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Stability of Semiclassical Bound States of Nonlinear Schrödinger Equations with Potentials

Yong-Geun Oh
Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720, USA

Abstract. In this paper, we study the Lyapunov stabilities of some "semiclassical" bound states of the (nonhomogeneous) nonlinear Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2}\Delta\psi+V\psi-|\psi|^{p-1}\psi,\quad 1\leqq p<1+\frac{4}{n}.$$

We prove that among those bound states, those which are "concentrated" near local minima (respectively maxima) of the potential V are stable (respectively unstable). We also prove that those bound states are positive if $\hbar>0$ is sufficiently small.

1. Introduction

In [W.a] and [FW], the following nonlinear Schrödinger equation (abbreviated as NLS) on $\mathbf{R}$

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{1}{2}\hbar^2\frac{d^2\psi}{dx^2}+V\psi-|\psi|^2\psi \tag{1}$$

was proposed to study to stabilize linear modes concentrated near local minima for sufficiently small $\hbar>0$ for potentials bounded below. Unlike the linear case, Floer and Weinstein proved the existence of solutions of (1) for sufficiently small $\hbar>0$, which is localized near each nondegenerate critical point of V for all time. We call these solutions "semiclassical solutions." In [O3], the present author generalized the existence result for arbitrary potentials with mild restrictions on the oscillations of V at infinity. Let us briefly summarize the existence result in [FW] and [O3]: If we rescale time and space by $t\to\hbar s$ and $x\to\hbar y$, then rewriting s by t, Eq. (1) becomes

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}+V_\hbar\psi-|\psi|^2\psi, \tag{2}$$

where $V_\hbar(y)=V(\hbar y)$. Without loss of generalities, we assume that 0 is the critical point we are considering and that $V(0)=0$. Then as $\hbar\to 0$, $V_\hbar\to 0$ uniformly over all compact subsets of **R**. In this sense, we consider the standard nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial y^2} - |\psi|^2\psi \tag{3}$$

as a limit of Eq. (2). It is well-known that among the standing solutions of (2), i.e., the solutions of the type $e^{-iEt}u(y)$ for a fixed $E<0$, there is the unique ground state with center 0, $u_0(y)=\sqrt{-2E}\,\mathrm{sech}\sqrt{-2E}y$. Now we are ready to state the main existence result in [FW] and [O3].

Theorem (Existence). *Let V be a potential which is bounded below and $(V-a)>0$ and $(V-a)^{-1/2}$ is uniformly Lipschitz for some $a\in\mathbf{R}$. Moreover we assume that 0 is a nondegenerate critical point and $V(0)=0$ and $V-E>\varepsilon>0$ for some $\varepsilon>0$. Then there is some $\hbar_0>0$ such that for all $\hbar$ with $0<\hbar<\hbar_0$, the equation*

$$-\frac{1}{2}\frac{d^2u}{dy^2} + (V_\hbar - E)u - u^3 = 0 \tag{4}$$

has a solution of the type $u_0+\phi_\hbar$ with the estimates;

$$\|\phi_{z,\hbar}\|_\hbar^2 \leqq K(e^{-\mu_1\rho} + \sup_{|y|<\rho}|V(\hbar y+z)-V(0)|^2 + e^{-\mu_2/\hbar})$$

for any ρ with $0<\rho<1/\hbar$ and $|z|<\hbar^\nu$, $\nu>1$, where $\|\cdot\|_\hbar$ to be defined later. Moreover $\phi_{z,\hbar}$ depends on $\hbar$ in the C^1-sense.

From the point of view of Hamiltonian systems, NLS is an infinite dimensional Hamiltonian system whose generating Hamiltonian is the energy functional

$$E(\phi):=\tfrac{1}{2}\langle\phi, H_\hbar\phi\rangle - \tfrac{1}{4}\int|\phi|^4dy,$$

where $H_\hbar:= -(1/2)(d^2/dy^2)+(V_\hbar-E)$ as a quadratic form whose domain is given by

$$Q(H_\hbar):=\left\{\phi\in L^2(\mathbf{R})\left|\frac{1}{2}\int\left|\frac{d\phi}{dy}\right|^2 + \langle(V_\hbar-E)\phi,\phi\rangle dy<\infty\right.\right\}.$$

(Note that under the assumption on V and E in Theorem (Existence), the differential expression $-(1/2)(d^2/dy^2)+(V_\hbar-E)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R})$ and so it has the unique self-adjoint extension). Moreover, NLS has a natural S^1 – symmetry (i.e., phase rotations) in general and an additional translational symmetry for constant potentials. The conserved quantity. corresponding to the S^1-action is nothing but the probability density

$$L(\phi):=\tfrac{1}{2}\langle\phi,\phi\rangle = \int|\phi|^2dy.$$

In this point of view, the bound states found above correspond to relative equilibria with respect to the S^1-symmetry. Thus a natural question to ask is the orbital stability of the bound states under the flow of NLS. The main theorem of this paper is the following:

Theorem (Stability). *Assume that (1) has a global flow on $Q(H_\hbar)$ and that V satisfies the conditions in Theorem (Existence). Then when 0 is a local minimum (respectively* *maximum), then the bound state found in Existence Theorem is (Lyapunov) orbitally stable (respectively unstable) if $\hbar$ is sufficiently small.*

Here we will not attempt to make the assumption more clear because the Cauchy problem of NLS itself is still to be investigated for general unbounded potentials. However we should mention that the present author has established the existence of the global flow on $Q(H)$ (for Cauchy problem $\hbar$ is irrelevant as long as it is positive) for potentials satisfying the condition that $|D^\alpha V|$ *are bounded for all* $|\alpha| \geqq 2$ (e.g., *harmonic potential*) in [O2]. Therefore in those cases, this theorem is a rigorous theorem but this theorem will still hold as long as the global flow on $Q(H)$ exists.

For the case when $V=0$, it has been proven in [LS, CL] and [W.m] that the ground state of NLS is orbitally stable with respect to the angular and translational symmetries, i.e. the shape of the wave packet is stable near the ground state. But in our case, the translational symmetry is irrelevant and so we also have to take care of the "spatial stability" as well as "shape stability." To take care of this spatial stability, we exploit the nonlinear Ehrenfest's law established in [O2].

Now let us summarize the contents of the present paper. In Sect. 2, we establish an easy estimate on the error term $\phi_{z,\hbar}$. In Sect. 3, we study spectral properties of the real and imaginary parts of the linearized operator. As a corollary, we prove (Theorem 3.5) that the bound states are positive if $\hbar$ is sufficiently small. In Sect. 4, we prove that the semiclassical bound states are stable if 0 is a local minimum. For the stability, we use a priori estimates on the linearized operator established in Sect. 2 and then use a standard procedure of the "Energy–Casimir Method" (see [HMRW]) to get Lyapunov stability. In Sect. 5, we prove that the semiclassical bound states are Lyapunov unstable if 0 is a local maximum. For the instability, we follow the idea behind Ehrenfest's law mentioned above to construct a Lyapunov function to get instability. This kind of instability result is a new phenomenon for the nonlinear Schrödinger equation with potentials, which does not appear in the case when V is constant.

Around the same time, Grillakis–Shatah–Strauss also [GSS] obtained the stability result modulo the fact established in Propositions 3.4 and 3.6 in the present paper. And after this work was done, we got a preprint of Grillakis [G] where he studied spectral properties of Schrödinger type operators (a similar result on symplectic matrices was obtained previously by the present author and his collaborators [OSKM]). With his results and some general arguments on getting Lyapunov instability from spectral instability, he was able to get the same instability result, if the spectral properties established in Propositions 3.4 and 3.6 in the present paper were given. But our proofs are quite different from theirs in both cases and in particular our instability proof is more intuitive and explicit.

2. Preliminaries

Since we study the stability of "semiclassical" bound states, i.e., only for small $\hbar > 0$, it is crucial to have control of the perturbation term $\phi_\hbar$ in Theorem (Existence). Following [O3], we control the perturbation term $\phi_{z,\hbar}$ by the following norm:

Definition 2.1.

$$\| \phi \|_\hbar^2 = \int \left| \frac{d^2\phi}{dy^2} \right|^2 dy + \int (V_\hbar - E)^2 |\phi|^2 dy.$$

Now the following lemma will be essential for later discussions.

Lemma 2.2 *Let $\phi_{z,\hbar}$ be as in Theorem (Existence). Then we have for any fixed* $0 < \varepsilon < 1$,

$$\| \phi_{z,\hbar} \|_\hbar^2 \leqq K_1^2 \hbar^{4(1-\varepsilon)}$$

if $0 < \hbar < \hbar_4$ *for some* $\hbar_4 > 0$ *where* K_1 *depends only on* ε.

