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On the Cauchy Problem for the Discrete Boltzmann Equation with Initial Values in $L_1^+(\mathbb{R})$

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Abstract. We prove a global existence theorem for a discrete velocity model of the Boltzmann equation when the initial values $\varphi_i(x)$ have finite entropy and, for some constant $\alpha > 0$, $(1+|x|^\alpha)\varphi_i(x) \in L_1^+(\mathbb{R})$.

1. Introduction

The discrete kinetic theory is concerned with the analysis of systems of gas particles with a finite set of selected velocities, and provides a useful substitute for the Boltzmann equation, in terms of a system of semilinear hyperbolic equations. This system defines the space-time evolution of the number of densities associated with every chosen velocity. Generally the discrete kinetic theory only takes into account the binary collisions.

These models, known as discrete velocity models of the Boltzmann equation, have been studied for some time now, but were introduced by Maxwell.

The general model is written in the form:

$$\frac{\partial f_i}{\partial t} + \vec{v}_i \cdot \nabla_x f_i = G_i(\underset{\sim}{f}, \underset{\sim}{f}) - f_i L_i(\underset{\sim}{f}) \qquad i = 1, 2, \ldots, r, \tag{1.1}$$

where $\vec{v}_i$, $i = 1, 2, \ldots, r$ is the set of the admissible velocities, and $\underset{\sim}{f}(\vec{x}, t) = \{f_1(\vec{x}, t), f_2(\vec{x}, t), \ldots, f_r(\vec{x}, t)\}$ is the r-component vector whose i^{th} component represents the density of the particles with velocity $\vec{v}_i$ is the position $\vec{x}$ at the time t. Both $\vec{x}$ and $\vec{v}$ are referred to an inertial reference frame S with unit vectors $\vec{i}, \vec{j}, \vec{k}$.

In the system (1.1) the gain term G_i and the loss term L_i are defined through the expressions:

$$\begin{cases} G_i(\underset{\sim}{f}, \underset{\sim}{f})(\vec{x}, t) = \dfrac{1}{2} \sum\limits_{j,k,m} A_{ij}^{km} f_k(\vec{x}, t) f_m(\vec{x}, t), \\ L_i(\underset{\sim}{f})(\vec{x}, t) = \dfrac{1}{2} \sum\limits_{j,k,m} A_{ij}^{km} f_j(\vec{x}, t). \end{cases} \tag{1.2}$$

The quantities A_{ij}^{km} are nonnegative constants, linked to the probability that two particles with velocities $\vec{v}_i$ and $\vec{v}_j$ collide and come out of the collision with velocities $\vec{v}_k$ and $\vec{v}_m$. Such constants obey the reversibility hypothesis, $A_{ij}^{km}=A_{km}^{ij}$, and also that of physical consistency $A_{ij}^{kk}=A_{ij}^{mm}=A_{ii}^{km}=A_{jj}^{km}=0$.

Since the collision conserves the momentum and the energy, the constants A_{ij}^{km} are different from zero when

$$\vec{v}_i+\vec{v}_j=\vec{v}_k+\vec{v}_m, \qquad v_i^2+v_j^2=v_k^2+v_m^2,$$

with $i, j, k, m=1, 2, \ldots, r$. Naturally, this condition limits the number of discrete velocity models that can be constructed. Cabannes [4] and Gatignol [6] may help to give a more in-depth study of the various types of models.

Due to the form of the collisional operator, the system (1.1) possesses a certain number of collisional invariants, that is to say:

$$\sum_i \xi_i \frac{1}{2} \sum_{j,k,m} A_{ij}^{km}(f_k f_m - f_i f_j)=0 \tag{1.3}$$

at least, when $\underset{\sim}{\xi}$ is the vector with all equal components, and

$$\underset{\sim}{\xi}_i=\{v_{1,i}; v_{2,i}; \ldots; v_{r,i}\}, \qquad i=1,2,3; \qquad \underset{\sim}{\xi}=\{v_1^2; v_2^2; \ldots; v_r^2\}.$$

This induces, at least from a formal point of view, a priori bounds on the solution. Likewise, as in the classical kinetic theory, if the Boltzmann H-function is defined as:

$$H(\underset{\sim}{f})(t)=\sum_i \int_{\mathbb{R}^3} f_i(\vec{x}, t)\log f_i(\vec{x}, t)d\vec{x}, \tag{1.4}$$

it is proved that, at least on a formal level, the H-function of the solution to the Cauchy problem for the system (1.1) decreases with time.

In spite of its apparent simplicity (with respect to the complete Boltzmann equation), the Cauchy problem to the system (1.1) has not been solved, except in particular cases. Progress in this field has been achieved essentially in the last fifteen years.

The state of research is noticeably different according to the Cauchy problem, whether studied for initial data which depends on a single spatial variable, or for more than one spatial dimension. In the latter case, results on the existence and uniqueness of solutions have been obtained essentially for initial values close to a known solution to the system (1.1). When the identically vanishing solution is considered, the theorems of existence and uniqueness are proved with initial data satisfying suitable smallness conditions [7, 8, 13, 15], i.e. for perturbations of the vacuum.

For initial values close to the equilibrium solutions (globally Maxwellian functions), the corresponding theorem of existence and uniqueness has been proved by Kawashima [11]. In the same paper, an interesting result of existence and uniqueness for initial data close to a spatially homogeneous distribution can be found.

The limitations which are imposed on the initial data, as in the "smallness" condition for the problem of the perturbation of the vacuum, are governed by the fact that the techniques used (fixed-point theorems, iteration scheme of Kaniel and

Shinbrot [10]) are applied to bounded initial data, and in practice they do not take into consideration the loss term of the collisional operator, whose presence implies, through (1.3), the conservation of the mass.

Such an a priori bound to the solution can be used studying the problem in a L_1-setting, where, however, the presence of the quadratic terms on the right-hand side of (1.1) causes a lot of difficulties in proving existence theorems in these spaces.

The results are noticeably better when the system (1.1) is studied with initial data which depends on a single spatial variable. In such a case, in fact, when the initial data are uniformly bounded and integrable, it has been shown, first by Nishida and Mimura [12] for a sufficiently small mass, and second by Crandall and Tartar [14] for masses of any other size, that regular solutions to the Cauchy problem exist globally in time. Crandall and Tartar's argument lies in proving that the problem admits a local solution, and that, as a consequence to the H-theorem (which for uniformly bounded data can be shown to hold rigorously), the solution can be extended globally in time. It should be noted that, for the validity of the H-theorem, the form of the collisional operator plays a fundamental role. Crandall and Tartar's method was originally applied to the Broadwell model, which is one of the more simple discrete velocity models. Then, the same method was applied by Cabannes [4] to more complex models.

Recently, Beale [1, 2] has studied, still with initial data in $L_1 \cap L_\infty$, the asymptotic behaviour of the solution of the system (1.1), showing that this solution manifests a trend towards a state in which each component is a running wave without interactions. Cabannes [5] and Bony [3] have found other results on the asymptotic behaviour of the solution.

In 1984, Illner [9] appropriately adapting the iteration scheme of Kaniel and Shinbrot in the study of the Cauchy problem for the Broadwell model, showed the existence of a global mild solution for initial data in L_1, with small L_1-norm. In addition, if these initial values have a finite entropy, he showed that a global solution exists, independently of the size of the L_1-norm, when the Boltzmann H-theorem holds. Even if it seems reasonable to expect the H-theorem as a consequence of the kinetic equations, Illner did not give a rigorous proof of it, but only a semi-formal discussion.

In this paper the Illner program is completed, showing that the Cauchy problem for a general discrete velocity model of the Boltzmann equation, in one spatial dimension, has a unique global mild solution, provided that the initial values have finite entropy, and belong to a weighted L_1-space. More precisely, let $\varphi_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, \ldots, r$, be the initial densities of the gas, and let us suppose that, for given constant $\alpha > 0$,

$$(1 + |x|^\alpha)\varphi_i(x) \in L_1(\mathbb{R})$$

and

$$\varphi_i(x) \log \varphi_i(x) \in L_1(\mathbb{R}), \qquad i = 1, 2, \ldots, r.$$

Furthermore, for any couple $(\vec{v}_i, \vec{v}_j)$ of possible pre-collisional velocities, let $(\vec{v}_i - \vec{v}_j) \circ \vec{i} \neq 0$.

In Sect. 2, it will be shown (Theorem 1), through a classical fixed-point theorem, that the Cauchy problem for the system (1.1) has a unique local solution.

In Sect. 3, to show rigorously the H-theorem for the solution, we will verify (Lemma 2) that for nonnegative initial data, the local solution is nonnegative and that (Lemma 3) the solution can be uniformly bounded in its interval of existence, from below and above. These bounds are determined by the initial data. Keeping in mind these results, and making use of those techniques already introduced by the author in [16] for the study of the complete Boltzmann equation, the H-theorem will be shown first for the solution of the Cauchy problem for the system (1.1) when the initial data are opportunely bounded away from zero (Lemma 6), and second for any other initial data.

The main result derives from the local existence theorem and from the argument of Crandall and Tartar (Theorem 3).

Other problems are left open. We have not investigated the asymptotic behaviour of the solution. The problem of the extension of the existence theorem, where $(\vec{v}_i - \vec{v}_j) \circ \vec{i}$ vanishes for some couples $(\vec{v}_i, \vec{v}_j)$ of incoming velocities, is also left open. This last problem may be solved using the techniques of Bony [3].

