of vorticity q can be constructed.

The strategy is the same as the one used in [2] to construct quantum kinks: we directly construct Euclidean Green functions

$$S_{lmn}(x_1, q_1, \dots, x_l, q_l, z_1, \dots, z_m, \alpha_1, \mathcal{L}_1, \dots, \alpha_n, \mathcal{L}_n)$$

for vortex fields and local gauge invariant fields. From $\{S_{l,m,n}\}$ we obtain the vortex sectors \mathcal{H}_q , satisfying $Q\mathcal{H}_q = q\mathcal{H}_q$, representation of the Poincaré group on \mathcal{H}_q , and local field operators $v_q(x), \hat{\phi}^2(z), \hat{W}_\alpha(\mathcal{L})$, by the Osterwalder–Schrader reconstruction theorem. We proceed as in [2], where more details can be found.

Construction of Vortex Correlation Functions. We choose n points, $\underline{x} = \{x_1 \dots x_n\}$ in Euclidean space-time, \mathbf{R}^3 , and define

$$M_{\underline{x}} = \mathbf{R}^3 \setminus \{x_1 \dots x_n\}. \quad (4.8)$$

The manifold $M_{\underline{x}}$ is simply connected and its second cohomology group is given by

$$H^2(M, \mathbf{Z}) = \underbrace{\mathbf{Z} \oplus \mathbf{Z} \oplus \dots \oplus \mathbf{Z}}_{n\text{-times}}.$$

Hence, non trivial $U(1)$ bundles exist on $M_{\underline{x}}$, by the classification theorem 3.1. Such bundles are uniquely characterized by a set of integers $\underline{q} = \{q_1 \dots q_n\}$ and the corresponding harmonic form

$$\begin{aligned} F_h(\underline{x}, \underline{q}) &= 2\pi \sum_i q_i * d\Delta^{-1} \delta_{x_i}, \\ &\equiv \sum_i 2\pi q_i v_{\nu\rho}^\mu \left(\int \frac{\partial}{\partial x^\mu} \Delta^{-1}(x, y) \delta^{(3)}(x_i - y) d^3 y \right) dx^\nu \wedge dx^\rho. \end{aligned} \quad (4.9)$$

From now on we consider only non-vanishing q_i satisfying the neutrality condition $\sum_i q_i = 0$. (If $\sum_i q_i \neq 0$, the classical action diverges.) We call such principal bundles *vortex bundles* and we denote them by $P(\underline{x}, \underline{q})$.

As a remarked in Sect. 3, any closed form, F_0 , cohomologous to $F_h(\underline{x}, \underline{q})$ is the curvature of a connection A_0 of the principal $U(1)$ bundle $P(\underline{x}, \underline{q})$.

We define a preliminary form of the action functional on the connections, \tilde{A} , of the principal bundle $P(\underline{x}, \underline{q})$ and the sections, ϕ , of the associated bundle $E(\underline{x}, \underline{q})$, with fiber \mathbf{C} , by:

$$\begin{aligned} S_{\lambda q}^{(0)}(\tilde{A}, \phi) &= \int_{M_{\underline{x}}} d^3 x \left\{ \frac{1}{2e^2} (F(\tilde{A}))^2(x) + \frac{1}{2} |\nabla_{\tilde{A}} \phi|^2(x) + V(|\phi|)(x) \right\} \\ &= \int_{M_{\underline{x}}} d^3 x \left\{ \frac{1}{2e^2} (F_0 + dA)^2(x) + \frac{1}{2} |\nabla_{A+A_0} \phi|^2(x) + V(|\phi|)(x) \right\}, \end{aligned} \quad (4.10)$$

where $V(\cdot)$ is the polynomial

$$V(x) = \lambda x^4 - m^2 x^2.$$

Notice that $|\nabla_{A+A_0} \phi|^2(x)$ does not depend on the choice of A_0 , for fixed F_0 , due

to gauge invariance. As it stands the functional (4.10) is not well defined, due to the infinite self energy of the curvature F_0 .

To compute the necessary counterterm let us take

$$F_0 = F_h \equiv F_h(\underline{x}, \underline{q}), \tag{4.11}$$

and restrict the integration to the complement of small balls $S_\delta(x_i)$ of radius δ around each $x_i \in \underline{x}$. The cross term

$$\int_{\mathbf{R}^3 \setminus S_\delta(\underline{x})} F_h^* \wedge dA$$

vanishes, since $\delta F_h^* = 0$ and the fields have rapid fall off at ∞ .

The quadratic term in F_h is easily evaluated:

$$\int_{\mathbf{R}^3 \setminus S_\delta(\underline{x})} d^3 y F_h^2(y) = \int_{\mathbf{R}^3 \setminus S_\delta(\underline{x})} d^3 y (2\pi * d\Delta^{-1} \sum_i q_i \delta_{x_i})^2(y) \longrightarrow \sum_{\substack{\delta \downarrow 0 \\ i \neq j}} 4\pi^2 q_i q_j \Delta^{-1}(x_i, x_j) + c(\delta)$$

with

$$c(\delta) = \sum_i 4\pi q_i^2 \int_{\partial S_\delta(x_i)} *d\Delta^{-1} \delta_{x_i} \wedge \Delta^{-1} \delta_{x_i} \longrightarrow \sum_i 4\pi q_i^2 \Delta^{-1}(x_i, x_i)^2. \tag{4.12}$$

[The behaviour at infinity does not cause trouble, because

$$(\Delta^{-1} \delta_{x_i})(y) \sim \frac{1}{y} \quad (d\Delta^{-1} \delta_{x_i})(y) \sim \frac{1}{y^2},$$

hence $\int_{|x|=R} *d\Delta^{-1} \delta_{x_i} \wedge \Delta^{-1} \delta_{x_i}$ is uniformly bounded in R .] Hence we define a regularized action functional by

$$S_{\underline{x}, \underline{q}}^{(\delta)}(\tilde{A}, \phi) = \left\{ \int_{\mathbf{R}^3 \setminus S_\delta(\underline{x})} d^3 x \left[\frac{1}{2e^2} (F_h + dA)^2(x) + \frac{1}{2} |\nabla_{A+A_h} \phi|^2(x) + V(|\phi|)(x) \right] - c(\delta) \right\}. \tag{4.13}$$

For $\sum_i q_i = 0$ we define the (formal) n -vortex correlation functions of the non-compact abelian Higgs model by:

$$\begin{aligned} S_n(x_1, q_1, \dots, x_n, q_n) \\ \equiv S_n(\underline{x}, \underline{q}) &= \lim_{\delta \downarrow 0} \left[\frac{1}{Z} \int_{\mathcal{M}'} D[\tilde{A}] \int_{\mathcal{Q}(E)} D\phi e^{-S_{\underline{x}, \underline{q}}^{(\delta)}(\tilde{A}, \phi)} \right]_{\text{ren}} \\ &\equiv \left[\frac{1}{Z} \int_{\mathcal{M}'} D[\tilde{A}] \int_{\mathcal{Q}(E)} D\phi e^{-S_{\underline{x}, \underline{q}}(\tilde{A}, \phi)} \right]_{\text{ren}} = \left[\frac{1}{Z} \int_{\mathcal{C}'} DA \int_{\mathcal{Q}(E)} D\phi e^{-S_{\underline{x}, \underline{q}}(A+A_h, \phi)} \right]_{\text{ren}}, \end{aligned} \tag{4.14}$$

where: \mathcal{M}' is the distributional completion of the space of gauge orbits of connections (see Sect. 3) for the vortex bundle $P(\underline{x}, \underline{q})$, \mathcal{C}' is the space of equivalence classes of the 1-currents,

$$[A] = \{A' \mid d(A - A') = 0\}, \tag{4.15}$$

$\mathcal{Q}(E)$ is the space of section distributions of the associated bundle $E(\underline{x}, \underline{q})$. The

normalization factor in (4.14) is given by

$$Z = \int_{\mathcal{C}'} DA \int_{\mathcal{D}'_0} D\phi e^{-S(A,\phi)}, \tag{4.16}$$

where \mathcal{D}'_0 is the space of complex distributions in \mathbf{R}^3 . Finally the subscript “ren” indicates a multiplicative renormalization, $\prod^n c(\varepsilon)^{q_i^2}$, where $c(\varepsilon)$ is constant which diverges, as a regulator, ε , is removed.

Remark 4.2. Let $A \subset \mathbf{R}^3$ be a finite volume, and denote by $\mathcal{C}'(A)$ the space of equivalence classes of currents in \mathcal{C}' whose support is contained in A . Then to the formal expression

$$D[A] \exp \left\{ -\frac{1}{2e^2} \int_A (dA)^2(x) d^3x \right\}$$

there corresponds a well defined Gaussian measure on $\mathcal{C}'(A)$ whose covariance is given by

$$C_{A,A} = e^2 (\delta d)_{A,D}^{-1},$$

where the subscript D indicates 0-Dirichlet b.c. at ∂A . This choice of b.c. corresponds to the requirement that A behaves as a pure gauge on ∂A .

Furthermore if $A + A_h$ is a regular form in Δ , to the formal expression

$$D\phi \exp \left\{ \left[-\frac{1}{2} \int_A |\nabla_{A+A_h} \phi|^2 + |\phi|^2 \right] (x) d^3x \right\}$$

there corresponds the product of a Gaussian measure with covariance

$$C_{\phi,A}(A + A_h) = (\nabla_{A+A_h}^\dagger \nabla_{A+A_h} + 1)_{A}^{-1}$$

on $\mathcal{D}(E)(A)$, the space of section distributions in $\mathcal{D}(E)'$ whose support is contained in A , and the determinant

$$\det \left(\frac{\nabla_{A+A_h}^\dagger \nabla_{A+A_h} - 1}{\nabla^\dagger \nabla + 1} \right)^{-1}. \tag{4.17}$$

Here E is the bundle associated to the principal bundle in which A_h is a connection. If $A_h = 0$, then E is the trivial bundle and $\mathcal{D}(E)' = \mathcal{D}'_0$.

Notice that by defining formally (see [1-2]):

$$D(x_1, q_1, \dots, x_l, q_l; F_h) = [e^{-[S_{\Sigma, q}(A + A_h, \phi) - S(A, \phi)]}]_{\text{ren}}, \tag{4.18}$$

one can rewrite (4.14) as

$$\langle D(x_1, q_1, \dots, x_n, q_n) \rangle, \tag{4.18i}$$

where $\langle \cdot \rangle$ denotes the expectation value in the measure (4.2). The variable (4.18) is called a *disorder field*.

A full set of (formal) correlation functions for the non-compact abelian Higgs model is defined by:

$$\begin{aligned} & S_{e,m,n}(x_1, q_1, \dots, x_l, \dots, q_l, z_1, \dots, z_m, \alpha_1, \Sigma(\mathcal{L}_1), \dots, \alpha_n, \Sigma(\mathcal{L}_n)) \\ &= \left\langle D(x_1, q_1, \dots, x_e, q_e; F_h) \prod_{i=1}^m :|\phi|^2:(z_i) \prod_{j=1}^n W_{\alpha_j}(\mathcal{L}_j) \exp \left(\alpha_j \int_{\Sigma(\mathcal{L}_j)} F_h \right) \right\rangle \end{aligned} \tag{4.19}$$

if $\Sigma_i q_i = 0$, and correlation functions with non-vanishing total charge q are defined by introducing a compensating charge q and removing it to infinity [1–2]. In (4.19), $\Sigma(\mathcal{L}_j)$ is a surface whose boundary is given by \mathcal{L}_j . If \mathcal{L}_j is a flat loop, $\Sigma(\mathcal{L}_j)$ is the flat surface whose boundary is given by \mathcal{L}_j ; to simplify the notation we write \mathcal{L}_j in the left-hand side of (4.19) instead of $\Sigma(\mathcal{L}_j)$.

Furthermore we set

$$\begin{aligned} :|\phi|^2(\underline{z}): &\equiv \prod_{i=1}^m :|\phi|^2(z_i):, \\ W_{\underline{z}}(\underline{\mathcal{L}}) &\equiv \prod_{j=1}^n W_{\alpha}(\mathcal{L}_j), \\ D(\underline{x}, q) &\equiv D(x_1, q_1, \dots, x_l, q_l). \end{aligned}$$

The left-hand side of (4.19) is then written as

$$\langle D(\underline{x}, q; F_h) :|\phi|^2(\underline{z}): W_{\underline{z}}(\underline{\mathcal{L}}) \exp \left\{ i \int_{\underline{\Sigma}} F_h \right\} \rangle. \tag{4.20}$$

The correlation functions (4.20) do not change if we substitute F_h by a different curvature F_0 in the same cohomology class. Then, in fact, $F_0 - F_h = d\alpha$, where α is a globally defined 1-form. Performing the change of variables $A \rightarrow A + \alpha$, we obtain:

$$\langle D(\underline{x}, q; F_0) :|\phi|^2(\underline{z}): W_{\underline{z}}(\underline{\mathcal{L}}) \exp \left\{ i \int_{\underline{\Sigma}} F_0 \right\} \rangle = \langle D(\underline{x}, q; F_h) :|\phi|^2(\underline{z}): W_{\underline{z}}(\underline{\mathcal{L}}) \exp \left\{ i \int_{\underline{\Sigma}} F_h \right\} \rangle. \tag{4.21}$$

In view of (4.21) we omit the reference to the field F_h in the notation for the disorder variable, i.e. we write $D(\underline{x}, q)$.

Remark 4.3. We can use the above freedom to choose in (4.19) a field F_0 which coincides with F_h in a neighborhood of \underline{x} and vanishes outside some compact region K .

With this choice of F_0 , 0-Dirichlet b.c. on $[A]$ implies that \tilde{A} behaves asymptotically as a pure gauge.

Ultraviolet Behaviour. Ultraviolet singularities for Wilson loop expectations have been discussed in [22, 23], in particular one expects an upper bound

$$\exp \frac{1}{[d(\underline{\mathcal{L}})]^p}, \quad \text{as } d(\underline{\mathcal{L}}) \downarrow 0, \tag{4.22}$$

where $d(\underline{\mathcal{L}})$ is the minimal distance between any two of the loops and p is positive and arbitrarily small.

Define $F_h = F_h(\underline{x}, q)$ and

$$\left[\int_{M_{\underline{x}}} d^3 y (F_h^2)(y) \right]_{\text{reg}} = \lim_{\delta \downarrow 0} \left\{ \int_{\mathbb{R}^3 \setminus S_{\delta}(\underline{x})} d^3 F_h^2(y) - c(\delta) \right\}, \tag{4.23}$$

then the leading ultraviolet singularity of correlation functions of disorder fields

is expected to come from the term

$$\exp \left\{ -\frac{1}{2e^2} \left[\int_{M_x} F_h^2(x) d^3x \right]_{\text{res}} \right\} = \exp \left\{ -\frac{1}{2e^2} \sum_{i \neq j} \frac{4\pi^2 q_i q_j}{|x_i - x_j|} \right\}. \tag{4.24}$$

The determinant (4.17), renormalized, gives a subleading logarithmic contribution to the exponential as $|x_i - x_j| \downarrow 0$; see [2] for a discussion of an analogous problem in $d = 2$. Equation (4.24) shows that in term of the x 's the correlation functions of disorder fields may have *ultradistributional singularities* like

$$\exp \left(\frac{1}{d(x)} \right) \text{ as } d(x) \downarrow 0, \tag{4.25}$$

where $d(x)$ is the minimal distance between any two x 's.

