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Existence of Excited States for a Nonlinear Dirac Field

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Abstract. We prove the existence of infinitely many stationary states for the following nonlinear Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi + (\bar{\psi}\psi)\psi = 0.$$

Seeking for eigenfunctions splitted in spherical coordinates leads us to analyze a nonautonomous dynamical system in $\mathbf{R}^2$. The number of eigenfunctions is given by the number of intersections of the stable manifold of the origin with the curve of admissible datum. This proves the existence of infinitely many stationary states, ordered by the number of nodes of each component.

1. Introduction

We study the existence of stationary states for the following nonlinear Dirac equation

$$i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu\psi - m\psi + F(\bar{\psi}\psi)\psi = 0. \tag{1.1}$$

The notation is the following. ψ is defined on $\mathbf{R}^4$ with values in $\mathbf{C}^4$, $\partial_\mu = \partial/\partial x_\mu$, m is a positive constant, $\bar{\psi}\psi = (\gamma^0\psi, \psi)$, where $(\ ,\ )$ is the usual scalar product in $\mathbf{C}^4$, and the γ^μ's are the 4×4 matrices of the Pauli-Dirac representation, given by

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad\text{and}\quad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \quad\text{for}\quad k = 1, 2, 3,$$

where

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Finally, $F:\mathbf{R}\to\mathbf{R}$ models the nonlinear interaction.

We are interested in stationary states, or localized solutions of (1.1), that is solutions ψ of the form $\psi(t,x) = e^{i\omega t}\varphi(x)$, where $t = x_0$ and $x = (x_1, x_2, x_3)$. In addition, we seek finite energy solutions, and so we want φ to be at least square integrable.

The problem of the existence of stationary states for (1.1) arises in several models of particle physics (see for example [1, 4, 5] and the references therein). For the sake of simplicity we specialize here to the model case $F(x)=x$ (the Soler model of extended Fermions, see [4]), even though the method applies as well to more general nonlinearities (see below).

Clearly, the equation for $\varphi:\mathbf{R}^3\to\mathbf{C}^4$ is

$$i\sum_{k=1}^{3}\gamma^k\partial_k\varphi-m\varphi+\omega\gamma^0\varphi+(\bar{\varphi}\varphi)\varphi=0. \tag{1.2}$$

We seek solutions that are separable in spherical coordinates, of the form

$$\varphi(x)=\begin{bmatrix} v(r)\begin{bmatrix}1\\0\end{bmatrix} \\ iu(r)\begin{bmatrix}\cos\theta\\ \sin\theta\, e^{i\Phi}\end{bmatrix}\end{bmatrix}. \tag{1.3}$$

Here, $r=|x|$, and (θ,Φ) are the angular parameters.

The Dirac equation then turns to a nonautonomous planar differential system, in the r variable, which is (compare [4, 7])

$$u'+(2/r)u=v(v^2-u^2-(m-\omega)), \tag{1.4}$$

$$v'=u(v^2-u^2-(m+\omega)). \tag{1.5}$$

In order to avoid solutions with singularity at the origin, due to the term $(2/r)u$ in (1.4), we impose

$$u(0)=0. \tag{1.6}$$

Since we are interested in finite energy solutions of (1.2), we seek solutions of (1.4)–(1.5), which fulfill

$$|u(r)|+|v(r)|\to 0, \quad \text{as} \quad r\to+\infty. \tag{1.7}$$

For every given x, there exists a local solution (u_x, v_x) of (1.4)–(1.6), with initial datum $v(0)=x$. The problem is to find x, such that the corresponding solution is global (i.e. defined for all $r\geqq 0$), and satisfies (1.7). Our main result is the following.

Theorem 1.1. *Assume $0<\omega<m$. There exists an increasing and bounded sequence $(x_n)_{n\geqq 0}$ of positive numbers, with the following properties. For every $n\geqq 0$,*

(i) *the solution (u_n, v_n) of (1.4)–(1.6) with $v_n(0)=x_n$ is a global solution,*

(ii) *both u_n and v_n have exactly n zeroes on $(0,+\infty)$,*

(iii) *(u_n, v_n) converges exponentially to $(0,0)$, as $r\to+\infty$.*

Theorem 1.1 calls for several comments.

The first existence result for stationary states of (1.1) was obtained by Cazenave and Vazquez [1]. Under some assumptions on F (satisfied in particular for $F(x)=x$) they proved the existence of a solution without nodes (positive u and v), which is essentially the solution (u_0, v_0) of Theorem 1.1. Later, that result was extended to a wider class of nonlinearities by Merle [3].

Theorem 1.1 is stated in the special case where $F(x)=x$ in Eq. (1.1). However, the method applies, without any modification, to more general situations. In particular, Theorem 1.1 is still valid (all properties except boundedness of the sequence $(x_n)_{n\geqq 0}$) when F satisfies the following assumptions. $F\in C^1(\mathbf{R},\mathbf{R})$, $F(0)=0$, F is increasing on $(0,+\infty)$, $F(x)>m+\omega$ for x large, $F'(F^{-1}(m-\omega))>0$, and $F(x)\leqq 0$ for $x\leqq 0$. These assumptions are basically the assumptions in [1]. For the boundedness of the sequence $(x_n)_{n\geqq 0}$, we need for example that $F(x)\geqq\delta(\mathrm{Log}(x))^\beta$, for x large, and where $\beta>1$, $\delta>0$. As pointed out in [1], the condition $\omega\in(0,m)$ is almost necessary for the existence of even one solution of (1.4)–(1.7) (see Remark 4.2 of [1]). Let us also mention that it does not restrict the generality to consider only the case $v(0)>0$. Indeed, if (u,v) is a solution of (1.4)–(1.5), then $(-u,-v)$ is also a solution.

The numerical experiments performed on system (1.4)–(1.5) indicate the following [6]. First, starting from x larger than some x^*, the solutions blow-up (compare Proposition 3.1). It is also observed that the global solutions converge to one of the rest points of the system, which are $(0,0)$, $(0,(\mathrm{m}-\omega)^{1/2})$ and $(0,-(m-\omega)^{1/2})$. The solutions wind around the origin (in the plane (v,u)) before converging to a rest point. Note that $(0,\pm(m-\omega)^{1/2})$ are stable rest points while $(0,0)$ is a saddle point. The set of positive initial data for which the solution turns $n/2$ times (n being an integer) around $(0,0)$ before converging to $(0,\pm(m-\omega)^{1/2})$ seems to be an interval of the form (x_{n-1},x_n). The only solutions with positive initial data converging to $(0,0)$ appear to be those starting from x_n, and they have n nodes. We prove these properties here, except uniqueness of the n-nodes solution with positive initial datum.

In the proof of Theorem 1.1, we consider (1.4)–(1.5) as a non-autonomous planar dynamical system (r being the time variable), and we follow essentially the scheme suggested by the numerical results. For every $n\geqq 0$, we construct an open, non-empty set I_n of initial data for which the solution turns $n/2$ times around the origin and then remains trapped near one of the stable rest points. Next, we show that the solutions with initial data in I_n are bounded, uniformly in $r\geqq 0$ and in the initial datum. Finally, we show that the solution with initial datum $\mathrm{Sup}(I_n)$ is the expected n-nodes solution. The boundedness of the sequence $(x_n)_{n\geqq 0}$ follows from a blow-up result (Proposition 3.1).

Theorem 1 raises some open questions, apart from the uniqueness problem. For example, the sequence $(x_n)_{n\geqq 0}$ is bounded, but we do not know whether the corresponding sequence $(u_n,v_n)_{n\geqq 0}$ of solutions is bounded or not (the numerical experiments indicate that it is unbounded). A related question is the following. Consider the limit, say x_∞, of $(x_n)_{n\geqq 0}$. Does the solution of (1.4)–(1.6) blow-up in a finite time when $v(0)\geqq x_\infty$?

Notice the importance of the term $(2/r)u$ in (1.4). Indeed, it is the non-autonomous term that allows the existence of infinitely many solutions (the same phenomenon appears in the semilinear elliptic problems, see [2]). More surprisingly, it is also the non-autonomous term (although being linear) that makes some solutions blow-up in a finite time (when this term is removed, all the solutions are global solutions).

The paper is organized as follows. In Sect. 2, we introduce the notation and we collect some basic properties of system (1.4)–(1.5). In Sect. 3, we study the blowingup of solutions and in Sect. 4, we establish the main boundedness property. Finally, in Sect. 5, we complete the proof of Theorem 1.1

2. Some Preliminary Results

In order to analyze the dynamical system (1.4) and (1.5), we recall that:

$$u' + (2/r)u = v(v^2 - u^2 - (m - \omega)), \tag{1.4}$$

$$v' = u(v^2 - u^2 - (m + \omega)), \tag{1.5}$$

and in all the sequel, we assume $0 < \omega < m$. It will be useful to keep in mind the velocity field for various values of r (as in Figs. 1 and 2). We will consider the Hamiltonian system associated with (1.4)–(1.5), which is

$$u' = v(v^2 - u^2 - (m - \omega)), \qquad v' = u(v^2 - u^2 - (m + \omega)); \tag{2.1}$$

where $'$ denotes the differentiation with respect to the r variable. The corresponding Hamiltonian is given by

$$H(u, v) = \tfrac{1}{2}[\tfrac{1}{2}(v^2 - u^2)^2 - m(v^2 - u^2) + \omega(v^2 + u^2)], \quad \text{for} \quad u, v \in \mathbf{R}. \tag{2.2}$$

There exists two constants, k and K, such that (see [1, Proof of Lemma 2.1])

$$H(u, v) \geqq k(u^2 + v^2) - K, \quad \text{for any} \quad u, v \in \mathbf{R}. \tag{2.3}$$

