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Relating Kac–Moody, Virasoro and Krichever–Novikov Algebras

José Alberty, Anne Taormina and Pierre van Baal
Theory Division CERN, CH-1211 Geneva 23, Switzerland

Abstract. We demonstrate that the Kac–Moody and Virasoro-like algebras on Riemann surfaces of arbitrary genus with two punctures introduced by Krichever and Novikov are in two ways linearly related to Kac–Moody and Virasoro algebras on S^1. The two relations differ by a Bogoliubov transformation, and we discuss the connection with the operator formalism.

1. Introduction

Two-dimensional conformal field theories [1] have been considerably developed recently. In particular, they are relevant in the study of string multiloop amplitudes, which amount to the contribution of higher genus Riemann surfaces to partition functions and expectation values. The application of powerful mathematical results in algebraic geometry and in complex analysis on Riemann surfaces has led to a rather detailed understanding of the multiloop structure, especially in the operator formalism which uses punctured Riemann surfaces to describe scattering amplitudes [2, 3, 4].

On the other hand, the older algebraic approach consists of using the Kac–Moody and Virasoro algebras to describe the Hilbert space of a closed string. These algebras are then defined on S^1. They can be naturally extended to the Riemann sphere CP_1 with punctures at $z=0$ and $z=\infty$. This allows, for instance, the algebraic construction of the non-interacting string partition function.

Krichever and Novikov [5] introduced a natural extension of these algebras to the interacting string theory by formulating the algebras on a Riemann surface Σ of arbitrary genus g with two punctures at $P_\pm$. Note that for $g \geq 1$, these punctures cannot be moved to two specified points by conformal transformations as was the case for CP_1. The Kac–Moody algebra is defined as that of Lie algebra-valued meromorphic functions on Σ which are holomorphic outside $P_\pm$. Similarly, the Virasoro algebra is given by the algebra of meromorphic vector fields on Σ, holomorphic outside $P_\pm$. As in the operator formalism, one associates a set of local complex coordinates $z_\pm$ with the punctures $P_\pm$ that vanish at the puncture. As for the sphere, the radial parameter is related to the time parameter τ, such that P_- corresponds to $\tau=\infty$ and P_+ to $\tau=-\infty$. A coordinateindependent way of defining τ is through the use of the third kind differential [5]

$$\tau = \mathrm{Re} \int_{Q_0}^{Q} d\ln\left[\frac{E(Q, P_-)}{E(Q, P_+)}\right], \tag{1}$$

where E is the prime form [6]. The contours C_τ of constant τ give a snapshot of the string. In particular, C_τ is not necessarily connected and the snapshots depend on the choice of punctures. However for $|\tau|$ big enough, i.e. for Q close to $P_\pm, C_\tau$ will be a circle which we generically denote by $C_\pm$. The orientation of $C_\pm$ is positive with respect to $P_\pm$. As was shown by Krichever and Novikov, the restriction of these algebras to $C_\pm$ gives the Kac–Moody and Virasoro algebras on S^1. These algebras specify the Hilbert spaces relevant for the ingoing and outgoing string states.

The natural question to ask is in what sense the algebras formulated by Krichever and Novikov contain information on the interactions taking place between $\tau = -\infty$ and $\tau = \infty$. A strong argument against such a dynamical content of the algebra is that an algebra is a local structure. The current algebras considered by Krichever and Novikov [5] are restrictions of the algebra of Lie algebra-valued meromorphic functions on Σ (in the sense that the poles of the meromorphic functions are restricted to the punctures) with a central extension defined in the following [7]. Let Γ be the space of Lie algebra-valued meromorphic functions on Σ and $\hat{\Gamma}$ its central extension defined by

$$\hat{\Gamma} = \Omega_1/d\Omega_0 \times \Gamma, \tag{2}$$

where Ω_n is the space of meromorphic n-forms. Let us label a generic element of the extension by $A = (a, F)$, where a is a non-exact one-form and F is in Γ. The commutator of two such elements is given by

$$[(a, F), (b, G)] = (k\,\mathrm{Tr}(FdG - GdF), [F, G]). \tag{3}$$

One easily verifies that the Jacobi identity is satisfied since

$$[(a, F), [(b, G), (c, H)]] + \text{cyclic} = (-kd\,\mathrm{Tr}(H[G, F]), 0) = 0. \tag{4}$$