O.S. Reconstruction Theorem and Vortex Sectors. We now assume that the correlation functions $\{S_{l,m,n}\}$ satisfy the O.S. axioms in the form given in [23]. In particular, the required distributional properties are satisfied if the bounds (4.22) (4.25) hold (together with temperedness in z). Osterwalder–Schrader positivity and clustering will be justified by using a lattice approximation at the end of the section.

We recall that clustering in the x 's in particular, implies that all correlation functions with non-vanishing total charge are zero, since they are defined by removing a compensating charge to infinity. From the reconstruction theorem of [1–2, 23, 24] it then follows that $S_{l,m,n}$ are the Euclidean Green functions of local vortex field operators $v_q(x)$, Wilson loops operators $\widehat{W}_\alpha(\mathcal{L})$ and Higgs field operators $|\widehat{\phi}|^2(z)$ and the physical Hilbert space reconstructed from (4.19), \mathcal{H} , decomposes into orthogonal sectors \mathcal{H}_q ,

$$\mathcal{H} = \bigoplus_{q \in \mathbf{Z}} \mathcal{H}_q. \tag{4.26}$$

We now sketch how the sectors \mathcal{H}_q are defined. Let \mathcal{F}_+ denote the polynomial algebra of euclidean fields generated by the fields: $|\phi|^2(z)$, $W_\alpha(\mathcal{L}) \exp \{i\alpha \int_{\Sigma(\mathcal{L})} F_h\}$ with support in $\{x^0 > 0\}$. Then, to each element $F \in \mathcal{F}_+$, we associate a vector $|F\rangle$ in \mathcal{H}_0 . The scalar product between such states is defined by

$$\langle F|G\rangle = \langle (\Theta F)G\rangle, \tag{4.26i}$$

where Θ is the Osterwalder–Schrader involution: the product of a reflection in the time zero plane, r , and a complex conjugation.

Let φ denote the map

$$\varphi: F \in \mathcal{F}_+ \rightarrow |F\rangle \in \mathcal{H}_0,$$

then $\varphi(\mathcal{F}_+)$ is dense in \mathcal{H}_0 .

Let z and \mathcal{L} belong to the time zero plane, then the field operators $\widehat{\phi}^2(z)$, $\widehat{W}_\alpha(\mathcal{L})$ are defined on $\varphi(F_+)$ by

$$\begin{aligned} \widehat{\phi}(z)|F\rangle &= |:\phi|^2(z):F\rangle, \\ \widehat{W}_\alpha(\mathcal{L})|F\rangle &= |W_\alpha(\mathcal{L})F\rangle. \end{aligned}$$

The sector \mathcal{H}_q is the closure of the set of states

$$v_q(f)|F\rangle$$

with $F \in \mathcal{F}_+$ and f a test function with support in $\{x^0 > 0\}$.

The scalar product between such states is defined by

$$\langle v_q(x)F|v_{q'}(x')G\rangle = \langle D(rx, -q, x', q')(\Theta F)G\rangle. \tag{4.26ii}$$

The decomposition (4.26) then follows easily from the vanishing of charged correlation functions, in fact

$$\langle v_q(f)F|v_{q'}(f')F'\rangle = 0 \quad \text{if } q \neq q'.$$

The space \mathcal{H} carries a unitary representation of the universal covering of the Poincaré group which factorises on the sectors \mathcal{H}_q , (see [1–2]). By clustering, the vacuum $\Omega \equiv |1\rangle$ is the unique Poincaré invariant state in \mathcal{H} , and it belongs to \mathcal{H}_0 . According to standard definitions [25], the sector \mathcal{H}_0 is a vacuum sector and the sectors $\mathcal{H}_q, q \neq 0$, are *soliton sectors*: they are called *vortex sectors*.

We now show that $v_q(x)$ and $\hat{W}_\alpha(\mathcal{L})$ satisfy the *dual algebra*: for \mathcal{L}, x in the $t = 0$ plane, $x \notin \text{supp } \mathcal{L}$,

$$\hat{W}_\alpha(\mathcal{L})v_q(x) = \begin{cases} e^{i\alpha q}v_q(x)\hat{W}_\alpha(\mathcal{L}) & x \subset \text{int } \Sigma \\ v_q(x)\hat{W}_\alpha(\mathcal{L}) & x \notin \text{int } \Sigma \end{cases} \tag{4.27}$$

where $\text{int } \Sigma$ is the interior of the flat surface Σ whose boundary is given by \mathcal{L} .

To prove (4.33) we consider the Euclidean Green functions

$$S_{l,m,n}(\dots q, x, \dots, \alpha, \mathcal{L}^{(-\varepsilon)}, \dots) \quad \text{and} \quad S_{e,m,n}(\dots q, x, \dots, \alpha, \mathcal{L}^{(\varepsilon)}, \dots),$$

where x, \mathcal{L} are as above and the index (ε) denotes a translation by ε in the time direction.

From the definition (4.19):

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} S_{l,m,n}(\dots x, q, \dots, \alpha, \mathcal{L}^{(\varepsilon)}, \dots) \\ &= \lim_{\varepsilon \downarrow 0} \exp\left\{i\alpha \int_{\Sigma(\varepsilon)} F_h \exp\right\} \left\{-i\alpha \int_{\Sigma(-\varepsilon)} F_h\right\} S_{l,m,n}(\dots, x, q, \dots, \alpha, \mathcal{L}^{(-\varepsilon)}, \dots) \\ &= \lim_{\varepsilon \downarrow 0} S_{l,m,n}(\dots, x, q, \dots, \alpha, \mathcal{L}^{(-\varepsilon)}, \dots) \cdot \begin{cases} e^{i\alpha q}, & x \subset \text{int } \Sigma \\ 1, & x \notin \Sigma \end{cases} \end{aligned}$$

The dual algebra (4.27) follows by standard arguments of axiomatic field theory.

Vorticity of Sectors. We now define the vorticity operator Q and we show that the sectors \mathcal{H}_q are eigenstates of the operator $e^{i\alpha Q}, \alpha \in \mathbf{R}$, corresponding to the eigenvalues $e^{i\alpha q}$, i.e.

$$e^{i\alpha Q} \mathcal{H}_q = e^{i\alpha q} \mathcal{H}_q. \tag{4.28}$$

We define

$$e^{i\alpha Q} \equiv w\text{-}\lim_{R \nearrow \infty} \frac{\hat{W}_\alpha(C_R)}{\langle W_\alpha(C_R) \rangle} \tag{4.29}$$

where C_R is a circle in the time zero plane of radius R , centered at the origin.

Since the linear span of the vectors $\{v(f)\mathcal{F}_+ : \text{supp } f \subset \{x^0 > 0\}\}$ is dense in \mathcal{H}_q , to prove (4.28) we only need to compute

$$\begin{aligned} \langle v_q(x)F|e^{i\alpha Q}|v_q(y)G\rangle &= \lim_{R \nearrow \infty} \frac{\langle v_q(x)F|\widehat{W}_\alpha(C_R)|v_q(y)G\rangle}{\langle \Omega|\widehat{W}_\alpha(C_R)\Omega\rangle} \\ &= \lim_{R \nearrow \infty} \frac{\left\langle (\Theta F)W_\alpha(C_R)\exp\left\{i\alpha \int_{\Sigma_R} F_0\right\}GD(rx, -q, y, q)\right\rangle}{\langle W_\alpha(C_R)\rangle}, \end{aligned} \quad (4.30)$$

where $F, G \in \mathcal{F}_+$, Σ_R is the surface bordered by C_R , and we choose F_0 of compact support. By clustering

$$(4.30) = \lim_{R \nearrow \infty} \exp\left\{i\alpha \int_{\Sigma_R} F_0\right\} \langle (\Theta F)GD(rx, -q, y, q)\rangle = e^{i\alpha q} \langle v_q(x)F|v_q(y)G\rangle.$$

Remark 4.4. For the lattice model we have shown in [1] that *the vortex of charge 1 is a massive, stable particle* by showing that the vortex propagator has a spectral representation

$$\langle v_1(0)\Omega, v_1(x)\Omega\rangle = \int d\rho(a, \vec{k}) e^{-ax^0} e^{i\vec{k}\cdot\vec{x}},$$

where $d\rho(a, \vec{k})$ is a positive measure on $[0, \infty] \times [-\pi, \pi]^2$ and

$$d\rho(a, \vec{0}) = c\delta(a - m_s) + d\rho'(a)$$

with $\text{suppl } \rho' \subseteq [m_s + \mu_s, \infty]$, $m_s > 0$ being the vortex mass and $\mu_s > 0$ the upper gap.

A Semi-Classical Expansion. As for kinks in ϕ^4 [2] one might ask if correlation functions of vortices may be analyzed with the help of an asymptotic semi-classical expansion. Here we just outline the scheme of such an analysis for the two-point function $S_2(x, q, y, -q) \equiv S_2(\underline{x}, \underline{q})$.

We start by looking for the critical points of the classical covariant action (4.13) in the limit $\delta \downarrow 0$ on some Sobolev space of connections, \tilde{A} , of $P(\underline{x}, \underline{q})$ and sections, ϕ , of $E(\underline{x}, \underline{q})$. It turns out that it is easier to work with an F_0 which is harmonic near $\{\underline{x}\}$, and of compact support. We also add a Feynman gauge fixing term in $A = \tilde{A} - A_0$. The minimizer (\tilde{A}_c, ϕ_c) of (4.22), as $\delta \downarrow 0$, can be written in terms of the minimizer $(A_c = \tilde{A}_c - A_0, \phi_c)$ of the action

$$\begin{aligned} S_c(A, \varphi) &= \int_{M_x} \left\{ \frac{1}{2e^2} [(A\Delta A)(x) + 2^* A \wedge \delta F(A_0)(x)] \right. \\ &\quad \left. + \frac{1}{2} |\nabla_{A_0} \phi + iA\phi|^2(x) + V(|\phi|)(x) \right\} d^3x. \end{aligned} \quad (4.31)$$

The Feynman gauge fixing still leaves a global $U(1)$ symmetry in the action (4.31); to fix it we require e.g. that $\int_K A = 0$, where K is the support of $F(A_0)$.

Once one has constructed the minimizer (A_c, ϕ_c) one expands $S_c(A, \phi)$ around (A_c, ϕ_c) , i.e.

$$S_c(A_c + \phi, \phi_c + \varphi) = S_c(A_c, \phi_c) + \mathcal{Q}_{A_c, \phi_c}(a, \varphi) + O(\psi^3),$$

where $\mathcal{Q}_{A_c, \phi_c}(a, \varphi)$ denotes a quadratic form in (a, ϕ) and $\psi = a, \varphi$.

One must prove that the quadratic form

$$\mathcal{Q}_{A_c, \phi_c}([a, \varphi]) \tag{4.32}$$

has no zero modes when defined on the gauge equivalence classes

$$\begin{aligned} [A_c, \phi_c] &= \{(A'_c, \phi'_c): A'_c - A_c = d\lambda, \phi'_c = e^{-i\lambda} \phi_c\}, \\ [a, \varphi] &= \{(a', \varphi'): a' = a, \varphi' = e^{-i\lambda} \varphi\}. \end{aligned} \tag{4.33}$$

One can then consider the mean field theory described by a Gaussian measure over the equivalence classes (4.33) with mean 0 and covariance $\mathcal{Q}_{A_c, \phi_c}^{-1}$.

The Lattice Model. Vortices on a lattice with finite lattice spacing have been constructed in [1]. Here we only make some remarks concerning Osterwalder–Schrader positivity and cluster properties of correlation functions involving vortices, which may be useful to support the corresponding assumptions made for the formal continuum correlation functions. On a finite lattice Λ with lattice spacing ε , the action of the non-compact abelian Higgs model, with Feynman gauge fixing term, is given by

$$\begin{aligned} S^\varepsilon(A, \phi) &= \varepsilon^3 \left[\sum_p \frac{1}{2e^2} (d^\varepsilon A)_p^2 + \frac{1}{2} \sum_{\langle xy \rangle} |\nabla_A^\varepsilon \phi|_{\langle xy \rangle}^2 \right. \\ &\quad \left. + \sum_x \lambda |\phi_x|^2 - \frac{1}{4} m^2 |\phi_x|^2 - \frac{1}{2} \delta m^2(\varepsilon) |\phi_x|^2 + E(\varepsilon) + \sum_x \frac{1}{2e^2} (\delta^\varepsilon A)_x^2 \right] \end{aligned} \tag{4.34}$$

where $d^\varepsilon, \delta^\varepsilon, \nabla_A^\varepsilon$ are, respectively, the lattice exterior differential, codifferential ($\equiv *d^\varepsilon*$) and covariant differential:

$$(\nabla_A^\varepsilon \phi)_{\langle xy \rangle} \equiv \varepsilon^{-1} (\phi_x - e^{i\varepsilon A_{\langle xy \rangle}} \phi_y),$$

$\delta m^2(\varepsilon)$ and $E(\varepsilon)$ are mass and vacuum energy counterterms.

In the lattice approximation of the covariant action functional, $S_{\underline{x}, q}^\varepsilon(A, \phi)$ the points $\underline{x} = \{x_1 \cdots x_n\}$ are in the dual lattice; $S_{\underline{x}, q}^\varepsilon(A, \phi)$ can be obtained from (4.40) by the substitution:

$$d^\varepsilon A \rightarrow d^\varepsilon A + F_h^\varepsilon, \quad \nabla_A^\varepsilon \rightarrow \nabla_{A+A_h^\varepsilon}^\varepsilon, \tag{4.35}$$

where F_h^ε and A_h^ε are constructed as follows. Let ω be an integer lattice 2-form satisfying

$$d^\varepsilon \omega = \sum_i q_i (\delta_{x_i})^*, \tag{4.36}$$

where δ_{x_i} is a 0-form on the dual lattice defined by

$$\delta_{x_i}(x) = \begin{cases} 1 & x_i = x \\ 0 & \text{otherwise} \end{cases}.$$

Then we set

$$F_h^\varepsilon = 2\pi \delta^\varepsilon (\Delta^\varepsilon)^{-1} \sum_i q_i (\delta_{x_i})^*, \tag{4.37}$$

$$A_h^\varepsilon = 2\pi \delta^\varepsilon (\Delta^\varepsilon)^{-1} \omega,$$

where $\Delta^\varepsilon \equiv d^\varepsilon \delta^\varepsilon + \delta^\varepsilon d^\varepsilon$ is the lattice laplacian.

Correlation functions of vortices are then defined by

$$\langle D(\underline{x}, \underline{q}) \rangle^\varepsilon = \lim_{\Lambda \nearrow \mathbb{Z}^3} \frac{\int \prod_{\langle \lambda \nu \rangle \in \Lambda} dA_{\langle \lambda \nu \rangle} \prod_{x \in \Lambda} d\phi_x e^{-S_{x,q}^c(A,\phi)}}{\int \prod_{\langle \lambda \nu \rangle \in \Lambda} dA_{\langle \lambda \nu \rangle} \prod_{x \in \Lambda} d\phi_x e^{-S^c(A,\phi)}}. \tag{4.38}$$

To prove O.S. positivity one may use the Hodge decomposition on the lattice:

$$2\pi\omega = 2\pi d^\varepsilon \delta^\varepsilon (\Delta^\varepsilon)^{-1} \omega + 2\pi \delta^\varepsilon d^\varepsilon (\Delta^\varepsilon)^{-1} \omega = d^\varepsilon A_h^\varepsilon + F_h^\varepsilon, \tag{4.39}$$

and make the field redefinition

$$A^\varepsilon \rightarrow A^\varepsilon - A_h^\varepsilon.$$

Then we obtain the expression for $\langle D(\underline{x}, \underline{q}) \rangle^\varepsilon$ discussed in [1]. One easily shows that this expression is invariant under the transformation

$$A \rightarrow A + 2\pi \xi, \quad \omega \rightarrow \omega - d^f \xi, \tag{4.40}$$

where ξ is an arbitrary integer lattice one-form.