2. The Discrete Velocity Model. Local Existence

The formulation of the initial value problem for a one dimensional in space discrete velocity model of the Boltzmann equation is the following:

$$\begin{cases} \dfrac{\partial f_i}{\partial t} + v_i \dfrac{\partial f_i}{\partial x} = G_i(\underset{\sim}{f}, \underset{\sim}{f}) - f_i L_i(\underset{\sim}{f}) \\ f_i(x,0) = \varphi_i(x); \qquad i = 1, 2, \dots, r. \end{cases} \tag{2.1}$$

All the quantities appearing in (2.1) were defined in the introduction. In addition, $x = \vec{x} \cdot \vec{i} \in \mathbb{R}$, $v_i = \vec{v}_i \cdot \vec{i} \in \mathbb{R}$. In this paper, instead of the formulation (2.1), after integration along the characteristics, we shall consider the weaker form:

$$\begin{cases} \hat{f}_i(x,t) = \varphi_i(x) + \int\limits_0^t \{\hat{G}_i(\underset{\sim}{f}, \underset{\sim}{f})(x,s) - \hat{f}_i(x,s) \hat{L}_i(\underset{\sim}{f})(x,s)\} ds \\ i = 1, 2, \dots, r, \end{cases} \tag{2.2}$$

where, for a given vector $\underset{\sim}{g}(x,t)$, we denoted:

$$\hat{g}_i(x,t) = g_i(x + v_i t, t). \tag{2.3}$$

We will start to solve (2.2) with a few definitions and hypotheses. For each constant $p \geqq 1$ and $\alpha \geqq 0$, let $L_{p,\alpha}$ denote:

$$L_{p,\alpha} = \{f : (1 + |x|^\alpha) f(x) \in L_p(\mathbb{R})\}, \tag{2.4}$$

and, for every fixed $T > 0$, let $\mathscr{B}_{p,\alpha}$ be the Banach space:

$$\mathscr{B}_{p,\alpha} = \{\underset{\sim}{f}(x,t) : \hat{f}_i(x,t) \in C[(0,T); L_{p,\alpha}]; i = 1, 2, \dots, r\} \tag{2.5}$$

endowed with the norm:

$$|||\underset{\sim}{f}|||_{p,\alpha} = \sum_{i=1}^{r} \|\hat{f}_i\|_{p,\alpha}, \tag{2.6}$$

where

$$\|f_i\|_{p,\alpha}=\left\{\int_{-\infty}^{\infty}\left[(1+|x|^{\alpha})\sup_{t\leqq T}|\hat{f}_i(x,t)|\right]^p dx\right\}^{1/p}. \tag{2.7}$$

In all this paper we shall consider only discrete velocity models such that the set of the admissible velocities satisfy the following:

Hypothesis. For every couple $(\vec{v}_i,\vec{v}_j)$ of velocities entering in a collision,

$$|v_i-v_j|>0, \tag{2.8}$$

The above condition is satisfied by many models, as the plane $2r$-velocity model by Gatignol [6], when r is odd, and the spatial eight-velocity model by Cabannes [4]. Other models, like the Broadwell model, are not included.

The proof of the local existence theorem will apply the principle of contraction mapping to (2.2). As a first step toward this, we need a preliminary result.

Let $\mathbb{A}$ be the operator acting on $\mathscr{B}_{p,\alpha}$, whose components are

$$(\widehat{\mathbb{A}\underset{\sim}{f}})_i(x,t)=\varphi_i(x)+\int_0^t\{\hat{G}_i(\underset{\sim}{f},\underset{\sim}{f})(x,s)-\hat{f}_i(x,s)\hat{L}_i(\underset{\sim}{f})(x,s)\}ds. \tag{2.9}$$

Then we prove

Lemma 1. *Let* $\varphi_i(x)\in L_{p,\alpha}$, $i=1,2,\ldots,r$. *Then, if the Hypothesis holds,* $\mathbb{A}$ *maps* $\mathscr{B}_{p,\alpha}$ *into* $\mathscr{B}_{p,\alpha}$ *for every fixed* $T>0$.

Proof. Let $\underset{\sim}{f}\in\mathscr{B}_{p,\alpha}$. Then, for every i and $t<T$:

$$\begin{aligned}|(\widehat{\mathbb{A}\underset{\sim}{f}})_i(x,t)|&\leqq|\varphi_i(x)|\\&+\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\int_0^T|\hat{f}_k(x+(v_i-v_k)s,s)|\,|\hat{f}_m(x+(v_i-v_m)s,s)|ds\\&+\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\int_0^T|\hat{f}_i(x,s)|\,|\hat{f}_j(x+(v_i-v_j)s,s)|ds.\end{aligned}$$

Now, for every $x\in\mathbb{R}$, let us set:

$$F_i(x)=\sup_{t\leqq T}|\hat{f}_i(x,t)|;\qquad i=1,2,\ldots,r. \tag{2.10}$$

Since $\underset{\sim}{f}\in\mathscr{B}_{p,\alpha}$, $F_i(x)\in L_{p,\alpha}$, and for each $x\in\mathbb{R}$ and $t\leqq T$:

$$|\hat{f}_i(x,t)|\leqq F_i(x).$$

Moreover, for any given constant c,

$$|\hat{f}_i(x+ct,t)|\leqq\sup_{s\leqq T}|f_i(x+v_is+ct,s)|=F_i(x+ct). \tag{2.11}$$

By (2.11) we get:

$$\begin{aligned}\sup_{t\leqq T}|(\widehat{\mathbb{A}\underset{\sim}{f}})_i(x,t)|&\leqq|\varphi_i(x)|\\&+\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\int_0^T F_k(x+(v_i-v_k)s)F_m(x+(v_i-v_m)s)ds\\&+\frac{1}{2}F_i(x)\sum_{j,k,m}A_{ij}^{km}\int_0^T F_j(x+(v_i-v_j)s)ds.\end{aligned}$$

Let us multiply both sides of the above inequality by $(1+|x|^\alpha)$ and evaluate the L_p-norm. By applying the Minkowski inequality, we get:

$$\begin{aligned}\|(\underset{\sim}{\mathbb{A}}f)_i\|_{p,\alpha} \leqq \|\varphi_i\|_{p,\alpha} &+ \frac{1}{2}\sum_{j,k,m} A_{ij}^{km} \\ &\times \left\| \int_0^T F_k(x+(v_i-v_k)s)F_m(x+(v_i-v_m)s)ds \right\|_{p,\alpha} \\ &+ \frac{1}{2}\sum_{j,k,m} A_{ij}^{km} \left\| F_i(x) \int_0^T F_j(x+(v_i-v_j)s)ds \right\|_{p,\alpha}.\end{aligned} \tag{2.12}$$

Let us consider first the last term on the right-hand side of (2.12). By using the Hölder inequality we obtain:

$$\begin{aligned}\int_0^T F_j(x+(v_i-v_j)s)ds &= (v_i-v_j)^{-1} \int_x^{x+(v_i-v_j)T} F_j(s)ds \\ &\leqq \frac{1}{2} T^{\frac{p-1}{p}} |v_i-v_j|^{-1/p} \|F_j\|_{p,0}.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{2}\sum_{j,k,m} A_{ij}^{km} \left\| F_i(x) \int_0^T F_j(x+(v_i-v_j)s)ds \right\|_{p,\alpha} \\ &\quad \leqq \frac{1}{4} T^{\frac{p-1}{p}} \|F_i\|_{p,\alpha} \sum_{j,k,m} A_{ij}^{km} |v_i-v_j|^{-\frac{1}{p}} \|F_j\|_{p,0}.\end{aligned}$$

In order to find an upper bound for the second term on the right-hand side of (2.12), we need a simple further inequality. First, consider that, when v_i-v_k or v_i-v_m are equal to zero, the upper bound has been found above. Therefore, suppose that $v_i-v_k \neq 0$ and at the same time $v_i-v_m \neq 0$ (the Hypothesis implies that v_i-v_k and v_i-v_m cannot be simultaneously zero). In this case it is a simple matter to show that, for every $x \in \mathbb{R}$ and $t \leqq T$:

$$x^2 \leqq c_{i,k,m}\{[x+(v_i-v_k)t]^2 + [x+(v_i-v_m)t]^2\}, \tag{2.13}$$

the constant $c_{i,k,m}$ being defined as follows:

$$c_{i,k,m} = \max\left\{\frac{1}{2}; \frac{(v_i-v_k)^2+(v_i-v_m)^2}{(v_k-v_m)^2}\right\}. \tag{2.14}$$

Inequality (2.13) allows us to conclude that, for a suitable constant $c_{i,k,m}^{(\alpha)}$,

$$\begin{aligned}1+|x|^\alpha \leqq c_{i,k,m}^{(\alpha)}\{&[1+|x+(v_i-v_k)t|^\alpha] \\ &+[1+|x+(v_i-v_m)t|^\alpha]\}.\end{aligned}$$