There is a map from the space $\Omega_1/d\Omega_0$ (now restricted to forms holomorphic outside of $P_\pm$) to the complex numbers, defined by the map

$$a \in \Omega_1 \to \frac{1}{2\pi i}\oint_{C_+} a = -\frac{1}{2\pi i}\oint_{C_-} a, \tag{5}$$

which leads to the central extension

$$\gamma_k(F, G) = \frac{k}{\pi i}\oint_{C_+} \mathrm{Tr}(FdG) \tag{6}$$

used by Krichever and Novikov. (They only consider $U(1)$ current algebras, for which the traces in Eqs. (3, 4, 6) are replaced by a factor $\frac{1}{2}$). Somewhat more complicated is the central extension for meromorphic vector fields on Σ, for which we refer to the appendix of ref. [7].

2. The Case of Current Algebras

Although the central extension was formulated independent of the genus g, constructing a basis for the algebras will crucially depend on the global structure of the Riemann surface, and in particular on its genus g. The key tool here is the Weierstrass gap theorem [6] which states that meromorphic functions f having a pole of order n at a given point cannot be extended holomorphically outside that point for g values of n between 1 and $2g$ (for a generic point, $n = 1, 2, \ldots, g$).

The basis of meromorphic functions holomorphic outside of $P_\pm$ is constructed by specifying the order of the poles (or zeros) at P_+ and P_-, such that there are g zeros outside $P_\pm$ whose positions are fixed by requiring the function to be single-valued. This is the famous Jacobi inversion theorem [6]. The above needs two remarks. First, the constant function is clearly an element of the algebra. The addition of a constant to a meromorphic function is viewed as a gauge freedom. Indeed, if one takes a meromorphic function with poles both at P_+ and P_- (a situation required to occur due to the Weierstrass gap theorem), it cannot be uniquely specified by giving the order of the poles at $P_\pm$. Instead one requires there to be $g+1$ zeros outside of $P_\pm$ for the g functions having poles at both P_+ and P_-. They will then have a one-parameter freedom corresponding precisely to the addition of a constant. The basis is therefore specified by

$$\begin{aligned} A_j(z_\pm) &\sim z_\pm^{\pm j - g/2}(1 + O(z_\pm)), \quad |j| > g/2, \\ &\sim z_\pm^{\pm j - g/2 \pm 1/2 - 1/2}(1 + O(z\pm)), \quad |j| \leqq g/2, \quad j \neq g/2, \\ A_{g/2} &= 1, \end{aligned} \tag{7}$$

where j is half-integer for g odd and integer for g even. (However, note that one can just as well define $B_n = A_{n+g/2}$ with n integer for the basis.) The second remark is that for each given order j, there are special points $P_\pm$ for which the inversion theorem [6] is not valid. This is easily demonstrated for the torus case ($g = 1$) and we illustrate it with $P_\pm = \pm z_0$. The Weierstrass σ-function can be used to factorize arbitrary meromorphic functions. (For higher genus, the prime form is used for this purpose [6].) Single-valuedness is easily seen to imply up to a constant

$$A_j = \frac{\sigma^{j-1/2}(z - z_0)}{\sigma^{j+1/2}(z + z_0)}\sigma(z + 2jz_0), \quad |j| \neq 1/2. \tag{8}$$

When $2jz_0$ equals $\pm z_0$ modulo periods, the zero which is supposed to occur outside of $P_\pm$ will actually coincide with it. For these special values of z_0 one has to modify the specification of the basis. Our arguments relating the algebras of Krichever and Novikov to the algebras on S^1 are independent of the detailed choice of the basis and we therefore will not dwell any further on this point.