Therefore one can arrange ω in such a way that O.S. positivity becomes manifest. E.g. for the two-point correlation $\langle D(x, -q; y, q) \rangle^\varepsilon$ function this is shown in Fig. 1. Clustering of correlation functions involving vortices can be proved easily, using a Combined Low and High Temperature Cluster Expansion [1, 26] for e, λ small, m large, $em \gg \lambda$.

Using the slightly more complicated expansion around a Gaussian of [27], one can prove the same result for the larger domain of coupling constants given by e, λ small, $m, e^2/\lambda = O(1)$. With both these expansions, however, the estimates are *not* uniform in the lattice spacing ε .

To really justify our continuum construction, starting from the lattice model, one should prove

1. estimates yielding clustering, uniformly in ε .
2. convergence of the lattice approximation ($\varepsilon \downarrow 0$) for the correlation functions involving disorder fields, Wilson loops and scalar fields (suitably renormalized);
3. existence and euclidean invariance of the thermodynamic limit.

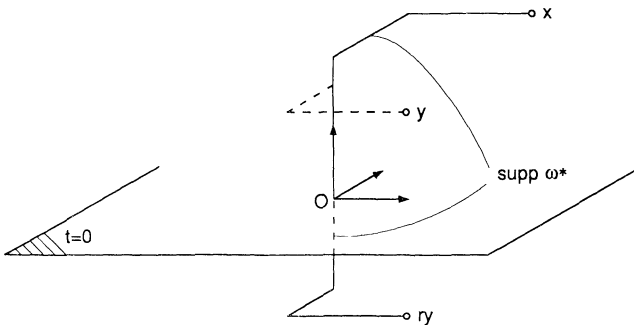


Fig. 1.

In principle the methods of [19], suitably modified to take into account vortices, should suffice to prove point 1).

Final Comments

1. In Appendix 1 we sketch a formal construction of vortex sectors using as basic variables the Wilson loops (\equiv holonomy) instead of the connections.
2. Our construction of vortices can be generalized to $SU(N)/\mathbf{Z}_N$ Higgs models, by choosing the reference connection A_0 in the $U(1)^{N-1}$ subgroup of $SU(N)$. The classification theorem for such vortex sectors is given by Theorem 3.1b). A rigorous construction of vortex sectors in the lattice approximation has been obtained in [1].

5. Anyons

We now pass to the quantum theory of *electrically charged vortices*, or *anyons*, in the non-compact abelian Higgs model with Chern–Simons term. Our construction will be presented in the formal continuum limit. The rigorous construction in the lattice approximation is sketched in Sect. 7.

The classical action of the model is given by

$$S_\mu(A, \phi) = S(A, \phi) + S_\mu^{\text{C.S.}}(A), \tag{5.1}$$

where $S(A, \phi)$ is defined as in (4.1) and

$$S_\mu^{\text{C.S.}} = \frac{i\mu}{4\pi} \int_{\mathbf{R}^3} (A \wedge dA)(x) d^3x \tag{5.2}$$

is the Chern–Simons term.

The corresponding quantum measure is formally given by

$$\frac{1}{Z_\mu} D[A] D\phi e^{-S_\mu(A, \phi)}, \tag{5.3}$$

where

$$Z_\mu = \int_{\mathcal{C}'} D[A] \int_{\mathcal{Q}(E')} D\phi e^{-S_\mu(A, \phi)}$$

(see (4.23))

Remark 5.1. The model without matter fields can be rigorously formulated as a Gaussian theory.

In a finite volume Λ , the (complex) covariance of the gauge equivalence classes $[A]$ is given by

$$C_{A, \mu}^\Lambda = e^2 \left(\delta d + \frac{i\mu e^2}{2\pi} *d \right)_{\Lambda, D}^{-1}. \tag{5.4}$$

In the limit $\Lambda \uparrow \mathbf{R}^3$ we can write the Fourier transform of (5.4) as

$$(C_{A, \mu})_{\rho\sigma}(p) = \frac{e^2}{p^2 + \frac{\mu^2 e^4}{4\pi^2}} \left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2} + \frac{i\mu e^2}{2\pi} \varepsilon_{\rho\sigma\tau} \frac{p^\tau}{p^2} \right). \tag{5.5}$$

Equation (5.5) shows that the pure gauge theory with Chern–Simons term describes a massive excitation of mass $(|\mu|e^2)/2\pi$; see [28].

Notice that, with 0-Dirichlet b.c. on $[A]$, the action (5.1) is gauge invariant. Moreover, in spite of the “i” in (5.2), the measure (5.3) is formally Osterwalder–Schrader positive. In fact, if Θ denotes the O.S. involution (see (4.26)):

$$\Theta(iA \wedge dA)(x) = r(-iA \wedge dA)(x) = iA \wedge dA(rx),$$

where the last equality follows from the fact that $A \wedge dA$ is a pseudoscalar.

Therefore, the vacuum sector of the model, $\mathcal{H}_0^{(\mu)}$, can be reconstructed as in the previous section, using the O.S. reconstruction theorem. However, when A is coupled to a Higgs scalar $\mathcal{H}_0^{(\mu)}$ does not contain all physical states of the model, as suggested by the existence of the classical vortex solutions, for arbitrary values of μ (see Sect. 2).

Actually, we claim the following: In the Higgs phase of the model, the Hilbert space of states, $\mathcal{H}^{(\mu)}$, of the Q.F.T. determined by the measure (5.3) has a decomposition into orthogonal sectors:

$$\mathcal{H}^{(\mu)} = \bigoplus_{q \in \mathbb{Z}} \mathcal{H}_q^{(\mu)}. \quad (5.6)$$

The sectors $\mathcal{H}_q^{(\mu)}$ with $q \neq 0$ are soliton sectors of vorticity q , which, furthermore, carry an electric charge $q_e = \mu q$, as expressed by the equation

$$e^{i2\pi Q_E} \mathcal{H}_q^{(\mu)} = e^{i2\pi \mu q} \mathcal{H}_q^{(\mu)}, \quad (5.7)$$

where Q_E is the electric charge to be defined more precisely; (see Remark 5.2).

There exist *non-local* field operators, the *anyon fields*, which map the vacuum sector, $\mathcal{H}_0^{(\mu)}$, into the vortex sector $\mathcal{H}_q^{(\mu)}$.

Remark 5.2. In the Higgs phase, integer charges are screened, and the electric charge operator cannot be defined as the generator of the one-parameter group of constant gauge transformations (see e.g. [29]). Only periodic functions of the electric charge are well defined operators.

Therefore we use $e^{i2\pi Q_E}$ to characterize the electric charge of the vortex sectors.

To construct the anyon sectors, $\mathcal{H}_q^{(\mu)}$, we proceed in analogy with Sect. 4. We choose n points, \underline{x} , in \mathbf{R}^3 , define $M_{\underline{x}} = \mathbf{R}^3 \setminus \{\underline{x}\}$, and, on $M_{\underline{x}}$, we consider non-trivial $U(1)$ bundles characterized by magnetic charges \underline{q} , $q_i \neq 0$, and denoted by $P(\underline{x}, \underline{q})$, i.e. vortex bundles.

We now propose to define the action functional with Chern–Simons term for connections, \tilde{A} , on those non-trivial bundles and sections ϕ of the associated bundles, in analogy with (4.13).

Construction of the Chern–Simons Term for Vortex Bundles. In order to define a Chern–Simons term on non-trivial bundles, we need to specify a reference connection A_0 , (see e.g. [30]). Given a connection \tilde{A} on the bundle and the reference connection A_0 , a natural definition would be

$$S_{\mu}^{\text{C.S.}}(\tilde{A}, A_0) = \frac{i\mu}{4\pi} \int (\tilde{A} - A_0) \wedge d(\tilde{A} - A_0).$$

However, for the physical interpretation of our construction it is more

convenient to add to A_0 a globally defined one-form α , defined as follows. Consider the vortex bundle $P(\underline{x}, q)$. We denote by E_x^+ (respectively E_x^-) the flux distribution 2-form of a source of magnetic charge 1 at x such that the support of E_x^\pm is given by a cone with apex at x , in the positive (respectively negative) time half-space. Let

$$(\underline{x}, \underline{E}, \underline{q}) = (\{x_i, E_{x_i}^-, q_i\}, \{x_j, E_{x_j}^+, q_j\}) \quad i = 1, \dots, r; \quad j = r + 1, \dots, n. \quad (5.8)$$

then for $\sum_{k=1}^n q_k = 0$ and $x_i^0 < x_j^0, \forall i < j$, we define the field $F(\underline{x}, \underline{E}, \underline{q})$ by

$$F(\underline{x}, \underline{E}, \underline{q}) = \sum_{i=1}^r q_i E_{x_i}^- + \sum_{j=r+1}^n q_j E_{x_j}^+.$$

For a more precise definition see Sect. 7.

Since $F(\underline{x}, \underline{E}, \underline{q})$ is in the same cohomology class as $F(A_0)$, there exists an exact form $\alpha_{A_0}(\underline{x}, \underline{E}, \underline{q})$ such that

$$F(A_0) - F(\underline{x}, \underline{E}, \underline{q}) = d\alpha_{A_0}(\underline{x}, \underline{E}, \underline{q}), \quad (5.10)$$

and we define a Chern–Simons term by

$$S_\mu^{-\text{c.s.}}(\tilde{A}, A_0 - \alpha_{A_0}(\underline{x}, \underline{E}, \underline{q})) = \frac{i\mu}{4\pi} \int (\tilde{A} - A_0 + \alpha_{A_0}) \wedge d(\tilde{A} - A_0 + \alpha_{A_0}). \quad (5.11)$$

Anyon Correlation Functions. Defining $A = \tilde{A} - A_0$, the regularized action functional is given by:

$$S_{\mu, \underline{x}, \underline{q}}^\delta(A, \phi, A_0, \underline{E}) = \left\{ \int_{\mathbf{R}^3 \setminus S_\delta(\underline{x})} \frac{1}{2e^2} (F(A_0) + dA)^2(x) + \frac{1}{2} |\nabla_{A+A_0} \phi|^2(x) + V(|\phi|)(x) + \frac{i\mu}{4\pi} (A + \alpha_{A_0}(\underline{x}, \underline{E}, \underline{q})) \wedge d(A + \alpha_{A_0}(\underline{x}, \underline{E}, \underline{q}))(x) - c(\delta) \right\}, \quad (5.12)$$

where $c(\delta)$ is a counterterm for the self-energy of $F(A_0)$, as in (4.12).

The n -vortex correlation functions, with $\sum_i q_i = 0$ and $x_1^0 < \dots < x_r^0 < 0 \leq x_{r+1}^0 < \dots < x_n^0$ are given by

$$S_n(\underline{x}, \underline{E}, \underline{q}) = \lim_{\delta \downarrow 0} [Z_\mu^{-1} \int_{\mathcal{C}'} D[A] \int_{D(E')} D\phi e^{-S_{\mu, \underline{x}, \underline{q}}^\delta}]_{\text{ren}} i\nu \equiv [Z_\mu^{-1} [\int_{\mathcal{C}'} D[A] \int_{\mathcal{Q}(C')} e^{-S_{\mu, \underline{x}, \underline{q}}(A, \phi, A_0, \underline{E})}]_{\text{ren}}]. \quad (5.13)$$

As the notation suggests, the definition (5.13) is independent of the choice of A_0 , within the class connections of the vortex bundle $P(\underline{x}, q)$.

To display the physical meaning of (5.12), (5.13), we use Eq. (5.10), and rewrite (5.13) as:

$$\left\langle D(\underline{x}, \underline{q}, F(A_0)) \exp \left\{ \frac{i\mu}{2\pi} \int A \wedge [F(\underline{x}, \underline{E}, \underline{q}) - F(A_0)] \right\} \right\rangle_\mu \exp \left\{ -\frac{i\mu}{4\pi} \int \alpha \wedge d\alpha \right\}. \quad (5.14)$$

Equation (5.14) can be interpreted as follows:

$D(\underline{x}, \underline{q}, F(A_0))$ introduces a vorticity flux along $\text{supp } F(A_0)$ the term

$$\exp \left\{ \frac{i\mu}{2\pi} \int A \wedge [F(\underline{x}, \underline{E}, \underline{q}) - F(A_0)] \right\} \quad (5.15)$$

introduces an electric flux along

$$\text{supp}(F(\underline{x}, \underline{E}, q) - F(A_0))$$

(this justifies the use of notation “ E ”, see also Remark 5.3) and the term

$$\exp -\frac{i\mu}{4\pi} \int \alpha \wedge d\alpha \quad (\alpha = \alpha_{A_0}(\underline{x}, \underline{E}, q))$$

is a topological term.

Remark 5.3. A possible limiting choice of $F(A_0)$ is obtained by shrinking $F(A_0)$ to a current with support on a set of lines joining the points \underline{x} in a neutral way (see Fig. 2). They describe vorticity flux lines, ending at the magnetic sources at \underline{x} , together with electric flux lines which spread at \underline{x} in shapes described by the electric distributions \underline{E} .

To simplify notations we formally define a disorder field

$$D(\underline{x}, \underline{E}, q) = [\exp - [S_{\mu, \underline{x}, q}(A, \phi, \underline{E}) - S_\mu(A, \phi)]]_{\text{ren}} \tag{5.16}$$

so that

$$S_n(\underline{x}, \underline{E}, q) = \langle D(\underline{x}, \underline{E}, q) \rangle_\mu. \tag{5.17}$$

We now enlarge the domain of definition of anyon correlation functions to a Euclidean invariant domain. The correlation functions (5.13) are formally translation-invariant, so we need to consider only the euclidean rotations \mathcal{R} . Given a euclidean rotation $\mathcal{R}_{\vec{e}}(\varphi)$, by an angle φ around some axis \vec{e} , we approximate it with a (truncating) family of diffeomorphisms of \mathbf{R}^3 , given by:

$$\mathcal{R}_{\vec{e}}^R(\varphi)x = \mathcal{R}_{\vec{e}}(\varphi_r^R)x \quad R \in \mathbf{R}_+,$$

where φ_r^R is a smooth monotonic decreasing function of φ satisfying

$$\varphi_r^R = \begin{cases} \varphi, & r < R, \\ 0, & r > R + 1. \end{cases}$$

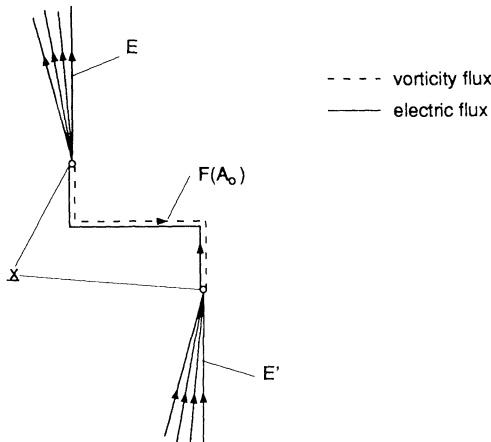


Fig. 2.