We recall (see [1, Lemma 2.4]) that for any $x \in \mathbf{R}$, there exists a unique solution $(u_x, v_x) \in C^1([0, R_x), \mathbf{R}^2)$ of (1.4)–(1.6) such that $v_x(0) = x$. The solution is defined on the maximal interval $[0, R_x)$, with $|u_x| + |v_x| \to +\infty$ as $r \uparrow R_x$ if $R_x < \infty$. Furthermore, we have

$$(u_x, v_x) \text{ depends continuously on } x \text{ in } C^1([0, R], \mathbf{R}^2), \text{ for any } R < R_x. \tag{2.4}$$

The continuous dependence in $C([0, R], \mathbf{R}^2)$ is stated in [1] (Lemma 2.5), and the continuous dependence in $C^1([0, R], \mathbf{R}^2)$ follows from the equations. For conve-

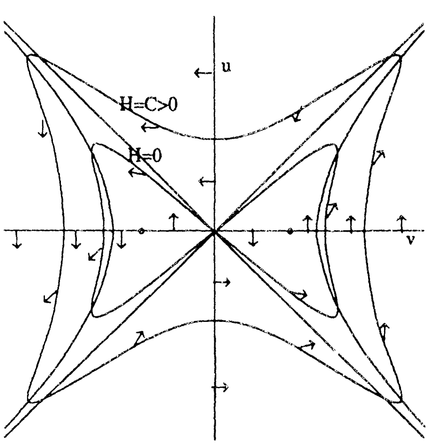

Fig. 1. The velocity field at $r > 1/\omega$

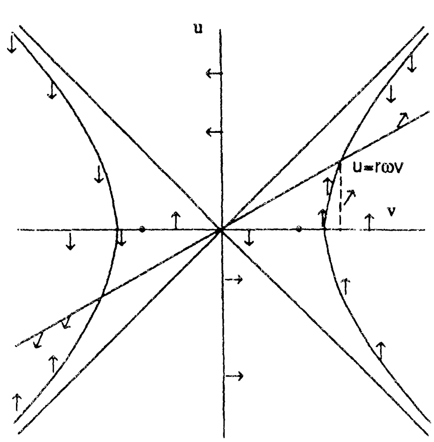

Fig. 2. The velocity field at $r < 1/\omega$

nience, we define the function H_x, for $x \in \mathbf{R}$, by

$$H_x(r) = H(u_x(r), v_x(r)), \quad \text{for every } r \in (0, R_x).$$

We have (see [1, Lemma 2.6])

$$(H_x)'(r) = (2/r)u_x^2(r)\,[v_x^2(r) - u_x^2(r) - (m+\omega)], \quad \text{for every } x \in \mathbf{R} \text{ and } r \in (0, R_x); \quad (2.5)$$

and so, the sign of the variation of H_x, at some time $r > 0$, is determined only by the region of the plane where $(u_x, v_x)(r)$ belongs. We shall need the following identities, which are easily obtained from (1.4)–(1.5) and hold for $x \in \mathbf{R}$ and $r \in (0, R_x)$:

$$(v_x^2 - u_x^2)' = (4/r)u_x^2 - 4\omega u_x v_x, \quad (2.6)$$

$$(u_x/v_x)' = (1/v_x^2)\,[(v_x^2 - u_x^2)^2 + (m-\omega)(u_x^2 - v_x^2) + 2\omega u_x^2 - (2/r)u_x v_x], \quad \text{if} \quad v_x \neq 0, \quad (2.7)$$

$$(v_x/u_x)' = (1/u_x^2)\,[-(v_x^2 - u_x^2)^2 - (m-\omega)(u_x^2 - v_x^2) - 2\omega u_x^2 + (2/r)u_x v_x], \quad \text{if} \quad u_x \neq 0. \quad (2.8)$$

In the following lemmas, we collect some basic properties, of a geometric nature, for solutions of (1.4)–(1.5).

Lemma 2.1. *Let* $x \neq 0$. *If for some* $r_0 > 0$, *we have* $u_x(r_0) = 0$, *then* $v_x(r_0) \notin \{0, \pm(m-\omega)^{1/2}\}$ *and* $u_x'(r_0) \neq 0$. *If, for some* $r_0 > 0$, *we have* $v_x(r_0) \in \{0, \pm(m-\omega)^{1/2}\}$, *then* $u_x(r_0) \neq 0$ *and* $v_x'(r_0) \neq 0$.

Proof. First, observe that the only rest points of (1.4)–(1.5) are $(0,0)$, $(0,(m-\omega)^{1/2})$ and $(0, -(m-\omega)^{1/2})$. Furthermore, for $r_0 > 0$, the Cauchy problem for (1.4)–(1.5) is locally well-posed for any initial datum $(u_0, v_0) \in \mathbf{R}^2$, for both $r \geqq r_0$ and $r \leqq r_0$. Thus, a rest point cannot be reached in a finite time. Hence the result.

Lemma 2.2. *Let* $x \in \mathbf{R}$. *Assume* $R_x = +\infty$, *and* $(u_x, v_x) \to (0, \pm(m-\omega)^{1/2})$ *as* $r \to +\infty$. *Then* u_x *has infinitely many zeroes.*

Proof. It is equivalent to show that (u_x, v_x) cannot converge to $(0, \pm(m-\omega)^{1/2})$, while being in one of the half-planes $\{u > 0\}$ or $\{u < 0\}$. It is proved in [1] (Proof of Lemma 2.10) that (u_x, v_x) cannot converge to $(0, (m-\omega)^{1/2})$, while being in the half-plane $\{u > 0\}$. By symmetry, (u_x, v_x) cannot converge to $(0, -(m-\omega)^{1/2})$, while being in the half-plane $\{u < 0\}$. The same proof applies to the two other situations.

Lemma 2.3. *Let* $x \neq 0$. *Assume that, for some* $r_0 > 0$, *we have* $v_x(r_0) = u_x(r_0)$. *Then* $r_0 > 1/\omega$.

Proof. We can assume that $x > 0$, and that r_0 is the first zero of $v_x - u_x$. Let $h(r) = (v_x - u_x)(r)$. We have

$$h'(r_0) \leqq 0. \quad (2.9)$$

Since $h(r_0) = 0$, we get from (1.4)–(1.5),

$$h'(r_0) = 2((1/r_0) - \omega)v_x(r_0). \quad (2.10)$$

Furthermore, $v_x(r_0) \neq 0$ by Lemma 2.1, and even $v_x(r_0) > 0$. Indeed, if $v_x(r_0) < 0$, then v_x has a first zero, say ϱ, in $(0, r_0)$. Thus $0 = v_x(\varrho) > u_x(\varrho)$, and $v_x'(\varrho) \leqq 0$. This is ruled out by (1.4)–(1.5); and so $v_x(r_0) > 0$. Therefore, from (2.9)–(2.10), we obtain $r_0 \geqq 1/\omega$.

It remains to show that $r_0 \neq 1/\omega$. Indeed, if $r_0 = 1/\omega$, we get from (1.4)–(1.5) and (2.10),

$$h'(r_0)=0; \qquad h''(r_0) = -2\omega^2 v_x(r_0) < 0 .$$

This is again impossible, since $h>0$ on $(0, r_0)$, and $h(r_0)=0$. Thus, $r_0 > 1/\omega$; and so the proof of Lemma 2.3 is complete.

Lemma 2.4. *Let* $x \neq 0$. *Assume that* $(u_x, v_x) \in \{v^2 - u^2 \geqq m+\omega\}$ *on some interval* $[r_0, r_1) \subset [0, R_x)$. *Then* $r_1 - r_0 < 3/2\omega$.

Proof. Because v_x is continuous, we can assume, say, $v_x > 0$ on $[r_0, r_1)$. Thus, we have

$$-1 < u_x/v_x < 1 , \quad \text{on} \quad [r_0, r_1) . \tag{2.11}$$

From (2.7), we get, on $[r_0, r_1)$,

$$(u_x/v_x)' \geqq (1/v_x^2)\,[2\omega v_x^2 - (2/r) u_x v_x];$$

and so

$$\left(r^2 \frac{u_x}{v_x} \right)' \geqq 2\omega r^2 . \tag{2.12}$$

Integrating (2.12) between r_0 and r_1, and using (2.11), we get

$$r_1^2 + r_0^2 \geqq (2\omega/3)(r_1^3 - r_0^3) = (2\omega/3)(r_1 - r_0)(r_1^2 + r_1 r_0 + r_0^2) \geqq (2\omega/3)(r_1 - r_0)(r_1^2 + r_0^2) .$$

This proves Lemma 2.4.