By virtue of the last inequality, and by applying the Fubini's theorem, we get

$$\left\{\int_{-\infty}^{\infty}\left[(1+|x|^\alpha)\int_0^T F_k(x+(v_i-v_k)s)\right.\right.$$
$$\left.\left.\times F_m(x+(v_i-v_m)s)ds\right]^p dx\right\}^{1/p}$$
$$\leqq\left\{\int_{-\infty}^{\infty}\left[c_{i,k,m}^{(\alpha)}\int_0^T(1+|x+(v_i-v_k)s|^\alpha)\right.\right.$$
$$\left.\left.\times F_k(x+(v_i-v_k)s)F_m(x+(v_i-v_m)s)ds\right]^p dx\right\}^{1/p}$$
$$+\left\{\int_{-\infty}^{\infty}\left[c_{i,k,m}^{(\alpha)}\int_0^T(1+|x+(v_i-v_m)s|^\alpha)\right.\right.$$
$$\left.\left.\times F_m(x+(v_i-v_m)s)F_k(x+(v_i-v_k)s)ds\right]^p dx\right\}^{1/p}$$
$$=c_{i,k,m}^{(\alpha)}\left\{\int_{-\infty}^{\infty}[(1+|x|^\alpha)F_k(x)]^p\left[\int_0^T F_m(x+(v_i-v_j)s)ds\right]^p dx\right\}^{1/p}$$
$$+c_{i,k,m}^{(\alpha)}\left\{\int_{-\infty}^{\infty}[(1+|x|^\alpha)F_m(x)]^p\left[\int_0^T F_k(x+(v_i-v_j)s)ds\right]^p dx\right\}^{1/p}$$
$$\leqq c_{i,k,m}^{(\alpha)}[\|F_k\|_{p,\alpha}\|F_m\|_{p,0}+\|F_m\|_{p,\alpha}\|F_k\|_{p,0}]\,|v_i-v_j|^{-\frac{1}{p}}\cdot\tfrac{1}{2}T^{\frac{p-1}{p}}.$$

Finally,

$$\|(\mathbb{A}\underset{\sim}{f})_i\|_{p,\alpha}\leqq\|\varphi_i\|_{p,\alpha}+\frac{1}{2}T^{\frac{p-1}{p}}\sum_{j,k,m}A_{ij}^{km}|v_i-v_j|^{-\frac{1}{p}}$$
$$\times\{\|F_i\|_{p,\alpha}\|F_j\|_{p,0}+\|F_k\|_{p,\alpha}\cdot\|F_m\|_{p,0}+\|F_m\|_{p,\alpha}\|F_k\|_{p,0}\}.\qquad\square$$

From now on, μ shall denote the Lebesgue measure on $\mathbb{R}$, and $\mathscr{L}(\mathbb{R})$ the class of all Borel sets.

Given a vector function $\underset{\sim}{g}(x,t)\in\mathscr{B}_{p,\alpha}$, let us set:

$$\lambda_{p,\alpha}(g_i)(\beta)=\sup_{B\in\mathscr{L}(\mathbb{R});\,\mu(B)=\beta}\left\{\int_B\left[(1+|x|^\alpha)\sup_{t\leqq T}|\hat{g}_i(x,t)|\right]^p dx\right\}^{1/p}.\qquad(2.15)$$

Let us introduce the subset $\mathscr{D}_{p,\alpha}$ of $\mathscr{B}_{p,\alpha}$ by:

$$\mathscr{D}_{p,\alpha}=\left\{\underset{\sim}{f}\in\mathscr{B}_{p,\alpha}:\sum_i\lambda_{p,\alpha}(f_i)(\beta)\leqq 2\sum_i\lambda_{p,\alpha}(\varphi_i)(\beta);\forall\beta>0\right\}.$$

It is straightforward to show that $\mathscr{D}_{p,\alpha}$ is a closed convex set. The closedness property of $\mathscr{D}_{p,\alpha}$ can be shown by contradiction. Let in fact $\{\underset{\sim}{f_n}\}_{n\geqq 1}$ be a sequence of $\mathscr{D}_{p,\alpha}$, which converges in $\mathscr{B}_{p,\alpha}$ to $\underset{\sim}{f}$, and suppose that $\underset{\sim}{f}\notin\mathscr{D}_{p,\alpha}$.

This means that there exists a set $\tilde{B}\in\mathscr{L}(\mathbb{R})$, with $\mu(\tilde{B})=\beta$, such that

$$\sum_i\left\{\int_B\left[(1+|x|^\alpha)\sup_{t\leqq T}|\hat{f}_i(x,t)|\right]^p dx\right\}^{1/p}>c_\beta=2\sum_i\lambda_{p,\alpha}(\varphi_i)(\beta).$$

Let $\chi(B)$ denote the characteristic function of the set B. Then, if we set $\underset{\sim}{g}_n=\underset{\sim}{f}_n\cdot\chi(B)$, $\underset{\sim}{g}_n$ converges in $\mathscr{B}_{p,\alpha}$ to $\underset{\sim}{g}=\underset{\sim}{f}\chi(B)$, with $|||\underset{\sim}{g}_n|||_{p,\alpha}\leqq c_\beta$, whenever $|||\underset{\sim}{g}|||_{p,\alpha}>c_\beta$, and the contradiction arises.

The local existence result for the system (2.2) is contained in the following:

Theorem 1. *Let $\varphi_i\in L_{p,\alpha}$, $i=1,2,\ldots,r$. Then, if the Hypothesis holds, there exists a time $T_{p,\alpha}>0$ such that, when $T\leqq T_{p,\alpha}$, $\mathbb{A}$ is a contraction mapping from $\mathscr{D}_{p,\alpha}$ into $\mathscr{D}_{p,\alpha}$.*

Proof. In Lemma 1, we proved that $\mathbb{A}$ maps $\mathscr{B}_{p,\alpha}$ into $\mathscr{B}_{p,\alpha}$. Let now $\underset{\sim}{f}\in\mathscr{D}_{p,\alpha}$. If $B\in\mathscr{L}(\mathbb{R})$, and $\mu(B)=\beta$,

$$\left\{\int_B\left[(1+|x|^\alpha)\sup_{t\leqq T}|(\hat{\mathbb{A}}\underset{\sim}{f})_i(x,t)|\right]^p dx\right\}^{1/p}\leqq\lambda_{p,\alpha}(\varphi_i)(\beta)$$
$$+\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\left\{\int_B\left[(1+|x|^\alpha)\int_0^T F_k(x+(v_i-v_k)s)\right.\right.$$
$$\left.\left.\times F_m(x+(v_i-v_m)s)ds\right]^p dx\right\}^{1/p}$$
$$+\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\left\{\int_B\left[(1+|x|^\alpha)F_i(x)\int_0^T F_j(x+(v_i-v_j)s)ds\right]^p dx\right\}^{1/p}.$$

Let us set

$$V=\max_{i,j;i\neq j}|v_i-v_j|. \tag{2.16}$$

If we proceed as in Lemma 1, we obtain the bound:

$$\int_0^T F_j(x+(v_i-v_j)s)ds\leqq 2T^{\frac{p-1}{p}}|v_i-v_j|^{-\frac{1}{p}}\sum_i\lambda_{p,\alpha}(\varphi_i)(VT).$$

Thus

$$\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\left\{\int_B\left[(1+|x|^\alpha)F_i(x)\int_0^T F_j(x+(v_i-v_j)s)ds\right]^p dx\right\}^{1/p}$$
$$\leqq 2\left\{\sum_{j,k,m}A_{ij}^{km}|v_i-v_j|^{-\frac{1}{p}}\sum_i\lambda_{p,\alpha}(\varphi_i)(VT)\cdot T^{\frac{p-1}{p}}\right\}\sum_i\lambda_{p,\alpha}(\varphi_i)(\beta),$$

and

$$\left\{\int_B\left[(1+|x|^\alpha)\int_0^T F_k(x+(v_i-v_k)s)F_m(x+(v_i-v_m)s)ds\right]^p dx\right\}^{1/p}$$
$$\leqq 8c_{i,k,m}^{(\alpha)}|v_i-v_j|^{-\frac{1}{p}}\cdot T^{\frac{p-1}{p}}\sum_i\lambda_{p,\alpha}(\varphi_i)(VT)\sum_i\lambda_{p,\alpha}(\varphi_i)(\beta).$$

By grouping all these inequalities, and by taking the sum over i:

$$\sum_i\lambda_{p,\alpha}((\mathbb{A}\underset{\sim}{f})_i)(\beta)\leqq\sum_i\lambda_{p,\alpha}(\varphi_i)(\beta)$$
$$\times\left\{1+T^{\frac{p-1}{p}}\sum_i\lambda_{p,\alpha}(\varphi_i)(VT)\right.$$
$$\left.\times\sum_{i,j,k,m}A_{ij}^{km}|v_i-v_j|^{-\frac{1}{p}}(2+8c_{i,k,m}^{(\alpha)})\right\}.$$

Let us fix a constant $\delta<1$. By virtue of the absolute continuity of $\lambda_{p,\alpha}(\varphi_i)(\beta)$ with respect to β, when T is sufficiently small,

$$T^{\frac{p-1}{p}} \sum_i \lambda_{p,\alpha}(\varphi_i)(VT) \times \sum_{i,j,k,m} A_{ij}^{km}|v_i-v_j|^{-\frac{1}{p}}(2+8c_{i,k,m}^{(\alpha)}) \leqq \delta < 1\,. \tag{2.17}$$

Denoting by $T_{p,\alpha}$ the supremum of the times in (2.17), we conclude that, when $T \leqq T_{p,\alpha}$, $\mathbb{A}$ maps $\mathscr{D}_{p,\alpha}$ into $\mathscr{D}_{p,\alpha}$.