Setting

$$E^{(\varphi_{\vec{v}}^R)}(\underline{x}) = E(\mathcal{R}_{\vec{v}}^R(\varphi)\underline{x}),$$

we define

$$S_n(\mathcal{R}_{\vec{v}}(\varphi)\underline{x}, \underline{E}^{(\varphi_{\vec{v}})}, \underline{q}) \equiv \lim_{R \uparrow \infty} S_n(\mathcal{R}_{\vec{v}}^R(\varphi)\underline{x}, \underline{E}^{(\varphi_{\vec{v}}^R)}, \underline{q}), \tag{5.18}$$

and we assume that the limit (5.18) exists, independently of the choice of the truncating sequence. [The symbol \vec{v} will be omitted if \vec{v} is along the x^0 -axis.]

Finally, in the notation of Sect. 4, a full set of correlation functions of the model is given by

$$S_{l,m,n}(\underline{x}, \underline{E}, \underline{q}; \underline{z}; \Sigma(\mathcal{L})) \equiv \langle D(\underline{x}, \underline{E}, \underline{q}); |\phi|^2; (\underline{z}) W_{\vec{v}}(\mathcal{L}) \exp \{i \int_{\Sigma(\mathcal{L})} F(A_0)\} \rangle_{\mu}. \tag{5.19}$$

Remark 5.4. The assumptions made after (5.18) can be proved in the lattice approximation and, by explicit computations, in the absence of matter fields, as we now sketch.

We want to prove that if

$$d\alpha_R \sim \frac{1}{R^2}, \quad \text{for } \text{dist}(\text{supp } d\alpha_R, \underline{x}) \sim R \quad \text{as } R \nearrow \infty,$$

then

$$\left\langle D(\underline{x}, \underline{E}, \underline{q}) \exp \left\{ -\frac{i\mu}{4\pi} \int \alpha_R \wedge d\alpha_R + 2(A + \alpha_0) \wedge d\alpha_R \right\} \right\rangle_{\mu, R \nearrow \infty}^0 \rightarrow \langle D(\underline{x}, \underline{E}, \underline{q}) \rangle_{\mu}^0, \tag{5.20}$$

where $\langle \cdot \rangle_{\mu}^0$ denotes the expectation value with respect to the Gaussian measure with covariance (5.4–5), and $\alpha_0 = \alpha_{A_0}(\underline{x}, \underline{E}, \underline{q})$.

By identifying

$$\alpha_R = [\alpha_{A_0}(\mathcal{R}_{\vec{v}}^R(\varphi)\underline{x}, \underline{E}^{(\varphi_{\vec{v}}^R)}, \underline{q}) - \alpha_{A_0}(\mathcal{R}_{\vec{v}}(\varphi)\underline{x}, \underline{E}^{(\varphi_{\vec{v}})}, \underline{q})], \quad \alpha_0 = \alpha_{A_0}(\mathcal{R}_{\vec{v}}(\varphi)\underline{x}, \underline{E}^{(\varphi_{\vec{v}})}, \underline{q})$$

it follows from (5.20) the independence of the choice of the truncating sequence, in the limit (5.18) for the Gaussian model.

To do the computation, it is convenient to make the change of variable

$$A \rightarrow A + \alpha_R + \alpha_0,$$

then

$$\begin{aligned} & \left\langle D(\underline{x}, \underline{E}, \underline{q}) \exp \left\{ -\frac{i\mu}{4\pi} \int \alpha_R \wedge d\alpha_R + 2d\alpha_R \wedge A \right\} \right\rangle_{\mu}^0 \\ &= \exp \left\{ -\frac{1}{2e^2} \int [F(\underline{x}, \underline{E}, \underline{q}) + d\alpha_R]^2 \right\} \left\langle \exp \left\{ -\frac{1}{e^2} \int [F(\underline{x}, \underline{E}, \underline{q}) + d\alpha_R] \wedge *dA \right\} \right\rangle_{\mu}^0 \\ &= \exp \left\{ -\frac{1}{2e^2} \int [(d\alpha_R)^2 + 2d\alpha_R \wedge *F(\underline{x}, \underline{E}, \underline{q})] \right\} \\ & \cdot \left\langle \exp \left\{ \frac{1}{2e^2} \int \delta d\alpha_R \wedge \left(\delta d + \frac{i\mu e^2}{2\pi} *d \right)^{-1} *[\delta d\alpha_R + 2\delta F(\underline{x}, \underline{E}, \underline{q})] \right\} \right\rangle_{\mu}^0. \end{aligned}$$

The first exponential term is easily evaluated: it is $\mathcal{O}\left(\frac{1}{R}\right)$. In the second term we rewrite

$$\left(\delta d + \frac{i\mu e^2}{2\pi} *d\right)^{-1} = (\delta d)^{-1} \left(\delta d + \frac{\mu^2 e^4}{4\pi^2}\right)^{-1} \left(\delta d + \frac{i\mu e^2}{2\pi} *d\right),$$

then the leading term in the exponential is

$$\frac{i\mu}{4\pi} \int \alpha_R \wedge \left(\delta d + \frac{\mu^2 e^4}{2\pi^2}\right)^{-1} *(\delta d \alpha_R + 2\delta F(\underline{x}, \underline{E}, \underline{q})) = i\mu \mathcal{O}\left(\frac{1}{R}\right).$$

O.S. Reconstruction and Vortex Sectors. We assume that the correlation functions (5.19) satisfy the following properties:

0. continuity with respect to local variations of the electric flux distributions \underline{E} ;
1. distribution property: $S_{l,m,n}$ are continuous functions in $\underline{x}, z, \underline{\mathcal{L}}$ if their mutual distances are bounded, and they have singularities not worse than ultradistributions, as some distance tends to zero;
2. euclidean invariance;
3. O.S. positivity for \underline{x} as in (5.13);
4. clustering.

Remark 5.5. Osterwalder–Schrader positivity formally holds due to the arbitrariness of $F(A_0)$ and the specific choice we made for $F(\underline{x}, \underline{E}, \underline{q})$.

Furthermore, translation invariance, O.S. positivity and clustering can be proved rigorously in the lattice approximation (see Sect. 7). Clustering renders the assumption about the limits (5.18) and the assumption of euclidean invariance plausible.

From these assumptions it follows that we can apply to (5.19) the O.S. reconstruction theorem, in the form discussed in [23–24], and obtain the Hilbert space of states, $\mathcal{H}^{(\mu)}$, the vacuum Ω and a representation of the universal covering of the Poincaré group (see also Sect. 6).

Local fields $|\hat{\phi}|^2(\vec{x})$, $\hat{W}_\alpha(\mathcal{L})$ can be reconstructed as in Sect. 5, and in addition one can define *non-local anyon field operators*

$$v(x, E, q), \quad x^0 > 0, \quad E = E^+.$$

Let $F \in \mathcal{F}_+$, and

$$v(f, E, q) |F\rangle = \int d^3x f(x) v(x, E, q) \Omega,$$

where f is a test-function with $\text{supp } f \subset \{x^0 > 0\}$.

The scalar product between such vectors is given by

$$\langle v(x', E', -q') F | v(x, E, Q) G \rangle = \langle D(rx', rE', -q', x, E, q) (\Theta F) G \rangle_\mu. \quad (5.21i)$$

A similar definition holds for products of anyons field operators

$$v(f^{(n)}, \underline{E}, \underline{q}) = \int d^2\underline{x} v(x_1, E_1, q_1) \cdots v(x_n, E_n, q_n) f^{(n)}(\underline{x}), \quad (5.21ii)$$

where f is a test function with $\text{supp } f \subset \{x_i^0 > 0\}$, $i = 1 \cdots n$. The set of vectors $\{v(f^{(n)}, \underline{E}, \underline{q}) |F\rangle, F \in \mathcal{F}_+\}$ is dense in $\mathcal{H}^{(\mu)}$ by construction.

Clustering implies that all charged correlation functions vanish. In view of the definition (5.21) this implies that $\mathcal{H}^{(\mu)}$ decomposes into orthogonal sectors, \mathcal{H}_q , labelled by the total vorticity q . For $q \neq 0$ they are soliton sectors. The operator $v(f, E, q)$ maps the vacuum sector $\mathcal{H}_0^{(\mu)}$ to the soliton sector $\mathcal{H}_q^{(\mu)}$.

The Electric Charge of the Vortex Sectors $\mathcal{H}_q^{(\mu)}$. We now define more precisely the operator $e^{i2\pi Q_E}$ and prove the equation

$$e^{i2\pi Q_E} \mathcal{H}_q^{(\mu)} = e^{i2\pi\mu q} \mathcal{H}_q^{(\mu)}. \tag{5.22}$$

For a planar loop \mathcal{L} we introduce the disorder field variable, $\mathcal{D}_{2\pi}(\mathcal{L})$, in the non-compact abelian Higgs models as follows. Let $\Sigma(\mathcal{L})(\equiv \Sigma)$ denote both the flat surface Σ , whose boundary is given by \mathcal{L} , and the corresponding integer current (see Sect. 3).

Then we define

$$\mathcal{D}_{2\pi}(\mathcal{L}) = \frac{\exp \left\{ -\frac{1}{2e^2} \int d^3x [(F(A_0) + dA - 2\pi d\Sigma)^2 - (F(A_0) + dA)^2](x) \right\}}{\left\langle \exp \left\{ -\frac{1}{2e^2} \int d^3x [(F(A_0) + dA - 2\pi d\Sigma)^2 - (F(A_0) + dA)^2](x) \right\} \right\rangle_{\mu}}. \tag{5.23}$$

The relation of $\mathcal{D}_{2\pi}(\mathcal{L})$ with the electric charge Q_E is easily seen by computing expectation values such as

$$\langle \mathcal{D}_{2\pi}(\mathcal{L}) \prod_i W_{\alpha_i}(\mathcal{L}_i) \rangle$$

in the $\mu = 0$ model.

By making the change of variables:

$$A \rightarrow A - 2\pi\Sigma, \tag{5.24}$$

one obtains

$$\langle \mathcal{D}_{2\pi}(\mathcal{L}) \prod_i W_{\alpha_i}(\mathcal{L}_i) \rangle = \prod_i \exp \{ i\alpha_i 2\pi \oint_{\mathcal{L}_i} \Sigma \} \langle \prod_i W_{\alpha_i}(\mathcal{L}_i) \rangle. \tag{5.25}$$

In fact, all the terms in the action involving the matter field ϕ are invariant under (5.24), since $\oint \Sigma \in \mathbf{Z}$, see Eq. (4.4i). Equation (5.25) shows that $\mathcal{D}_{2\pi}(\mathcal{L})$ measures $e^{i2\pi Q_E(\Sigma)}$, where $Q_E(\Sigma)$ is the electric charge contained in $\Sigma(\mathcal{L})$.

Therefore we define

$$e^{i2\pi Q_E} = \lim_{R \nearrow \infty} \mathcal{D}_{2\pi}(C_R), \tag{5.26}$$

where C_R is the circle of radius R , centered at the origin, in time zero plane. By definition

$$e^{i2\pi Q_E} \Omega = \Omega, \tag{5.27}$$

i.e. the vacuum has 0-electric charge mod \mathbf{Z} .

Remark 5.6. Notice that if we substitute in (5.25) 2π by a real $\lambda \notin 2\pi\mathbf{Z}$, the operator $\mathcal{D}_{\lambda}(\Sigma)$ becomes ill-defined, since the matter action is not invariant under the

transformation

$$A \rightarrow A - \lambda \Sigma.$$

Since the set of vectors $v(f^{(n)}, \underline{E}, \underline{q})|F\rangle, F \in \mathcal{F}_+,$ with $\sum_i q_i = q,$ is dense in $\mathcal{H}_q^{(\mu)},$ we only need to compute

$$\begin{aligned} & \langle v(\underline{x}', \underline{E}', \underline{q}')F | e^{i2\pi Q_E} | v(\underline{x}, \underline{E}, \underline{q})G \rangle \\ &= \lim_{R \rightarrow \infty} \langle D(r\underline{x}', r\underline{E}', \underline{q}', \underline{x}, \underline{E}, \underline{q}) \mathcal{D}_{2\pi}(C_R)(\Theta F)G \rangle_{\mu}. \end{aligned} \quad (5.28)$$

Using (5.14) we can rewrite the expectation value on the left-hand side of (5.28) as

$$\left\langle (\underline{x}, \underline{q}, F(A_0)) \exp \left\{ -\frac{i\mu}{2\pi} \int A \wedge d\alpha_0 \right\} \mathcal{D}_{2\pi}(C_R)(\Theta F)G \right\rangle_{\mu} \exp \left\{ -\frac{i\mu}{4\pi} \int \alpha_0 \wedge d\alpha_0 \right\}, \quad (5.29)$$

where

$$\alpha_0 \equiv \alpha_{A_0}(r\underline{x}', r\underline{E}', -\underline{q}', \underline{x}, \underline{E}, \underline{q}).$$

We now perform the charge of variable

$$A \rightarrow A - 2\pi \Sigma_R,$$

where $\Sigma_R = \Sigma(C_R)$ and (5.29) becomes

$$\begin{aligned} & \left\langle D(\underline{x}, \underline{q}, F(A_0)) \exp \left\{ -\frac{i\mu}{2\pi} \int A \wedge d\alpha_0 \right\} \exp \left\{ -i\mu \int_{\Sigma_R} dA \right\} \right\rangle_{\mu} \\ & \cdot \exp \left\{ -i\mu \int_{\Sigma_R} d\alpha_0 \right\} \exp \left\{ -\frac{i\mu}{2\pi} \int \alpha_0 \wedge d\alpha_0 \right\}. \end{aligned} \quad (5.30)$$

One easily shows that

$$\lim_{R \rightarrow \infty} \exp \left\{ -i\mu \int_{\Sigma_R} d\alpha_0 \right\} = e^{i\mu 2\pi q}, \quad (5.31)$$

This can be understood in terms of Fig. 2 since $-\int_{x^0=0} d\alpha_0$ measures the electric flux through the time zero plane.

We now choose $F(A_0)$ of compact support. Then by clustering and (5.30), (5.31) we obtain

$$\langle v(\underline{x}', \underline{E}', \underline{q}')F | e^{i2\pi Q_E} | v(\underline{x}, \underline{E}, \underline{q})G \rangle = e^{i\mu 2\pi q} \langle v(\underline{x}', \underline{E}', \underline{q}')F | v(\underline{x}, \underline{E}, \underline{q})G \rangle.$$

6. Spin and Statistics of Electrically Charged Vortices

Under very general assumptions on relativistic quantum physics (locality, relativistic spectrum condition, existence of one-particle states), it has been shown in [3, 40] that, in four or more dimensions, massive fields and particles can carry either integral or half-integral spin and obey either Bose or Fermi statistics. Fields and particles of integer spin obey Bose statistics, those with half-integer spin obey Fermi statistics. The notion of field used in this analysis is very general: ‘‘Fields’’

are localizable in space-like cones of arbitrarily small opening angle, but generally are not localizable in compact regions.