Lemma 2.5. *There exists a function* $F \in C(\mathbf{R}, \mathbf{R})$, *with the following property. If* $x \in \mathbf{R}$ *is such that* $R_x > 1/\omega$ *and* $(u_x, v_x) \in \{v^2 - u^2 \geqq m+\omega\}$, *on some interval* $[r_0, r_1) \subset [1/\omega, R_x)$, *then* $r_1 < R_x$ *and*

$$(|u_x| + |v_x|)(r) \leqq F((|u_x| + |v_x|)(r_0)), \quad \textit{for every } r \in [r_0, r_1) .$$

Proof. By continuity and symmetry, we can assume, $(u_x, v_x) \in \{v^2 - u^2 \geqq m+\omega, v>0\}$ for $r \in [r_0, r_1)$. Let $D^+ = \{v^2 - u^2 \geqq m+\omega,\ v \geqq 0,\ u \geqq 0\}$, and $D^- = \{v^2 - u^2 \geqq m+\omega,\ v>0,\ u \leqq 0\}$. Observe that, from (1.5), $u_x' > 0$ on $[r_0, r_1)$; and so, if $(u_x, v_x)(\varrho) \in D^+$, for some $\varrho \in [r_0, r_1)$, then $(u_x, v_x)(r) \in D^+$, for all $r \in [\varrho, r_1)$. Thus, we can assume that $(u_x, v_x)(r) \in D^-$ on some interval $[r_0, r_2]$, and $(u_x, v_x)(r) \in D^+$ on $[r_2, r_1)$. On $(r_0, r_2]$, we have $u_x' > 0$ and $v_x' < 0$. Therefore,

$$(|u_x| + |v_x|)(r) \leqq (|u_x| + |v_x|)(r_0), \quad \text{for every } r \in [r_0, r_2] . \tag{2.13}$$

Next, $(v_x^2 - u_x^2)$ is nonincreasing on $[r_2, r_1)$, by (2.6). Thus, by (1.4)–(1.5),

$$(u_x + v_x)' \leqq (u_x + v_x)\,[(v_x^2 - u_x^2)(r_2)], \quad \text{on} \quad [r_2, r_1) . \tag{2.14}$$

Finally, by Lemma 2.5, we have $r_1 - r_2 \leqq 3/2\omega$. Therefore, (2.13) and (2.14) yield

$$(|u_x| + |v_x|)(r) \leqq (|u_x| + |v_x|)(r_0) \exp((3/2\omega)\,[(|u_x| + |v_x|)(r_0)]^2), \quad \text{for every } r \in [r_0, r_1) .$$

Hence the bound on $(|u_x| + |v_x|)$; and so $r_1 < R_x$, unless $r_1 = R_x = +\infty$. This is ruled out by Lemma 2.4.

Lemma 2.6. *Let $x \neq 0$. Assume that for some $r_0 > 0$, we have $v_x(r_0) \leqq u_x(r_0)$ and $u_x(r_0) > 0$ (respectively, $u_x(r_0) \leqq v_x(r_0)$ and $u_x(r_0) < 0$). Then there exists $r_0 < r_1 < R_x$, such that $|u_x| > 0$, on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > (m-\omega)^{1/2}$. In addition, either $v_x(r_0) > 0$ (respectively, $v_x(r_0) < 0$), and then v_x has exactly one zero in (r_0, r_1), or else $v_x(r_0) \leqq 0$ (respectively $v_x(r_0) \geqq 0$), and then $|v_x| > 0$, on (r_0, r_1).*

Proof. Assume, for example, $v_x(r_0) \leqq u_x(r_0)$ and $u_x(r_0) > 0$. Suppose first that $v_x(r_0) \geqq -u_x(r_0)$. Because the velocity field on $\{v = -u, u > 0\}$ points towards $\{v < -u\}$, (u_x, v_x) can only enter $\{-u \leqq v \leqq u,\ u > 0\}$ by crossing the diagonal $\{u = v\}$; and so, by Lemma 2.3, $r_0 > 1/\omega$.

For $r \geqq r_0$, and while (u_x, v_x) belongs to $\{-u \leqq v \leqq u, u > 0\}$, we have by (2.8)

$$(v_x/u_x)' \leqq (1/u_x^2)((2/r_0)u_x v_x - 2\omega u_x^2) \leqq -2(\omega - (1/r_0)) < 0\,.$$

Thus, (u_x, v_x) exits $\{-u \leqq v \leqq u, u > 0\}$ in a finite time, by crossing the u-axis once, if $v_x(r_0) > 0$, and then, by crossing the diagonal $\{v = -u, u > 0\}$ (note that in the region $\{-u \leqq v \leqq u, u > 0\}$, the trajectory remains trapped in the set $\{H(u, v) \leqq H_x(r_0)\}$ due to (2.5)). Therefore, we can assume $v_x(r_0) < -u_x(r_0)$.

Suppose now that $v_x^2(r_0) - u_x^2(r_0) < m + \omega$. While (u_x, v_x) belongs to the region

$$D = \{v^2 - u^2 \leqq m + \omega, u + v \leqq 0, u \geqq 0\}\,,$$

we have, by (2.5), $H'_x \leqq 0$; and so (u_x, v_x) is bounded. We claim that (u_x, v_x) must exit D in a finite time. Indeed, assume that (u_x, v_x) remains in D for $r \geqq r_0$. We have $v'_x \leqq 0$, thus v_x has a negative limit as $r \to +\infty$. It is not difficult to show that u_x also has a limit (compare [1], proof of Lemma 2.10). The limit (k, h) of (u_x, v_x) is therefore a rest point of (1.4)–(1.5) in the half-plane $\{v < 0\}$. Thus, $(k, h) = (0, -(m-\omega)^{1/2})$. This is ruled out by Lemma 2.2. Therefore, (u_x, v_x) must exit D in a finite time. Next, (u_x, v_x) cannot exit D by crossing the diagonal $\{u = -v,\ u > 0\}$, since the velocity field on $\{v = -u,\ u > 0\}$ points towards $\{v < -u\}$. If (u_x, v_x) exits D by crossing the v-axis, then $u'_x < 0$; and so $v_x < -(m-\omega)^{1/2}$, which is the desired estimate. Therefore, we can suppose that (u_x, v_x) exits D by crossing the hyperbola $\{v^2 - u^2 = m + \omega\}$. When (u_x, v_x) belongs to $D' = \{v^2 - u^2 \geqq m + \omega,\ v < 0,\ u \geqq 0\}$, we have by (2.6)

$$(v_x^2 - u_x^2)' \leqq 0;$$

and so, (u_x, v_x) cannot exit D' by crossing again the hyperbola $\{v^2 - u^2 = m + \omega\}$. By Lemmas 2.5 and 2.4, we know that (u_x, v_x) must exit D' in a finite time by crossing the v-axis; and when it does so, we have $v_x \leqq -(m+\omega)^{1/2}$. Hence the result.

Corollary 2.7. *Let $x \neq 0$. Assume that, for some $r_0 > 0$, we have $v_x(r_0) = 0$. Then there exists $r_1 > r_0$, $r_1 < +\infty$, such that $|u_x| > 0$, $|v_x| > 0$ on (r_0, r_1), and $u_x(r_1) = 0$, $|v_x(r_1)| > (m-\omega)^{1/2}$.*

Proof. If $u_x(r_0) > 0$, we have $v_x(r_0) \leqq u_x(r_0)$, and if $u_x(r_0) < 0$, we have $u_x(r_0) \leqq v_x(r_0)$. Thus, we can apply Lemma 2.6.

Lemma 2.8. *Let $x \neq 0$ be such that $R_x = +\infty$. Assume that for some $r_0 > 0$, we have $|u_x| > 0$ on $[r_0, +\infty)$. Then, we have the following.*

(i) $u_x v_x > 0$ *on* $(r_0, +\infty)$,

(ii) *there exists* C, *such that* $0 < |u_x(r)| < |v_x(r)| < C\exp(-(1/2)(m-\omega)r)$, *for* $r \in (r_0, +\infty)$.

Proof. Assume, for example, $u_x > 0$ on $[r_0, +\infty)$. Then, by Lemma 2.6, we have

$$0 < u_x < v_x, \quad \text{on} \quad [r_0, +\infty).$$

This proves (i). By (2.6), and the above inequality, we have

$$(v_x^2 - u_x^2)' < 0, \quad \text{for } r \text{ large}.$$

Therefore, by Lemma 2.4, we have

$$v_x^2 - u_x^2 < m + \omega, \quad \text{for } r \text{ large};$$

and so, $u_x' < 0$ and $v_x' < 0$, for r large. Thus, (u_x, v_x) has a limit, as $r \to +\infty$, which is a rest point of (1.4)–(1.5). By Lemma 2.2, this limit is $(0, 0)$. Therefore, for r large, we have

$$0 < v_x^2 - u_x^2 < (1/2)(m-\omega).$$

It follows from (1.4)–(1.5) that we have then

$$(u_x + v_x)' \leqq -(1/2)(m-\omega)(u_x + v_x);$$

from which the exponential decay follows.

Corollary 2.9. *Let* $x \neq 0$. *Assume that* $R_x > 1/\omega$ *and that, for some* $r_0 \geqq 1/\omega$, *we have* $u_x(r_0) = 0$. *Then, one of the following properties holds:*

(i) $|v_x(r_0)| < (m-\omega)^{1/2}$, *and there exists* $r_1 \in (r_0, R_x)$ *such that* $u_x v_x < 0$ *on* (r_0, r_1), $0 < |u_x| < |v_x|$ *on* (r_0, r_1), $|v_x(r_1)| > (m-\omega)^{1/2}$, *and* $u_x(r_1) = 0$;

(ii) $|v_x(r_0)| > (m-\omega)^{1/2}$, *and there exists* $r_1 \in (r_0, R_x)$ *such that* $u_x v_x > 0$ *on* (r_0, r_1), $0 < |u_x| < |v_x|$ *on* (r_0, r_1), $|v_x(r_1)| < (m-\omega)^{1/2}$, *and* $u_x(r_1) = 0$;

(iii) $|v_x(r_0)| > (m-\omega)^{1/2}$, *and there exists* $r_1 \in (r_0, R_x)$ *such that* $u_x v_x > 0$ *on* (r_0, r_1), *and* $v_x(r_1) = 0$;

(iv) $|v_x(r_0)| > (m-\omega)^{1/2}$, $R_x = +\infty$, $u_x v_x > 0$ *on* $(r_0, +\infty)$, *and* $0 < |u_x| < |v_x| < C\exp(-(1/2)(m-\omega)r)$, *on* $(r_0, +\infty)$.