It remains to prove that $\mathbb{A}$ is a contraction mapping on $\mathscr{D}_{p,\alpha}$. To this end, let $\underset{\sim}{f}, \underset{\sim}{g} \in \mathscr{D}_{p,\alpha}$. Then

$$\begin{aligned}
&|(\mathbb{A}\hat{\underset{\sim}{f}})_i(x,t)-(\mathbb{A}\hat{\underset{\sim}{g}})_i(x,t)| \\
&\leqq \frac{1}{2}\sum_{j,k,m} A_{ij}^{km}\int_0^T \{|\hat{f}_k(x+(v_i-v_k)s,s) \\
&\quad -\hat{g}_k(x+(v_i-v_k)s,s)|\,|\hat{f}_m(x+(v_i-v_m)s,s)|+|\hat{f}_m(x+(v_i-v_m)s,s) \\
&\quad -\hat{g}_m(x+(v_i-v_m)s,s)|\,|\hat{g}_k(x+(v_i-v_k)s,s)|\}ds \\
&\quad +\frac{1}{2}\sum_{j,k,m} A_{ij}^{km}\int_0^T \{|\hat{f}_i(x,s)-\hat{g}_i(x,s)|\,|\hat{f}_j(x+(v_i-v_j)s,s)| \\
&\quad +|\hat{f}_j(x+(v_i-v_j)s,s)-\hat{g}_j(x+(v_i-v_j)s,s)|\,|\hat{g}_i(x,s)|\}ds\,.
\end{aligned}$$

If $i \neq m \neq k$,

$$\begin{aligned}
&\left\{\int_{-\infty}^{\infty}\left[(1+|x|^\alpha)\int_0^T |\hat{f}_k(x+(v_i-v_k)s,s)\right.\right. \\
&\quad \left.\left. -\hat{g}_k(x+(v_i-v_k)s,s)|\,|\hat{f}_m(x+(v_i-v_m)s,s|ds\right]^p dx\right\}^{1/p} \\
&\leqq 2c_{i,k,m}^{(\alpha)}\{\|f_k-g_k\|_{p,\alpha}\cdot 2|v_i-v_j|^{-\frac{1}{p}}T^{\frac{p-1}{p}}\}\sum_i \lambda_{p,\alpha}(\varphi_i)(VT)\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\{\int_{-\infty}^{\infty}\left[(1+|x|^\alpha)\int_0^T |\hat{f}_i(x,s)-\hat{g}_i(x,s)|\,|\hat{f}_j(x+(v_i-v_j)s,s)|ds\right]dx\right\}^{1/p} \\
&\leqq \|f_i-g_i\|_{p,\alpha}T^{\frac{p-1}{p}}|v_i-v_j|^{-\frac{1}{p}}\cdot 2\cdot\sum_i \lambda_{p,\alpha}(\varphi_i)(VT)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\int_{-\infty}^{\infty}\left[(1+|x|^\alpha)\int_0^T |\hat{f}_j(x+(v_i-v_j)s,s)\right.\right. \\
&\quad \left.\left. -\hat{g}_j(x+(v_i-v_j)s,s)|\,|\hat{g}_i(x,s)|ds\right]^p dx\right\}^{1/p} \\
&\leqq \left\{\int_{-\infty}^{\infty}\left[(1+|x|^\alpha)\sup_{t\leqq T}|\hat{f}_j(x,t)-\hat{g}_i(x,t)|\right.\right. \\
&\quad \left.\left.\times\int_0^T (1+|x-(v_i-v_j)s|^\alpha)|\hat{g}_i(x-(v_i-v_j)s,s)|ds\right]^p dx\right\}^{1/p} \\
&\leqq \|f_j-g_j\|_{p,\alpha}|v_i-v_j|^{-\frac{1}{p}}T^{\frac{p-1}{p}}\cdot 2\cdot\sum_i \lambda_{p,\alpha}(\varphi_i)(VT)\,.
\end{aligned}$$

Therefore

$$|||\mathbb{A}\underset{\sim}{f}-\mathbb{A}\underset{\sim}{g}|||_{p,\alpha}\leqq|||\underset{\sim}{f}-\underset{\sim}{g}|||_{p,\alpha}\cdot T^{\frac{p-1}{p}}\sum_i \lambda_{p,\alpha}(\varphi_i)(VT)$$
$$\times\sum_{i,j,k,m} A_{ij}^{km}|v_i-v_j|^{-\frac{1}{p}}(2+8c_{i,k,m}^{(\alpha)}),$$

so that, if $T\leqq T_{p,\alpha}$,

$$|||\mathbb{A}\underset{\sim}{f}-\mathbb{A}\underset{\sim}{g}|||_{p,\alpha}\leqq\delta|||\underset{\sim}{f}-\underset{\sim}{g}|||. \quad \square$$

Via the contraction mapping theorem, we stated that system (2.2) has a local in time solution, provided that the initial value $\underset{\sim}{\varphi}$ is in $L_{p,\alpha}$. Let us remark that the result of Theorem 1 is quite general with respect to what we need to prove the main theorem of Sect. 5.

3. Nonnegative Solutions

In this section we shall prove that the local solution to system (2.2) is nonnegative when the initial values are. This result will be achieved by means of a classical argument that has been applied before to nonnegative bounded initial values.

Let us suppose that $\varphi_i(x)\in L_{p,\alpha}\cap L_\infty$; $i=1,2,\ldots,r$. Then, it is a simple matter to verify that, for every $T>0$, $\mathbb{A}$ maps $\ell_{p,\alpha}$ into $\ell_{p,\alpha}$, if

$$\ell_{p,\alpha}=\left\{\underset{\sim}{f}\in\mathscr{B}_{p,\alpha}:\sup_{t\leqq T}|\hat{f}_i(x,t)|\in L_{p,\alpha}\cap L_\infty;\ i=1,2,\ldots,r\right\}. \tag{3.1}$$

This implies that the local solution $\underset{\sim}{f}(x,t)$ we found in Theorem 1 lies in $\ell_{p,\alpha}$. Let us set

$$K=\max_{i\leqq r}\operatorname*{ess\,sup}_{x\in\mathbb{R}}\sup_{t\leqq T}|\hat{f}_i(x,t)|, \tag{3.2}$$

and introduce, for $\gamma>0$, the functions $\phi_i(x,t)=f_i(x,t)\exp(\gamma t)$. Obviously, system (2.1) can be written in equivalent form in terms of the functions ϕ_i,

$$\begin{cases}\dfrac{\partial\phi_i}{\partial t}+v_i\dfrac{\partial\phi_i}{\partial x}=\gamma\phi_i+e^{-\gamma t}\{G_i(\underset{\sim}{\phi},\underset{\sim}{\phi})-\phi_iL_i(\underset{\sim}{\phi})\}\\ \phi_i(x,0)=\varphi_i(x);\qquad i=1,2,\ldots,r\end{cases} \tag{3.3}$$

or, in integral form

$$\begin{aligned}\hat{\phi}_i(x,t)=\varphi_i(x)+\int_0^t e^{-\gamma s}\cdot\frac{1}{2}\sum_{j,k,m}A_{ij}^{km}\hat{\phi}_k(x+(v_i-v_k)s,s)\\ \times\hat{\phi}_m(x+(v_i-v_m)s,s)ds\\ +\int_0^t\hat{\phi}_i(x,s)\left[\gamma-\frac{1}{2}e^{-\lambda s}\sum_{j,k,m}A_{ij}^{km}\hat{\phi}_j(x+(v_i-v_j)s,s)\right]ds.\end{aligned} \tag{3.4}$$

Choose $\gamma\geqq\frac{1}{2}K\sum_{i,j,k,m}A_{ij}^{km}$; then, since the functions into the integral appearing in (3.4) are nonnegative at time $t=0$, both these functions and the densities ϕ_i remain positive for $t>0$.

By virtue of the equivalence of systems (2.2) and (3.4), it follows that, if $0 \leqq \varphi_i(x) \in L_{p,\alpha} \cap L_\infty$, the local solution to system (2.2) is nonnegative. Keeping this in mind, for given nonnegative initial values $\varphi_i(x) \in L_{p,\alpha}$, let us introduce the sequences

$$\varphi_i^{(n)}(x) = \min\{n; \varphi_i(x)\}; \qquad i = 1, 2, \ldots, r; \qquad n \geqq 1,$$

and let $\underset{\sim}{f}^{(n)}$ be the local solution to system (2.2) with initial datum $\underset{\sim}{\varphi}^{(n)}$. Since $\varphi_i^{(n)}(x) \leqq \varphi_i(x)$, according to the result of Theorem 1, this solution exists at least in the same interval $[0, T_{p,\alpha}]$ in which system (2.2) has a solution with $\underset{\sim}{\varphi}$ as initial datum. Moreover,

$$|||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{p,\alpha} \leqq |||\underset{\sim}{\varphi}^{(n)} - \underset{\sim}{\varphi}||_{p,\alpha} + \delta |||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{p,\alpha} \tag{3.5}$$

with $\delta < 1$.

The above inequality is easily derived if we consider that $\underset{\sim}{f}^{(n)}$ and $\underset{\sim}{f}$ are functions of $\mathscr{D}_{p,\alpha}$, and the operator $\mathbb{A}$ is a contraction mapping on $\mathscr{D}_{p,\alpha}$. On the other hand, with a simple application of the dominated convergence theorem we can prove that $|||\underset{\sim}{\varphi}^{(n)} - \underset{\sim}{\varphi}|||_{p,\alpha} \to 0$, as $n \to \infty$, so that by (3.5), $|||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{p,\alpha} \to 0$. This proves that $f_i(x,t) \geqq 0$ a.s.

In conclusion we have:

Lemma 2. *Let $0 \leqq \varphi_i(x) \in L_{p,\alpha}$, $i = 1, 2, \ldots, r$. Then, the solution to system* (2.2) *is nonnegative in its interval of existence.* □

Owing to the positivity of the local solution, we can derive some useful bounds, from above and below, for the solution itself.

In Theorem 1, we did not take advantage of the minus sign in front of the loss term $f_i L_i(\underset{\sim}{f})$. This means that, in effect, we proved Theorem 1 also for the system:

$$\begin{cases} \dfrac{\partial f_i}{\partial t} + v_i \dfrac{\partial f_i}{\partial x} = G_i(\underset{\sim}{f}, \underset{\sim}{f}) + f_i L_i(\underset{\sim}{f}) \\ f_i(x, 0) = \varphi_i(x) \qquad i = 1, 2, \ldots, r. \end{cases} \tag{3.6}$$

so that this system has a unique local solution $\underset{\sim}{f}^+$ in the interval $[0, T_{p,\alpha}]$.