Proof. By Theorem (Existence), we have

$$\| \phi_{z,\hbar} \|_\hbar^2 \leqq K_2 \left(e^{-\mu_1 \rho} + \sup_{|y| < \rho} | V(\hbar y + z) - V(0) |^2 + e^{-\mu^2/\hbar} \right)$$

for any ρ with $0 < \rho < 1/\hbar$. Obviously, the first and the third terms satisfy the required estimates since they are of order $O(\hbar^\infty)$. All we have to estimate is the middle term. Since V is non-degenerate at 0 and

$$V(y) - V(0) = \tfrac{1}{2} V''(0) y^2 + 0(y^3)$$

as $|y| \to 0$, we have

$$| V(y) - V(0) | \leqq K_3 | V''(0) | y^2$$

if $|y| < r$ for a sufficiently small $r > 0$. Now,

$$\sup_{|y| < \rho} | V(\hbar y + z) - V(0) |^2 = \sup_{|x| < \rho\hbar} | V(x + z) - V(0) |^2 \leqq \sup_{|x| < |z| + \rho\hbar} | V(x) - V(0) |^2 .$$

Recall that $|z| < \hbar^\mu$ for $\mu > 1$ and ρ is arbitrary as long as $0 < \rho < 1/\hbar$. Setting $\rho = \hbar^{-\varepsilon}$, $\varepsilon > 0$, we have

$$\begin{aligned} \sup_{|x| < |z| + \rho\hbar} (V(x) - V(0))^2 &\leqq \sup_{|x| < |z| + \rho\hbar} K_3^2 | V''(0) | y^4 = K_3^2 | V''(0) |^2 (|z| + \rho\hbar)^4 \\ &\leqq K_3^2 | V''(0) |^2 (\hbar^\mu + \hbar^{1-\varepsilon})^4 \end{aligned}$$

if $\hbar$ is sufficiently small. Hence,

$$\sup_{|y| < \rho} | V(\hbar y + z) - V(0) |^2 \leqq K_4 \hbar^{4(1-\varepsilon)}$$

for some $K_4 > 0$ since $\mu > 1$. Since ε is arbitrary in $0 < \varepsilon < 1$, we are done.

Q.E.D.

It is quite confusing to directly consider the stability in the original NLS,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{1}{2} \hbar^2 \frac{\partial^2 \psi}{\partial x^2} + V\psi - |\psi|^2 \psi,$$

as we want study the stability properties for sufficiently small $\hbar$ and we have to estimate many quantities with respect to $\hbar$, and so we rescale the time and space variables by $t = \hbar s$ and $x = \hbar y$. Replacing s by t, we get the rescaled equation

$$i\frac{d\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial y^2} + V_h\psi - |\psi|^2\psi. \tag{5}$$

3. Spectral Properties of $L^{\hbar}_{\pm}$

Lemma 3.1. i) *The operator* $L^0_+ = -(1/2)(d^2/dx^2) + \lambda - 3u_0^2$ *has just one negative eigenvalue* -3λ *and one dimensional kernel in* H^1. *Moreover the corresponding eigenspaces are spanned by* u_0^2 *and* u_0' *respectively.*
ii) *The operator* $L^0_- = -(1/2)(d^2/dx^2) + \lambda - u_0^2$ *has one dimensional kernel in* H^1, *which is spanned by* u_0.
iii) *Neither operator has positive eigenvalues, and*

$$\inf\mathrm{ess}\,(L^0_+) = \inf\mathrm{ess}\,(L^0_-) = \lambda.$$

Proof. Recall that $u_0(x) = \sqrt{2\lambda}\,\mathrm{sech}\sqrt{2\lambda}x$. It is enough to prove the proposition for $\lambda = 1$ by the standard rescaling procedure. Then, the statements in i), ii) and the first part of iii) are well-known (see [T]). The second statement of iii) comes from the fact (see e.g., [A]) that $\lambda - 3u_0^2$ and $\lambda - u_0^2$ approach λ as $x \to \pm\infty$.

Q.E.D.

Corollary 3.2. i) *Let* v *be orthogonal to the eigenspaces of* L^0_+, i.e., $v \perp \mathrm{span}\,\{u_0', u_0^2\}$. *Then*

$$\langle L^0_+ v, v\rangle \geqq \lambda\langle v, v\rangle.$$

ii) *Let* v *be orthogonal to the eigenspace of* L^0_-, span $\{u_0\}$. *Then,*

$$\langle L^0_- v, v\rangle \geqq \lambda\langle v, v\rangle.$$

Proof. These come from Lemma 3.1 and the mini-max principle (see [ReS]). More specifically,

$$\inf_{v\perp\{\text{eigenspaces}\}} \frac{\langle L^0_\pm v, v\rangle}{\langle v, v\rangle} = \inf\mathrm{ess}\,(L^0_\pm) = \lambda.$$

Hence, we are done. Q.E.D.

Definition 3.3.

$$L^h_+ = -\frac{1}{2}\frac{d^2}{dx^2} + (V_h - E) - 3u_h^2,$$

$$L^h_- = -\frac{1}{2}\frac{d^2}{dx^2} + (V_h - E) - u_h^2.$$

Note that L^h_+ (respectively L^h_-) is the real (respectively imaginary) part of the linearized operator of (4) at u_h. Now the following propositions are crucial for both positivity and Lyapunov stability of the semiclassical bound states.

Proposition 3.4. *There exists $\hbar_5 > 0$ such that if $\hbar < \hbar_5$, then $L^{\hbar}_-$ has no negative spectrum and one dimensional kernel spanned by $u_\hbar$. Moreover, the spectrum besides zero is positive away from zero, uniformly in $\hbar$.*

Proof. We know that $u_\hbar$ satisfies the equation,

$$L^{\hbar}_- u_\hbar = -\frac{1}{2}\frac{d^2 u_\hbar}{dx^2} + (V_\hbar - E)u_\hbar - u_\hbar^2 u_\hbar = 0,$$

i.e., $u_\hbar$ is in $\ker L^{\hbar}_-$. To prove the proposition, again by the mini-max principle we have only to prove

$$\inf_{u \perp u_\hbar} \frac{\langle L^{\hbar}_- v, v\rangle}{\langle v, v\rangle} > \mu > 0, \tag{6}$$

where μ does not depend on $\hbar$ if $\hbar$ is sufficiently small. Let $v \perp u_\hbar$, i.e., $\langle v, u_\hbar\rangle = 0$ and decompose

$$v = v_\perp + v_\parallel,$$

where $v_\parallel = (\langle u_0, v\rangle / \langle u_0, u_0\rangle) u_0$ and $v_\perp = v - v_\parallel$. By definition, $\langle u_0, v_\perp\rangle = 0$. Since $L^{\hbar}_- = L^0_- + V_\hbar + (u_0^2 - u_\hbar^2)$, we have

$$\langle L^{\hbar}_- v, v\rangle = \langle L^0_- v, v\rangle + \langle V_\hbar v, v\rangle + \langle (u_0^2 - u_\hbar^2) v, v\rangle. \tag{7}$$

Recall that we have assumed that

$$V - E = V + \lambda > \varepsilon > 0,$$

and so

$$V_\hbar - E > \varepsilon > 0 \quad \text{for all} \quad \hbar > 0.$$

Now,

$$\begin{aligned}\langle L^0_- v, v\rangle &= \langle L^0_- v_\perp, v_\perp\rangle + \langle L^0_- v_\parallel, v_\parallel\rangle = \langle L^0_- v_\perp, v_\perp\rangle \\ &\geqq \lambda \langle v_\perp, v_\perp\rangle \text{ (by Corollary 3.2 ii))} \\ &\geqq \lambda\langle v, v\rangle - \lambda\langle v_\parallel, v_\parallel\rangle. \end{aligned} \tag{8}$$

And,

$$v_\parallel = \frac{\langle u_0, v\rangle}{\langle u_0, u_0\rangle} u_0 = \frac{\langle u_0 - u_\hbar, v\rangle}{\langle u_0, u_0\rangle} u_0,$$

since we assume that $\langle u_\hbar, v\rangle = 0$. Therefore,

$$\|v_\parallel\| = \left|\frac{\langle u_0 - u_\hbar, v\rangle}{\langle u_0, u_0\rangle}\right| \cdot \|u_0\| \leqq \frac{\|u_0 - u_\hbar\|}{\|u_0\|}\|v\|. \tag{9}$$

By Lemma 2.2, we have $\|u_0 - u_\hbar\|_\hbar \to 0$ as $\hbar \to 0$. Moreover, by the Sobolev imbedding theorem $D(H_\hbar) \subsetneq H^2 \subsetneq L^\infty$, we have $\|u_0 - u_\hbar\|_{L^\infty} \to 0$ as $\hbar \to 0$ and in particular

$$\|u_\hbar\|_{L^\infty} < K_5$$

for some $K_5 > 0$. Now we choose $\hbar > 0$ so small that

$$\|u_0 - u_\hbar\| < \sqrt{\frac{\varepsilon}{2}}\|u_0\|, \tag{10}$$

$$\|u_0 - u_\hbar\|_{L^\infty} < \frac{\varepsilon}{8K_5}. \tag{11}$$

From (9) and (10), we have

$$\|v_\parallel\|^2 = \langle v_\parallel, v_\parallel\rangle < \frac{\varepsilon}{2\lambda}\langle v, v\rangle.$$