Proof. Assume for example $v_x(r_0) > 0$. If $v_x(r_0) < (m-\omega)^{1/2}$, then $u_x'(r_0) < 0$. Applying Lemma 2.6, we get (i), except the property $0 < |u_x| < |v_x|$ on (r_0, r_1). Observe that $H_x(r_0) < 0$, and that on (r_0, r_1), we have $H_x' \leqq 0$ and $v_x' \geqq 0$, until possibly (u_x, v_x) crosses the hyperbola $\mathscr{H} = \{v^2 - u^2 = m + \omega\}$; and so, until then, we have $H_x < 0$, from which it follows easily that $0 < |u_x| < |v_x|$. Now, the velocity field on $\mathscr{H} \cap \{u < 0\}$ points towards $\{v^2 - u^2 > m + \omega\}$. Thus, if (u_x, v_x) crosses $\mathscr{H}$, it cannot come back in the region $\{v^2 - u^2 \leqq m + \omega\}$ on (r_0, r_1). Hence (i).

Suppose now $v_x(r_0) > (m-\omega)^{1/2}$, and let $\varrho = \operatorname{Sup}\{r \in (r_0, R_x), u_x v_x > 0 \text{ on } (r_0, r_1)\}$. If $\varrho < R_x$, we have either $v_x(\varrho) = 0$, in which case we get (ii), or else $u_x(\varrho) = 0$. In the last case, we have $0 < |u_x| < |v_x|$ on (r_0, ϱ), by Lemma 2.6; hence (iii). Assume now $\varrho = R_x$. By Lemmas 2.4 and 2.5, there exists $\tau > r_0$, such that $v_x^2(\tau) - u_x^2(\tau) < m + \omega$. Since $\tau > 1/\omega$, the velocity field prevents the solution from entering the region $\{v^2 - u^2 > m + \omega\}$ for $r \geqq \tau$. Thus, $H_x(r) \leqq H_x(\tau)$, for $r \in [\tau, R_x)$; and so $R_x = +\infty$. Applying Lemma 2.8, we obtain (iv).

Lemma 2.10. *Let $x \neq 0$. Assume that v_x has at least two zeroes. Then, between two consecutive zeroes of v_x, u_x has an odd number of zeroes.*

Proof. Assume for example that $v_x > 0$ on (r_0, r_1), and $v_x(r_0) = v_x(r_1) = 0$. Because $v'_x < 0$ when $v_x = 0$, $u_x > 0$, and $v'_x > 0$ when $v_x = 0$, $u_x < 0$, we have $u_x(r_0)u_x(r_1) < 0$. Hence the result.

Lemma 2.11. *Let $x \neq 0$. Assume that $R_x \geqq 1/\omega$, and that u_x has a finite number of zeroes. Then $R_x = +\infty$, and $|u_x| + |v_x| \to 0$ as $r \to +\infty$.*

Proof. In the region $\{v^2 - u^2 \leqq m + \omega\}$, H_x is nonincreasing, by (2.5); and so, by (2.3), (u_x, v_x) cannot blow-up. Now, the region $\{v^2 - u^2 > m + \omega\}$ is the union of two connected, open components D_1 and D_2, where $D_1 = \{v^2 - u^2 > m + \omega, v > 0\}$ and $D_2 = \{v^2 - u^2 > m + \omega, v < 0\}$. Considering the velocity field on the boundary of D_1, and for $r \geqq 1/\omega$, it is not difficult to show that (u_x, v_x) can enter D_1 only in the half-plane $\{u < 0\}$, and can exit D_1 only in the half-plane $\{u > 0\}$. Thus, when the solution crosses D_1 after $r = 1/\omega$, u_x has at least one zero. By symmetry, the same holds for D_2; and so, (u_x, v_x) can only cross D_1 or D_2 a finite number of times. Thus, by Lemma 2.5, we have $R_x = +\infty$. Note that v_x cannot vanish after the last zero of u_x, by Corollary 2.7. Therefore, we can apply Lemma 2.8, from which the result follows.

3. A Blow-Up Result

In this section, we prove that for $|x|$ large enough, the solution (u_x, v_x) blows up in a finite time, and remains in the region $\{v^2 - u^2 > m + \omega\}$. This shows the existence of an a-priori bound on admissible initial data for the solutions of (1.4)–(1.7). Our main result is the following.

Proposition 3.1. *For every $\tau > 0$, there exists $B(\tau)$ such that if $|x| \geqq B(\tau)$, then $R_x < \tau$ and $v_x^2 - u_x^2 > m + \omega$ for $r \in (0, R_x)$.*

The basic argument is the construction of a trapping region for the solutions of (1.4)–(1.5), in which they blow up in a finite time. Before proceeding to the proof, we shall establish two preliminary lemmas.

Lemma 3.2. *Let $x > (m + \omega)^{1/2}$, and let $h_x = \sup\{r \in [0, R_x),\ v_x^2 - u_x^2 \geqq m + \omega$ on $[0, r]\}$. Then, we have*

$$u_x/v_x \geqq (2\omega/3)r, \quad \textit{for every } r \in [0, h_x).$$

Proof. Since $x > (m + \omega)^{1/2}$, we have $h_x > 0$. On $[0, h_x)$, inequality (2.12) is satisfied. Integrating (2.12) between 0 and $r \in [0, h_x)$ yields the result.

Lemma 3.3. *Let $C_1, C_2 > 0$, $\beta > 1$, $\theta > 0$. Let $f \in C([0, \theta])$ satisfy for $r \in [0, \theta]$, $f(r) \geqq C_2$ and $f'(r) \geqq C_1 r f(r)\,(\mathrm{Log}(f(r)/C_2))^\beta$. Then*

$$f(0) \leqq C_2 \exp\left[\left(\frac{2}{C_1(\beta - 1)\theta^2}\right)^{\frac{1}{\beta - 1}}\right].$$

Proof. Let $g = \mathrm{Log}(f/C_2)$. We have

$$g'(r) \geqq C_1 r g(r)^\beta, \quad \text{for} \quad r \in [0, \tau]. \tag{3.1}$$

The result follows from integrating (3.1).

Proof of Proposition 3.1 (See Fig. 3). By symmetry, it is enough to consider the case $x > 0$. Let $\tau > 0$, with $\tau\omega < 1$, and let (x_τ, y_τ) be the solution of the algebraic system

$$y_\tau = \tau\omega x_\tau, \quad x_\tau^2 - y_\tau^2 = m + \omega, \quad x_\tau > 0.$$

We set $\sigma = \tau/2$. Let $C > 0$, and $\beta > 1$, and consider the set

$$\Delta = \{(u, v) \in \mathbf{R}^2,\ v > x_\tau,\ u > 0,\ v^2 - u^2 > (m + \omega) + C(\mathrm{Log}(v/x_\tau))^\beta\}.$$

Choose C small enough, so that

$$C^2\beta(\mathrm{Log}(x/x_\tau))^{2\beta-1} \leqq 2\omega x^2, \quad \text{for} \quad x \geqq x_\tau; \tag{3.2}$$

$$C(\mathrm{Log}(x/x_\tau))^\beta \leqq (m + \omega)(x^2/x_\tau^2 - 1), \quad \text{for} \quad x \geqq x_\tau. \tag{3.3}$$

We claim that if C is as above, and if $x > x_\tau$, then (u_x, v_x) cannot exit Δ for $r \leqq \sigma$. Indeed, assume that there exists $x > x_\tau$, such that $R_x > \sigma$ and (u_x, v_x) exits Δ, at $r_0 \in (0, \sigma)$. Because $u_x'(0) > 0$, $(u_x, v_x) \in \Delta$ for $r > 0$ and small. In Δ we have $v_x' > 0$, so $v_x > x$; and if $u_x(r_0) = 0$, we have $u_x'(r_0) > 0$. Therefore, (u_x, v_x) must exit Δ, by crossing its upper boundary

$$\Gamma = \{(u, v) \in \mathbf{R}^2,\ v > x_\tau,\ u > 0,\ v^2 - u^2 = (m + \omega) + C(\mathrm{Log}(v/x_\tau))^\beta\}.$$

Thus we have

$$(v_x^2 - u_x^2 - (m + \omega) - C(\mathrm{Log}(v_x/x_\tau))^\beta|_{r=r_0} = 0, \tag{3.4}$$

$$(v_x^2 - u_x^2 - (m + \omega) - C(\mathrm{Log}(v_x/x_\tau))^\beta)'|_{r=r_0} \leqq 0. \tag{3.5}$$

From (3.5), (2.6), (1.5), and (3.4), we get

$$(4/r_0)u_x(r_0)v_x(r_0) - 4\omega v_x(r_0)^2 - C^2\beta(\log(v_x(r_0)/x_\tau))^{2\beta-1} \leqq 0. \tag{3.6}$$

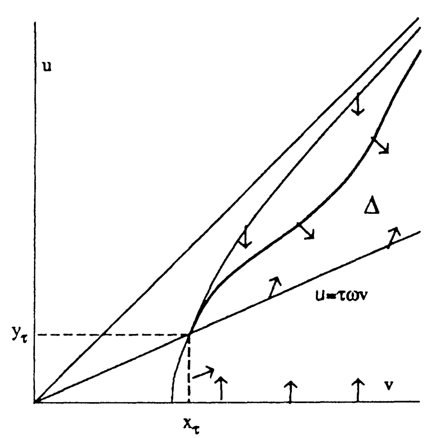

Fig. 3. The trapping region Δ

Note that we have $1/r_0 \geqq 2/\tau$, and so (3.6) and (3.2) yield

$$(8/\tau)u_x(r_0) \leqq 6\omega v_x(r_0). \tag{3.7}$$

Since $(u_x(r_0), v_x(r_0)) \in \Gamma$, (3.3) yields

$$v_x(r_0)^2 - u_x(r_0)^2 \leqq (m+\omega)(v_x(r_0)/x_\tau)^2.$$

Thus, since $m+\omega = x_\tau^2 - y_\tau^2$,

$$[v_x(r_0)^2 - u_x(r_0)^2]/v_x(r_0)^2 \leqq [x_\tau^2 - y_\tau^2]/x_\tau^2. \tag{3.8}$$

Next, by (3.8), we have

$$u_x(r_0) \geqq (y_\tau/x_\tau)v_x(r_0) = \tau\omega v_x(r_0). \tag{3.9}$$

Putting together (3.7) and (3.9), we get a contradiction; and so, Δ is a trapping region for $r \in (0, \sigma)$.