In three space-time dimensions, however, an analysis of the projective representations of the Poincaré group shows that elementary relativistic free particles can have arbitrary real spin. For particles created by fields localizable in compact regions, the analysis of D.H.R. [31] excludes the possibility of spin $s \notin \frac{1}{2}\mathbf{Z}$. The upshot of this analysis is in fact that, also in $d = 3$, such particles can carry only integral and half-integral spin, obey Bose or Fermi statistics and the spin-statistics theorem holds. For particles created by fields not localizable in compact regions, spin need not be integral or half-integral. No direct information on statistics is obtained from the analysis in [3].

The electrically charged vortices of the non-compact abelian Higgs model cannot be created by fields localized in compact regions. This is a consequence of Gauss' law: one can measure the electric charge by operating at an arbitrary distance from the localization region of the charge (see [29, 32]); therefore the fields carrying the electric charge cannot be localized in bounded regions. However the analysis of [3] still applies. It then follows that the interpolating fields of the electrically charged vortices can be localized in space-like cones (the fields $v(x, E, q)$, which we constructed in the previous section, have weaker localization properties). In this section, we show that particles created by such fields have spin given by $\mu/2 \bmod \mathbf{Z}$, and obey Bose statistics if μ is an even integer, Fermi statistics if μ is an odd integer, and, for $\mu \notin \mathbf{Z}$, they obey so-called "intermediate (Θ) statistics" [33]. Statistics manifests itself most clearly in the analysis of asymptotic states.

The main features (localization properties, construction of asymptotic states spin-statistics connection) of the following analysis of anyons are model independent. They apply e.g. to the anyons in the $O(3)$ non-linear σ model with Hopf term, discussed in [8a] in connection with high T_c superconductivity.

Relativistic Free Particles in $d = 3$. By definition, the Hilbert space of states of a free particle carries a projective irreducible representation of the Poincaré group $\mathcal{P}_+^1 \cong \mathbf{R}^3 \otimes SO(2, 1)$. Bargmann has shown in [34] that all the projective representations of \mathcal{P}_+^1 are induced by unitary representations of the universal covering of \mathcal{P}_+^1 which is isomorphic to $\mathbf{R}^3 \otimes \widetilde{SO(2, 1)}$, where $\widetilde{SO(2, 1)}$ is the universal covering of $SO(2, 1)$. Topologically $SO(2, 1) \cong S_1 \times \mathbf{R}^2$, where S_1 corresponds to the group of rotations in the two-dimensional space, hence $\pi_1(SO(2, 1))\mathbf{Z}$, and topologically $\widetilde{SO(2, 1)} \cong \widetilde{S}_1 \times \mathbf{R}^2 \cong \mathbf{R} \times \mathbf{R}^2$.

It follows from a general theorem of Mackey on the representations of semi-direct products of groups [35], that all irreducible representations of

$$\mathbf{R}^3 \times \widetilde{SO(2, 1)}$$

are classified by

- an orbit of a point p in \mathbf{R}^3 under the action of the group $\widetilde{SO(2, 1)}$,
- an irreducible representation of the subgroup of $\widetilde{SO(2, 1)}$ leaving invariant a point on a fixed orbit.

For a fixed orbit, all these subgroups are isomorphic to the same group, called the little group.

Massive particles correspond to orbits characterized by

$$V_m = \{p^\mu: p^\mu p_\mu = m^2 > 0, p^0 > 0\}, \quad (6.1)$$

where $m \in \mathbf{R}_+$ is identified with the mass of the particle: The spin characterizes the irreducible representations of the little group. In $d=3$ dimensions, this is the covering group of rotations and it is isomorphic to \mathbf{R} . All unitary irreducible representations of \mathbf{R} are of the form:

$$\varphi \in \mathbf{R} \rightarrow e^{i2\pi s\varphi}, \quad s \in \mathbf{R},$$

where s is identified with the *spin* of the particle.

Massless particles correspond to the orbit

$$V_0 = \{p^\mu; p^\mu p_\mu = 0, p^0 > 0\}.$$

The little group is also isomorphic to \mathbf{R} and its irreducible representations are classified by a real number, the *helicity*. Therefore, in $d=3$ dimensions, relativistic free particles can have arbitrary real spin (if $m > 0$) and helicity (if $m = 0$).

The “Spin” of Anyon States. (See also Appendix 2 for details.) We now show that the space of one-anyon states of vorticity q carries a unitary representations of the covering group of rotations of $\text{spin}(\mu q^2/2) \bmod \mathbf{Z}$. Let $U(\varphi)$ denote the representative of an element, φ , of the little group on $\mathcal{H}_q^{(\mu)}$. Then under the assumptions made after (5.16),

$$\langle v(\underline{x}', \underline{E}', \underline{q}') F | U(\varphi) | v(x, E, q) \Omega \rangle \equiv \lim_{R \nearrow \infty} \langle (\Theta F) D(r \underline{x}', r \underline{E}', \underline{q}', \mathcal{R}^R(\varphi) x, E^{(\varphi^R)}, q) \rangle_\mu \quad (6.2i)$$

$$\begin{aligned} &= e^{-i\pi\mu q^2 n} \langle (\Theta F) D(r \underline{x}', r \underline{E}', -\underline{q}', \mathcal{R}(\varphi) x, E^{(\varphi)}, q) \rangle_\mu \\ &= e^{-i\pi\mu q^2 n} \langle v(\underline{x}', \underline{E}', \underline{q}') F | v(x, E, q) \Omega \rangle, \end{aligned} \quad (6.2ii)$$

where n is an integer such that

$$\varphi = \varphi_0 + 2\pi n, \quad \varphi_0 \in [0, 2\pi].$$

To prove (6.2ii) we first note that $E^{(\varphi)} - E^{(\varphi^R)}$ is exact, i.e.

$$E^{(\varphi)} - E^{(\varphi^R)} = d\alpha^R,$$

where α^R is a smooth form, then, for R large enough, we can rewrite the expectation value in (6.2i) as:

$$\left\langle (\Theta F) D(\dots, \mathcal{R}(\varphi) x, E^{(\varphi)}, q) \exp\left\{\frac{i\mu}{2\pi} \int A \wedge d\alpha^R\right\} \right\rangle_\mu \exp\left\{-\frac{i\mu}{4\pi} \int \alpha^R \wedge d\alpha^R\right\}.$$

We assume that the term

$$\exp\left\{\frac{i\mu}{2\pi} \int A \wedge d\alpha^R\right\},$$

does not give any contribution in the limit $R \nearrow \infty$, due to clustering and decay properties of $d\alpha^R$. This can be justified as follows. Integrating out the matter field, one expects that the effective action for $[A]$ has the same long distance behaviour of a gaussian theory with a complex covariance like (5.4–5). For this covariance, the claim is easily checked as in Remark 5.5, since $d\alpha^R \sim 1/R^2$ as $R \nearrow \infty$.

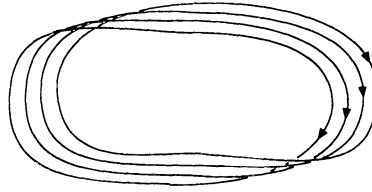


Fig. 3. Flux lines of $d\alpha_R^*$

We now compute

$$\int \alpha_R \wedge d\alpha_R.$$

Let Σ_1, Σ_2 be two integer currents corresponding to two surfaces whose boundary is given by two flux lines of $*d\alpha_R$ (recall $*d\alpha_R$ is a vector!), see Fig. 3. Then one easily realizes that

$$\int \alpha_R \wedge d\alpha_R = 4\pi^2 q^2 \int \Sigma_1 \wedge d\Sigma_2 = 4\pi^2 \int_{\Sigma_1} d\Sigma_2 = 4\pi^2 q^2 n \quad (\text{see Appendix 2}).$$

In particular for $\varphi = 2\pi$, one finds

$$U(2\pi)|v(x, E, q)\Omega\rangle = e^{-in\mu q^2}|v(x, E, q)\Omega\rangle, \tag{6.3}$$

i.e. the space of one-anyon states carries representation of spin $(\mu/2)q^2 \bmod \mathbf{Z}$ of the covering of the rotation group. It is easy to generalize this result to an arbitrary n -anyon state $v(x, E, q)|\Omega\rangle$.

Let us assume that $\text{supp } E_i \cap E_j = \emptyset$. Then one can define 1-forms $\alpha_i, i = 1, \dots, n$, by

$$E_i^{(\varphi)} - E_i^{(\varphi^R)} = d\alpha_i^R,$$

where $\varphi = 2\pi$.

We need to compute

$$\lim_{R \rightarrow \infty} \sum_{i,j} \int d\alpha_i^R \wedge \alpha_j^R.$$

One easily realizes that if $\Sigma_{i,1}, \Sigma_{i,2}$ are two surfaces whose boundary is given by two flux lines of $*d\alpha_i^R$, then for R sufficiently large,

$$\sum_{i,j} \int d\alpha_i^R \wedge \alpha_j^R = \sum_{i,j} 4\pi^2 q_i q_j \int \Sigma_{i,1} \wedge d\Sigma_{j,2} = \sum_{i,j} 4\pi^2 q_i q_j,$$

i.e. the “spin” of the representation of the covering group of rotations is given by

$$\frac{\mu}{2} \left(\sum_{i=1}^n q_i^2 + \sum_{i \neq j}^n q_i q_j \right) = \frac{\mu}{2} \left(\sum_{i=1}^n q_i \right)^2 \tag{6.4}$$

in the space of n -anyon states of charges $\{q_1 \dots q_n\}$. The spin of an n -anyon state is *not* the sum of the spins of the individual anyons. Besides the single anyon contributions, $(\mu/2)q_i^2$, corresponding to a self-braiding of the flux lines within each electric flux distribution E_i , there are also contributions, $\mu q_i q_j$ corresponding to the braiding of distinct electric distributions (see Fig. 4).

This point will be discussed more formally later on; see Appendix 2.

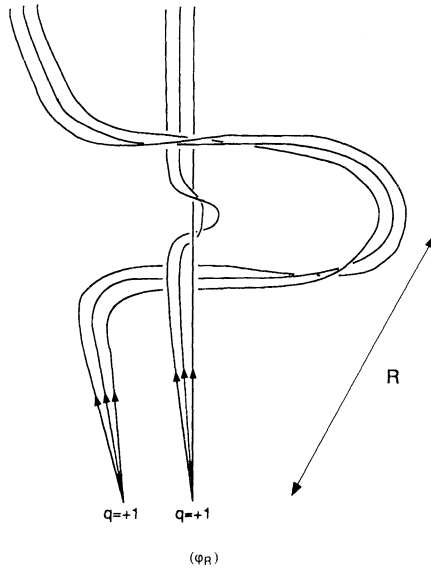


Fig. 4. Flux lines of E

Asymptotic Anyon States. On the lattice one can prove (see Sect. 7) that the field operators $v(x, E, \pm 1)$ couple the vacuum to a stable one-particle state of mass $m_s > 0$, and the spectral representation of the two-point function has a non-vanishing upper gap $\mu_s > 0$, i.e.

$$\langle v(0, E, \pm 1)\Omega | v(0, E, \pm 1)\Omega \rangle = \int d\rho(a, \vec{k}) e^{-a|x^0|} e^{i\vec{k}\vec{x}}, \tag{6.5i}$$

where $d\rho(a, \vec{k})$ is a finite measure with

$$\text{supp } d\rho(a, \vec{0}) = m_s \cup [m_s + \mu_s, \infty), \quad \mu > 0. \tag{6.5ii}$$

Assuming a non-vanishing upper gap also in the continuum theory, there are in principle no obstructions against applying Haag–Ruelle collision theory and constructing asymptotic anyon states. We now sketch how that works. One first constructs euclidean anyon correlation functions in which the support of each electric flux distribution E is shrunk to a wedge contained in a two-dimensional plane. To obtain well defined correlations one needs to renormalize the self-energy of the flux distribution E , since the self-energy of a distribution E localized in a cone diverges logarithmically, as the cone shrinks to a two-dimensional wedge. From these correlation functions one attempts to reconstruct Wightman distribution field operators

$$v^M(x, E(x), q) \tag{6.6}$$

localized in space-like planes in the three-dimensional Minkowski space; (here α is the boost parameter corresponding to the boost mapping the time zero plane to the plane where $E(x)$ is supported).

We briefly sketch how these Wightman distributions are obtained: let E denote

a distribution with support contained in the half-plane

$$\{x^0 = 0, x^1 > 0\}$$

and source at x . Let $E(x)$ denote the distribution obtained from E through a rotation by an angle $\tilde{\alpha}$ around the x^2 -axis, and let $x(\tilde{\alpha})$ denote the position of its source. For $\tilde{\alpha}_1 < \tilde{\alpha}_2 \dots < \tilde{\alpha}_n$ the correlation functions

$$S_n(x_1(\tilde{\alpha}), E_1(\tilde{\alpha}), q_1, \dots, x_n(\tilde{\alpha}), E_n(\tilde{\alpha}), q_n)$$

can be formally written as

$$\langle \Omega, v(x_1, E_1, q_1) e^{-(\tilde{\alpha}_2 - \tilde{\alpha}_1)M_1} v(x_2, E_2, q_2) \dots e^{-(\tilde{\alpha}_n - \tilde{\alpha}_{n-1})M_1} v(x_n, E_n, q_n) \Omega \rangle,$$

where M_1 is the generator of boosts in the (x^0, x^1) -plane. Wightman distributions can be obtained by analytic continuation in the $\tilde{\alpha}$ -parameters.

Let g be a test function for ultradistributions in \mathbf{R}^3 , and $h \in \mathcal{S}(\mathbf{R})$. Then

$$v_{g,h,E}(q) \equiv \int d^3x g(x) \int d\alpha h(\alpha) v^M(x, E(\alpha), q) \tag{6.7}$$

are well defined operators, with support in space-like cones $\mathcal{C} = \mathcal{C}(E, g, h)$. Due to Eq. (6.5), a Haag–Ruelle collision theory in the form developed by Buchholz and Fredenhagen [3] can be developed for the fields $v_{g,h,E}(q), q = \pm 1$.

Here we sketch the key points for the reader’s convenience: In a Lorentz system where the time axis is in the direction of a time-like unit vector e we consider the subspace $\mathcal{L}(\mathcal{C}, e) \subset \mathcal{S}(\mathbf{R}^3)$ of functions f whose Fourier transform $\tilde{f}(p)$ satisfies the conditions:

1. $\text{supp } \tilde{f}(p) \cap \text{spec } \{P^\mu\} \subset V_m \equiv \{p^\mu : p_\mu p^\mu = m^2, p^0 > 0\}$,
2. for $p \in \text{supp } \tilde{f}(p)$

$$p - (p \cdot e)e \in \text{int}(\mathcal{C})_{-a}, \tag{6.8}$$

where a is the apex of \mathcal{C} , $\text{int}(\mathcal{C})_{-a}$ denotes the interior of \mathcal{C} , translated by the vector a . Let us denote by $U(x)$ the translation operator in $\mathcal{H}^{(\mu)}$, and define

$$v_{g,h,E}(p, q) = (2\pi)^{-1/2} \int d^3x e^{-ipx} U(x) v_{g,h,E}(q) U(x)^\dagger. \tag{6.9}$$

Then for

$$\varepsilon_e(p) = \sqrt{(p \cdot e)^2 - p^2 + m^2}, \tag{6.10i}$$

and $f \in \mathcal{L}(\mathcal{C}, e)$,

$$v_{g,h,E}(f, q|te) = \int d^2p \tilde{f}(p) e^{i(p \cdot e - \varepsilon_e(p))t} v_{g,h,E}(p, q) \tag{6.10ii}$$

is an operator which creates from Ω a one-particle state. It is essentially localized in the region $\mathcal{C}_{t(e+s)} \subset \mathcal{C}_{te}$, where $s \in \mathcal{C}_{-a}$ is a fixed vector. (For simplicity we omit henceforth the explicit reference to E, h and g .)