Useful in our further analysis will be the set of meromorphic one-forms $d\omega_i$, holomorphic outside of $P_\pm$ and dual to the basis A_i in the following sense [5]:

$$\frac{1}{2\pi i}\oint_{C_+} A_i d\omega_j = -\frac{1}{2\pi i}\oint_{C_-} A_i d\omega_j = \delta_{ij}. \tag{9}$$

If $\phi(z)$ is the chiral scalar field on the Riemann surface Σ, Krichever and Novikov define the operator expansion as follows:

$$d\phi = \alpha_n d\omega_n. \tag{10}$$

The chiral creation-annihilation operators are given, for f a meromorphic function holomorphic outside of $P_\pm$, by [3]

$$a[f] = \frac{1}{2\pi i} \oint_{C_+} f\, d\phi = -\frac{1}{2\pi i} \oint_{C_-} f\, d\phi, \tag{11}$$

such that $\alpha_n = a[A_n]$. One easily verifies that $d\omega_{g/2}$ is exactly the third kind differential used to define τ on Σ with two punctures [5]. $\alpha_{g/2}$ is then naturally identified with the momentum flowing through Σ from P_+ to P_-. The commutation relations (for central extension $k = 1$) follow from the general result [3, 8]

$$[a[f], a[g]] = \frac{1}{2\pi i} \oint_{C_+} f\, dg = -\frac{1}{2\pi i} \oint_{C_-} f\, dg, \tag{12}$$

such that

$$[\alpha_n, \alpha_m] = \frac{1}{2\pi i} \oint_{C_+} A_n dA_m = -\frac{1}{2\pi i} \oint_{C_-} A_n dA_m \equiv \gamma_{nm}. \tag{13}$$

Note that $\alpha_{g/2}$ commutes with all α_n, as it should for the momentum operator.

We can now easily construct a linear transformation which relates this algebra to the standard $U(1)$ Kac–Moody algebra. For this we expand the chiral field with respect to the coordinates at the punctures

$$d\phi(z_+) = \sum_n a_n z_+^{-n} \frac{dz_+}{z_+},$$

$$d\phi(z_-) = -\sum_n b_n z_-^{n} \frac{dz_-}{z_-}, \tag{14}$$

such that from Eq. (12) one finds

$$[a_n, a_m] = [b_n, b_m] = m\delta_{n+m,0}. \tag{15}$$

From this it follows that

$$\alpha_n = \frac{1}{2\pi i} \oint_{C_+} A_n d\phi = \frac{1}{2\pi i} \oint_{C_+} \frac{A_n(z_+)}{z_+^{m+1}} dz_+ a_m$$

$$= -\frac{1}{2\pi i} \oint_{C_-} A_n d\phi = \frac{1}{2\pi i} \oint_{C_-} \frac{A_n(z_-)}{z_-^{-m+1}} dz_- b_m, \tag{16a}$$

or

$$\alpha_n = A_{nm} a_m = B_{nm} b_m, \tag{16b}$$

with A_{nm} and B_{nm} respectively the Laurent coefficients of A_n at z_+ and z_-. (Observe that $A_{g/2,n} = B_{g/2,n} = \delta_{0,n}$ such that indeed the ingoing and outgoing momenta a_0 and b_0 are equal.)

It may be instructive to verify by explicit computation that Eq. (13) is satisfied when using (15) and (16) only (C_z is a contour around z)

$$[\alpha_n, \alpha_m] = -\sum_k k A_{nk} A_{m,-k} = -\oint_{C_+} \frac{dz}{2\pi i} \oint_{C_+} \frac{dw}{2\pi i} A_n(z) A_m(w) \sum_k \frac{k}{wz}\left(\frac{w}{z}\right)^k$$

$$= -\oint_{C_+} \frac{dz}{2\pi i} \oint_{C_+} \frac{dw}{2\pi i} A_n(z) A_m(w) \frac{\theta_{|z|>|w|} - \theta_{|w|>|z|}}{(z-w)^2}$$

$$= \oint_{C_+} \frac{dz}{2\pi i} \oint_{C_z} \frac{dw}{2\pi i} \frac{A_n(z) A_m(w)}{(z-w)^2} = \oint_{C_+} \frac{dz}{2\pi i} A_n(z) \frac{dA_m(z)}{dz} = \gamma_{nm}. \tag{17}$$

We can easily extend the result to non-abelian Kac–Moody algebras where the chiral commutation relations in terms of the Lie algebra-valued functions are given by (see Eq. (6))

$$[a[f], a[g]] = a[[f, g]] \pm \frac{k}{\pi i} \oint_{C_\pm} \mathrm{Tr}(f\, dg). \tag{18}$$