Consider now τ, β, and C as above. Let $x > x_\tau$, and $\varrho = \min\{R_x, \sigma\}$. We have $(u_x, v_x) \in \Delta$ for $r \in (0, \varrho)$, thus by (1.5) and Lemma 3.2, we obtain

$$v_x' \geqq (2\omega/3)Crv_x(\mathrm{Log}(v_x/x_\tau))^\beta, \quad \text{on} \quad (0, \varrho).$$

Therefore, by Lemma 3.3, if x is large enough, we have $\varrho < \sigma$. Hence the result.

4. A Boundedness Property

We introduce the sets I_n, A_n, and E_n, (see Fig. 4) defined for $n \in \mathbf{N}$ by

$I_n = \{x > (m-\omega)^{1/2}$, there exists $r_{x,n} \in (0, R_x)$ such that both u_x and v_x have exactly n zeroes on $(0, r_{x,n})$, and $u_x(r_{x,n}) = 0\}$,

$A_n = \{x > (m-\omega)^{1/2}$, $R_x = +\infty$, both u_x and v_x have exactly n zeroes on $(0, +\infty)$, and (u_x, v_x) satisfies (1.7)$\}$,

$E_n = I_n \cup A_n$.

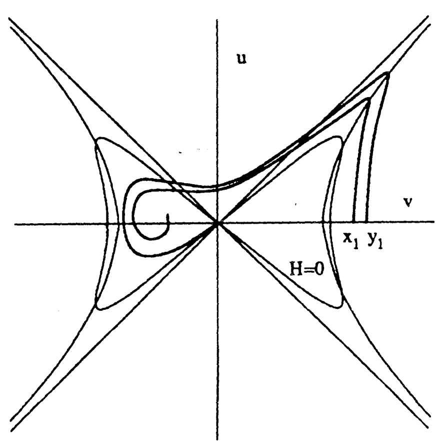

Fig. 4. Trajectories with x_1 in I_1 and y_1 in A_1

For $x \in A_n$, we set

$$r_{x,n} = +\infty .$$

Finally, we define the sets I, A, and E, by

$$I = \bigcup_{n \geqq 0} I_n , \quad A = \bigcup_{n \geqq 0} A_n , \quad E = \bigcup_{n \geqq 0} E_n .$$

In this section, we shall establish a boundedness property for solutions of (1.4)–(1.5), with initial data in E_n. Our main result is the following.

Proposition 4.1. *Let $n \in \mathbf{N}$, and assume $E_n \neq \emptyset$. Then*

$$\operatorname{Sup}_{x \in E_n} \ \operatorname{Sup}_{r \in [0, r_{x,n})} |u_x(r)| + |v_x(r)| < \infty .$$

Before proceeding to the proof of Proposition 4.1, we need some preliminary lemmas, where we set topological features of the sets E_n, and properties of trajectories with initial data in E_n. We begin with

Lemma 4.2. *For every $n \geqq 0$, I_n is an open subset of* $((m-\omega)^{1/2}, +\infty)$.

Proof. Observe that if $v_x = 0$ (respectively $u_x = 0$), then by Lemma 2.1, we have $v'_x \neq 0$ (respectively $v'_x \neq 0$). Therefore, the result follows from (2.4).

Next, we recall the following result of [1] (steps 1 and 3 of the proof of Theorem 3.1, and Lemma 3.4), which concerns the case $n = 0$.

Lemma 4.3. *I_0 is a nonempty set,* $\operatorname{Sup} I_0 \in A_0$, *and*

$$\operatorname{Sup}_{x \in I_0} \ \operatorname{Sup}_{r \in [0, r_{x,0})} |u_x(r)| + |v_x(r)| < \infty .$$

The argument given in [1] applies as well to show the following result.

Lemma 4.4. *We have* $\operatorname{Sup}_{x \in A_0} \ \operatorname{Sup}_{r \in [0, r_{x,0})} |u_x(r)| + |v_x(r)| < \infty .$

We now consider the case $n \geqq 1$. In the following lemma, we describe some topological properties of (u_x, v_x), when $x \in I_n$.

Lemma 4.5. *Let $n \geqq 1$, and $x \in I_n$. Then,*

(i) $R_x > 1/\omega$,

(ii) *the first zero of $u_x v_x$ in $(0, r_{x,n})$ is a zero of v_x,*

(iii) *the $2n$ zeroes of $u_x v_x$ in $(0, r_{x,n})$ are alternatively one zero of v_x and one zero of u_x,*

(iv) *the last two zeroes of $u_x v_x$ in $(0, r_{x,n}]$ are zeros of u_x,*

(v) *we have* $|v_x(r_{x,n})| < (m-\omega)^{1/2}$.

Proof. Let r_0 be the first zero of v_x. By Corollary 2.7, u_x has at least n zeroes in $(r_0, r_{x,n}]$. If u_x has a first zero, say, ϱ_0, in $(0, r_0)$, we have $u'_x(\varrho_0) < 0$, and so $0 < v_x(\varrho_0) < (m-\omega)^{1/2}$. By Corollary 2.9, u_x must have another zero in $(0, r_0)$; and so u_x has at least $n+2$ zeroes in $(0, r_{x,n}]$, which is impossible. This proves (ii). Therefore, (u_x, v_x) must cross the diagonal $\{u = v\}$ in $(0, r_0)$. Hence (i), by Lemma 2.3. Next, note that, by Corollary 2.7, a zero of u_x follows every zero of v_x; hence (iii) and (iv). (v) follows from (iii), (iv), and Corollary 2.9.

A similar result holds for solutions with initial data in A_n. More precisely, we have.

Lemma 4.6. *Let $n \geqq 1$, and $x \in A_n$. Then, the first zero of $u_x v_x$ in $(0, r_{x,n})$ is a zero of v_x. The $2n$ zeroes of $u_x v_x$ in $(0, r_{x,n})$ are alternatively one zero of v_x and one zero of u_x. Furthermore, for some $r_0 > 0$, we have $u_x v_x > 0$ on $(r_0, +\infty)$, and there exists C, such that*

$$(|u_x| + |v_x|)(r) \leqq C \exp(-(1/2)(m-\omega)r), \quad \text{for} \quad r \geqq 0.$$

Proof. The alternate character of the zeroes of u_x and v_x is proved with the argument of the proof of Lemma 4.5. The other properties follow from Lemma 2.8.

The following corollary is an immediate consequence of Lemmas 4.5 and 4.6.

Corollary 4.7. *We have $I \cap A = \emptyset$. Furthermore, for $k \neq j$, we have $I_k \cap I_j = \emptyset$.*

Next, observe that, from Proposition 3.1 and property (i) of Lemma 4.5, we have.

Lemma 4.8. *The set E is bounded.*

Finally, we will need the following two results.

Lemma 4.9. *For any $K > 0$ and $n \in \mathbf{N}$, there exists $C(K, n)$ with the following property. Assume $x > 0$ is such that $R_x > 1/\omega$. Assume furthermore that there exists $r_0, r_1 \in [1/\omega, R_x]$, such that $H_x(r_0) \leqq K$, and u_x has at most n zeroes on (r_0, r_1). Then*

$$(|u_x| + |v_x|)(r) \leqq C(K, n), \quad \text{for} \quad r \in (r_0, r_1).$$

Proof. Considering the velocity field on $\{v^2 - u^2 = m + \omega\}$, for $r > 1/\omega$, we observe that (u_x, v_x) can enter $D_1 = \{v^2 - u^2 > m + \omega,\ v > 0\}$, only in the half-plane $\{u < 0\}$. Similarly, (u_x, v_x) can exit D_1, only in the half-plane $\{u > 0\}$. Therefore, when (u_x, v_x) crosses D_1, u_x has at least one zero. The same holds for $D_2 = \{v^2 - u^2 > m + \omega$, $v < 0\}$; and so (u_x, v_x) crosses D_1 or D_2, at most n times on (r_0, r_1). When (u_x, v_x) crosses the set $\{v^2 - u^2 \leqq m + \omega\}$, we have $H'_x \leqq 0$, and so (u_x, v_x) is bounded. Therefore, the result follows from Lemma 2.5.

Lemma 4.10. *For every $n \geqq 1$, and $K > 0$, there exists $\tau(K, n) > 0$, with the following property. Suppose $x \in E_n$ and $H_x(\varrho) = K$, for some $\varrho \in [0, r_{x,n})$. Then $\varrho \leqq \tau(K, n)$.*

Proof. Let $x \in \mathbf{R}$ and $\varrho > 0$ be as in the assumptions of the lemma. By Lemma 2.5 of [1], there exists a function θ, with the following property. For any $T > 0$, if ϱ is larger than $\theta(T, u_x(\varrho), v_x(\varrho))$, then (u_x, v_x) will remain close to the solution (u, v) of (2.1), which fulfills $(u, v)(\varrho) = (u_x, v_x)(\varrho)$, on the interval $[\varrho, \varrho + T]$. It is easy to check that $\theta(T, a, b)$ is uniform in (a, b) such that $H(a, b) = K$; and so we will write $\theta(T, K)$. Note that (u, v) is a Hamiltonian motion, which is periodic and that the zeroes of u and those of v alternate. Thus, if T large enough, and $\varrho > \theta(T, K)$, (u_x, v_x) will have at least $n+1$ alternate zeroes of u_x and v_x, in $[\varrho, \varrho + T]$; and so, by Lemmas 4.5 and 4.6, $x \notin E_n$.