Substitute this into (9) and then we get

$$\langle L_-^0 v, v\rangle \geqq \lambda\langle v, v\rangle - \frac{\varepsilon}{2}\langle v, v\rangle = \left(\lambda - \frac{\varepsilon}{2}\right)\langle v, v\rangle. \tag{12}$$

Moreover,

$$|\langle (u_0^2 - u_\hbar^2)v, v\rangle| \leqq \|u_0^2 - u_\hbar^2\|_{L^\infty}\langle v, v\rangle \leqq \|u_0 - u_\hbar\|_{L^\infty}\cdot\|u_0 + u_\hbar\|_{L^\infty}\langle v, v\rangle$$
$$\leqq \frac{\varepsilon}{8K_5}2K_5\langle v, v\rangle = \frac{\varepsilon}{4}\langle v, v\rangle. \tag{13}$$

Here, we used (10) and (11) for the second inequality. Now substitute (12) and (13) into (7) and then,

$$\begin{aligned}\langle L_-^\hbar v, v\rangle &\geqq \left(\lambda - \frac{\varepsilon}{2}\right)\langle v, v\rangle + \langle V_\hbar v, v\rangle - \frac{\varepsilon}{4}\langle v, v\rangle \\ &= \langle(\lambda + V_\hbar)v, v\rangle - \tfrac{3}{4}\varepsilon\langle v, v\rangle \\ &\geqq \varepsilon\langle v, v\rangle - \tfrac{3}{4}\varepsilon\langle v, v\rangle = \frac{\varepsilon}{4}\langle v, v\rangle.\end{aligned}$$

For the second inequality, we used the assumption $\lambda + V_\hbar > \varepsilon > 0$. Therefore,

$$\inf_{|v \perp u_\hbar|} \frac{\langle L_-^\hbar v, v\rangle}{\langle v, v\rangle} \geqq \frac{\varepsilon}{4} > 0$$

if we choose $\hbar > 0$ so that (10) and (11) are satisfied. Hence 0 is the lowest eigenvalue of $L_-^\hbar$ and $\inf\mathrm{ess}(L_-^\hbar) \geqq (3\varepsilon/4) > 0$ where ε does not depend on $\hbar$ if $\hbar$ is sufficiently small. Q.E.D.

With this proposition, it is immediate to get the following theorem.

Theorem 3.5 *The semiclassical bound states found in* [FW] *and* [O3] *are positive if* $0 < \hbar < \hbar_5$ *so that* $L_-^\hbar$ *satisfies the properties in Proposition 3.4.*

Proof. From Proposition 3.4, we have proved that $y_\hbar$ is the ground state of the Schrödinger operator

$$L_-^\hbar = -\frac{1}{2}\frac{d^2}{dx^2} + (V_\hbar + \lambda) - u_\hbar^2.$$

The theorem then follows from the general fact on Schrödinger operators that the ground state is positive (see [ReS]). Q.E.D.

Now we study the operator $L^{\hbar}_{+}$.

Proposition 3.6 *There exists some $\hbar_6 > 0$ such that if $0 < \hbar < \hbar_6$, then $L^{\hbar}_{+}$ has no kernel and*

i) *when the critical point is a local minimum, then $L^{\hbar}_{+}$ has one negative eigenvalue near -3λ and one positive eigenvalue near* 0,
ii) *when the critical point is a local maximum, then $L^{\hbar}_{+}$ has one negative eigenvalue near -3λ and one negative eigenvalue near* 0.

And all the remaining spectra besides the ones above are positive away from 0 *uniformly in $\hbar$*, i.e.,

$$\langle L^{\hbar}_{+} v, v\rangle \geqq K_6 \langle v, v\rangle \tag{14}$$

for some $K_6 > 0$ if v is orthogonal to the above eigenspaces.

Proof. Let us first prove the last statement. This can be proved in the same way as the proof of Proposition 3.4 using i) of Corollary 3.2. More specifically, consider

$$\inf_{v \perp \{u_\hbar^2, u_\hbar'\}} \frac{\langle L^{\hbar}_{+} v, v\rangle}{\langle v, v\rangle}.$$

Note that $u_\hbar \to u_0$ in H^2 because

$$\| u_\hbar - u_0 \|_{H_2} \leqq C \| u_\hbar - u_0 \|_\hbar \to 0,$$

where C does not depend on $\hbar$ and hence $u_\hbar \to u_0$ in H^2 from Lemma 2.2. Therefore the projection of v onto span $\{u_0^2, u_0'\}$ can be made arbitrarily small uniformly over v if $v \perp \{u_\hbar^2, u_\hbar'\}$ and $\| v \| = 1$. Now apply the similar arguments as the proof of Proposition 3.4 using Corollary 3.2 to get the last statement, since the eigenspaces of $L^{\hbar}_{+}$ are close to span $\{u_\hbar^2\}$ or span $\{u_\hbar'\}$.

Now for i) and ii), it is easy to prove the existence of one negative eigenvalue near each of -3λ and 0 from the perturbation theory of Schrödinger operators noting that $L^{\hbar}_{+} \to L^{0}_{+}$ in the strong resolvent sense (see [ReS] or [K]). To prove the remaining statements, we need the following lemmas.

Lemma 3.7. *There exists some $\hbar_7 > 0$ such that if $0 < \hbar < \hbar_7$, we have*

$$\langle L^{\hbar}_{+} u_\hbar', u_\hbar'\rangle = \tfrac{1}{2} V''(0)\hbar^2 \langle u_\hbar', u_\hbar'\rangle + O(\hbar^3).$$

Lemma 3.8. *Let $u \in Q(H_\hbar)$. Then there exists some $\hbar_8 > 0$ such that if $0 < \hbar < \hbar_8$,*

$$|\langle L^{\hbar}_{+} u_\hbar', u\rangle| \leqq K_8 \hbar^2 \|u_\hbar'\| \cdot \| u \|$$

for some $K_8 > 0$.

Assuming these two lemmas for the moment, we first prove i). If 0 is a local minimum (nondegenerate), then $V''(0) > 0$, and so

$$\langle L^{\hbar}_{+} u_\hbar', u_\hbar'\rangle \geqq \tfrac{1}{4} V''(0)\hbar^2 \langle u_\hbar', u_\hbar'\rangle. \tag{15}$$

Now, let $\langle u, u_\hbar^2\rangle = 0$ and decompose $u = u_\parallel + u_\perp$, where $u_\parallel$ is a multiple of $u_\hbar'$ and $u_\perp$ is the orthogonal to $u_\hbar'$. Then

$$\langle u_\perp, u_\hbar^2\rangle = 0$$

since $\langle u_\hbar', u_\hbar^2\rangle = 0$. Now,

$$\begin{aligned}
\langle L_+^\hbar u, u\rangle &= \langle L_+^\hbar(u_\parallel + u_\perp), u_\parallel + u_\perp\rangle \\
&= \langle L_+^\hbar u_\parallel, u_\parallel\rangle + 2\langle L_+^\hbar u_\parallel, u_\perp\rangle + \langle L_+^\hbar u_\perp, u_\perp\rangle \\
&\geqq \tfrac{1}{4} V''(0)\hbar^2\langle u_\parallel, u_\parallel\rangle - 2K_8\hbar^2 \|u_\parallel\|\cdot\|u_\perp\| + C\langle u_\perp, u_\perp\rangle \\
&\geqq \tfrac{1}{4} V''(0)\hbar^2\langle u_\parallel, u_\parallel\rangle + C\langle u_\perp, u_\perp\rangle - K_8(\hbar^3\langle u_\parallel, u_\parallel\rangle + \hbar\langle u_\perp, u_\perp\rangle) \\
&= (\tfrac{1}{2} V''(0) - K_8\hbar)\hbar^2\langle u_\parallel, u_\parallel\rangle + (C - K_8\hbar)\langle u_\perp, u_\perp\rangle \\
&\geqq \min\{(\tfrac{1}{4} V''(0) - K_8\hbar)\hbar^2, C - K_8\hbar\}(\langle u_\parallel, u_\parallel\rangle + \langle u_\perp, u_\perp\rangle) \\
&= (\tfrac{1}{4} V''(0) - K_8\hbar)\hbar^2\langle u, u\rangle,
\end{aligned}$$

if $\hbar$ is sufficiently small. Here, the first inequality comes from Lemmas 3.7, 3.8 and Eq. (15). Therefore,

$$\inf_{u\perp u_\hbar^2} \frac{\langle L_+^\hbar u, u\rangle}{\langle u, u\rangle} \geqq O(\hbar^2) > 0.$$

Hence, the second eigenvalue is positive again by mini-max principle.