The chiral field is written as

$$d\phi = \alpha_n^a T_a d\omega_n \tag{19}$$

with T_a a basis of the Lie algebra such that

$$[T_a, T_b] = i f_{ab}{}^c T_c, \quad \mathrm{Tr}(T_a T_b) = \tfrac{1}{2}\delta_{ab}, \tag{20}$$

which leads to the algebra

$$[\alpha_n^a, \alpha_m^b] = i f^{ab}{}_c d_{nm}^s \alpha_{n+m+s}^c + k \delta^{ab} \gamma_{nm} \tag{21}$$

with

$$d_{nm}^s = \pm \frac{1}{2\pi i} \oint_{C_\pm} A_n A_m d\omega_{n+m+s}. \tag{22}$$

Note that one can easily show that γ_{nm} is zero for $|n+m| > r$ and that d_{nm}^s is zero for $|s| > r$ with r of the order of g. (The detailed behaviour is not interesting for us.) The fact that d_{nm}^s is non-zero for more than one value of s is interpreted by Krichever and Novikov as a generalized grading [5]. But as in the $U(1)$ case, this is just a consequence of the choice of the basis. The commutation relations (21) can be rederived by using

$$\alpha_n^a = A_{nm} a_m^a = B_{nm} b_m^a \tag{23a}$$

with A and B as in (16) and

$$[a_n^a, a_m^b] = i f^{ab}{}_c a_{n+m}^c + km\delta^{ab}\delta_{n+m,0}, \tag{23b}$$

$$[b_n^a, b_m^b] = i f^{ab}{}_c b_{n+m}^c + km\delta^{ab}\delta_{n+m,0}. \tag{23c}$$

Finally the use of the following "addition theorem"

$$\sum_l A_{n(p-l)} A_{ml} = \oint_{C_+} \frac{dz}{2\pi i} \oint_{C_+} \frac{dw}{2\pi i} \sum_l \frac{A_n(z)}{z^{p-l+1}} \frac{A_m(w)}{w^{l+1}}$$

$$= \oint_{C_+} \frac{dz}{2\pi i} \frac{1}{z^p} \oint_{C_+} \frac{dw}{2\pi i} \frac{A_n(z)}{z} \frac{A_m(w)}{w} \sum_l \left(\frac{z}{w}\right)^l$$

$$= \oint_{C_+} \frac{dz}{2\pi i} \frac{1}{z^{p+1}} \oint_{C_+} \frac{dw}{2\pi i} A_n(z) A_m(w) \frac{\theta_{|w|>|z|} - \theta_{|w|<|z|}}{w - z}$$
$$= \oint_{C_+} \frac{dz}{2\pi i} \frac{A_n(z) A_m(z)}{z^{p+1}} = \sum_s d^s_{n,m} \oint_{C_+} \frac{dz}{2\pi i} \frac{A_{n+m+s}(z)}{z^{p+1}}$$
$$= \sum_s d^s_{n,m} A_{n+m+s,p} \tag{24}$$

leads to

$$[\alpha^a_n, \alpha^b_m] = i f^{ab}{}_c A_{nk} A_{ml} a^c_{k+l} + k\delta^{ab} \gamma_{nm}$$
$$= i f^{ab}{}_c \sum_{s,q} d^s_{n,m} A_{n+m+s,q} a^c_q + k\delta^{ab} \gamma_{nm}, \tag{25}$$

which coincides with (21).

We have thus seen quite explicitly that the Kac–Moody operator algebras constructed by Krichever and Novikov on Σ are in a simple linear way related to Kac–Moody operator algebras on the circle, and hence the general grading is just a consequence of the choice of basis functions.

However, the linear transformation depends on the particular puncture considered. To understand how the information of Σ is encoded in the algebra we recall that the two circles $C_\pm$ have a well-defined interpretation. C_+ can be identified with an ingoing string state while C_- is identified with an outgoing string state. Since the string interacts, an ingoing string in a certain vibrating mode redistributes its excitation over the allowed states due to the interaction. This means that the b oscillators are related to the a oscillators by a Bogoliubov transformation which is uniquely determined by the interactions. This information is thus contained in the Krichever–Novikov algebra. After all, from Eq. (16) we simply obtain

$$b_m = (B^{-1})_{mk} A_{kn} a_n. \tag{26}$$

There is a nice and compact expression for this Bogoliubov transformation based on the observation

$$(B^{-1})_{mk} = -\frac{1}{2\pi i} \oint_{C_-} z_-^{-m} d\omega_k(z_-), \tag{27}$$

which follows from

$$\delta_{kl} = -\frac{1}{2\pi i} \oint_{C_-} A_l d\omega_k$$
$$= \frac{1}{2\pi i} \oint_{C_-} B_{lm} z_-^{-m} (B^{-1})_{nk} z_-^{n-1} dz_-$$
$$= B_{lm} (B^{-1})_{mk}. \tag{28}$$