As a consequence of the previous results, we have the following.

Corollary 4.11. *For every* $x>(m-\omega)^{1/2}$, $x\notin I$, *one of the following properties is satisfied:*

(i) $R_x\leqq 1/\omega$, *and* u_x, v_x *and* $[v_x^2-u_x^2-(m+\omega)]$ *are positive on* $(0,R_x)$,

(ii) $R_x>1/\omega$, *both* u_x *and* v_x *have infinitely many zeroes on* $(0,R_x)$, *and they are alternate,*

(iii) $x\in A_n$, *for some* $n\in\mathbf{N}$.

Proof. Let x be as above. We have $u_x'(0)>0$, and so $u_x>0$ and $v_x>0$, for some time. Let

$$\varrho=\operatorname{Sup}\{r\in(0,R_x),\ u_x>0 \text{ and } v_x>u_x, \text{ on } (0,r)\}.$$

Suppose first that $\varrho=R_x$. Then, either v_x is bounded, and therefore $x\in A_0$, by Lemma 2.10 of [1]; or else, v_x is unbounded. In the latter case, v_x' is positive somewhere. Thus,

$$J=\{r\in[0,R_x),\ v_x^2(r)-u_x^2(r)\geqq(m+\omega)\}\neq\emptyset.$$

Therefore, by Lemma 2.9 of [1], J is either the interval $[0,R_x)$, or else some interval $[0,R]$, with $R<R_x$. In the latter case, v_x' is negative on $[R,R_x)$, and so v_x is bounded, which is a contradiction. Thus, $J=[0,R_x)$. Applying Lemma 2.5, we obtain $R_x\leqq 1/\omega$, and so we have property (i).

Suppose now that $\varrho<R_x$. Then, we have either $u_x(\varrho)=0$, or else $u_x(\varrho)=v_x(\varrho)$. In the first case, we have $x\in I_0$, which is a contradiction (by Corollary 2.7). Therefore $u_x(\varrho)=v_x(\varrho)$; and so, Lemma 2.6 implies that v_x has at least one zero, which is the first zero of u_xv_x. Furthermore, by Corollary 2.7, after every zero of v_x, the next zero of u_xv_x is a zero of u_x. A zero of v_x cannot be followed by two zeroes of u_x, since we would have $x\in I$, and this is ruled out by Corollary 4.7. Therefore, if v_x has infinitely many zeroes, (ii) is satisfied, and if v_x has a finite number of zeroes, we have (iii), by Lemma 2.11. Hence the result.

Proof of Proposition 4.1. From Lemmas 4.3 and 4.4, we only have to prove the property for $n\geqq 1$. Therefore, by Lemma 4.5, we have $R_x>1/\omega$. Now, we choose K large enough, so that the sets $\{H(u,v)<K\}$ and $\{v^2-u^2>m+\omega\}$ have a nonempty intersection.

Consider $x\in E_n$. By Lemma 4.5, we have $H_x(r_{x,n})<0$, if $x\in I_n$. Furthermore, $H_x(r)\to 0$ as $r\to+\infty$, if $x\in A_n$. Therefore, there exists $r\in(1/\omega,r_{x,n})$, with $H_x(r)<K$. We define the number ϱ, by

$$\varrho=1/\omega,\quad \text{if}\quad H_x(1/\omega)\leqq K;$$
$$\varrho=\operatorname{Inf}\{r\in((1/\omega),r_{x,n}),\ H_x(r)<K,\quad \text{if}\quad H_x(1/\omega)>K.$$

By Lemma 4.9, we have

$$(|u_x|+|v_x|)(r)\leqq C(K,n),\quad \text{for}\quad r\in[\varrho,r_{x,n}).$$

Therefore, it remains to bound the solution on $(0,\varrho)$. Note that, from Lemma 4.10, there exists $T>0$, depending only on n, such that $\varrho\leqq T$.

We shall bound the solution separately on $[0,1/\omega]$ and on $[1/\omega,\varrho]$.

Step 1. A bound on $[0, 1/\omega]$.

Let us show that for every $A>0$, there exists $B>0$, such that if $(|u_x| + |v_x|)(1/\omega) \leqq A$, then $(|u_x|+|v_x|)(r) \leqq B$, for $r \in [0, 1/\omega]$.

To prove this, we first observe that $(u_x, v_x)(r) \in \{0<u<v\}$ on $[0, 1/\omega]$. Indeed, by Lemmas 4.5 and 4.6, the first zero of $u_x v_x$ in $(0, r_{x,n})$ is a zero of v_x; and so, (u_x, v_x) has to cross the axis $\{u=v\}$ in the region $\{v>0\}$. By Lemma 2.3, this happens after $1/\omega$. Therefore, we also have $v_x > u_x$ on $[0, 1/\omega]$. By Lemma 2.9 of [1], there exists $\tau \in [0, 1/\omega]$, such that $(u_x, v_x) \in \{v^2 - u^2 \geqq m+\omega\}$ on $[0, \tau)$, and $(u_x, v_x) \in \{v^2 - u^2 < m+\omega\}$ on $(\tau, 1/\omega)$. Now, on $[0, \tau]$, we have $v_x' \geqq 0$; and so

$$0 \leqq u_x(r) \leqq v_x(r) \leqq v_x(\tau), \quad \text{for} \quad r \in [0, \tau]. \tag{4.1}$$

It remains to bound the solution on $(\tau, 1/\omega)$, in the case $\tau < 1/\omega$. Observe that

$$(v_x^2 - u_x^2)(\tau) = m+\omega \quad \text{and} \quad (v_x^2 - u_x^2)'(\tau) \leqq 0,$$

which implies

$$\tau \geqq (1/\omega)(u_x(\tau)/v_x(\tau)). \tag{4.2}$$

Now, $(u_x(\tau), v_x(\tau))$ is on the hyperbola $\{v^2 - u^2 = m+\omega\}$, and $H(u_x(\tau), v_x(\tau)) = K$. Therefore, there exists $\varepsilon > 0$, depending only on K, such that

$$u_x(\tau)/v_x(\tau) \geqq \varepsilon. \tag{4.3}$$

Let us set

$$f(r) = u_x(r) + v_x(r), \quad \text{for} \quad r \in [\tau, 1/\omega]. \tag{4.4}$$

From (1.4)–(1.5), we get,

$$\begin{aligned} f' &= -((2/r)+m)f + (2/r)v_x + \omega(v_x - u_x) + f(v_x^2 - u_x^2) \\ &\geqq -((2/r)+m)f, \quad \text{on} \quad [\tau, 1/\omega]. \end{aligned} \tag{4.5}$$

It follows from (4.5), (4.2), and (4.3), that

$$f' \geqq -((2\omega/\varepsilon)+m)f, \quad \text{on} \quad [\tau, 1/\omega]. \tag{4.6}$$

Integrating (4.6) between $r \in [\tau, 1/\omega]$ and $1/\omega$, we get

$$f(r) \leqq f(1/\omega)\exp(-(1/\omega)((2\omega/\varepsilon)+m)). \tag{4.7}$$

Putting together (4.1), (4.4), and (4.7), we obtain the desired estimate.

Step 2. A bound on $[1/\omega, \varrho]$.

Let us consider the regions Δ_1, Δ_2, Δ_3, and Δ_4, (see Fig. 5), defined by

$$\begin{aligned} \Delta_1 &= \{v^2 - u^2 > m+\omega, H(u,v) > K, v>0\}, \\ \Delta_2 &= \{v^2 - u^2 < m+\omega, H(u,v) > K, u>0\}, \\ \Delta_3 &= \{v^2 - u^2 > m+\omega, H(u,v) > K, v<0\}, \\ \Delta_4 &= \{v^2 - u^2 < m+\omega, H(u,v) > K, u<0\}. \end{aligned}$$

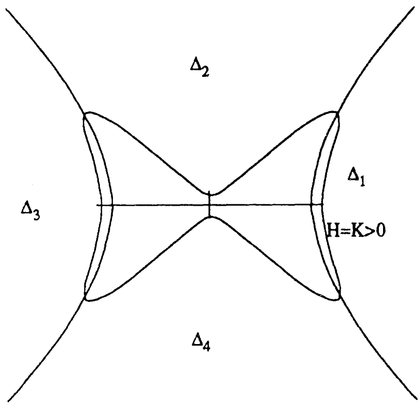

Fig. 5. The sets Δ_i, $i=1,2,3,4$

Assume that (u_x, v_x), enters Δ_1 at some time $t\in[1/\omega, \varrho]$. Considering the velocity field on the hyperbola $\{v^2-u^2=m+\omega\}$, we obtain that $u_x(t)<0$. By Lemma 2.4, (u_x, v_x) must exit Δ_1 in a finite time. The velocity field on the hyperbola $\{v^2-u^2=m+\omega\}$ forces (u_x, v_x) to exit Δ_1 in the half-plane $\{u>0\}$. Therefore, u_x has at least one zero in Δ_1. The same holds for Δ_3, by symmetry. A similar argument shows that, when (u_x, v_x) crosses Δ_2 or Δ_4, v_x has at least one zero. Therefore, (u_x, v_x) crosses at most n times each of the sets Δ_i. Thus, the proof of Proposition 4.1 will be complete, provided we show the following.

Claim. *For every* $i\in\{1,2,3,4\}$, *and for every* $T>0$, $A>0$, *there exists* $B>0$, *with the following property. Let* $x\in\mathbf{R}$, *be such that* $R_x>T$. *If on some interval* $[r_0, r_1]\subset[1/\omega, T]$, *we have* $(u_x, v_x)\in\Delta_i$, *and if* $(|u_x|+|v_x|)(r_1)\leqq A$, *then* $(|u_x|+|v_x|)(r)\leqq B$, *for every* $r\in[r_0, r_1]$.