Finally let us prove ii). When 0 is a local maximum, then $V''(0) < 0$ and so we have

$$\langle L_+^\hbar u_\hbar', u_\hbar'\rangle \leqq -|V''(0)|\hbar^2\langle u_\hbar', u_\hbar'\rangle < 0. \tag{16}$$

Note that $\langle u_\hbar', u_\hbar^2\rangle = 0$ and that if $\hbar$ is sufficiently small, then

$$\langle L_+^\hbar u_\hbar^2, u_\hbar^2\rangle < -C,$$

where C is independent of $\hbar$. Now by the Rayleigh–Ritz principle (see, e.g., [ReS]),

$$\text{the second eigenvalue of } L_+^\hbar \leqq -|V''(0)|\hbar^2 < 0$$

from (16), hence the proposition. Q.E.D.

Now we prove Lemmas 3.7 and 3.8

Proof of Lemma 3.7. Differentiate the equation

$$-\frac{1}{2}\frac{d^2u_\hbar}{dy^2} + (V_\hbar + \lambda)u_\hbar - u_\hbar^3 = 0$$

with respect to y; we get

$$-\frac{1}{2}\frac{d^2}{dy^2}\left(\frac{du_\hbar}{dy}\right) + (V_\hbar + \lambda)\frac{du_\hbar}{dy} - 3u_\hbar^2\frac{du_\hbar}{dy} = -\frac{dV_\hbar}{dy}u_\hbar,$$

i.e.,

$$L_+^\hbar u_\hbar' = -V_\hbar' u_\hbar. \tag{17}$$

Therefore,

$$\langle L_+^\hbar u_\hbar', u_\hbar'\rangle = -\langle V_\hbar' u_\hbar, u_\hbar'\rangle = -\int V_\hbar' u_\hbar u_\hbar' dy$$
$$= -\frac{1}{2}\int V_\hbar' \frac{d}{dy}(u_\hbar^2)dy = \frac{1}{2}\int \frac{d^2 V_\hbar}{dy^2} u_\hbar^2 dy \tag{18}$$

by integration by parts. Now,

$$\int \frac{d^2 V_\hbar}{dy^2} u_\hbar^2 dy = \hbar^2 \int V''(\hbar y) u_\hbar^2(y) dy$$
$$= \hbar^2 \int V''(\hbar y)\chi_\hbar^2(y) u_0^2(y) dy + \hbar^2 \int V''(\hbar y)(u_\hbar^2(y) - \chi_\hbar^2(y) u_0^2(y)) dy.$$

We can easily see using Lemma 2.2 that the second term is of order $O(\hbar^3)$. For the first term, since V is smooth and we cut-off u_0 by $\chi_\hbar$, and so that the integrand and its derivatives are uniformly bounded, $\int V''(\hbar y)\chi_\hbar^2(y)u_0^2(y)dy$ is smooth at $\hbar = 0$ as a function of $\hbar$. Hence,

$$\int V''(\hbar y) u_0^2(y) dy = \int V''(0) u_0^2(y) dy + O(\hbar^2),$$

as we assume that $V(0) = V'(0) = 0$. Therefore,

$$\langle L_+^\hbar u_\hbar', u_\hbar'\rangle = \tfrac{1}{2}\hbar^2 \int u_0^2 dy \cdot V''(0) + O(\hbar^3) = \tfrac{1}{2}\hbar^2 V''(0) < u_0, u_0 > + O(\hbar^3).$$

Since $|\langle u_\hbar', u_\hbar'\rangle - \langle u_0', u_0'\rangle| = O(\hbar^3)$, we have

$$\langle L_+^\hbar u_\hbar', u_\hbar'\rangle = \tfrac{1}{2} V''(0)\langle u_\hbar', u_\hbar'\rangle + O(\hbar^3). \qquad \text{Q.E.D.}$$

Proof of Lemma 3.8.

$$|\langle L_+^\hbar u_\hbar', u\rangle|^2 = \left|\left\langle -\frac{dV_\hbar}{dy} u_\hbar, u\right\rangle\right|^2 = \left|\int_{-\infty}^{\infty} \frac{dV_\hbar}{dy} u_\hbar u\, dy\right|^2$$
$$\leqq \int_{-\infty}^{\infty} \left|\frac{dV_\hbar}{dy} u_\hbar\right|^2 dy \cdot \int |u|^2 dy.$$

Now, we estimate $\int_{-\infty}^{\infty} \left|\frac{dV_\hbar}{dy} u_\hbar\right|^2 dy$,

$$\int \left|\frac{dV_\hbar}{dy} u_\hbar\right|^2 dy = \hbar^2 \int |V'(\hbar y) u_\hbar(y)|^2 dy = \hbar^2 \int |V'(\hbar y)\chi_\hbar(y) u_0(y)|^2 dy + O(\hbar^3),$$

as in the proof of Lemma 3.7. Moreover,

$$G(\hbar) := \int |V'(\hbar y)\chi_\hbar(y) u^0(y)|^2 dy$$

is a smooth function of $\hbar$. Then, we have

$$G(0) = G'(0) = 0$$

and

$$G''(0) = 2V''(0)^2 \int y^2 u_0^2(y) dy.$$

Therefore,

$$G(\hbar) = 2V''(0)^2 \int y^2 u_0^2(y)dy \cdot \hbar^2 + O(\hbar^3).$$

Hence,

$$\int \left| \frac{dV_\hbar}{dy} u_\hbar \right|^2 dy = 2V''(0)^2 \int y^2 u_0^2(y)dy \cdot \hbar^4 + O(\hbar^5),$$

and so we have

$$\sqrt{\int \left| \frac{dV_\hbar}{dy} u_\hbar \right|^2 dy} \leqq K_8 \hbar^2 \|u_\hbar'\|$$

if $\hbar$ is sufficiently small, as $\|u_\hbar'\| \sim \|u_0'\|$ in the obvious sense. Q.E.D.

4. Stability at Local Minima

With the study of spectral properties of the linearized operator in the previous section, the remaining argument on stability is quite standard. We will follow the standard procedure of "Energy-Casimir (or Energy-Momentum) method" (see [HMRW]) by adapting the arguments by Laedke–Spatschek [LS] and M. Weinstein [W.m].

If we view Eq. (1) as an (infinite dimensional) Hamiltonian system with S^1-symmetry (i.e., phase rotation), the solution $u_\hbar$ is a "relative equilibrium" with respect to this symmetry. It is a general fact that

$$\psi(t, y) = e^{-iEt} u_\hbar(y)$$

satisfies Eq. (1) with the initial condition $\psi(0, y) = u_\hbar(y)$. Since $u_\hbar$ is nontrivial, we must prove "orbital stability" rather than "point stability" of $u_\hbar$.

Definition 4.1.

$$\mathcal{O}_{u_\hbar} := \{u_\hbar e^{i\gamma} \in Q(H_\hbar) \,|\, \gamma \in \mathbf{R}\},$$

$$\begin{aligned} \rho^2_{\mathcal{O}_{u_\hbar}}(\phi) &:= \inf_{\gamma \in \mathbf{R}} (\langle (H_\hbar - E)(e^{i\gamma}\phi - u_\hbar), e^{i\gamma}\phi - u_\hbar \rangle) \\ &= \inf_{\gamma \in \mathbf{R}} (\tfrac{1}{2} \langle e^{i\gamma}\nabla\phi - \nabla u_\hbar, e^{i\gamma}\nabla\phi - \nabla u_\hbar \rangle \\ &\quad + \langle (V_\hbar - E)(e^{i\gamma}\phi - u_\hbar), e^{i\gamma}\phi - u_\hbar \rangle). \end{aligned}$$

Here we have chosen this distance because it is associated with the norm under which the energy functional is smooth.

Definition 4.2. The solution $u_\hbar$ is (Lyapunov) stable if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\rho_{\mathcal{O}_{u_\hbar}}(\psi(0)) < \delta$, $\rho_{\mathcal{O}_{u_\hbar}}(\psi(t)) < \varepsilon$ for all $t \in \mathbf{R}$, where $\psi(t)$ satisfies the time-dependent NLS *(1).*

This is just the standard definition of the Lyapunov (orbital) stability restricted to the solution $u_\hbar$ which we want to study.