Let $f \in \mathcal{L}(\mathcal{C}_1, e_1) \otimes \dots \otimes \mathcal{L}(\mathcal{C}_n, e_n)$ with $\mathcal{C}_1 + te_1, \dots, \mathcal{C}_n + te_n$ mutually space-like separated for large t and let $v_i(q_i), q_i = \pm 1$ be supported on $\mathcal{C}_i, i = 1 \dots n$. The asymptotic states are defined by

$$s - \lim_{t \rightarrow \pm \infty} v(f^{(n)}, q_1 \dots q_n|t) \Omega = |f^{(n)}, q_1 \dots q_n\rangle_\pm, \tag{6.11i}$$

where

$$v(f^{(n)}, q_1 \cdots q_n | t) = \int \prod_{i=1}^n d^3 p_i \tilde{f}^{(n)}(p_1 \cdots p_n) \exp \left\{ i \sum_j (p_j \cdot e_j - \varepsilon_{e_j}(p_j) t) \right\} \prod_{i=1}^n v_i(q_i). \quad (6.11 \text{ii})$$

We denote by $\mathcal{H}_{\pm}^{(\text{as})}$ the Hilbert spaces of states given by the closure of the linear span of vectors (6.11). The spaces $\mathcal{H}_{\pm}^{(\text{as})}$ carry a unitary representation $U_{\pm}^{\text{as}}(a, \Lambda)$, $a \in \mathbf{R}^3$, $\Lambda \in \widetilde{SO}(2, 1)$ of the universal covering of the Poincaré group defined by

$$U_{\pm}^{\text{as}}(a, \Lambda) | f^{(n)}, q_1, \dots, q_n \rangle_{\pm} = s\text{-}\lim_{t \rightarrow \pm \infty} U(a, \Lambda) v(f^{(n)}, q_1, \dots, q_n | t) \Omega. \quad (6.12)$$

In particular, if $U(2\pi)$ denotes the representation on $\mathcal{H}^{(\mu)}$ of the rotation by 2π around the time axis, then from (6.2-4), (6.11-12) it follows:

$$U_{\pm}^{\text{as}}(2\pi) | f^{(n)}, q_1, \dots, q_n \rangle_{\pm} = \exp \left\{ -i\mu\pi \left(\sum_i q_i \right)^2 \right\} | f^{(n)}, q_1, \dots, q_n \rangle_{\pm}. \quad (6.13)$$

For $n=1$, Eq. (6.13) shows that the spin of one-particle states is given by

$$s = \frac{\mu}{2} \bmod \mathbf{Z}. \quad (6.14)$$

It is quite plausible that generically the one-particle states are non-degenerate, i.e. the integer in (6.14) is fixed. Then the states $| f^{(n)}, q \rangle_{\pm}$, $q = \pm 1$ describes a relativistic free particle of spin $s \in \mu/2 + \mathbf{Z}$, the *anyon*.

In the momentum representation, the 1-particle Hilbert space is isomorphic to

$$L^2(V_m, d^3 p) \otimes \mathbf{C},$$

where \mathbf{C} is the spin space, and the wave functions, $f^{(1)}(p) \otimes u$, transforms under rotations as

$$U_{\pm}^{\text{as}}(\varphi) [f^{(1)}(p) \otimes u] = f^{(1)}(\mathcal{R}^{-1}(\varphi)p) \otimes e^{is\varphi} u. \quad (6.15)$$

Statistics of Anyons. In this section we discuss the statistics of anyon of spin $s = (\mu/2) \bmod \mathbf{Z}$ and prove a *spin-statistics connection*. Within the support of the wave functions $f^{(n)}(p_1, q_1, \dots, p_n, q_n)$ of the collision states constructed by (6.11), neither can a momentum p_i perform a complete rotation around the p_0 -axis nor can two momenta p_i, p_j , can be exchanged. This is because all the allowed 2-momenta \vec{p}_i have different directions

$$\varphi_i = \text{arctg} \frac{p_i^2}{p_i} \neq \varphi_j, \quad i \neq j.$$

This prevents one from deriving information on statistics of anyons by considering the collision states (6.11). The full Hilbert spaces $\mathcal{H}_{\pm}^{\text{as}}$, however, carry a representation of the universal covering of the group of rotation, defined by (6.12). We use such representations to obtain information on the statistics of anyons.

We consider states

$$|f^{(n)}, q, \dots, q\rangle \in \mathcal{H}^{\text{as}} (\equiv \mathcal{H}_+^{(\mu)\text{as}} \text{ or } \mathcal{H}_-^{(\mu)\text{as}}),$$

and denote by $f_q^{(n)}(p_1, \dots, p_n)$ the corresponding momentum space function, describing n -asymptotic identical anyons with vorticity q and three momenta $\{p_i\}$, $i = 1 \dots n$.

From (6.13) and (6.15) it follows:

$$\begin{aligned} U^{\text{as}}(2\pi)f_q^{(n)}(p_1, \dots, p_n) &= \exp\left\{i\mu\pi\left[\left(\sum_i q_i\right)^2 - ns\right]\right\} f_q(p_1, \dots, p_n) \\ &= e^{i\mu 2\pi(n(n-1)/2)} f_q(p_1, \dots, p_n). \end{aligned} \tag{6.16}$$

Therefore $f_q(p_1, \dots, p_n)$ is not a single valued function for $\mu \notin \mathbf{Z}$. We now show that it can be defined as a multivalued function (section) on the non-simply connected space

$$M_n = V_m^{\times n} \setminus D / \Sigma_n, \tag{6.17}$$

where

$$D = \{p_1, \dots, p_n : p_i = p_j \text{ for some } i \neq j\},$$

and Σ_n is the permutation group of n elements.

Wave functions on a non-simply connected space N can generally be defined only as sections of a flat complex vector bundle \mathcal{L} over N , with structure group $\pi_1(N)$, [36]. If the wave functions take values in \mathbf{C} , then \mathcal{L} is a line bundle, and all such line bundles with connection are classified by elements of

$$\text{Hom}(\pi_1(N), U(1)), \tag{6.18}$$

the set of homomorphisms from $\pi_1(N)$ to $U(1)$ corresponding to the holonomy group of the connection of the bundle.

Given the homomorphism $\chi \in \text{Hom}(\pi_1(N), U(1))$, the bundle \mathcal{L} can be constructed as follows. Let \tilde{N} be the universal covering space of N , with canonical projection

$$\pi: \tilde{N} \rightarrow N, \quad \tilde{x} \rightarrow x. \tag{6.19}$$

Let N_0 be a branch of \tilde{N} such that all other points in \tilde{N} can be reached from N_0 by an unique element, γ , of $\pi_1(N)$. Denote by N_γ the corresponding branch; then the transition functions of \mathcal{L} between N_0 and N_γ are given by $\chi(\gamma)$. A section ψ of \mathcal{L} , then corresponds to a function ψ_χ on \tilde{N} satisfying

$$\psi_\chi(\gamma x) = \chi(\gamma)\psi_\chi(x), \quad x \in N_0. \tag{6.20}$$

We henceforth identify ψ and ψ_χ .

Remark 6.1. A more formal definition of \mathcal{L} , independent of the choice of a branch, N_0 is as follows. On the set of couples $(\tilde{x}, z) \in \tilde{N} \times \mathbf{C}$ we define the equivalence relation \sim ,

$$(\gamma\tilde{x}, z) \sim (\tilde{x}, \chi(\gamma)z). \tag{6.21}$$

We identify \mathcal{L} with the space $\tilde{N} \times \mathbf{C}$ modulo the equivalence relation (6.21).

The Braid Group \mathcal{B}_n . In our case, $\pi_1(M_n)$ is isomorphic to the braid group \mathcal{B}_n , defined as follows. We choose n distinct points x_1, \dots, x_n in \mathbf{R}^2 and consider n maps $\gamma_1, \dots, \gamma_n$ from $[0, 1]$ to the slab $\mathbf{R}^2 \times [0, 1]$ in \mathbf{R}^3 with the properties:

a)

$$\gamma_i(0) = x_i \quad \gamma_i(1) = x_{\sigma(i)},$$

where σ is an arbitrary permutation of $\{1, \dots, n\}$.

b)

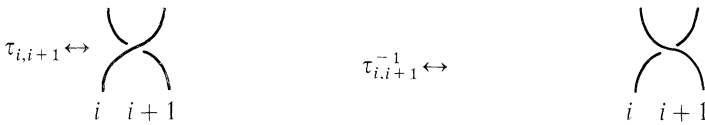
$$\dot{\gamma}_i^3(t) > 0 \quad \forall t \in [0, 1].$$

c) For

$$i \neq j \quad \gamma_i(t) \neq \gamma_j(s) \quad \forall t \in [0, 1], \quad s \in [0, 1].$$

We now identify all maps with properties a), b), c) related to each other by an ambient isotopy. The resulting object is \mathcal{B}_n , the braid group on n -strings.

Let $\tau_{i,i+1}, \tau_{i,i+1}^{-1} \in \mathcal{B}_n$ be the operation of once braiding the i^{th} with the $i+1^{\text{st}}$ string, as shown in Fig. 5:



The operations $\tau_{i,i+1}, i = 1, \dots, n-1$ are the generators of \mathcal{B}_n . They satisfy the following relations:

$$\begin{aligned} \tau_{i,i+1} \tau_{i+1,i+2} \tau_{i,i+1} &= \tau_{i+1,i+2} \tau_{i,i+1} \tau_{i+1,i+2}, \\ \tau_{i,i+1} \tau_{j,j+1} &= \tau_{j,j+1} \tau_{i,i+1} \quad \text{for } |i-j| \geq 2. \end{aligned}$$

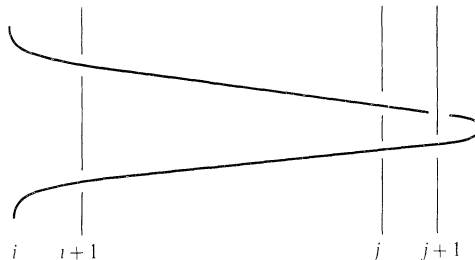
Every element $b \in \mathcal{B}_n$ can be represented as

$$b = \prod_{k=1}^m \tau_{i(k),i(k)+1}^{\sigma(k)}, \quad \sigma(k) = \pm 1.$$

The pure braid group \mathcal{P}_n is the subgroup generated by the elements

$$\gamma_{ij} = \tau_{i,i+1} \cdots \tau_{j,j+1}^2 \tau_{j-1,j}^{-1} \cdots \tau_{i,i+1}^{-1}; \tag{6.22i}$$

γ_{ij} pictorially is represented by



A rotation by 2π corresponds to the element of \mathcal{P}_n :

$$\mathcal{R}(\pi) = \prod_{i < j} \gamma_{ij}. \tag{6.22ii}$$

Following the general analysis presented above, we view $f_q^{(n)}$ as a section of a line bundle \mathcal{L} with base M_n and structure group \mathcal{B}_n . All such lines bundles are classified by elements

$$\chi_{\underline{\Theta}^{(n)}} \in \text{Hom}(\mathcal{B}_n, U(1)), \quad \underline{\Theta}^{(n)} = \{\Theta_i^{(n)}, i = 1, \dots, n, \Theta_i^{(n)} \in [0, 1]\},$$

defined by:

$$\chi_{\underline{\Theta}^{(n)}}: \tau_{j,j+1} \rightarrow e^{2\pi i \Theta_j^{(n)}} j = 1 \dots n - 1. \tag{6.23}$$

Since all anyons are identical, in our case

$$\underline{\Theta}^{(n)} = \underbrace{\{\Theta^{(n)}, \dots, \Theta^{(n)}\}}_{n\text{-times}}, \Theta^{(n)} \in [0, 1], \tag{6.24}$$

and we set

$$\chi_{\underline{\Theta}^{(n)}} = \chi_{\Theta^{(n)}}.$$

From (6.20) it follows that we can view $f_q^{(n)}$ also as a function on \tilde{M}_n . We denote by $f_q^{(n)}(b, \cdot)$ the branch of that function corresponding to the element $b \in \mathcal{B}_n$; it satisfies

$$f_q^{(n)}(b; p_{\sigma(1)}, \dots, p_{\sigma(n)}) = \chi_{\Theta^{(n)}}(b) f_q^{(n)}(1; p_1, \dots, p_n), \tag{6.25}$$

where σ is the permutation corresponding to b .

From (6.19) and (6.22–5) we obtain:

$$\begin{aligned} \exp\left\{i \frac{n(n-1)}{2} 2\pi\mu\right\} &= \chi_{\Theta^{(n)}}(\mathcal{R}(\pi)) = \chi_{\Theta^{(n)}}\left(\prod_{i < j} \gamma_{ij}\right) = \chi_{\Theta^{(n)}}\left(\prod_{i < j} \tau_{j,j+1}^2\right) \\ &= \exp\{i \Theta^{(n)} 2\pi n(n-1)\}, \end{aligned}$$

i.e.

$$\Theta^{(n)} = \frac{\mu}{2} \text{ mod } \frac{\mathbf{Z}}{n(n-1)}. \tag{6.26i}$$

Let us now imagine that the n -anyon wave function $f_q^{(n)}$ is peaked around some set $\{\bar{p}_i\}, i = 1, \dots, n$ of three momenta with $|\bar{p}_m - \bar{p}_{m+1}|$ small and $|\bar{p}_i - \bar{p}_j|$ large for $i, j = 1, \dots, n, (i, j) \neq (m, m+1)$. As time increases the particles m and $m+1$ become more and more separated from the others. By clustering one expects that an exchange of the particles $m, m+1$ is not affected by the presence of the other particles, if the exchange is obtained by changing the momenta in a neighborhood of \bar{p}_m and \bar{p}_{m+1} . I.e.

$$\chi_{\Theta^{(n)}}(\tau_{m,m+1}) = \chi_{\Theta^{(2)}}(\tau_{m,m+1}).$$

Hence

$$\Theta^{(n)} = \Theta^{(2)} \equiv \Theta,$$

and from (6.26)

$$\Theta = \frac{\mu}{2} \text{ mod } \mathbf{Z}. \tag{6.26ii}$$

From Eqs. (6.25) and (6.26) it follows that the momentum space wave functions $f_q^{(n)}(\cdot)$.