Proof. If $i=1$, or $i=3$, H_x is nondecreasing on $[r_0, r_1]$; and the result follows from (2.3). Now, assume for example $i=2$, the case $i=4$ being treated similarly. We split Δ_2 in four subdomains δ_1, δ_2, δ_3, and δ_4, defined by (see Fig. 6)

$$\delta_1=\{(u,v)\in\Delta_2, u\leqq v\},$$
$$\delta_2=\{(u,v)\in\Delta_2, 0\leqq v\leqq u\},$$
$$\delta_3=\{(u,v)\in\Delta_2, 0\leqq -v\leqq u\},$$
$$\delta_4=\{(u,v)\in\Delta_2, u\leqq -v\}.$$

Considering the velocity field on the axes $\{u=v\}$, and $\{u=-v\}$, for $r>1/\omega$, we obtain easily that (u_x, v_x) can only move from δ_1 to δ_2, from δ_2 to δ_3, and from δ_3 to δ_4. Therefore, we can assume, without restricting the generality, that $(u_x, v_x)(r_0)\in\delta_1$, and $(u_x, v_x)(r_1)\in\delta_4$. We denote by τ_i, for $i=1,2,3$, the time when (u_x, v_x) moves from δ_i to δ_{i+1}.

On $[\tau_3, r_1]$, we have $v_x<0$, $v_x'<0$, and $u_x\leqq -v_x$. Thus,

$$(|u_x|+|v_x|)(r)\leqq 2A, \quad \text{on} \quad [\tau_3, r_1]. \tag{4.8}$$

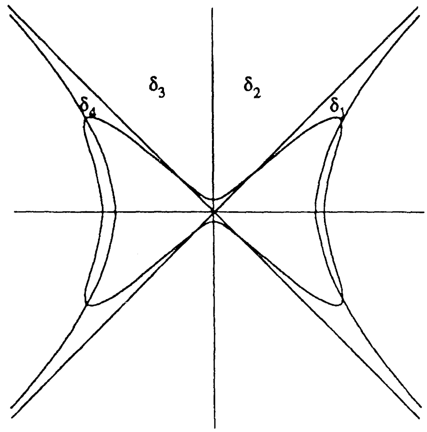

Fig. 6. The sets δ_i, $i=1,2,3,4$

Next, in δ_3 we set

$$g(r)=u_x(r)-v_x(r), \quad \text{for} \quad r\in[\tau_2,\tau_3].$$

Considering (1.4)–(1.5), we obtain

$$g'=-(2/r)g-(2/r)v_x+\omega(v_x+u_x)+mg-(v_x^2-u_x^2)g, \quad \text{on} \quad [\tau_2,\tau_3];$$

from which we get

$$g'\geqq -2\omega g. \tag{4.9}$$

Integrating (4.9) between $r\in[\tau_2,\tau_3]$ and τ_3, yields

$$g(r)\leqq g(\tau_3)\exp(2\omega T), \quad \text{for every } r\in[\tau_2,\tau_3]. \tag{4.10}$$

Putting together (4.8) and (4.10), we obtain

$$(|u_x|+|v_x|)(r)\leqq 2A\exp(2\omega T), \quad \text{on} \quad [\tau_2,r_1]. \tag{4.11}$$

Then, in δ_2, for $r\in[\tau_1,\tau_2]$, we set

$$h(r)=u_x^2(r)-v_x^2(r).$$

Applying (2.6), we obtain

$$h'\geqq 4\omega(u_xv_x-u_x^2)\geqq -4\omega h, \quad \text{on} \quad [\tau_1,\tau_2]. \tag{4.12}$$

Integrating (4.12) between $r\in[\tau_1,\tau_2]$ and τ_2, and applying (4.11), we obtain

$$h(r)\leqq h(\tau_2)\exp(4\omega T)\leqq 4A^2(\exp(4\omega T))^2, \quad \text{for every } r\in[\tau_1,\tau_2]. \tag{4.13}$$

Considering (1.4) and (4.13), we obtain

$$u_x'\geqq -(2/r)u_x-Cv_x\geqq -(C+2\omega)u_x, \quad \text{for every } r\in[\tau_1,\tau_2], \tag{4.14}$$

where $C=4A^2(\exp(4\omega T))^2$. Since $u_x\geqq v_x$ on $[\tau_1,\tau_2]$, integrating (4.14) between $r\in[\tau_1,\tau_2]$ and τ_2, and applying (4.11), gives

$$(|u_x|+|v_x|)(r)\leqq 2A\exp(2\omega T)\exp((C+\omega)T), \quad \text{on} \quad [\tau_1,r_1]. \tag{4.15}$$

Finally, in δ_1, we set

$$k(r)=u_x(r)+v_x(r), \quad \text{for} \quad r\in[r_0,\tau_1].$$

We apply the argument of the proof of Step 1 to obtain

$$k'\geqq -(m+2\omega)k, \quad \text{on} \quad [r_0,\tau_1];$$

and so

$$k(r)\leqq k(\tau_1)\exp(-(m+2\omega)T), \quad \text{for every } r\in[r_0,\tau_1]. \tag{4.16}$$

Putting together (4.15) and (4.16), completes the proof of the claim, hence the proof of Proposition 4.1.

5. Proof of Theorem 1.1

We begin with two preliminary observations.

Lemma 5.1. *Let $n\geqq 0$. Assume $A_n\neq\emptyset$, and let x belong to the closure of A_n. Then $x\in \bigcup_{0\leqq j\leqq n} A_j$.*

Proof. By Corollary 4.7, we have $I\cap A=\emptyset$. Thus, since I is open (Lemma 4.2), $x\notin I$. By (2.4) and Proposition 4.1, (u_x, v_x) is bounded. This proves $R_x=+\infty$. If u_x and v_x have more than n zeroes, then it is the same for x' close to x. This is impossible, since x belongs to the closure of A_n. Therefore, the result follows from Corollary 4.11.

Lemma 5.2. *Let $n\geqq 0$. Assume $I_n\neq\emptyset$, and let x belong to the boundary of I_n. Then $x\in \bigcup_{0\leqq j\leqq n} A_j$.*

Proof. By Proposition 4.1, $I_n\neq\mathbf{R}$, so its boundary is nonempty. By Corollary 4.7, and since I_n is open, $x\notin I_n$. Therefore, since I is also open, $x\notin I$. Now, we apply Corollary 4.11. Property (i) is ruled out by Proposition 4.1, while property (ii) is ruled out by (2.4). Thus, x belong to some A_j. By continuous dependence, again, we must have $j\leqq n$.

In order to show that I_n is nonempty, we need the following lemmas.

Lemma 5.3. *For every $C>0$, there exists $T>0$, with the following property. Let $x\neq 0$ be such that $R_x\geqq T$. Assume that for some $\varrho\geqq T$, we have $v_x(\varrho)=0$ and $|u_x(\varrho)|\leqq C\exp(-(1/2)(m-\omega)\varrho)$. Then, there exists $\theta>\varrho$, such that $|u_x|>0$ and $|v_x|\leqq(m-\omega)^{1/2}$ on (ϱ,θ), $|v_x(\theta)|=(m-\omega)^{1/2}$, and $H_x(\theta)<0$.*

Proof (See Fig. 7). Let x and ϱ be such that $\varrho\leqq R_x$, and $v_x(\varrho)=0$, and

$$|u_x(\varrho)|\leqq C\exp(-(1/2)(m-\omega)\varrho). \tag{5.1}$$

Assume, for example, $u_x(\varrho)>0$. Then, v_x will become negative, before u_x vanishes. If ϱ is large enough, we have $u_x(\varrho)\leqq(1/4)(m-\omega)^{1/2}$. We define the number $R>0$, by

$$R=\operatorname{Sup}\{r>0, u_x+|v_x|\leqq(1/2)(m-\omega)^{1/2} \text{ on } [\varrho,r]\}. \tag{5.2}$$

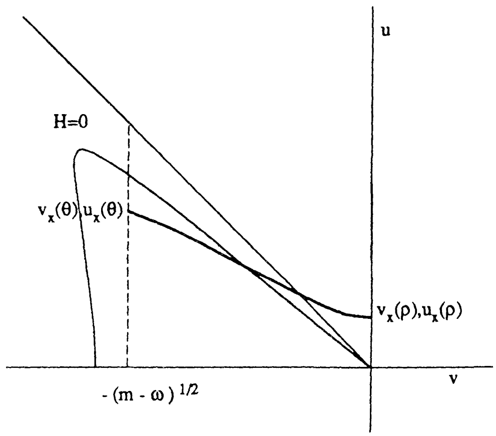

Fig. 7. Notation for Lemma 5.3

By Corollary 2.7, we have $R<R_x$. Thus,

$$(u_x+|v_x|)(R)=(1/2)(m-\omega)^{1/2}. \tag{5.3}$$

Observe that on $[\varrho,R]$ we have $v_x'\leqq 0$, and so $v_x\leqq 0$. Furthermore, on the set $(-(m-\omega)^{1/2},0)\times\{0\}$, the velocity field points upward. Therefore, on $[\varrho,R]$, we have $u_x>0$, and

$$(u_x-v_x)'\leqq 2(m+\omega)(u_x-v_x). \tag{5.4}$$

Integrating (5.4) between ϱ and R, we get

$$(u_x+|v_x|)(R)\leqq u_x(\varrho)\exp[2(m+\omega)(R-\varrho)]. \tag{5.5}$$

Putting together (5.1), (5.3), and (5.5), we obtain

$$2(m+\omega)(R-\varrho)\geqq \mathrm{Log}[(1/2C)(m-\omega)^{1/2}]+[(m-\omega)/2]\varrho.$$