Let us write the perturbed orbit as

$$\phi(t) = u_h + u(t) + iv(t)$$

and set $w(t) = u(t) + iv(t)$, where u, v are real. Now consider the function

$$F_h(\phi) := \underbrace{\tfrac{1}{2}\langle \phi, H_h\phi \rangle - \tfrac{1}{4}\int |\phi|^4 dy}_{\text{"Energy"}} - \underbrace{\frac{E}{2}\int |\phi|^2 dy}_{\text{"Momentum"}}.$$

Recalling the definitions of $E(\phi)$ and $L(\phi)$, we have

$$F_h(\phi) = E_h(\phi) - EL(\phi).$$

Since the energy and momentum are conserved, this F_h is also conserved under the flow of (1). As in the Energy–Casimir method, we control the distance function $\rho_{\mathcal{O}_{u_h}}$ with this conserved function. A little computation using that u_h is a solution of the time independent NLS (4) gives

$$\begin{aligned} F_h(u_h + w) - F_h(u_h) &= \tfrac{1}{2}\int \langle (u_h + w), (H_h - E)(u_h + w) \rangle - \tfrac{1}{4}\int |u_h + w|^4 dy \\ &\quad - \tfrac{1}{2}\int \langle u_h, (H_h - E)u_h \rangle + \tfrac{1}{4}\int |u_h|^4 dy \\ &= \tfrac{1}{2}\langle L_+^h u, u \rangle + \tfrac{1}{2}\langle L_-^h v, v \rangle \\ &\quad - \tfrac{1}{4}\int (4u_h u^3 + u^4 4u_h uv^2 + 2u^2v^2 + v^4) dy \\ &\geqq \tfrac{1}{2}\langle L_+^h u, u \rangle + \tfrac{1}{2}\langle L_-^h v, v \rangle - C_1 \| w \|_{Q(H)}^3 - C_2 \| w \|_{Q(H)}^4, \end{aligned}$$

since $Q(H) \subsetneq H^1 \subsetneq L^3$ or L^4. We write

$$\delta^2 F_h(u_h) = \begin{pmatrix} L_+^h & 0 \\ 0 & L_-^h \end{pmatrix}.$$

As we showed in the previous section, L_-^h has one dimensional kernel and L_+^h has one negative eigenvalue with the corresponding eigenspace one dimensional A priori, these eigenvalues obstruct the point stability, but since we want to prove orbital stability we will use some constraints to overcome this problem. These constraints naturally come in from the S^1-symmetry. The constraint will be removed using conservation of the corresponding momentum function, i.e., L^2-norm.

First, let us consider initial conditions which satisfy

$$\int |\psi(0)|^2 dy = \int u_h^2 dy, \tag{19}$$

and so

$$\int |\psi(t)|^2 dy = \int u_h^2 dy$$

for all t by the conservation of L^2-norm. This constraint will take care of the negative eigenvalues of L_+^0. We still have to take care of the kernel of L_-^0, and so we introduce another constraint which naturally comes in from the fact that we want to prove orbital stability.

Lemma 4.3. *Assume that*

$$\rho_{\mathcal{O}_{u_h}}^2(\phi) = \langle (H_h - E)(\phi - u_h), \phi - u_h \rangle$$

i.e., *ϕ realizes the minimum distance between ϕ-orbit and u_h-orbit. Then if $\phi = u + iv$,* *u, v real,*

$$\int v(y)\cdot u_\hbar^3(y)dy = 0. \tag{20}$$

Proof. Differentiate $\langle (H_\hbar - E)(e^{i\gamma}\phi - u_\hbar), e^{i\gamma}\phi - u_\hbar \rangle$ with respect to γ at $\gamma = 0$. Since the above function of γ attains its minimum at $\gamma = 0$ from the hypothesis, we have

$$\begin{aligned} 0 &= \frac{d}{d\gamma}\bigg|_{\gamma=0} \langle (H_\hbar - E)(e^{i\gamma}\phi - u_\hbar), e^{i\gamma}\phi - u_\hbar \rangle \\ &= 2\mathrm{Re}(i\langle (H_\hbar - E)\phi, \phi - u_\hbar \rangle) \\ &= 2\mathrm{Re}(i\langle (H_\hbar - E)\phi, -u_\hbar \rangle) = 2\mathrm{Re}(i\langle \phi, -(H_\hbar - E)u_\hbar \rangle) \\ &= 2\mathrm{Re}(i\langle u + iv, -(H_\hbar - E)u_\hbar \rangle) = 2\langle v, (H_\hbar - E)u_\hbar \rangle \\ &= 2\langle v, u_\hbar^3 \rangle = 2\int v(y)\cdot u_\hbar^3 dy. \end{aligned}$$

Here, we used the self-adjointness of $H_\hbar - E$ for the third equality and the fact that $u_\hbar$ satisfies

$$(H_\hbar - E)u_\hbar - u_\hbar^3 = 0$$

for the seventh equality. Q.E.D.

Now, we prove that $L_-^\hbar$ is positive definite under the constraint (20).

Proposition 4.4. *Assume that $v \in Q(H_\hbar)$ satisfies (20). Then, there is a constant $K_9 > 0$, $\hbar_9 > 0$ such that*

$$\langle L_-^\hbar v, v \rangle \geqq K_9 \langle v, v \rangle_{Q(H_\hbar)}$$

if $0 < \hbar < \hbar_9$.

Proof. Let us first prove that

$$\langle L_-^\hbar v, v \rangle \geqq C\langle v, v \rangle \tag{21}$$

for some $C > 0$ if v satisfies (20) and $\hbar$ is sufficiently small. Note that we already proved that

$$\langle L_-^\hbar v, v \rangle \geqq 0,$$

and equality holds if v is a scalar multiple of $u_\hbar$, for $0 < \hbar < \hbar_5$. Let $0 < \hbar < \hbar_5$ be a fixed constant. Assume that

$$\inf_{\substack{v \neq 0 \\ \langle v, u_\hbar^3 \rangle = 0}} \frac{\langle L_-^\hbar v, v \rangle}{\langle v, v \rangle} = 0.$$

Then the above infimum is attained by some nontrivial u with $\langle u, u_\hbar^3 \rangle = 0$, and u satisfies the Euler–Lagrange equation

$$L_-^\hbar u = \eta u_\hbar^3$$

for some $\eta \in \mathbf{R}$. Indeed, let v_i be a minimizing sequence with $\| v_i \| = 1$ and $\langle L_-^\hbar v_i, v_i \rangle \downarrow 0$. Then,

$$0 < \langle (H_\hbar - E)v_i, v_i \rangle \leqq \int u_\hbar^2 v_i^2 dx + \eta \tag{22}$$

eventually for all $\eta > 0$. Since $\|v_i\| = 1$ and $u_\hbar^2$ is bounded, $\|v_i\|_{Q(H_\hbar)}$ is bounded. Therefore v_i converges weakly to some u in $Q(H_\hbar)$. Since $u_\hbar$ has an exponential decay, we have

$$\int u_\hbar^2 v_i^2 \, dx \to \int u_\hbar^2 u^2 \, dx.$$

Moreover u satisfies $\langle u, u_\hbar^3 \rangle = 0$ by the weak convergence. Now recall by the choice of E in Theorem (Existence) that

$$\langle (H_\hbar - E)v, v \rangle \geqq \varepsilon \langle v, v \rangle$$

for all $v \in Q(H_\hbar)$, and in particular

$$\langle (H_\hbar - E)v_i, v_i \rangle \geqq \varepsilon \langle v_i, v_i \rangle = \varepsilon.$$

Then from (22) and if we choose $\eta < \frac{1}{3}\varepsilon$, we have

$$\int u_\hbar^2 v_i^2 \, dx > \tfrac{1}{2}\varepsilon$$

eventually and so

$$\int u_\hbar^2 u^2 \, dx > \tfrac{1}{2}\varepsilon,$$

hence u is not zero. Then it follows from the lower semi-continuity of the quadratic form $\langle L_-^\hbar \cdot, \cdot \rangle$ that

$$0 \leqq \langle L_-^\hbar u, u \rangle \leqq \liminf \langle L_-^\hbar v_i, v_i \rangle = 0$$

and thus a nontrivial $u/\|u\|$ attains the minimum and it satisfies the Euler–Lagrange equation. Now by rewriting $u/\|u\|$ by u, we have

$$0 = \langle u, L_-^\hbar u_\hbar \rangle = \langle L_-^\hbar u, u_\hbar \rangle = \eta \langle u_\hbar^3, u_\hbar \rangle.$$

Therefore, $\eta = 0$ and so $L_-^\hbar u = 0$ which implies $u = C u_\hbar$ for some C. However, it contradicts the fact that $u \neq 0$ and $\langle u, u_\hbar^3 \rangle = 0$. Therefore,

$$\inf_{\substack{v \neq 0 \\ \langle v, u_\hbar^3 \rangle = 0}} \frac{\langle L_-^\hbar v, v \rangle}{\langle v, v \rangle} := C_1 > 0$$

for some $C_1 > 0$. Hence we have proved (21). Next

$$\begin{aligned} \langle L_-^\hbar v, v \rangle &= \left\langle -\frac{1}{2}\frac{d^2 v}{dy^2} + (V_\hbar + \lambda)v - u_\hbar^2 v, v \right\rangle = \langle (H_\hbar - u_\hbar^2)v, v \rangle \\ &= \langle H_\hbar v, v \rangle - \langle u_\hbar^2 v, v \rangle \geqq \langle H_\hbar v, v \rangle - C_2 \langle v, v \rangle \end{aligned} \tag{23}$$

for some $C_2 > 0$, since $u_\hbar$ is uniformly bounded. From (21) and (23)

$$\langle L_-^\hbar v, v \rangle \geqq \langle H_\hbar v, v \rangle - \frac{C_2}{C_1} \langle L_-^\hbar v, v \rangle.$$

Therefore,

$$\langle L_-^\hbar v, v \rangle \geqq \frac{1}{1 + \dfrac{C_2}{C_1}} \langle H_\hbar v, v \rangle.$$

Setting $C_0 = 1/(1 + C_2/C_1)$, we are done. Q.E.D

Now let us take care of the negative eigenvalue of $L_+^\hbar$ using the constraint (19). As in [W.m], we first consider the constraint

$$\langle u_\hbar, u\rangle = 0, \tag{24}$$

which is the first order approximation of (19).