Consequently,

$$b_m = -\frac{1}{2\pi i} \oint_{C_-} z_-^{-m} \oint_{C_+} \frac{dz_+}{2\pi i} z_+^{-n-1} \Delta(z_+, z_-) a_n, \tag{29}$$

with

$$\Delta(z,w)=\sum_k A_k(z)d\omega_k(w), \tag{30}$$

which satisfies

$$d\phi(z)=\frac{1}{2\pi i}\oint_C d\phi(w)\Delta(w,z), \tag{31}$$

where C is any curve homologous to C_+.

3. The Connection with the Operator Formalism

We can now make contact with the operator formalism [9]. Let us first concentrate on the puncture P_+. We will closely follow the elegant formalism developed in ref. [3] which allows one to work entirely in a chiral sector (for fixed loop momenta). The state $|\Sigma, P_+\rangle$ obtained by integrating over the Riemann surface minus the disk around P_+ is annihilated by all the operators $a[f]$ for f holomorphic outside P_+, i.e. for $f\in H^0(\Sigma-P_+)$. As a basis we can choose $f_n=A_{-n-g/2}, n=1,2,\ldots$. However, as observed in ref. [3], this is not a complete set of commuting operators and this does not specify the state $|\Sigma, P_+\rangle$ uniquely. In fact, $a[h]$ (for h having constant shifts around the non-trivial homology cycles and holomorphic outside P_+) will commute with all the above annihilation operators. Modulo $H^0(\Sigma-P_+)$ there are $2g$ such functions (e.g. for the torus, the two functions are easily seen to be the functions z and $\zeta(z)$, where ζ is the Weierstrass ζ-function).

Given a choice of marking (a canonical homology basis $\{A_i,B_i\}_{\{i=1,\ldots,g\}}$, where only A_i and B_i intersect), one chooses h_{A_i} and h_{B_i} such that

$$\begin{aligned} a[h_{A_i}]|\Sigma,P_+\rangle &= p_i|\Sigma,P_+\rangle,\\ a[h_{B_i}]|\Sigma,P_+\rangle &= \frac{1}{2\pi i}\frac{\partial}{\partial p_i}|\Sigma,P_+\rangle. \end{aligned} \tag{32}$$

We will need the properties [3]:

$$\begin{aligned} &\oint_{B_j} dh_{A_i} = -\oint_{A_j} dh_{B_i}=\delta_{ij},\\ &\frac{1}{2\pi i}\oint_{C_+} h_D d\omega=\frac{1}{2\pi i}\oint_D d\omega,\quad D=A_i \quad\text{or}\quad D=B_i, \end{aligned} \tag{33}$$

where $d\omega$ is an arbitrary one-form, holomorphic on $\{\Sigma-P_+\}$.

We can likewise consider the puncture P_-. A basis for $H^0(\Sigma-P_-)$ is now given by $\tilde{f}_n=A_{n+g/2}, n=1,2,\ldots$, and the additional $2g$ non-single-valued functions denoted by $\tilde{h}$ satisfy the same properties as in Eq. (33) if we replace C_+ by C_-. We observe that $h_D-\tilde{h}_D$ is single valued and

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_+}(h_D-\tilde{h}_D)d\omega &= \frac{1}{2\pi i}\oint_{C_+} h_D d\omega = \frac{1}{2\pi i}\oint_D d\omega\\ &= \frac{1}{2\pi i}\oint_{C_-}\tilde{h}_D d\omega = -\frac{1}{2\pi i}\oint_{C_-}(h_D-\tilde{h}_D)d\omega \end{aligned} \tag{34}$$

for any holomorphic one-form $d\omega$. Therefore, we can choose the following for the Krichever–Novikov basis:

$$\begin{aligned} A_{g/2-i} &= 2\pi i(h_{A_i} - \tilde{h}_{A_i}), \\ d\omega_{g/2-i} &= d\lambda_i, \quad i = 1, 2, \ldots, g, \end{aligned} \tag{35}$$

where $d\lambda_i$ form the basis of the holomorphic one-forms such that

$$\oint_{A_i} d\lambda_j = \delta_{ij}, \quad \oint_{B_i} d\lambda_j = \tau_{ij}, \tag{36}$$

where τ_{ij} is the symmetric period matrix. Our choice for the g basis elements in Eq. (35) deviates by an irrelevant $S\ell(g, C)$ transformation from the choice specified by Eq. (7). As we observed before, $A_{g/2} = 1$, $\alpha_{g/2}$ corresponds to the momentum flowing through the diagram and $d\omega_{g/2}$ is the third kind differential and is identified with $d\tau$.

Having completed a description of the Krichever–Novikov basis in terms of functions associated to each of the punctures, we can also express the state vector for the two-punctured Riemann surface in terms of the Krichever–Novikov basis. This state vector was already given by Alvarez–Gaumé et al. [2] in terms of the prime form and the holomorphic differentials as a vector in $\mathscr{H}_+ \otimes \mathscr{H}_-$ with $\mathscr{H}_\pm$ the standard Hilbert spaces associated with $P_\pm$. But we prefer working in $\mathscr{H}_+ \otimes \mathscr{H}_-^\dagger$, because P_- corresponds to $\tau \to +\infty$. In this way, the operators b_m with $m > 0$ naturally act as creation operators in $\mathscr{H}_-^\dagger$ as is suggested in Eq. (14). If $|0_\pm\rangle$ are the standard vacua in $\mathscr{H}_\pm$, one finds [9] for the state vector in $\mathscr{H}_+ \otimes \mathscr{H}_-^\dagger$:

$$\begin{aligned} \langle \Sigma, P_\pm \rangle \equiv \Bigg\langle 0_- \Bigg| \exp\Bigg(-\frac{1}{2} \sum_{j,k=\{+,-\}} \oint_{C_j} \frac{dz_j}{2\pi i} \oint_{C_k} \frac{dw_k}{2\pi i} \phi_+(z_j) \frac{\partial^2 \ln E(z_j, w_k)}{\partial z_j \partial w_k} \phi_+(w_k) \Bigg) \\ \cdot \exp\Bigg(i\pi \sum_{\alpha,\beta=1}^{g} p_\alpha \tau_{\alpha\beta} p_\beta + \sum_{\alpha=1}^{g} p_\alpha \sum_{j=\{+,-\}} \oint_{C_j} dz_j \phi_+(z_j) \lambda_\alpha(z_j) \Bigg) \Bigg| 0_+ \Bigg\rangle, \end{aligned} \tag{37}$$

where ϕ_+ is the creation part of the chiral field (depending on the puncture). This equation follows from the requirements:

$$\begin{aligned} a[f_n]\langle \Sigma, P_\pm \rangle &= a[\tilde{f}_n]\langle \Sigma, P_\pm \rangle = 0, \\ a[h_{A_i}]\langle \Sigma, P_\pm \rangle &= a[\tilde{h}_{A_i}]\langle \Sigma, P_\pm \rangle = p_i \langle \Sigma, P_\pm \rangle, \\ a[h_{B_i}]\langle \Sigma, P_\pm \rangle &= a[\tilde{h}_{B_i}]\langle \Sigma, P_\pm \rangle = \frac{1}{2\pi i} \frac{\partial}{\partial p_i} \langle \Sigma, P_\pm \rangle, \end{aligned} \tag{38}$$

which can be seen to be equivalent to [3]:

$$\frac{\partial \phi(z)}{\partial z} \langle \Sigma, P_\pm \rangle = \left[-\sum_{j=\{+,-\}} \oint_{C_j} \frac{dw_j}{2\pi i} \phi_+(w_j) \frac{\partial^2 \ln E(w_j, z)}{\partial w_j \partial z} + 2\pi i \sum_\alpha p_\alpha \lambda_\alpha(z) \right] \langle \Sigma, P_\pm \rangle, \tag{39a}$$

$$\frac{\partial}{\partial p_\alpha} \langle \Sigma, P_\pm \rangle = \left[2\pi i \sum_\beta \tau_{\alpha\beta} p_\beta + \sum_{j=\{+,-\}} \oint_{C_j} \phi_+(z_j) d\lambda_\alpha(z_j) \right] \langle \Sigma, P_\pm \rangle. \tag{39b}$$