- are symmetric for μ an *even integer*, i.e. the anyons are *bosons*,
- are antisymmetric for μ an *odd integer*, i.e. the anyons are *fermions*,
- are sections of a flat complex line bundle \mathcal{L} over M_n characterized by $\chi_{\mu/2} \in \text{Hom}(\pi_1(M_n), U(1))$ for $\mu \notin \mathbf{Z}$, i.e. they change by a factor $e^{i\mu\pi}$ (respectively $e^{-i\mu\pi}$) under an exchange of two momenta obtained by a curve in M_n ,

$$\{p_i(t), p_j(t), t \in [0, 1]: \\ p_i(0) = p_i, p_j(0) = p_j, p_i(1) = p_j, p_j(1) = p_i\},$$

anticlockwise (respectively clockwise) oriented, not enclosing any $p_k, k = 1, \dots, n$. This last transformation property is called “ Θ statistics” in the physical literature, with statistics parameter $\Theta = \mu/2$. Hence, the statistics of anyons smoothly interpolates between Bose and Fermi statistics, as μ varies from 0 to 1. Furthermore, by comparing (6.26) with (6.19) we obtain the following *spin-statistics connection*: anyons with spin $s = (\mu/2) \bmod \mathbf{Z}$ obey a Θ statistics with statistics parameter $\Theta = (\mu/2) \bmod \mathbf{Z}$. Finally notice that Eq. (6.25–6) combined with Eq. (6.19) show that M_n is the maximal extension of the support of $f_q^{(n)}$ for $\mu \notin 2\mathbf{Z}$, i.e. there is a *generalized exclusion principle*; the wave functions of n identical anyons of non-integer spin vanishes if two momenta are equal.

7. Anyons on the Lattice

In this section we discuss the lattice approximation of the anyon sectors of the non-compact abelian Higgs model. We analyze their particle structure and show that the anyon field operator of charge ± 1 couples the vacuum to a stable massive one-particle state for a suitable choice of the coupling constants. This one-particle state has electric charge $\pm \mu$ and vorticity ± 1 .

To define the lattice approximation of the Chern–Simons term (5.2) we introduce a wedge product on the lattice, partially following [37].

The Wedge Product on the Lattice. We start by introducing our notation. We identify an oriented p -cell, c_p , on the lattice \mathbf{Z}^d with a pair $(x, \underline{\mu})$, where $x \in \mathbf{Z}^d$ and

$$\underline{\mu} = \{\mu_1, \dots, \mu_p \mid \mu_i \in I = \{\pm 0, \pm 1, \dots, \pm d - 1\}, \mu_i \neq \pm \mu_j\}.$$

More precisely for $c_p = (x, \underline{\mu})$,

$$\text{supp } c_p = \left\{ y = x + \sum_{\mu \in I} \zeta^\mu e_\mu, 0 \leq \zeta^\mu \leq 1, \zeta^\mu = 0 \text{ unless } \mu \in \underline{\mu} \right\}, \quad (7.1)$$

where e_μ is the unit vector in direction μ , and the orientation of c_p is given by the sign of

$$\wedge e_{\underline{\mu}} \equiv e_{\mu_1} \wedge \dots \wedge e_{\mu_p}. \quad (7.2)$$

To every positively oriented cell there corresponds a pair $(x, \underline{\mu})$, where $\mu_i \in \{0, 1, \dots, d - 1\}$ and $\mu_i < \mu_{i+1}$.

We define

$$e_\mu = \sum_{\mu \in \underline{\mu}} e_\mu.$$

For oriented cells a cup product, \wedge , is defined by

$$(x, \underline{\mu}) \wedge (y, \underline{\mu}') = \begin{cases} \delta_{x+e_{\mu}, y} (x, \underline{\mu} \cup \underline{\mu}') & \text{if } \wedge e_{\underline{\mu} \cup \underline{\mu}'} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

For example in $d = 3$

$$\begin{array}{ccc} \bullet \xrightarrow{x} & & \bullet \xrightarrow{y} \\ \downarrow y & \wedge & \downarrow x \\ \bullet & & \bullet \end{array} = \begin{array}{ccc} & x & y \\ & \nearrow & \searrow \\ & \square & \\ & \nwarrow & \nearrow \\ \bullet & & \bullet \end{array},$$

$$(x, 1) \wedge (y, 2) = (x, 1, -2).$$

In the following $-c_p$ denotes a cell with the same support of c_p and opposite orientation. Notice that e.g.

$$\begin{array}{ccc} & x & \\ \bullet \longleftarrow & & \bullet \longleftarrow \\ (x, -1) & \neq & -(x, 1). \end{array}$$

Let $c_p = (x, \underline{\mu})$, then we denote by c_p^j the cell obtained by reflection in the x^0 -plane containing x , i.e. changing $+0 \leftrightarrow -0$ if ± 0 are contained in $\underline{\mu}$. We define a map in the group of oriented cells by $j: c_p \rightarrow c_p^j$. A real lattice p -form (or form of rank p) is a map, A , from p -cells to \mathbf{R} satisfying $A(-c_p) = -A(c_p)$.

Let \mathcal{C}_p denote the set of positively oriented cells; then two possible bases for real forms are given by

$$1) \{e^{c_p}, c_p \in \mathcal{C}_p\}, \tag{7.4}$$

$$2) \{e^{c_p^j}, c_p^j \in j(\mathcal{C}_p)\}, \tag{7.5}$$

where for $c_p = (x, \underline{\mu})$, $c_p^j = (y, \underline{\mu}') \in \mathcal{C}_p$

$$e^{c_p}((y, \underline{\mu}')) = \delta_{x,y} \delta_{\underline{\mu}, \underline{\mu}'}, \quad e^{-c_p} = -e^{c_p},$$

and similar definitions for c_p^j .

A real p -form A can then be expressed as:

$$A = \sum_{c_p \in \mathcal{C}_p} A(c_p) e^{c_p}, \quad A(c_p) \in \mathbf{R}, \tag{7.6}$$

$$= \sum_{c_p^j \in j(\mathcal{C}_p)} A(c_p^j) e^{c_p^j}, \quad A(c_p^j) \in \mathbf{R}. \tag{7.7}$$

The exterior product between a p -form and a q -form that we define depends on the choice of the basis. We denote by \wedge the exterior product relative to the basis (7.4) and by \wedge_j the exterior product relative to the basis (7.5). Let

$$c_p = (x, \underline{\mu}) \in \mathcal{C}_p, c_q = (y, \underline{\mu}') \in \mathcal{C}_q; c_p^j(x, \underline{\mu}) \in j(\mathcal{C}_p), \quad c_q^j = (y, \underline{\mu}') \in j(\mathcal{C}_q),$$

then

$$e^{c_p} \wedge e^{c_q} = \begin{cases} e^{c_p \wedge c_q} & \text{if } \wedge e_{\underline{\mu} \cup \underline{\mu}'} \neq 0, \quad y = x + e_{\underline{\mu}}, \\ 0 & \text{otherwise} \end{cases} \tag{7.8}$$

$$e^j_p \wedge_j e^j_q = \begin{cases} e^{c^j_p \wedge c^j_q}, & \text{if } \wedge e_{\tilde{\mu} \cup \tilde{\mu}'} \neq 0, \quad y = x + e_{\tilde{\mu}}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.9)$$

By linearity this defines the exterior products $A \wedge B, A \wedge_j B$ of arbitrary lattice forms A and B . The exterior differential, d , of a lattice p -form A is defined by

$$dA(c_{p+1}) = \sum_{c_p \in \tilde{\partial}c_p} A(c_p).$$

Then we have

$$\begin{aligned} d(A \wedge B) &= dA \wedge B + (-1)^{\text{rank } A} A \wedge dB, \\ d(A \wedge_j B) &= dA \wedge_j B + (-1)^{\text{rank } A} A \wedge_j dB, \end{aligned} \quad (7.10)$$

but in general

$$\begin{aligned} A \wedge B &\neq (-1)^{\text{rank } A} B \wedge A, \\ A \wedge_j B &\neq (-1)^{\text{rank } A} B \wedge_j A. \end{aligned} \quad (7.11)$$

Lattice Chern–Simons Term. In view of (7.11), a possible choice is

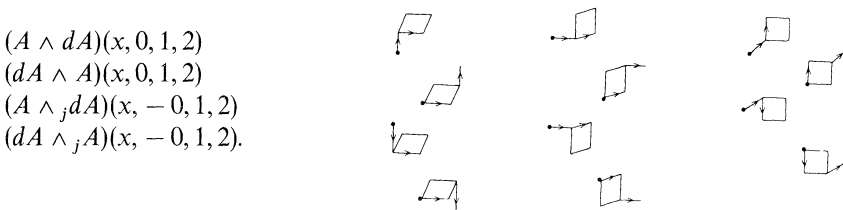
$$\tilde{S}_\mu^{\text{C.S.}} = \frac{i\mu}{8\pi} \sum_{c_3 \in \mathcal{C}_3} (A \wedge dA + dA \wedge A)(c_3).$$

However, this choice of the Chern Simons term is incompatible with O.S. positivity. Therefore we introduce the \wedge_j exterior product and write

$$S_\mu^{\text{C.S.}} = \frac{i\mu}{16\pi} \sum_{c_3 \in \mathcal{C}_3} (A \wedge dA + dA \wedge A - A \wedge_j dA - dA \wedge_j A)(c_3). \quad (7.12)$$

This choice preserves O.S. positivity provided the time zero plane is a lattice plane.

Let us look pictorially to the contributions to (7.12):



It follows that

$$\Theta \left[i \sum_{\substack{c_3 \in \mathcal{C}_3 \\ c_3 \in \{x^0 \geq 0\}}} (A \wedge dA + dA \wedge A)(c_3) \right] = -i \sum_{\substack{c_3 \in \mathcal{C}_3 \\ c_3 \in \{x^0 \leq 0\}}} (A \wedge_j dA + dA \wedge_j A)(c_3),$$

and hence O.S. positivity of $\exp\{-S_\mu^{\text{C.S.}}\}$.

Anyon Correlation Functions. The definitions introduced so far permit us to pass to a lattice approximation of the anyon correlation functions (5.13). In a finite lattice, A , with unit lattice spacing, with Feynman gauge fixing term, the action

of our model is given by:

$$S_\mu(A, \phi) = \sum_p \frac{1}{2e^2} (dA)_p^2 + \sum_{\langle xy \rangle} \frac{1}{2} |\nabla_A \phi|_{\langle xy \rangle}^2 + \sum_x \lambda |\phi_x|^4 - \frac{1}{4} m^2 |\phi_x|^2 + S_\mu^{\text{C.S.}}(A) + \sum_x \frac{1}{2e^2} (\delta A)_x^2, \tag{7.13}$$

where the notations of (4.34–38) are used without ε . To define Green functions of anyons we modify the action as follows (compare with (5.12)).

It is convenient to choose $F(A_0)$ to be some integer 2-form satisfying

$$d\omega = \sum_i q_i \delta_{x_i}^*,$$

where x_i are points in the dual lattice where anyons are located and q_i the corresponding charge. Let E_x^+ (respectively E_x^-), $x \in \mathbf{Z}_{1/2}^3$, be a real 2-form with support on an infinite connected sublattice C_x , containing x , contained in the $\{x^0 > 0\}$ (respectively $\{x^0 < 0\}$) half-space. Further we assume that

$$dE_x^\pm = \delta_x^*$$

and

$$E_x^\pm(p) \sim d(x, p)^{-2} \quad \text{as } d(x, p) \nearrow \infty,$$

where $d(x, p)$ denotes the distance between x and p . Let $\bar{\Lambda}$ be a sublattice of Λ , and let $(\underline{x}, \underline{E}, \underline{q})$ be define as in (5.8), then we define $F^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q})$ by

$$F^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q})(p) = \begin{cases} \left(\sum_{i=1}^r q_i E_{x_i}^- + \sum_{i=r+1}^n q_i E_{x_i}^+ \right)(p) & p \in \text{int } \bar{\Lambda} \\ 0 & p \notin \bar{\Lambda} \end{cases}$$

and

$$dF^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q}) = \sum_i q_i \delta_{x_i}^*.$$

It follows that there exists a lattice one form $\alpha^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q})$ such that

$$\omega - F^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q}) = d\alpha^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q}). \tag{7.14}$$

Then $S_{\mu, \underline{x}, \underline{q}}^{\bar{\Lambda}}(A, \phi, \underline{E})$ is defined by performing the following substitution in (7.13):

$$dA \rightarrow dA + \omega, \\ S_\mu^{\text{C.S.}}(A) \rightarrow S_\mu^{\text{C.S.}}(A + \alpha^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q})).$$

For $\sum_i q_i = 0$, $x_{i+1}^0 > x_i^0$, correlation functions of anyons are then defined by

$$S_n(\underline{x}, \underline{E}, \underline{q}) = \lim_{\bar{\Lambda} \nearrow \mathbf{Z}^3} \lim_{\Lambda \nearrow \mathbf{Z}^3} \frac{Z_{\mu, \Lambda}^{\bar{\Lambda}}(\underline{x}, \underline{E}, \underline{q})}{Z_{\mu, \Lambda}} \\ \equiv \lim_{\bar{\Lambda} \nearrow \mathbf{Z}^3} \lim_{\Lambda \nearrow \mathbf{Z}^3} \frac{\int \prod_{\langle xy \rangle \in \Lambda} dA_{\langle xy \rangle} \prod_{x \in \Lambda} d\phi_x e^{-S_{\mu, \underline{x}, \underline{q}}^{\bar{\Lambda}}(A, \phi, \underline{E})}}{\int \prod_{\langle xy \rangle \in \Lambda} dA_{\langle xy \rangle} \prod_{x \in \Lambda} d\phi_x e^{-S_\mu(A, \phi)}}. \tag{7.15}$$

One can easily construct also a lattice approximation of the order-disorder correlation functions (5.19). Following [1, 26], one can prove that such correlation functions satisfy a lattice version of the Osterwalder–Schrader axioms [1], including clustering, at least for

$$e, \lambda \text{ small, } m \text{ large, } m \gg \lambda. \tag{7.16}$$

This permits us to reconstruct

- a Hilbert space of physical states $\mathcal{H}^{(\mu)}$, containing the vacuum; by clustering it decomposes into orthogonal anyon sectors $\mathcal{H}_q^{(\mu)}$,
- a self-adjoint transfer matrix T and a unitary representation of spatial lattice translations,
- non-local anyon field operators $v(x, E, q)$.

Defect Representation of Anyon Green Functions. We now sketch how to derive a representation of anyon Green functions in terms of defects. This representation is a first step in the proof of clustering and in the particle structure analysis (see [26] for more details). The defects of the non-compact abelian Higgs model are the (Abrikosov) vortices. They can easily be exhibited by decomposing the gauge field as

$$A_{\langle xy \rangle} = \theta_{\langle xy \rangle} + 2\pi\sigma_{\langle xy \rangle}; \quad \theta_{\langle xy \rangle} \in (-\pi, \pi), \quad \sigma_{\langle xy \rangle} \in \mathbf{Z}. \tag{7.17}$$

Defects in the partition functions are described by the \mathbf{Z} -valued 2-form

$$v = d\sigma. \tag{7.18}$$

Since v is closed its support is given by a set of loops.