Proposition 4.5. *Assume that V has a local minimum at $x = 0$. Then*

$$\langle L_+^\hbar u, u\rangle \geqq C_3\langle u, u\rangle_{Q(H_\hbar)}$$

for some $C_3 > 0$ and u with $\langle u, u_\hbar\rangle = 0$ if $\hbar$ is sufficiently small.

Proof. First, we prove that

$$m_\hbar := \inf_{\langle u_\hbar, u\rangle = 0} \frac{\langle L_+^\hbar u, u\rangle}{\langle u, u\rangle} > 0. \tag{25}$$

Suppose that $m_\hbar \leqq 0$. Since $\langle L_+^\hbar \cdot, \cdot\rangle$ is again weakly lower semicontinuous because $L_+^\hbar = L_-^\hbar - 2u_\hbar^2$ and so it is a compact perturbation of $L_-^\hbar$, we can prove that $m_\hbar$ is realized by some $a_\hbar$ with $\langle a_\hbar, u_\hbar\rangle = 0$ and $\| a_\hbar \| = 1$, by the similar argument as in Proposition 4.4. Then it satisfies the following Euler–Lagrange equation,

$$L_+^\hbar a_\hbar = m_\hbar a_\hbar + \zeta u_\hbar \tag{26}$$

for some $\zeta \neq 0 \in \mathbf{R}$. Now, we claim that $\zeta \neq 0$ and

$$m_\hbar > \lambda_\hbar, \tag{27}$$

where $\lambda_\hbar$ is the lowest eigenvalue of $L_+^\hbar < 0$. Suppose $m_\hbar \leqq \lambda_\hbar$ (which implies $m_\hbar = \lambda_\hbar$). Therefore,

$$\langle L_+^\hbar a_\hbar, a_\hbar\rangle = \lambda_\hbar,$$

since the dimension of the eigenspace of $\lambda_\hbar$ is one dimensional, and so $a_\hbar$ is the normalized eigenfunction. However, we know that u_0^2 is the eigenfunction of the negative eigenvalue of L_+^0 and $L_+^\hbar \to L_+^0$ in the strong resolvent sense. Moreover, we know from Lemma 2.2 that $u_\hbar \to u_0$ in $D(H_\hbar)$. Therefore $a_\hbar \sim C u_\hbar^2$, $C \neq 0$ if $\hbar$ is sufficiently small and so

$$\langle u_\hbar, a_\hbar\rangle \sim C\langle u_\hbar, u_\hbar^2\rangle \sim C\langle u_0, u_0^2\rangle > 0,$$

which contradicts the assumption $\langle u_\hbar, a_\hbar\rangle = 0$. Therefore, we get (27) and thus $\lambda_\hbar < m_\hbar \leqq 0$, since we have proved in Proposition 3.6 i) that $\lambda_\hbar$ is the only negative eigenvalue if $0 < \hbar < \hbar_6$. Therefore, $L_+^\hbar - m_\hbar$ is one to one. Now from (26)

$$a_\hbar = (L_+^\hbar - m_\hbar)^{-1}(\zeta u_\hbar).$$

Substituting this into (24), we have

$$0 = \langle u_\hbar, (L_+^\hbar - m_\hbar)^{-1}(u_\hbar) L\rangle.$$

Decompose $u_\hbar$ into $u_\hbar = u_\parallel + u_\perp$, where $u_\parallel$ is the part parallel to the eigenvector with the eigenvalue $\lambda_\hbar$. Then

$$0 = (\lambda_\hbar + |m_\hbar|)^{-1} \int u_\parallel^2 dy + \int u_\perp (L_+^\hbar + |m_\hbar|)^{-1} u_\perp dy$$
$$\leqq (\lambda_\hbar)^{-1} \int u_\parallel^2 dy + \int u_\perp (L_+^\hbar)^{-1} u_\perp dy = \int u_\hbar (L_+^\hbar)^{-1} u_\hbar dy.$$

In other words, we have

$$\int u_\hbar (L_+^\hbar)^{-1} u_\hbar dy \geqq 0 \tag{28}$$

if $\hbar$ is sufficiently small. On the other hand, we have

$$L_-^0 u_0 = -\frac{1}{2}\frac{d^2}{dy^2} + \lambda u_0 - u_0^3 = 0 \tag{29}$$

and $u_0 = \sqrt{2\lambda}\operatorname{sech}\sqrt{2\lambda}y$. Differentiate (29) with respect to $\sqrt{\lambda}$ to get

$$L_+^0 \frac{\partial u_0}{\partial\sqrt{\lambda}} + 2\sqrt{\lambda}u_0 = 0 \quad \text{i.e., } (L_+^0)^{-1} u_0 = -\frac{2}{\sqrt{\lambda}}\frac{\partial u_0}{\partial\sqrt{\lambda}}.$$

Now

$$\int u_0 (L_+^0)^{-1} u_0 = -\frac{2}{\sqrt{\lambda}} \int \frac{\partial u_0}{\partial\sqrt{\lambda}} u_0 dy = -4\frac{\partial}{\partial\lambda}\int u_0^2 dy$$
$$= -4\frac{\partial}{\partial\lambda}\int 2\lambda \operatorname{sech}^2\sqrt{2\lambda}y\, dy = -8\frac{\partial}{\partial\lambda}\int \lambda \operatorname{sech}^2 \sqrt{2\lambda}y\, dy$$
$$= -8\frac{d}{d\lambda}(\sqrt{2\lambda}\cdot\lambda) = -8\sqrt{2}\cdot\tfrac{3}{2}\sqrt{\lambda} = -12\sqrt{2\lambda} < 0.$$

However, this contradicts (28) if $\hbar$ is sufficiently small since $u_\hbar \to u_0$ in H^2 and $L_+^\hbar \to L_0^\hbar$ in the strong resolvent sense. Hence, we have proved (25). Now applying the same argument as in Proposition 4.4, we are done. Q.E.D.

Once we have Propositions 4.5 and 4.4, the remaining argument to get Lyapunov stability is a standard argument using Sobolev inequalities (see e.g., [W.m]). Hence, we have proved the orbital stability with the constraint (19). Now let us remove this constraint using the fact that $\phi_{z,\hbar}$ depends on $\hbar$ in the C^1-sense (see Theorem (Existence)). Denote the bound states for the nonlinear eigenvalue $E(= -\lambda)$ by $u_\hbar^\lambda$. If $\psi(0)$ is close to $u_\hbar^\lambda$, then $\int|\psi(0)|^2 dy$ is close to $\int |u_\hbar^\lambda|^2 dy$ too. Moreover, note that $u_\hbar^\lambda$ is a C^1-function of $\hbar$ and λ and so $\int |u_\hbar^\lambda|^2 dy$ is differentiable. Now

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda}\int |u_\hbar^\lambda|^2 dy \sim \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=\lambda}\int |u_\hbar^\lambda|^2 dy = -3\sqrt{2\lambda} < 0,$$

if $\hbar$ is sufficiently small. Therefore if $\hbar$ is sufficiently small, then $\partial/\partial\lambda|_{\lambda=\lambda}\int|u_\hbar^\lambda|^2 dy \neq 0$, and so by the implicit function theorem, we can find some $\tilde{\lambda}$ near λ so that

$$\int |\psi(0)|^2 dy = \int |u_\hbar^{\tilde{\lambda}}|^2 dy.$$

Now applying the previous argument to $u_\hbar^{\tilde{\lambda}}$, we have the following main theorem of this section.

Theorem 4.6. *There exists some $\hbar_{10} > 0$ such that our solutions $u_\hbar$ found in Theorem (Existence) are Lyapunov stable if $0 < \hbar < \hbar_{10}$ and the critical point of V is a local minimum.*

5. Instability at Local Maxima

In this section, we prove the Lyapunov instability of the solution $u_\hbar$ when the critical point of V is a local maximum, without the restriction on V at ∞ if the global evolution exists (e.g., if V is quadratic at infinity (see [O1])).