Both the mild solutions to systems (2.2) and (3.6) can be found by iteration; the solution $\underset{\sim}{f}$ is the limit of the sequences

$$\hat{f}_{i,0}(x,t) = \varphi_i(x),$$

$$\hat{f}_{i,n}(x,t) = \varphi_i(x) + \int_0^t \{\hat{G}_i(\underset{\sim}{f}_{n-1}, \underset{\sim}{f}_{n-1}) - \hat{f}_{i,n-1} \hat{L}_i(\underset{\sim}{f}_{n-1})\}(x,s)ds,$$

whenever the solution $\underset{\sim}{f}^+$ is the limit of the sequences:

$$\hat{f}_{i,0}^+(x,t) = \varphi_i(x),$$

$$\hat{f}_{i,n}^+(x,t) = \varphi_i(x) + \int_0^t \{\hat{G}_i(\underset{\sim}{f}_{n-1}^+, \underset{\sim}{f}_{n-1}^+) + \hat{f}_{i,n-1}^+ \hat{L}_i(\underset{\sim}{f}_{n-1}^+)\}(x,s)ds.$$

Since $f_{i,n}(x,t) \leqq f_{i,n}^+(x,t)$, $n \geqq 0$, taking the limits we have, for every $x \in \mathbb{R}$ and $t \leqq T_{p,\alpha}$:

$$0 \leqq f_i(x,t) \leqq f_i^+(x,t); \qquad i = 1, 2, \ldots, r. \tag{3.7}$$

Consider now the initial value problem (2.2) with initial value $\underset{\sim}{\varphi}_1$ such that $\varphi_{1,i}(x) \leqq \varphi_i(x)$. Then, system (3.6) has a unique solution $\underset{\sim}{f}_1$ at least in the same interval $[0, T_{p,\alpha}]$, and in this interval $f_{1,i}(x,t) < f_{1,i}^+(x,t)$. Therefore, system (2.2), with initial value $\underset{\sim}{\varphi}_1$, has a solution $\underset{\sim}{f}_1$ that satisfies

$$0 \leqq f_{1,i}(x,t) \leqq f_i^+(x,t); \qquad i = 1, 2, \ldots, r. \tag{3.8}$$

Finally, by virtue of the positivity, consider that the solution to system (2.2) satisfies, in $[0, T_{p,\alpha}]$:

$$\begin{aligned} \hat{f}_i(x,t) &\geqq \varphi_i(x) \exp\left\{ -\int_0^t \sum_{j,k,m} A_{ij}^{km} \hat{f}_j(x + (v_i - v_j)s, s) ds \right\} \\ &\geqq \varphi_i(x) \exp\left\{ -\int_0^{T_{p,\alpha}} \sum_{j,k,m} A_{ij}^{km} F_j^+(x + (v_i - v_j)s) ds \right\} \\ &\geqq \varphi_i(x) \exp\{-D\}, \end{aligned} \tag{3.9}$$

the function F_i^+ and the constant D being defined as follows:

$$\begin{aligned} F_i^+(x) &= \sup_{t \leqq T_{p,\alpha}} \hat{f}_i^+(x,t), \\ D &= 2 \sum_{i,j,k,m} A_{ij}^{km} \|\varphi_i\|_{p,0}. \end{aligned} \tag{3.10}$$

Let us group all we have derived into the following:

Lemma 3. *Let $0 \leqq \varphi_i(x) \in L_{p,\alpha}$, $i = 1, 2, \ldots, r$. Then, the local solution $\underset{\sim}{f}$ to system* (2.2) *satisfies, in its interval of existence:*

$$\varphi_i(x) \exp\{-D\} \leqq \hat{f}_i(x,t) \leqq \hat{f}_i^+(x,t); \qquad i = 1, 2, \ldots, r, \tag{3.11}$$

where D is given by (3.10) *and $\underset{\sim}{f}^+$ is the solution of system* (3.6) *with initial values $\varphi_i(x)$, $i = 1, 2, \ldots, r$. Moreover, let $0 \leqq \varphi_{1,i}(x) \leqq \varphi_i(x)$, $i = 1, 2, \ldots, r$. Then, if $\underset{\sim}{f}_1$ is the local solution of system* (2.2) *with $\underset{\sim}{\varphi}_1$ as initial value:*

$$f_{1,i}(x,t) \leqq f_i^+(x,t); \qquad i = 1, 2, \ldots, r. \quad \square \tag{3.12}$$

4. The *H*-Theorem

As a second step toward obtaining global solutions to the initial value problem (2.1), we need to show that the local solution satisfies the Boltzmann H-theorem. As explained in the introduction, given the solution $\underset{\sim}{f}(x,t)$ of system (2.2), the H-function is defined by:

$$H(\underset{\sim}{f})(t) = \sum_i \int_{-\infty}^{\infty} f_i(x,t) \log f_i(x,t) dx. \tag{4.1}$$

The Boltzmann H-theorem states that H is a nonincreasing function of the time.

In the rest of the paper we will fix $0 < \alpha < 1$. In this section, we will prove that, if $0 \leqq \varphi_i(x) \in L_{1,\alpha}$, and $\varphi_i \log \varphi_i \in L_1$, the H-function of the solution to system (2.2) is uniformly bounded, in its interval of existence, by $H(\underset{\sim}{\varphi})$. To prove this, we need some preliminary lemmas. The first is of independent interest.

Lemma 4. *Let* $\varphi_i(x) \in L_{1,\alpha}$, $i = 1, 2, \ldots, r$, *and in addition*

$$0 \leqq \varphi_i(x) \leqq c(1+x^2)^{-1}, \qquad i = 1, 2, \ldots, r$$

for some constant c. *Then, in* $[0, T_{1,\alpha}]$:

$$\hat{f}_i(x,t) \leqq k_c(1+x^2)^{-1}; \qquad i = 1, 2, \ldots, r; \qquad k_c < \infty. \tag{4.2}$$

Proof. The result follows from a standard domain of dependence argument. By virtue of the finite velocity of propagation, the solution at the point (x,t) does not know what the initial data look like outside of the interval $(x - VT_{1,\alpha}, x + VT_{1,\alpha})$. Let us break the real line into three intervals: $(-\infty, -\varrho)$; $[-\varrho, +\varrho]$ and $(+\varrho, +\infty)$, where $\varrho > 0$ is a suitable constant to be determined later.

Since the solution $\underset{\sim}{f}(x,t)$, where $x \geqq \varrho + VT_{1,\alpha}$ does not depend on the values of the initial datum outside the interval $(\varrho, +\infty)$, we will study the Cauchy problem (2.2) first when $\varphi_i(x)$ is modified outside that interval. We choose these initial values, say $\varphi_i^\varrho(x)$, in order to satisfy the inequalities

$$\varphi_i^\varrho(x) \leqq h_\varrho(x); \qquad i = 1, 2, \ldots, r, \tag{4.3}$$

where

$$\begin{cases} h_\varrho(x) = c(1+\varrho^2)^{-1} & \text{if} \quad -\varrho \leqq x \leqq \varrho \\ h_\varrho(x) = c(1+x^2)^{-1} & \text{if} \quad x < -\varrho \quad \text{or} \quad x > \varrho. \end{cases} \tag{4.4}$$

To this point, consider that, for all constants a, b, with $b \neq 0$, the following inequalities hold:

$$\int_0^t h_\varrho(x+as) h_\varrho(x+bs)\, ds \leqq h_\varrho(x) \int_0^{T_{1,\alpha}} h_\varrho(x+bs)\, ds \tag{4.5}$$

if $x \in [-\varrho, \varrho]$, whenever

$$\begin{aligned} &\int_0^t h_\varrho(x+as) h_\varrho(x+bs)\, ds \\ &\quad \leqq \int_0^{T_{1,\alpha}} h_\varrho(x) \frac{h_\varrho(x+as)}{h_\varrho(x)} h_\varrho(x+bs)\, ds \\ &\quad \leqq h_\varrho(x) \left(\frac{aT_{1,\alpha}}{2} + \sqrt{1 + \frac{a^2 T_{1,\alpha}^2}{4}} \right) \cdot \int_0^{T_{1,\alpha}} h_\varrho(x+bs)\, ds, \end{aligned} \tag{4.6}$$

if $|x| \in (\varrho, +\infty)$.

This proves that the operator $\mathbb{A}$ maps $\mathscr{J}$ into $\mathscr{J}$, if

$$\mathscr{J} = \{\underset{\sim}{f} \in \mathscr{B}_{1,\alpha} : |\hat{f}_i(x,t)| \leqq d h_\varrho(x) \text{ for some constant } d;\ i = 1, 2, \ldots, r\}.$$

In fact, when the Hypothesis is satisfied, and $\underset{\sim}{f} \in \mathscr{J}$, the application of (4.5) and (4.6) gives:

$$\begin{aligned} &\int_0^t |\hat{f}_k(x+(v_i-v_k)s, s)|\, |\hat{f}_m(x+(v_i-v_m)s, s)|\, ds \\ &\quad \leqq \int_0^t d^2 h_\varrho(x+(v_i-v_k)s) h_\varrho(x+(v_i-v_m)s)\, ds \leqq d' h_\varrho(x), \end{aligned}$$

if

$$d' = d^2 \int_{-\infty}^{\infty} h_\varrho(s)ds \cdot \max_{i,j;\, i \neq j} \left\{ 1; \frac{|v_i - v_j| T_{1,\alpha}}{2} + \sqrt{1 + \frac{|v_i - v_j|^2 T_{1,\alpha}^2}{4}} \right. .$$

In the above inequality we fixed $v_i - v_m \neq 0$.