Using the fact that

$$p_\alpha\langle \Sigma, P_\pm\rangle = \frac{1}{2\pi i}\oint_C A_{g/2-\alpha}d\phi\langle \Sigma, P_\pm\rangle, \tag{40a}$$

we can rewrite Eq. (39a) as:

$$d\phi(z)\langle \Sigma, P_\pm\rangle = \oint_C \frac{dw}{2\pi i} d\phi(w)\Delta(w,z)\langle \Sigma, P_\pm\rangle, \tag{40b}$$

where C is any cycle homologous to C_+. Using Eq. (38), one shows that the splitting into the contributions of $P_\pm$ will automatically amount to replacing ϕ by the appropriate creation part.

Note that there is a relative minus sign in Eq. (39a), which arises due to the fact that

$$\oint_{C_+} dz h_{B_j}(z)\partial_z\partial_w \log E(z,w) = -2i\pi\lambda_j(w), \tag{41}$$

according to the definition of the prime form [6]. Another reason to see the need for this relative minus sign in Eq. (39a) is by rewriting for example

$$\left(-\oint_{C_+}\frac{dw}{2\pi i}\phi_+(w)\partial_w\partial_z \log E(w,z) + 2\pi i\sum_\alpha p_\alpha\lambda_\alpha(z)\right)|\Sigma, P_+\rangle \tag{42a}$$

as

$$\oint_{C_+}\frac{d\phi_+(w)}{2\pi i}\left[\partial_z \log E(w,z) + 2\pi i\sum_\alpha h_{B_\alpha}(w)\lambda_\alpha(z)\right]|\Sigma, P_+\rangle \tag{42b}$$

with the function within square brackets single-valued as a function of w.

Recognizing that Eq. (40) is implied by Eq. (31) establishes the connection between the Krichever–Novikov algebra and the operator formalism.

4. The Case of Virasoro Algebras

Let us now turn our attention to the generalization of the Virasoro algebra considered by Krichever and Novikov [5]. They consider the meromorphic vector fields on Σ, holomorphic outside of $P_\pm$, and specify the basis e_i such that there are g zeros outside of $P_\pm$ which are uniquely fixed by requiring single-valuedness. The Riemann–Roch theorem states that for $g > 1$ the number of poles minus the number of zeros of a meromorphic vector field is $2(g-1)$. The basis is specified by

$$e_i \sim z_\pm^{\pm i - 3g/2+1}(1 + O(z_\pm))\frac{\partial}{\partial z_\pm}, \tag{43}$$

except for special values of i and points $P_\pm$, but this is not important for the following. Observe that for the torus, $e_i = A_i\partial/\partial z$. If $T(z)$ is the stress-energy tensor on Σ and the covering of Σ is part of a projective structure (i.e. the transition functions are in $S\ell(2,C)$ such that the Schwarzian derivative vanishes [1,2,3]), then for a given meromorphic vector field ξ on Σ which is holomorphic outside of $P_\pm$ one has the Virasoro generator

$$L[\xi] = \pm \frac{1}{2\pi i} \oint_{C_\pm} \xi T \tag{44}$$

(remember that T is a two-form). The Virasoro algebra with central extension is given by

$$[L[\xi], L[\eta]] = L[[\xi,\eta]] \pm \frac{c}{24\pi i} \oint_{C_\pm} dz_\pm \xi(z_\pm) \frac{d^3\eta(z_\pm)}{dz_\pm^3}, \tag{45}$$

with $[\xi,\eta]$ the Lie derivative of the vector fields and the central term is given in local coordinates ($\xi = \xi(z_\pm)\partial/\partial z_\pm$).