5.1. Idea of the Proof. We first outline the idea behind the proof. We have proved Ehrenfest's law for NLS in [O2],

$$\frac{d}{dt}\langle \psi_t, x\psi_t\rangle = \langle \psi_t, p\psi_t\rangle,$$

$$\frac{d}{dt}\langle \psi_t, p\psi_t\rangle = -\langle \psi_t, V'\psi_t\rangle$$

if $\psi(0)\in Q(H)\cap\mathcal{S}$. From these equations, we can heuristically say that if ψ_t is localized, then it behaves like a particle under Newton's equation. Therefore, if our bound state is translated from the equilibrium position at a local maximum of V and it is localized for some time, then classical mechanics tells us that the wave packet should fall into a nearby well. From this heuristic argument, it is quite natural to choose either the "position expectation value" $X := \langle \psi_t, x\psi_t\rangle$ or the "momentum expectation value" $P := \langle \psi_t, x\psi_t\rangle$ as a Lyapunov function. Here we mean by the Lyapunov function a continuous function which proves the instability. However, although we believe it to be so, we have not been able to prove that X or P is in fact a Lyapunov function, due to the fact that the bound state has a "tail" which gives some difficulty in the proof. Instead, we prove the instability result from the following general "instability principle":

"*Instability Principle*". Let M be a symplectic manifold (which may be infinite dimensional) and H be a Hamiltonian function. Assume that $x_0\in M$ is a critical point of H (i.e. an equilibrium of the Hamiltonian vector field X_H) and that the Hessian $d^2H(x_0)$ has just one negative eigenvalue with all the remaining spectrum positive bounded away from zero. Then the equilibrium x_0 is unstable.

This is obviously true for the finite dimensional case because in this situation, the equilibrium is spectrally instable (see e.g. [O1]). However, for the infinite dimensional case we have to take care of some technicalities to make this heuristic principle a theorem.

Let us collect some facts which we have proved in the previous sections.

Proposition 5.1. *The function,*

$$F_h(\phi) := \tfrac{1}{2}\langle \phi, H_h\phi\rangle - \tfrac{1}{4}\int|\phi|^4 dx - \frac{E}{2}\int|\phi|^2 dx$$

has u_h as a critical point, and its Hessian

$$\delta^2 F_h(u_h) = \begin{pmatrix} L_+^h & 0 \\ 0 & L_-^h \end{pmatrix}$$

has only one negative direction if we restrict $\delta^2 F_\hbar$ *to the space*

$$\{w \in Q(H_\hbar) \mid \langle w, u_\hbar \rangle = 0 \text{ and } \langle w, iu_\hbar^3 \rangle = 0\}.$$

Corollary 5.2. *Consider the reduced space* $L^{-1}(L(u_\hbar))/S^1$ *and reduced Hamiltonian of* $F_\hbar$. *Then its linearization has only one negative eigenvalue, and all the remaining spectrum is positive and bounded away from zero.*

Proof of Proposition 5.1. Let $w = u + iv$. We have proved in Proposition 4.4 that

$$\langle L_-^\hbar v, v \rangle \geqq C_0 \langle v, v \rangle$$

for all $v \in Q(H)$ with $\langle v, u_\hbar^3 \rangle = 0$. Moreover, note that $\langle u_\hbar, u_\hbar' \rangle = 0$ and $\langle L_+^\hbar u_\hbar', u_\hbar' \rangle < 0$, i.e. $\lambda_\hbar < m_\hbar < 0$ from Lemma 3.7. Also we can prove that $\langle L_+^\hbar u, u \rangle > 0$ for any $u \in Q(H_\hbar)$ such that $\langle u, u_\hbar' \rangle = \langle u, u_\hbar \rangle = 0$ in the same way as we proved Proposition 4.5. Therefore, $L_+^\hbar$ has just one negative eigen-direction in the sense above for $\langle u, u_\hbar \rangle = 0$. Since

$$\langle \delta^2 F_\hbar(u_\hbar) w, w \rangle = \langle L_+^\hbar u, u \rangle + \langle L_-^\hbar v, v \rangle$$

and

$$\langle w, u_\hbar \rangle = \langle u, u_\hbar \rangle, \quad \langle w, iu_\hbar^3 \rangle = \langle v, u_\hbar^3 \rangle,$$

we have proved the proposition. Q.E.D.

Since the principle is not mathematically rigorous for NLS, we will follow [GSS] and find the Lyapunov function by hand to prove the instability using the basic idea behind the instability principle.

5.2. Construction of a Lyapunov Function. In this subsection, we assume $\hbar > 0$ so small that all the results proved in the previous sections are true. Once we assume this, we omit the subscript or superscript $\hbar$ from all variables, since $\hbar$ will not play any role in later discussions. In other words, we write $L_\pm$, a(not u), H and F for $L_\pm^\hbar$, $u_\hbar$, $H_\hbar$ and $F_\hbar$ respectively.

Note that the function

$$L(\phi) = \tfrac{1}{2} \|\phi\|_2^2 = \tfrac{1}{2} \langle \phi, \phi \rangle$$

restricted to $Q(H)$ is smooth, and so $L^{-1}(L_0)$, $L_0 = L(u_\hbar)$ is a smooth submanifold of $Q(H)$ with codimension one.

Definition 5.3. $S_{L_0} := L^{-1}(L_0)$

$$(\phi, \psi) := \operatorname{Re} \langle \phi, \psi \rangle,$$

$P_\phi :=$ the orthogonal projection of $Q(H)$ onto the tangent space

$$T_\phi S_{L_0} = \{v \in Q(H) \mid (v, \phi) = 0\}.$$

Following [GSS], we define the following:

Definition 5.4. In a tubular neighborhood $\mathcal{U}_\varepsilon$ of the S^1-orbit of a, define

$$\mathcal{U}_\varepsilon := \{\phi \in Q(H) \mid \rho_a^2(\phi) \leqq \varepsilon\},$$

$s(\phi)$:= the unique phase $s \in \mathbf{R}/2\pi$ such that $e^{is(\phi)} \cdot \phi$ realizes the infimum in Definition 4.1.

It is obvious that this function is well-defined and smooth if we choose ε sufficiently small.

Now, consider the following function on $\mathscr{U}_\varepsilon \subset Q(H)$,

$$A(\phi) := \operatorname{Im}\langle a', P_a(e^{is(\phi)} \cdot \phi - a)\rangle = \operatorname{Im}\langle a', P_a(e^{is(\phi)} \cdot \phi)\rangle.$$

Remark 5.5 Recall the definition of the momentum observable;

$$\vec{p} = \frac{1}{i}\frac{\partial}{\partial x} \quad \text{and} \quad P = \langle \psi_t, p\psi_t \rangle.$$

Then this function A is essentially the linearization of P at the solution $a(=u_h)$, if we omit the projection P_a.

We will spend the remaining section proving that this function A is a Lyapunov function near $a(=u_h)$, so that u_h is unstable. The proof of this is an adaptation of the instability proof in [GSS] in our context. In particular, we refer readers to [GSS] for several functional analytic technicalities which appear in our proof. We want to remark that not only our case but also that in [GSS] are in the context of the "instability principle" if we go down to the reduced space. In this light, it might be interesting to investigate under what conditions the heuristic principle could be made rigorous in a way, which encompasses both ours and [GSS]. In fact, all the statements following make sense in the general abstract symplectic context.

Lemma 5.6. *Let ∇A be the L^2-gradient of A with respect to $(\ ,\ )$. Then $\nabla A(a) = ia'$ and so $i \cdot \nabla A(a) = -a'$.*

Proof. Immediate by a direct computation. Q.E.D.

Now consider the differential equation

$$\frac{d\phi}{dt} = -i\nabla A(\phi). \tag{30}$$

By the definition of A, it is easy to see that the flow of (30) is well-defined in $Q(H)$ (see Lemma 4.6 [GSS]). Let $R(\lambda, \phi)$ be the flow map at time λ with the initial condition ϕ.

Lemma 5.7. *There exists a smooth function*

$$\Lambda : L^{-1}(L_0) \to \mathbf{R}$$

such that $E(R(\Lambda(\phi), \phi)) \geqq E(a)$ for all $\phi \in \mathscr{U}_\varepsilon$ such that $L(\phi) = L(a)$, with equality only for $\phi \in \mathcal{O}_a$, where $\mathcal{O}_a$:= the S^1 orbit of a.

Proof. Since F is S^1-invariant, we have

$$F(e^{is(\phi)} \cdot \phi) = F(\phi).$$

Writing $e^{is(\phi)}\cdot\phi$ as $M(\phi)$, we have

$$\begin{aligned} F(\phi) &= F(M(\phi)) \\ &= F(a) + \tfrac{1}{2}\langle \delta^2 F(a)(M(\phi)-a), M(\phi)-a\rangle + o(\|M(\phi)-a\|^2). \end{aligned} \tag{31}$$

We shall define $\lambda = \Lambda(\phi)$ to be the unique solution of the equation

$$f(\lambda,\phi) := (P_a(M(R(\lambda,\phi))-a), a') = (M(R(\lambda,\phi))-a, a') = 0. \tag{32}$$

Here we used the fact that $a \perp a'$ with respect to (,). We have

$$f(0,a) = (M(R(0,a))-a, a') = 0$$

and

$$\frac{\partial f}{\partial\lambda}(0,a) = \left(DM(a)\frac{\partial R}{\partial\lambda}(0,a), a' \right) = (DM(a)(a'), a').$$