Finally, for a suitable constant c_1 depending on $T_{1,\alpha}$ and on the velocities v_i,

$$\int_0^t \hat{G}_i(\underset{\sim}{f}, \underset{\sim}{f})(x,s)ds \leqq c_1 d^2 h_\varrho(x) \cdot \int_{-\infty}^{\infty} h_\varrho(s)ds . \tag{4.7}$$

In the same way, for another constant c_2 depending only on $T_{1,\alpha}$ and on the velocities v_i,

$$\int_0^t \hat{f}_i(x,s) \hat{L}_i(\underset{\sim}{f})(x,s)ds \leqq c_2 d^2 h_\varrho(x) \int_{-\infty}^{\infty} h_\varrho(s)ds . \tag{4.8}$$

By (4.7) and (4.8) we conclude that $\mathbb{A}$ maps $\mathscr{J}$ into $\mathscr{J}$. Since when $\varrho \to \infty$, $\int_{-\infty}^{\infty} h_\varrho(s)ds$ goes to zero, we can choose ϱ in such a way that $\mathbb{A}$ maps $\mathscr{G}$ into $\mathscr{G}$, and is a contraction mapping on $\mathscr{G}$, if

$$\mathscr{G} = \{ \underset{\sim}{f} \in \mathscr{J} : 0 \leqq \hat{f}_i(x,t) \leqq 2h_\varrho(x);\ i = 1, 2, \ldots, r \} .$$

This follows from the fact that c_1 and c_2 do not depend on ϱ, whereas $\int_{-\infty}^{\infty} h_\varrho(s)ds$ depends on ϱ and decreases with ϱ. We proved that the solution to problem (2.2) satisfies, in the interval $(\varrho, +\infty)$, and for ϱ sufficiently great, but finite,

$$\hat{f}_i(x,t) \leqq 2c(1+x^2)^{-1}; \quad i = 1, 2, \ldots, r; \quad t \in [0, T_{1,\alpha}] .$$

The identical bound holds when $x \in (-\infty; -\varrho)$.

Now, let $x \in [-(\varrho + VT_{1,\alpha}), +\varrho + VT_{1,\alpha}]$. Since the solution depends only on the initial values in the interval $[-(\varrho + 2VT_{1,\alpha}), \varrho + 2VT_{1,\alpha}]$, we can take these initial values to be zero outside this interval, without affecting the solution.

Suppose that we can find a set $B \in \mathscr{L}(\mathbb{R})$ of Lebesgue measure $\mu_0 > 0$, such that, for $x \in B$ and for some $t \leqq T_{1,\alpha}$ and $i \leqq r$:

$$\hat{f}_i(x,t) \geqq 2cr(\varrho + 2VT_{1,\alpha})^3 (1+x^2)^{-1} .$$

Then

$$\sup_{E \in \mathscr{L}(\mathbb{R});\, \mu(E) = \mu_0} \int_E \sup_{t \leqq T_{1,\alpha}} \hat{f}_i(x,t)dx$$

$$\geqq 2 \int_{\varrho + VT_{1,\alpha} - \mu_0/2}^{\varrho + VT_{1,\alpha}} 2cr(\varrho + 2VT_{1,\alpha})^3 (1+x^2)^{-1} dx \geqq \mu_0 \frac{2cr(\varrho + 2VT_{1,\alpha})^3}{1 + (\varrho + VT_{1,\alpha})^2}$$

whenever, for each $E \in \mathscr{L}(\mathbb{R})$, with $\mu(E) = \mu_0$:

$$\int_E \varphi_i(x)dx \leqq \int_{-\mu_0/2}^{\mu_0/2} c(1+x^2)^{-1} dx \leqq 2c\mu_0 ,$$

so that

$$\sum_i \lambda_{1,0}(\varphi_i)(\mu_0) \leqq 2cr\mu_0 .$$

Choosing $\varrho + VT_{1,\alpha} > 3$, at the same time we have

$$\sum_i \lambda_{1,0}(f_i)(\mu_0) \geqq 2cr\mu_0 \frac{27}{10} > 4cr\mu_0,$$

and this contradicts the result of Theorem 1. □

Before we consider the validity of the H-theorem, we must prove:

Lemma 5. *Let $0 \leqq \varphi_i(x) \in L_{1,\alpha}$, $i = 1, 2, \ldots, r$; $\alpha < 1$. Then, there exists a constant d such that, for every constant $b \leqq d$, the Cauchy problem (2.2), with initial values*

$$\psi_i(x) = \max\{\varphi_i(x); b(1+x^2)^{-1}\}$$

has a local solution in $[0, T_{1,\alpha}]$.

Proof. Let δ be the constant appearing in (2.16), and let us put:

$$d = \frac{1-\delta}{2}\left\{r \sup_{B \in \mathscr{L}(\mathbb{R});\, \mu(B) = VT_{1,\alpha}} \int_B \frac{1+|x|^\alpha}{1+x^2}\, dx \cdot k_\alpha\right\}^{-1}$$

$$k_\alpha = \sum_{i,j,k,m} A_{ij}^{km} |v_i - v_j|^{-1} (2 + 8c_{i,k,m}^{(\alpha)}).$$

Then,

$$k_\alpha \sum_i \lambda_{1,\alpha}(\psi_i)(VT_{1,\alpha}) \leqq k_\alpha \sum_i \lambda_{1,\alpha}(\varphi_i)(VT_{1,\alpha}) + \frac{1-\delta}{2} = \delta + \frac{1-\delta}{2} < 1.$$

Thus, the conclusions of Theorem 1 follow. □

Given the initial value $\underset{\sim}{\varphi}$, satisfying the hypotheses of Theorem 1, let us introduce the following sequences:

$$\begin{aligned} &\varphi_i^{(n)}(x) = n(1+x^2)^{-1} \quad \text{if} \quad \varphi_i(x) \geqq n(1+x^2)^{-1} \\ &\varphi_i^{(n)}(x) = \frac{1}{n} d(1+x^2)^{-1} \quad \text{if} \quad \varphi_i(x) \leqq \frac{1}{n} d(1+x^2)^{-1} \\ &\varphi_i^{(n)}(x) = \varphi_i(x) \quad \text{elsewhere}; \quad i = 1, 2, \ldots, r; \quad n \geqq 1. \end{aligned} \tag{4.9}$$

In (4.9) $d < 1$ is chosen in order to satisfy Lemma 5.

Let us remark that, by virtue of the definition (4.9), the Cauchy problem (2.2), with initial data $\varphi_i^{(n)}$, has a local solution in the interval $[0, T_{1,\alpha}]$.

Moreover, we have:

Lemma 6. *Let $\underset{\sim}{f}^{(n)}(x,t)$ be the local solution to the Cauchy problem (2.2), with initial datum $\underset{\sim}{\varphi}^{(n)}$. Then, $\underset{\sim}{f}^{(n)}(x,t)$ satisfies the H-theorem in its interval of existence.*

Proof. Let us set

$$\log^+ x = \log x \cdot \chi\{x \geqq 1\}, \quad \log^- x = -\log x \cdot \chi\{0 \leqq x < 1\}.$$

First, let us observe that $\sum_i f_i^{(n)} \log f_i^{(n)} \in L_1(\mathbb{R})$. In fact, for every $i \leqq r$,

$$\begin{aligned} \int_{-\infty}^{\infty} |f_i^{(n)}(x,t) \log f_i^{(n)}(x,t)| dx &= \int_{-\infty}^{\infty} |\hat{f}_i^{(n)}(x,t) \log \hat{f}_i^{(n)}(x,t)| dx \\ &= \int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^+ \hat{f}_i^{(n)}(x,t) dx \\ &\quad + \int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^- \hat{f}_i^{(n)}(x,t) dx. \end{aligned}$$

By virtue of the inequality:

$$x \log^- x \leqq y - x \log y$$

that holds when $0 < y \leqq 1$, $x > 0$, if we choose $y = \exp\{-(1+|x|^\alpha)\}$,

$$\int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^- \hat{f}_i^{(n)}(x,t)dx \leqq \int_{-\infty}^{\infty} \{e^{-(1+|x|^\alpha)} + (1+|x|^\alpha)\hat{f}_i^{(n)}(x,t)\}dx .$$

Moreover, by the result of Lemma 4, for some constant k_n,

$$\hat{f}_i^{(n)}(x,t) \leqq k_n(1+x^2)^{-1}, \tag{4.10}$$

and

$$\int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^+ \hat{f}_i^{(n)}(x,t)dx \leqq \int_{-\infty}^{\infty} k_n(1+x^2)^{-1} \log^+ k_n(1+x^2)^{-1}dx .$$

By the definition (4.9), for each fixed n, $\varphi_i^{(n)}(x) \leqq \psi_i(x)$ if

$$\psi_i(x) = \max\{\varphi_i(x); d(1+x^2)^{-1}\}.$$

This implies that, by virtue of Lemma 3, inequality (3.12), $f_i^{(n)}(x,t) \leqq f_i^+(x,t)$, where $\underset{\sim}{f}^+$ is the local solution to the Cauchy problem (3.6) with initial datum $\underset{\sim}{\psi}$. On the other hand, since $\varphi_i^{(n)}(x) \geqq \frac{d}{n}(1+x^2)^{-1}$, a new application of Lemma 3 gives

$$\hat{f}_i^{(n)}(x,t) \geqq \frac{d}{n}(1+x^2)^{-1} \exp\{-D\}, \tag{4.11}$$

the constant D being defined by (3.10) with ψ_i instead of φ_i. To this point, it is easy to bound the gain and loss terms. We get:

$$\begin{aligned}\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)})(x,t) &= \frac{1}{2} \sum_{j,k,m} A_{ij}^{km} \hat{f}_k^{(n)}(x+(v_i-v_k)t,t) \hat{f}_m^{(n)}(x+(v_i-v_m)t,t) \\ &\leqq \frac{1}{2} k_n^2 \sum_{j,k,m} A_{ij}^{km}(1+|x+(v_i-v_k)t|^2)^{-1} \times (1+|x+(v_i-v_m)t|^2)^{-1}\end{aligned}$$

for every $t \leqq T_{1,\alpha}$ and $x \in \mathbb{R}$.