Following Krichever and Novikov [5] one can introduce a basis of two-forms Ω_i dual to the vector fields e_i in the sense

$$\pm \frac{1}{2\pi i} \oint_{C_\pm} e_i \Omega_j = \delta_{ij}, \tag{46}$$

and expand the stress-energy tensor T in terms of Ω_j

$$T = L_j \Omega_j. \tag{47}$$

Then we clearly have $L_i = L[e_i]$ and

$$[L_i, L_j] = \sum_{s=-3g/2}^{3g/2} c_{ij}^s L_{i+j-s} + c\chi(e_i, e_j), \tag{48}$$

with the central term and the coefficients c_{ij}^s given respectively by

$$\chi(\xi,\eta) = \pm \frac{1}{24\pi i} \oint_{C_\pm} dz_\pm \xi(z_\pm) \frac{d^3\eta(z_\pm)}{dz_\pm^3}, \tag{49}$$

$$c_{ij}^s = \pm \frac{1}{2\pi i} \oint_{C_\pm} [e_i, e_j] \Omega_{i+j-s}. \tag{50}$$

As for the current algebras, the two punctures allow us to define two standard Virasoro algebras

$$\begin{aligned} T(z_+) &= K_i z_+^{-(i+2)} dz_+ \otimes dz_+, \\ T(z_-) &= I_i z_-^{+(i-2)} dz_- \otimes dz_-, \end{aligned} \tag{51}$$

such that $K_i = L[-z_+^{i+1}\partial/\partial z_+]$ and $I_i = L[z_-^{-i+1}\partial/\partial z_-]$ satisfy

$$[K_i, K_j] = (i-j)K_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0},$$

$$[I_i, I_j] = (i-j)I_{i+j} + \frac{c}{12}(i^3 - i)\delta_{i+j,0}. \tag{52}$$

A simple computation shows that

$$L_k = C_{kn} K_n = D_{kn} I_n, \tag{53a}$$

with C_{kn} and D_{kn} the Laurent coefficients of e_k at z_+ and z_- respectively,

$$C_{kn} = \frac{1}{2\pi i} \oint_{C_+} z_+^{-n-2} e_k dz_+ \otimes dz_+,$$
$$D_{kn} = -\frac{1}{2\pi i} \oint_{C_-} z_-^{+n-2} e_k dz_- \otimes dz_-. \tag{53b}$$

Krichever and Novikov [5] also wrote a Sugawara form for the Virasoro algebra. For this one needs to introduce a normal ordering. One possible choice would be

$$\begin{aligned} \text{“}\alpha_n \alpha_m\text{”} &= \alpha_n \alpha_m, \quad n \leqq m, \\ &= \alpha_m \alpha_n, \quad n > m, \end{aligned} \tag{54}$$

but there is a large amount of freedom which leads to finite constant shifts in the Virasoro generators given by [5]:

$$L_k^S = \tfrac{1}{2} \ell^k_{nm} \text{“}\alpha_n \alpha_m\text{”};$$
$$\ell^k_{nm} = \pm \frac{1}{2\pi i} \oint_{C_\pm} e_k d\omega_n \otimes d\omega_m. \tag{55}$$

Using Eqs. (10) and (14) to substitute $\alpha_n\, d\omega_n = a_n z_+^{-n} dz_+/z_+$, one finds:

$$L_k^S = \left(\frac{1}{2\pi i} \oint_{C_+} e_k z_+^{-(n+m)-2} \tfrac{1}{2} {:}a_n a_m{:} dz_+ \otimes dz_+ \right) + v_k, \tag{56}$$

with v_k a constant which occurs due to the normal orderings involved (: : is the standard normal ordering for $\mathscr{H}_+$). Using Eq. (53b), we therefore find:

$$L_k^S = C_{kn} K_n^S + v_k,$$
$$K_n^S = \frac{1}{2} \sum_{p+q=n} {:}a_p a_q{:}, \tag{57}$$

where K_n^S satisfies Eq. (52) for $c = 1$. A similar result is easily derived for the expansion with respect to P_-.

Finally, we observe that the constant v_k can be understood as coming from the Schwarzian derivative related to a particular choice of coordinates compatible with the chosen normal ordering.

5. Conclusion

In conclusion, we have shown that the generalized grading of the Krichever–Novikov algebra is a consequence of the choice of a globally defined basis for the meromorphic functions and vector fields, holomorphic outside of $P_\pm$.

Based on each puncture, there is a linear transformation between the Krichever–Novikov basis and the standard basis at $P_\pm$. The global nature of the Krichever–Novikov basis however does contain information on the genus of the Riemann surface, since one can use it to describe the operator formalism on the two-punctured Riemann surfaces.

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