Here,

$$DM(a)(a') = a' + i\langle ds(a), a'\rangle\cdot a.$$

Since $(a,a') = 0$, we have

$$\frac{\partial f}{\partial\lambda}(0,a) = (a', a') \neq 0.$$

Therefore, by the implicit function theorem and the equivariance of f, Λ is well defined and smooth in $\mathscr{U}_\varepsilon$ if ε is sufficiently small. Now, if we restrict ourselves to ϕ's such that $L(\phi) = L(a)$, then we may rewrite (31) as

$$F(\phi) = F(a) + \tfrac{1}{2}\langle \delta^2 F(a)(P_a(M(\phi)-a), P_a(M(\phi)-a))\rangle + o(\|P_a(M(\phi)-a)\|^2). \tag{33}$$

Into (33), we substitute $\tilde{\phi} = R(\Lambda(\phi),\phi)$, and then we have

$$\langle P_a(M(\tilde{\phi})-a), a\rangle = 0$$

and

$$\langle P_a(M(\tilde{\phi})-a), a'\rangle = 0 \text{ from (32)}.$$

Moreover since $M(\tilde{\phi})$ realizes the infimum in Definition 4.1,

$$\langle P_a(M(\tilde{\phi})-a), ia^3\rangle = 0$$

from Lemma 4.3. Then from Proposition 5.1,

$$\langle \delta^2 F(a)(P_a(M(\tilde{\phi})-a)), P_a(M(\tilde{\phi})-a)\rangle \geqq C\cdot\|M(\tilde{\phi})-a\|^2$$

for some $C > 0$. Therefore,

$$F(\tilde{\phi}) \geqq F(a) + \frac{C}{4}\|M(\tilde{\phi})-a\|^2 \tag{34}$$

if the tubular neighborhood $\mathscr{U}_\varepsilon$ is sufficiently small, i.e. ε is sufficiently small. Hence we have

$$E(\tilde{\phi}) \geqq E(a) + \frac{C}{4}\|M(\tilde{\phi})-a\|^2$$

as $F(\tilde{\phi}) = E(\tilde{\phi}) - EL(\tilde{\phi})$ and $L(\tilde{\phi}) = L(a)$, and so we are done. Q.E.D.

Lemma 5.8. *If ε is sufficiently small, we have*

$$E(a) < E(\phi) + \Lambda(\phi)\{E, A\}(\phi)$$

for $\phi \in \mathscr{U}_\varepsilon$ with $L(\phi) = L(a)$ and ϕ not in $\mathcal{O}_a$, where

$$\{E, A\}(\phi) := \operatorname{Im}\langle \nabla E, \nabla A\rangle(\phi).$$

Remark 5.9. This is the standard definition of the cannonical Poisson bracket. In [GSS], it is denoted as P.

Proof. Note that

$$\left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} E(R(\lambda, \phi)) = \left(\nabla E(\phi), \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} R(\lambda, \phi)\right) = (\nabla E(\phi), -i\nabla A(\phi))$$
$$= \operatorname{Im}\langle \nabla E(\phi), \nabla A(\phi)\rangle = \{E, A\}(\phi)$$

and

$$\left.\frac{\partial^2}{\partial \lambda^2}\right|_{\substack{\lambda=0\\ \phi=a}} E(R(\lambda, \phi)) = \left\langle \delta^2 F(a)\frac{\partial R}{\partial \lambda}(0, a), \frac{\partial R}{\partial \lambda}(0, a)\right\rangle + \left\langle \delta F(a), \frac{\partial^2 R}{\partial \lambda^2}(0, a)\right\rangle$$
$$= \langle \delta^2 F(a)a', a'\rangle < 0.$$

Here, for the first equality, we used the fact that L is invariant under the flow of $R(\lambda, \phi)$ due to the fact that A is S^1-invariant, and for the last inequality, we applied Lemma 3.7. By the Taylor expansion, we have

$$E(R(\lambda, \phi)) \leqq E(\phi) + \lambda\{E, A\}(\phi)$$

for all sufficiently small λ, ε. By combining this with Lemma 5.7, we have

$$E(a) < E(R(\Lambda(\phi), \phi)) \leqq E(\phi) + \Lambda(\phi)\{E, A\}(\phi)$$

for all ϕ not in $\mathcal{O}_a$. Q.E.D.

Now choose a smooth curve $\alpha:(-\delta, \delta) \to \mathscr{U}_\varepsilon$ such that $\alpha(0) = a$, $d\alpha/ds|_{s=0} = a'$ and $L(\alpha(s)) = L(a)$ which is certainly possible as $L^{-1}(L_0)$ is a smooth submanifold and $a' \in T_a(L^{-1}(L_0))$. Moreover, if we choose $\delta > 0$ sufficiently small, then $E(\psi(s))$ has a strict maximum at $s = 0$. Indeed,

$$\left.\frac{d}{ds}\right|_{s=0} E(\alpha(s)) = \left.\frac{d}{ds}\right|_{s=0} F(\alpha(s)) = \left\langle \delta F(a), \frac{d\alpha}{ds}(0)\right\rangle = 0,$$
$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\alpha(s)) = \left\langle \delta^2 F(a)\frac{d\alpha}{ds}(0), \frac{d\alpha}{ds}(0)\right\rangle + \left\langle \delta F(a), \frac{d^2\alpha}{ds^2}(0)\right\rangle$$
$$= \langle \delta^2 F(a)a', a'\rangle < 0.$$

Remark 5.10. We have chosen α so that it drops down into the "mountain pass" through $a(= u_h)$ in the reduced space $L^{-1}(L_0)/S^1$.

Lemma 5.11. *$\{E, A\}(\alpha(s))$ changes its sign from positive to negative at $s = 0$.*

Proof. From Lemma 5.8, we have

$$0 < E(a) - E(\alpha(s)) \leqq \Lambda(\alpha(s))\{E, A\}(\alpha(s))$$

for all $s \neq 0$. Thus, we have only to prove that $\Lambda(\alpha(s))$ changes its sign from positive to negative. Obviously, $\Lambda(\alpha(0)) = 0$ since $\alpha(0) = a$. Now it is enough to prove that

$$\left.\frac{d}{ds}\right|_{s=0} \Lambda(\alpha(s)) < 0.$$

Consider the equation

$$(M(R(\Lambda(\alpha(s)), \alpha(s)) - a, a')) = 0$$

and differentiate this with respect to s at $s = 0$. We get

$$\begin{aligned} 0 &= \left(DM(a)\left\{ D_\phi R(0, a)\left.\frac{d\alpha}{ds}\right|_{s=0} + D_\lambda R(0, a)\left.\frac{d\lambda(\alpha(s))}{ds}\right|_{s=0}\right\}, a'\right) \\ &= \left(DM(a)\left\{ a' + a'\left.\frac{d\Lambda(\alpha(s))}{ds}\right|_{s=0}\right\}, a'\right). \end{aligned}$$

On the other hand, we have

$$DM(a)a' = a' + i\langle ds(a), a'\rangle a.$$

Therefore we have

$$0 = \left\{\left.\frac{d\Lambda(\alpha(s))}{ds}\right|_{s=0} + 1\right\}(DM(a)a', a') = \left\{\left.\frac{d\Lambda(\alpha(s))}{ds}\right|_{s=0} + 1\right\}(a', a'),$$

since $(ia, a') = 0$. As $(a', a') \neq 0$, we have $d\Lambda(\alpha(s))/ds|_{s=0} = -1 < 0$. Q.E.D.

5.3. Proof of the Instability Theorem. Now we are ready to prove the main theorem in this section.

Theorem 5.12. *The function A is a Lyapunov function and so a is unstable.*

Proof. Let $\psi(0) = \alpha(s)$. Then $\{E, A\}(\psi(0)) = \{E, A\}(\alpha(s)) > 0$. We will prove that the flow of (1) eventually goes out of $\mathcal{U}_\varepsilon$ for any small $s < 0$.

Suppose that $\psi(t) \in \mathcal{U}_\varepsilon$ for all t. Since $E(\alpha(s))$ has a strict maximum at $s = 0$, we have

$$0 < E(a) - E(\psi_0) = E(a) - E(\psi(t)) \leqq \Lambda(\psi(t))\{E, A\}(\psi(t)).$$

Here we used the conservation of the energy E for the second equality. By letting ε be smaller if necessary, we may assume that $\Lambda(\psi(t)) < 1$. Therefore,

$$\{E, A\}(\psi(t)) > E(a) - E(\psi_0) = \varepsilon_0 > 0$$

for all t. Now

$$\frac{d}{dt} A(\psi(t)) = \{A, E\}(\psi(t)) = -\{E, A\}(\psi(t)) < -\varepsilon_0$$

for all t. Therefore,

$$A(\psi(t)) < A(\psi(0)) - \varepsilon_0 t$$

for all t. Hence A is a Lyapunov function. Indeed, we have

$$|A(\psi)| \leqq \| M(\psi) - a \| \cdot \| a' \| = \| a' \| \cdot \rho_{\mathcal{O}_a}(\psi).$$

Therefore if t is sufficiently large, then $\psi(t)$ eventually goes out of $\mathcal{U}_\varepsilon$ if we choose the initial condition in the arc α however small s is if $s \neq 0$ and so $\alpha(s) \neq a$.

Q.E.D.

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