If we add to the model a Chern–Simons term, for simplicity taken in the form

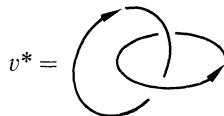
$$S_{\mu}^{\text{c.s.}}(A) = \frac{i\mu}{4\pi} \sum_{c_3} (A \wedge dA)(c_3),$$

then the decomposition (7.17) gives the following contributions to the exponential of the action:

$$1) \exp \left\{ -\frac{i\mu}{2} \sum_{c_3} (\theta \wedge v + \sigma \wedge d\theta)(c_3) \right\} = \exp \left\{ -\frac{i\mu}{2} \sum_{c_3} (\theta \wedge v + v \wedge \theta)(c_3) \right\} \tag{7.19}$$

$$2) \exp \left\{ -i\mu\pi \sum_{c_3} (\sigma \wedge v)(c_3) \right\} \tag{7.20}$$

and a term $\exp \{ -S_{\mu}^{\text{c.s.}}(\theta) \}$. The term (7.19) can be interpreted as giving a θ -electric flux μq to every vortex loop of vorticity q . The term (7.20) can be interpreted as a linking number for the vortex loops; actually $\sigma \wedge v$ gives the intersection number of v^* , considered as a 1-chain in the dual lattice. E.g. a configuration



has an intersection number -1 . Therefore we can represent the partition function Z_μ of the model with Chern–Simons term as a partition function of a gas of vortex loops carrying vorticity $q \in \mathbf{Z}$ and θ -electric flux μq . Using the decomposition (7.17) and defining

$$v = \omega + d\sigma, \tag{7.21}$$

one obtains a similar representation for the modified partition functions $Z_\mu(\underline{x}, \underline{E}, \underline{q})$, used in (7.15) to define anyon correlation functions.

For $\mu = 0$, the modified partition function $Z(\underline{x}, q)$ (recall: \underline{E} is present only in the Chern–Simons term) is the partition functions of a gas of closed vortex loops interacting with open line defects, v_ω , whose boundary is given by $\{\underline{x}\}$. An open line defect corresponds to the worldline of a vortex created at one end of the line and annihilated at the other one.

If a Chern–Simons term is added, the new contributions to e^{-S} are obtained from (7.19–20) by the substitution $\sigma \rightarrow \sigma + \alpha(\underline{x}, \underline{E}, \underline{q})$. In particular the term (see Eqs. (7.14), (7.21)):

$$\exp \left\{ \frac{i\mu}{2} \sum_{c_3} \left[\theta \wedge \left(-v_\omega + \frac{1}{2\pi} F(\underline{x}, \underline{E}, \underline{q}) \right) + \left(-v_\omega + \frac{1}{2\pi} F(\underline{x}, \underline{E}, \underline{q}) \right) \wedge \theta \right] (c_3) \right\}$$

is interpreted as giving a θ -electric flux to the open line v_ω which spreads out in $\text{supp}(1/2\pi)F(\underline{x}, \underline{E}, \underline{q})$. For coupling constants in the region (7.16), the gas of electrically charged vortices is dilute and fluctuations of θ and $|\phi|$ are strongly suppressed. We can set up a combined low and high temperature expansion [1, 26] and establish existence of the limits (7.15) and clustering of correlation functions.

Particle Structure Analysis. By resumming the closed defects one obtains a representation of the anyon Green functions in terms of fluctuating lines described by v^ω :

$$S_n(\underline{x}, \underline{E}, \underline{q}) = \sum_{v_\omega, dv_\omega = \sum_i q_i \delta x_i} z(v_\omega, \underline{E}).$$

For the two point function

$$S_2(0, E_0^-, -1, x, E_x^+, +1) = \langle v(0, E, 1)\Omega | v(x, E_x, 1)\Omega \rangle,$$

the open line defects $v_\omega \equiv v_{0,x}$ is a single random line joining 0 to x . One can analyze its fluctuations by means of an excitation expansion [26]. The output of this analysis is that $z(v_{0,x}, E_0^-, E_x^+)$ roughly behaves like the statistical weight of a simple random walk as $|x| \nearrow \infty$, and

$$\sum_{\vec{x}} \langle v(0, E_0, 1)\Omega | v(x, E_x, 1)\Omega \rangle \underset{x^0 \nearrow \infty}{\sim} e^{-m_s x^0} (1 + e^{-\mu_s x^0}) \tag{7.22}$$

with

$$m_s = \frac{2\pi^2}{e^2} + \mu \mathcal{O} \left(\frac{\lambda}{m^2} \right) + \mathcal{O} \left[\left| \ln \frac{\lambda}{e^2 m^2} \right| + \left| \ln \frac{1}{m^2} \right| \right], \tag{7.23}$$

$$\mu_s = \mathcal{O} \left[\left| \ln \frac{\lambda}{e^2 m^2} \right| + \left| \ln \frac{1}{m^2} \right| \right].$$

where μ_s is an estimate of the upper gap.

The decay (7.22) proves that in the range of coupling (7.16), $v(x, E, 1)$ the anyon field operator couples the vacuum Ω to a stable one particle state of mass m_s .

Appendix 1

A Loop-Space Construction. In this appendix we make some comments on a formal construction of vortices using as variables, instead of the connections \tilde{A} , the Wilson loop variables

$$W_\alpha(\mathcal{L}|\tilde{A}) \equiv \exp \left\{ i\alpha \oint_{\mathcal{L}} \tilde{A} \right\}. \quad (\text{A.1})$$

[More precisely we denote by $W_\alpha(\cdot|\tilde{A})$ the character α of the holonomy group of the connection \tilde{A}].

Since M_x is simply connected, for $\alpha \in \mathbf{Z}$ we can always write

$$W_\alpha(\mathcal{L}|\tilde{A} = A + A_0) = \exp \left\{ i\alpha \oint_{\mathcal{L}} A \right\} \exp \left\{ i\alpha \oint_{\Sigma(\mathcal{L})} F(A_0) \right\}.$$

We start with some topological considerations. Given any connection \tilde{A} in a principal $U(1)$ bundle over M_x , the Wilson loop variable defines a map

$$W_\alpha(\cdot|\tilde{A}): \Omega M_x \rightarrow U(1), \quad \mathcal{L} \rightarrow W_\alpha(\mathcal{L}|\tilde{A}) \quad (\text{A.2})$$

satisfying

$$W_\alpha(\mathcal{L}_1 \circ \mathcal{L}_2|\tilde{A}) = W_\alpha(\mathcal{L}_1|\tilde{A})W_\alpha(\mathcal{L}_2|\tilde{A}), \quad (\text{A.3i})$$

$$W_\alpha(*|\tilde{A}) = 1, \quad (\text{A.3ii})$$

where ΩM_x is the (based) loop space of M_x , the symbol \circ denotes the composition of loops and $*$ denotes the null-loop.

Equation (A.3) shows that $W(\cdot|\tilde{A})$ is a character of ΩM_x with respect to the group operation \circ . Since M_x is simply connected, it follows from a theorem of [38] that every irreducible character of ΩM_x , satisfying (A.3ii) [and some regularity conditions] can be written as the Wilson loop variable $W_\alpha(\cdot|\tilde{A})$ for some connection \tilde{A} of a $U(1)$ bundle over M_x and \tilde{A} is completely determined, modulo gauge transformations.

Let X denote the space of the characters of ΩM_x corresponding to $W_{\alpha=1}(\cdot|\cdot)$. Then we have the following homotopy classification theorem:

$$\pi_0(X) \stackrel{i_1}{\cong} H^1(\Omega M_x, \mathbf{Z}) \stackrel{i_2}{\cong} H^2(M_x, \mathbf{Z}). \quad (\text{A.4})$$

(See [39] for a proof, in a somewhat different context.)

To show how the isomorphism i_1 is defined, we need some preliminary definitions. Let Ωv denote a vector in ΩM_x at \mathcal{L} , i.e. a map $\Omega v: S^1 \rightarrow TM_x$ satisfying $\pi(\Omega v) = \mathcal{L}$, where TM_x is the tangent space of M_x and π is the canonical projection

$$\pi: TM_x \rightarrow M_x.$$

The vector Ωv at \mathcal{L} can be visualized as a vector field v defined on $\text{supp } \mathcal{L}$.

A $k + 1$ form Λ on $M_{\underline{x}}$ defines a k -form $\Omega\Lambda$ on $\Omega M_{\underline{x}}$ by

$$\Omega\Lambda(\Omega v_1, \dots, \Omega v_k)|_{\mathcal{L}} = \int_{\text{supp } \mathcal{L}} i_{v_1} \cdots i_{v_k} \Lambda.$$

The form $\Omega\Lambda$ satisfies the following properties: if Ωc_k is a k -chain of $\Omega M_{\underline{x}}$, and c_{k+1} is the corresponding $k + 1$ chain $M_{\underline{x}}$ representing its support, then

$$\int_{\Omega c_k} \Omega\Lambda = \int_{c_{k+1}} \Lambda.$$

The isomorphism i_1 is then constructed as follows: let $Y \in X, \tilde{A}$ be a connection such that $Y = W_1(\cdot|\tilde{A})$, then

$$[Y] \in \pi_0(X) \xrightarrow{i_1} \left[\frac{1}{2\pi} \Omega F(\tilde{A}) \right] \in H^1(\Omega M_{\underline{x}}, \mathbf{Z}), \tag{A.5}$$

where we identify the de Rham cohomology class with the integer cohomology class, by (3.5). The isomorphism i_2 is then defined by

$$\left[\frac{1}{2\pi} \Omega F(\tilde{A}) \right] \in H^1(\Omega M_{\underline{x}}, \mathbf{Z}) \xrightarrow{i_2} \left[\frac{1}{2\pi} F(\tilde{A}) \right] \in H^2(M_{\underline{x}}, \mathbf{Z}). \tag{A.6}$$

In view of the classification theorem 3.1a, the isomorphism $i_1 \circ i_2$ shows that there is a one-to one correspondence between homotopy classes of X and $U(1)$ -bundles over $M_{\underline{x}}$. Moreover the isomorphism i_1 shows that there is also a one-to one correspondence with the flat \mathbf{R} -bundles over $\Omega M_{\underline{x}}$ with integer holonomy, classified by $H^1(\Omega M_{\underline{x}}, \mathbf{Z})$.

Therefore to every vortex bundle $P(\underline{x}, \underline{q})$, there correspond:

- a non-trivial homotopy class of X ,
- a non-trivial flat \mathbf{R} -bundle over $\Omega M_{\underline{x}}$ with integer holonomy (whose sections are given by $(1/i) \ln W_1(\cdot|\tilde{A})$).

There is some analogy with the soliton bundles of the sine-Gordon model in $d = 2$, discussed in [2]. These were flat \mathbf{R} -bundles over $M_{\underline{x}}^{(2)} \equiv \mathbf{R}^2 \setminus \{\underline{x}\}$ with integer holonomy, classified by $H^1(M_{\underline{x}}^{(2)}, \mathbf{Z})$.

The dual spaces of the smooth sections, ϕ , of these bundles are the spaces on which is supported the functional measure which defines the soliton correlation functions. Moreover the holonomy being integer, $g \equiv e^{i\phi}$ defines a map from $M_{\underline{x}}^{(2)}$ to $U(1)$ and the homotopy classes of g are in one-to one correspondence with the soliton bundles.

Following [38], one can pursue such ideas to the level of expressing vortex correlation functions in terms of a formal functional integration over fields on $\Omega M_{\underline{x}}$.

Appendix 2

On the Monodromy of $S_n(\underline{x}, \underline{E}, \underline{q})$. In this appendix we show how the anyon correlation functions $S_n(\underline{x}, \underline{E}, \underline{q})$ transform, under variations of the electric distributions \underline{E} . A relation with the (pure) Braid group will appear. Let $\gamma(t): (\underline{x}^{\gamma(t)}, \underline{E}^{\gamma(t)})$, $t \in [0, 1]$ be a curve in the space, $\mathcal{E}^{\times n}$, of electric distributions. We consider a truncating sequence $\gamma^R(t): \underline{E} \rightarrow \underline{E}^{\gamma^R(t)}$, such that for all $E \in \underline{E}$, $S \in [0, \iota]$:

$$E^{\gamma R(t)}(y) = \begin{cases} E^{\gamma(t)}(y) & |y| < R \\ E^{\gamma(t-s)}(y) & |y| = R + s \\ E^{\gamma(0)}(y) & |y| > R + t. \end{cases}$$

Then we define:

$$S_n(\underline{x}^{\gamma(1)}, \underline{E}^{\gamma(1)}, \underline{q}) = \lim_{R \nearrow \infty} S_n(\underline{x}^{\gamma(1)}, \underline{E}^{\gamma R(1)}, \underline{q}). \tag{6.27}$$

For closed paths, γ , in the space $\mathcal{E}^{\times n}$ one finds

$$S_n(\underline{x}, \underline{E}^{\gamma(1)}, \underline{q}) = e^{i\Theta(\gamma)} S_n(\underline{x}, \underline{E}^{\gamma(0)}, \underline{q}) \tag{6.28}$$

for some $\Theta(\gamma) \in \mathbf{R}$.

Therefore, with respect to the arguments \underline{E} , the anyon correlation functions $S_n(\underline{x}, \underline{E}, \underline{q})$ are sections of a vector bundle $V^{(n)}$ over $\mathcal{E}^{\times n}$, with structure group $\Omega \mathcal{E}^{\times n}$, the loop group of $\mathcal{E}^{\times n}$, and transition functions with values in $U(1)$. In particular if we consider only curves in $\mathcal{E}^{\times n} \setminus \mathcal{D}^n$, where

$$\mathcal{D}^n = \{ \underline{E} = \{E_i\} : E_i \cap E_j \neq \emptyset, i \neq j \}, \tag{6.29}$$

then it is easy to show, using clustering, that $\Theta(\gamma)$ depends only on the homotopy class of

$$\gamma: S^1 \rightarrow \mathcal{E}^{\times n} \setminus \mathcal{D}^n.$$

One can easily show that

$$\pi_1(\mathcal{E}^{\times n} \setminus \mathcal{D}^n) \cong \mathcal{P}^n \times \underbrace{\mathbf{Z} \times \dots \times \mathbf{Z}}_{n\text{-times}}$$

where \mathcal{P}^n is the pure Braid group (see Eq. (6.22i)) and corresponds to curves in $\mathcal{E}^{\times n}$, where the distributions \underline{E} are translated, $\mathbf{Z} \times \dots \times \mathbf{Z}$ refers to the possible number of 2π rotations of $E_i^{\gamma(1)}$ with respect to $E_i^{\gamma(0)}$, $i = 1 \dots n$.

Therefore the curvature of the bundle $V^{(n)}$ is concentrated in \mathcal{D}^n ; in $\mathcal{E}^{\times n} \setminus \mathcal{D}^n$ the bundle is flat and $e^{i\Theta(\gamma)}$ in (6.28) gives the holonomy of the bundle.

If the homotopy class of γ is given by $[\gamma] = \gamma_{ij}$, where γ_{ij} is a generator of \mathcal{P}^n defined in (6.22i), then proceeding as in Eq. (6.2), one can show that $\Theta(\gamma) = -2\pi\mu q_i q_j$. The fields $v(\underline{x}, \underline{E}, \underline{q})$ are reconstructed from the correlation functions $S_n(\underline{x}, \underline{E}, \underline{q})$. With respect to the arguments \underline{E} , they are section of a bundle $\mathcal{V}^{(n)}$ over $\mathcal{E}^{\times n}$, isomorphic to $V^{(n)}$. Locally in \underline{E} the fields $v(\underline{x}, \underline{E}, \underline{q})$ can be written as

$$\prod_i v(x_i, E_i, q_i),$$

but as $\underline{E} = \{E_i\}$ varies, they change a section of the bundle $\mathcal{V}^{(n)}$. The individual fields $v(x_i, E_i, q_i)$ do not have well defined transformation properties under variations of \underline{E} . E.g. for rotations this easily follows from Eq. (6.4).

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