In addition, consider that, by (2.13)

$$(1+[x+(v_i-v_k)t]^2)(1+[x+(v_i-v_m)t]^2) \geqq \max\{1, c_{i,k,m}\}]^{-1}(1+x^2).$$

Thus:

$$\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)})(x,t) \leqq \frac{1}{2} k_n^2 \cdot \sum_{j,k,m} A_{ij}^{km} \max\{1, c_{i,k,m}\} \cdot (1+x^2). \tag{4.12}$$

With similar computations we get:

$$\hat{f}_i^{(n)}(x,t)\hat{L}_i(\underset{\sim}{f}^{(n)})(x,t) \leqq \frac{1}{2} k_n^2 \sum_{j,k,m} A_{ij}^{km} \cdot (1+x^2)^{-1}. \tag{4.13}$$

By (4.10), (4.11), (4.12), and (4.13), for every $t \leqq T_{1,\alpha}$ and $x \in \mathbb{R}$:

$$\begin{aligned}&\log^- \hat{f}_i^{(n)}(x,t)\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)})(x,t) + \hat{f}_i^{(n)}(x,t)\hat{L}_i(\underset{\sim}{f}^{(n)})(x,t)\}\\ &\quad \leqq d_{i,n}^2(1+x^2)^{-1}\log^-\left\{\frac{d}{n}\exp\{-D\}\cdot(1+x^2)^{-1}\right\},\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\log^+ \hat{f}_i^{(n)}(x,t)\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)})(x,t) + \hat{f}_i^{(n)}(x,t)\hat{L}_i(\underset{\sim}{f}^{(n)})(x,t)\}\\ &\quad \leqq d_{i,n}^2(1+x^2)^{-1}\log^+\{k_n(1+x^2)^{-1}\}\end{aligned} \tag{4.15}$$

with obvious meaning of the constants. The functions on the right-hand side of (4.14) and (4.15) are in $L_1(\mathbb{R}) \cap C_b(\mathbb{R})$.

If we multiply both sides of Eqs. (2.1) by $1 + \log \hat{f}_i^{(n)}(x,t)$, and take the sum over i, we get:

$$\begin{aligned}&\sum_i (1+\log \hat{f}_i^{(n)})\frac{d\hat{f}_i^{(n)}}{dt}\\ &\quad = \frac{d}{dt}\sum_i \hat{f}_i^{(n)}\log \hat{f}_i^{(n)}\\ &\quad = \sum_i (1+\log \hat{f}_i^{(n)})\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)}) - \hat{f}_i^{(n)}L_i(\underset{\sim}{f}^{(n)})\}.\end{aligned} \tag{4.16}$$

We integrate both sides of (4.16) with respect to the x variable obtaining:

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d}{dt}\sum_i \hat{f}_i^{(n)}\log \hat{f}_i^{(n)}dx\\ &\quad = \int_{-\infty}^{\infty}\sum_i (1+\log \hat{f}_i^{(n)})\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)}) - \hat{f}_i^{(n)}L_i(\underset{\sim}{f}^{(n)})\}dx.\end{aligned} \tag{4.17}$$

Let us remark that, for each $t \leqq T_{1,\alpha}$, the function into the integral on the right-hand side of (4.17) is integrable on $\mathbb{R}$ and is uniformly bounded by a function of $L_1(\mathbb{R}) \cap C_b(\mathbb{R})$ in consequence of the inequalities (4.14) and (4.15). Thus, we can interchange the integral with the derivative, and obtain:

$$\begin{aligned}\frac{d}{dt}H(\underset{\sim}{f}^{(n)})(t) = \int_{-\infty}^{\infty}&\sum_i (1+\log \hat{f}_i^{(n)}(x,t))\\ &\times\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)})(x,t)\\ &-\hat{f}_i^{(n)}(x,t)\hat{L}_i(\underset{\sim}{f}^{(n)})(x,t)\}dx.\end{aligned}$$

Now, by a well-known argument,

$$\begin{aligned}&\int_{-\infty}^{\infty}\sum_i (1+\log \hat{f}_i^{(n)})\{\hat{G}_i(\underset{\sim}{f}^{(n)}, \underset{\sim}{f}^{(n)}) - \hat{f}_i^{(n)}\hat{L}_i(\underset{\sim}{f}^{(n)})\}dx\\ &\quad = \frac{1}{8}\int_{-\infty}^{\infty}\sum_{i,j,k,m} A_{ij}^{km}\log\frac{f_i^{(n)}f_j^{(n)}}{f_k^{(n)}f_m^{(n)}}(f_k^{(n)}f_m^{(n)} - f_i^{(n)}f_j^{(n)})dx \leqq 0.\end{aligned}$$

This concludes the proof of the H-theorem for $\underset{\sim}{f}^{(n)}$. □

Theorem 2. *Let $0 \leqq \varphi_i(x) \in L_{1,\alpha}$, and in addition $\varphi_i \log \varphi_i \in L_1$, $i \leqq r$. Then the local solution of the Cauchy problem* (2.2) *satisfies*

$$H(\underset{\sim}{f})(t) \leqq H(\underset{\sim}{\varphi}) \tag{4.17}$$

for every $t \leqq T_{1,\alpha}$.

Proof. To start, let us observe that, in consequence of the definition (4.9),

$$|||\underset{\sim}{\varphi}^{(n)} - \underset{\sim}{\varphi}|||_{1,0} \to 0 .$$

Since $\underset{\sim}{f}^{(n)}$ and $\underset{\sim}{f}$ are in $\mathscr{D}_{1,\alpha}$, for $t \leqq T_{1,\alpha}$, we get

$$|||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{1,0} \leqq |||\underset{\sim}{\varphi}^{(n)} - \underset{\sim}{\varphi}|||_{1,0} + \delta |||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{1,0} ,$$

δ being defined as in Theorem 1, so that

$$|||\underset{\sim}{f}^{(n)} - \underset{\sim}{f}|||_{1,0} \to 0 .$$

Since $f_i^{(n)}$ converges in measure to f_i, we can apply Fatou's Lemma to obtain:

$$\begin{aligned} 0 &\leqq \int_{-\infty}^{\infty} \hat{f}_i(x,t) \log^+ \hat{f}_i(x,t) dx \\ &\leqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^+ \hat{f}_i^{(n)}(x,t) dx \\ &\leqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \sum_i \hat{f}_i^{(n)}(x,t) \log^+ \hat{f}_i^{(n)}(x,t) dx \\ &= \liminf_{n\to\infty} \left\{ H(\underset{\sim}{f}^{(n)})(t) + \sum_i \int_{-\infty}^{\infty} \hat{f}_i^{(n)}(x,t) \log^- \hat{f}_i^{(n)}(x,t) \right\} dx \\ &\leqq \liminf_{n\to\infty} \left\{ H(\underset{\sim}{\varphi}^{(n)}) + \sum_i \int_{-\infty}^{\infty} \{e^{-(1+|x|^\alpha)} + (1+|x|^\alpha) \hat{f}_i^{+}(x,t)\} \right\} dx \leqq c < \infty , \end{aligned}$$

c being a suitable constant.

In addition, since $\underset{\sim}{f} \in \mathscr{B}_{1,\alpha}$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \hat{f}_i(x,t) \log^- \hat{f}_i(x,t) dx \\ &\qquad \leqq \int_{-\infty}^{\infty} \{e^{-(1+|x|^\alpha)} + (1+|x|^\alpha) \hat{f}_i(x,t)\} dx < \infty . \end{aligned}$$

Thus we verified that $f_i \log f_i \in L_1$ for each $t \leqq T_{1,\alpha}$.

Let us set:

$$\sum_i \hat{f}_i^{(n)}(x,t) \log \hat{f}_i^{(n)}(x,t) - \sum_i \varphi_i^{(n)}(x) \log \varphi_i^{(n)}(x) ; \qquad t \leqq T_{1,\alpha} . \tag{4.18}$$

The above sequence converges in measure to

$$\sum_i (\hat{f}_i(t) \log \hat{f}_i(t) - \varphi_i \log \varphi_i) .$$

Let us remark that the functions defined by (4.18) can assume negative values, so that, in order to apply Fatou's lemma, we need to verify that:

$$\sum_i \hat{f}_i^{(n)}(x,t)\log \hat{f}_i^{(n)}(x,t) - \sum_i \varphi_i^{(n)}(x)\log\varphi_i^{(n)}(x) \geqq -h(x), \tag{4.19}$$

where $h(x)\geqq 0$, and $h(x)\in L_1(\mathbb{R})$.

By the result of Lemma 3, and by (4.9),

$$\sum_i \hat{f}_i^{(n)}(x,t)\log \hat{f}_i^{(n)}(x,t) \geqq -\sum_i \{e^{-(1+|x|^\alpha)} + (1+|x|^\alpha)\hat{f}_i^{+}(x,t)\},$$

whenever

$$\sum_i \varphi_i^{(n)}(x)\log\varphi_i^{(n)}(x) \leqq \sum_i \varphi_i(x)\log^+\varphi_i(x).$$

This means that inequality (4.19) is verified setting

$$h(x) = \sum_i \left\{e^{-(1+|x|^\alpha)} + (1+|x|^\alpha)\sup_{t\leqq T_{1,\alpha}} \hat{f}_i^{+}(x,t) + \varphi_i(x)\log^+\varphi_i(x)\right\}.$$

At this point, the result is achieved by the application of Fatou's lemma. □

5. Global Existence

In this final section, we will prove that, when the initial values for the Cauchy problem (2.2) stay in $L_{1,\alpha}$, for some $\alpha>0$, and in addition when they have finite entropy, the local solution can be extended globally. To do this, we shall use Theorem 1, coupling this result with an argument of Crandall and Tartar [14]. This argument was adapted to L_1-data, and for the Broadwell model, by Illner [9]. Our result is contained in the following:

Theorem 3. *Let* $0\leqq\varphi_i(x)\in L_{1,\alpha}$, $\alpha>0$, $i\leqq r$, *and, in addition*

$$\varphi_i\log\varphi_i\in L_1, \qquad i\leqq r.$$

Then, if the Hypothesis is satisfied, the Cauchy problem (2.2) *has a unique nonnegative mild solution.*

Proof. According to (2.13),

$$1+|x|^\alpha \leqq c_{i,k,m}^{(\alpha/2)}\{[1+|x+(v_i-v_k)t|^\alpha] + [1+|x+(v_i-v_m)t|^\alpha\},$$

so that, if

$$c_{i,k,m}^{*} = \max\left\{1; \frac{(v_i-v_k)^2+(v_i-v_m)^2}{(v_k-v_m)^2}\right\}, \tag{5.1}$$

$$1+|x|^\alpha \leqq c_{i,k,m}^{*}\{[1+|x+(v_i-v_k)t|^\alpha] + [1+|x+(v_i-v_m)t|^\alpha]\} \tag{5.2}$$

for every $x\in\mathbb{R}$, $t\in\mathbb{R}$. Let us set

$$k^{*} = \sum_{i,j,k,m} A_{ij}^{km}|v_i-v_j|^{-1}(2+8c_{i,k,m}^{*}) \tag{5.3}$$

and fix $\delta<1$. Since $\varphi_i \in L_{1,\alpha}$, $i \leqq r$, Theorem 1 assures the existence of a solution to the Cauchy problem (2.2) at least in the interval $[0, T_{1,\alpha}]$, where $T_{1,\alpha}$ is the solution of

$$\sum_i \lambda_{1,\alpha}(\varphi_i)(VT) = \frac{\delta}{k^*} \tag{5.4}$$

and, in this interval, $\underset{\sim}{f} \in \mathscr{B}_{1,\alpha}$.

If we consider that

$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} (1+|x|^{\alpha})\varphi_i(x)dx = 2 \int_{-\infty}^{\infty} \varphi_i(x)dx\,; \qquad i \leqq r\,,$$

owing to the definition of $\lambda_{1,\alpha}(g)(\beta)$, we can easily prove that, uniformly with respect to β,

$$\lim_{\alpha \to 0} \lambda_{1,\alpha}(\varphi_i)(\beta) = \lambda_{1,0}(\varphi_i)(\beta)\,; \qquad i \leqq r\,.$$

If $T_{1,0}$ is defined as the solution of

$$\sum_i \lambda_{1,0}(\varphi_i)(VT) = \frac{\delta}{k^*}\,,$$

it follows that $T_{1,\alpha}$ converges to $T_{1,0}$ as $\alpha \to 0$. This implies that, for every $\eta>0$, we can find $\alpha_1>0$ such that, when $\alpha \leqq \alpha_1$:

$$T_{1,\alpha} \geqq T_{1,0} - \eta\,. \tag{5.5}$$

Moreover, in the interval $[0, T_{1,\alpha_1}]$, we can apply Theorem 2 to conclude that in this interval the H-function of the solution is uniformly bounded by the H-function of the initial values.

Suppose that, in consequence of Theorem 2, the solution, in the interval $[0, T_{1,\alpha_1}]$ consists of uniformly absolutely continuous measures, in the sense that, for any $\varepsilon>0$, there exists a $\beta>0$ such that, for all $t \leqq T_{1,\alpha_1}$, and $B \in \mathscr{L}(\mathbb{R})$:

$$\mu(B) \leqq \beta \Rightarrow \int_B \sum_i \hat{f}_i(x,t)dx \leqq \varepsilon\,. \tag{5.6}$$

Then, we can apply Theorem 1 to the Cauchy problem (2.2), setting $\hat{f}_i(x, T_{1,\alpha_1})$, $i \leqq r$, as initial values, and we conclude the existence of the solution $\underset{\sim}{f} \in \mathscr{B}_{1,0}$, in the interval $[0, T_{1,\alpha_1} + T_{1,0}]$. On the other hand, if we apply Theorem 1 to the Cauchy problem (2.2), with initial values $\hat{f}_i(x, T_{1,\alpha_1})$, we conclude the existence of the solution $\underset{\sim}{f} \in \mathscr{B}_{1,\alpha_2}$, in the interval $[0, T_{1,\alpha_1} + T_{1,\alpha_2}]$. Again, if we choose α_2 sufficiently small, given the constant appearing in (5.5),

$$T_{1,\alpha_2} \geqq T_{1,0} - \frac{\eta}{2}\,. \tag{5.7}$$

So, we have the existence of a solution at least in the interval $[0, 2T_{1,0} - \eta(1+\frac{1}{2})]$. In this interval, the solution satisfies the hypotheses of Theorem 2, and therefore (4.17). With a repeated application of the previous arguments, we can find a sequence $\{\alpha_k\}_{k \geqq 1}$, such that $T_{1,\alpha_k} \geqq T_{1,0} - \frac{\eta}{2^k}$, and, after n steps, we prove the existence of a unique solution in the interval $\left[0, nT_{1,0}-\eta \sum_{k=1}^{n} 2^{-k}\right]$. Since $T_{1,0}$ is fixed at each step, we conclude that the solution exists globally.

It remains to verify that, under the hypotheses of Theorem 2, (5.6) follows. We will verify (5.6) by contradiction. Assume that (5.6) is violated, i.e. there is an $\varepsilon>0$ such that, for all $\delta>0$ there is a time $t \leqq T_{1,\alpha}$ and a Borel set $B \in \mathscr{L}(\mathbb{R})$, with $\mu(B)<\delta$, but $\int_B \sum_i \hat{f}_i(x,t)dx \geqq \varepsilon$. Since the solution satisfies (4.17), $H(\underset{\sim}{f})(t)$ is uniformly bounded,

$$\int_{-\infty}^{\infty} \sum_i \hat{f}_i(x,t) \log \hat{f}_i(x,t) dx \leqq H(\underset{\sim}{\varphi}). \tag{5.8}$$

On the other hand, in $[0, T_{1,\alpha}]$:

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}_i(x,t) \log^- \hat{f}_i(x,t) dx &\leqq \int_{-\infty}^{\infty} e^{-(1+|x|^\alpha)} dx + \|f_i^+\|_{1,\alpha} \\ &\leqq \int_{-\infty}^{\infty} e^{-(1+|x|^\alpha)} dx + 2|||\underset{\sim}{\varphi}|||_{1,\alpha},\end{aligned} \tag{5.9}$$

so that

$$\begin{aligned}&\int_{-\infty}^{\infty} \sum_i \hat{f}_i(x,t) \log^+ f_i(x,t) dx \\ &\leqq H(\underset{\sim}{\varphi}) + 2r|||\underset{\sim}{\varphi}|||_{1,\alpha} + r \int_{-\infty}^{\infty} e^{-(1+|x|^\alpha)} dx = c.\end{aligned} \tag{5.10}$$

By assumption, there are sequences $\delta_n \to 0$, $t_n \leqq T_{1,\alpha}$, and there are $B_n \in \mathscr{L}(\mathbb{R})$ such that $\mu(B_n) \leqq \delta_n$, but

$$\int_{B_n} \sum_i \hat{f}_i(x,t_n) dx \geqq \varepsilon.$$

By passing to subsequences, without loss of generality, we can assume that, for $i \leqq r$:

$$\int_{B_n} \hat{f}_i(x,t_n) dx \geqq \frac{\varepsilon}{r}.$$

From (5.10) we have:

$$\int_{B_n} \hat{f}_i(x,t_n) \log^+ \hat{f}_i(x,t_n) dx \leqq c.$$

Now, let $m \geqq 1$ arbitrary, and let

$$B_{n,1} = \{x \in B_n : \hat{f}_i(x,t_n) \geqq e^m\}, \qquad B_{n,2} = \{x \in B_n : \hat{f}_i(x,t_n) < e^m\}.$$

Then

$$\begin{aligned}\frac{\varepsilon}{r} &\leqq \int_{B_n} \hat{f}_i(x,t_n) dx = \int_{B_{n,1}} \hat{f}_i(x,t_n) dx + \sum_{B_{n,2}} \hat{f}_i(x,t_n) dx \\ &\leqq \frac{1}{m} \int_{B_{n,1}} \hat{f}_i(x,t_n) \log^+ f_i(x,t_n) dx + \delta_n e^m \leqq \frac{c}{m} + \delta_n e^m.\end{aligned}$$

Choose m such that $\frac{c}{m} < \frac{\varepsilon}{4r}$. Then choose n such that $e^m \delta_n < \frac{\varepsilon}{2r}$, so the contradiction $\frac{\varepsilon}{r} \leqq e^m \delta_n + \frac{c}{m} \leqq \frac{3}{4} \frac{\varepsilon}{r}$ results. □

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