Hence, for some $a>0$ and ϱ large enough

$$R\geqq(1+a)\varrho. \tag{5.6}$$

We claim that, if ϱ is large enough, then $H_x(R)<0$. To see this, observe first that from (2.2), we get

$$5(m+\omega)u^2\geqq 3(m-\omega)v^2,\ \text{for all } u,\ v \text{ such that } H(u,v)\geqq 0,$$
$$\text{and}\quad u^2+v^2\leqq(1/2)(m-\omega). \tag{5.7}$$

Let now

$$\tau=\mathrm{Sup}\{r\in[\varrho,R],H_x\geqq 0 \text{ on } [\varrho,r]\}. \tag{5.8}$$

From (5.7) and (5.8), we get

$$cu_x(r)\geqq -v_x(r),\quad \text{for}\quad r\in[\varrho,\tau], \tag{5.9}$$

where $c=[(5/3)((m+\omega)/(m-\omega))]^{1/2}$. Next, on $[\varrho, R]$, we have

$$v_x^2-u_x^2-(m+\omega)\leqq -(1/2)(m+\omega)\leqq -(1/2)(m-\omega),$$
$$\text{and}\quad v_x^2-u_x^2-(m-\omega)\leqq -(1/2)(m-\omega). \tag{5.10}$$

Therefore, we have, by (1.4) and (1.5),

$$(u_x-v_x)'\geqq -(2/r)u_x+(1/2)(m-\omega)(u_x-v_x). \tag{5.11}$$

Since $-u_x\geqq -(u_x-v_x)$, and if $\varrho>8/(m-\omega)$, we get from (5.11)

$$(u_x-v_x)(r)\geqq u_x(\varrho)\exp((1/4)(m-\omega)(r-\varrho)),\quad \text{for}\quad r\in[\varrho, R]. \tag{5.12}$$

Putting together (5.9) and (5.12), we obtain

$$u_x^2(r)\geqq ku_x^2(\varrho)\exp((1/2)(m-\omega)(r-\varrho)),\quad \text{for}\quad r\in[\varrho,\tau], \tag{5.13}$$

where $k=[1/(1+c)]^2$. Now, from (5.8), (5.10), and (2.5), we have

$$0\leqq H_x(\tau)\leqq H_x(\varrho)-(m+\omega)\int_\varrho^\tau \frac{1}{r}u_x(r)^2 \mathrm{d}r. \tag{5.14}$$

Next, observe that if ϱ is large enough, we have by (5.1)

$$H_x(\varrho)=(1/4)u_x(\varrho)^4+(1/2)(m+\omega)u_x(\varrho)^2\leqq (m+\omega)u_x(\varrho)^2. \tag{5.15}$$

Therefore, putting together (5.13), (5.14), and (5.15), we obtain

$$0\leqq 1-k^2\int_\varrho^\tau \frac{1}{r}\exp((1/2)(m-\omega)(r-\varrho))dr. \tag{5.16}$$

We deduce easily from (5.16), that

$$(k^2/\tau)[\exp((1/2)(m-\omega)(\tau-\varrho))-1]\leqq (m-\omega).$$

Therefore, for any $\alpha>0$, we get when ϱ is large enough

$$\tau\leqq(1+\alpha)\varrho. \tag{5.17}$$

Putting together (5.6) and (5.17), we obtain $\tau<R$. Since we have $H_x'<0$, on $[\tau, R]$, we get

$$H_x(R)<0,\quad \text{if } \varrho \text{ is large enough}. \tag{5.18}$$

To conclude, observe that by Corollary 2.7, v_x must decrease to some value less than $-(m-\omega)^{1/2}$, before u_x vanishes. Note also that when $|v_x|\leqq -(m-\omega)^{1/2}$, we have $H_x'<0$; and so, at the time when $v_x=-(m-\omega)^{1/2}$, we have $H_x<0$. Thus, the proof of Lemma 5.3 is complete.

Lemma 5.4. *There exists R with the following property. Assume $x\neq 0$ is such that $R_x\geqq R$. Assume further that for some $\varrho\geqq R$, we have $v_x(\varrho)=-(m-\omega)^{1/2}$, $u_x(\varrho)>0$ (respectively $v_x(\varrho)=(m-\omega)^{1/2}$, $u_x(\varrho)<0$), and $H_x(\varrho)<0$. Then, there exists $\tau>\varrho$ such that $|v_x|>0$ on (ϱ,τ) and such that u_x has two zeroes on (ϱ,τ).*

Proof. (See Fig. 8). Let x be as above, and assume, for exmaple, $u_x(\varrho)>0$. By Lemma 2.6, and Corollary 2.9, there exists $\varrho_1\in(\varrho, R_x)$, such that $v_x(\varrho_1)$

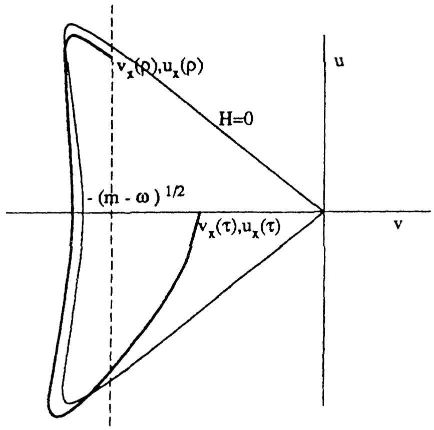

Fig. 8. Notation for Lemma 5.4

$= -(m-\omega)^{1/2}$, $v_x < -(m-\omega)^{1/2}$ on (ϱ, ϱ_1), and u_x has exactly one zero on (ϱ, ϱ_1).

Assume that $H_x(\varrho_1) < 0$. Let

$$\Sigma = \{H(u,v) < H_x(\varrho_1), 0 > v > -(m-\omega)^{1/2}, u < 0\}.$$

Observe that $H'_x(\varrho_1) < 0$, and $v'_x(\varrho_1) > 0$. Therefore, (u_x, v_x) will belong to Σ, for $r \in (\varrho_1, \varrho_1 + \varepsilon]$, where ε is some positive number. Since $H_x(\varrho_1) < 0$, Σ is bounded away from the u-axis. Furthermore, when $(u_x, v_x) \in \Sigma$, we have $H'_x < 0$, and $v'_x > 0$, by (1.5) and (2.5). Thus, (u_x, v_x) cannot exit Σ by crossing the line $\{v = -(m-\omega)^{1/2}\}$ or the curve $\{H(u,v) = H_x(\varrho_1)\}$. Finally, since Σ is bounded away from the u-axis, it follows from Corollary 2.9, that (u_x, v_x) exits Σ in a finite time, by crossing the v-axis. This is the desired result.

It remains to prove that $H_x(\varrho_1) < 0$. Note that for a fixed value of $H_x(\varrho)$, and for ϱ large enough, this is a consequence of the fact that the trajectory remains close to a trajectory of the Hamiltonian motion. Actually, we need an estimate on ϱ that does not depend on the value of $H_x(\varrho)$. To prove this, we consider two cases.

If $3\omega \geqq m$, then the sets $\{v^2 - u^2 \geqq (m-\omega)^{1/2}\}$ and $\{H(u,v) < 0\}$ are disconnected; and so, by (2.5), the set $\{H(u,v) < 0\}$ is a trapping region for the solutions of (1.4)–(1.5). Thus, we have $H_x(\varrho_1) < 0$.

If $3\omega < m$, then $H(0, (m+\omega)^{1/2}) < 0$, and the set $\{H(u,v) \leqq H(0, (m+\omega)^{1/2})\}$ is a trapping region. Thus, if for some $r \in [\varrho, \varrho_1]$, we have $H_x(r) \leqq H(0, (m+\omega)^{1/2})$, we get immediately $H_x(\varrho_1) < 0$. Therefore, it only remains to consider the case where x is such that $H_x(\varrho) < 0$, and

$$H(0, (m+\omega)^{1/2}) < H_x(r), \quad \text{for every } r \in [\varrho, \varrho_1]. \tag{5.19}$$

Let Λ be the set of such x's. Consider the set

$$J = \{z > 0, H(0, (m+\omega)^{1/2}) < H(z, -(m-\omega)^{1/2}) < 0\}.$$

Clearly, we have

$$u_x(\varrho) \in J, \quad \text{for every } x \in \Lambda. \tag{5.20}$$

Note that, since $3\omega < m$ there exists $\delta \in (0,(m-\omega)^{1/2})$ such that $H(0,-\delta) = H(0,(m+\omega)^{1/2})$. It is not too difficult to show that for $z \in J$, there exists $T^z > 0$, such that the solution (u^z, v^z) of (2.1), with $u^z(\varrho) = z$, $v^z(\varrho) = -(m-\omega)^{1/2}$, will satisfy

$$v^z(r) < -\delta, \quad \text{for} \quad t \in (\varrho, \varrho + T^z), \quad u^z(\varrho + T^z) < 0, \quad \text{and} \quad v^z(\varrho + T^z) = -\delta. \tag{5.21}$$

The set $\{0 \geqq H(u,v) \geqq H(0,(m+\omega)^{1/2}), v < -\delta\}$ is bounded away from the critical points of H, and so, there exists $M > 0$ such that

$$T^z \leqq M, \quad \text{for every } z \in J. \tag{5.22}$$

Note also that, since J is bounded,

$$(u^z(r), v^z(r)) \text{ is uniformly bounded, with respect to } r \in \mathbf{R}, \text{ and } z \in J. \tag{5.23}$$

Next, by Lemma 2.5 of [1], we know that for every $x \in \Lambda$, there exists $r_x > 0$, such that if $\varrho \geqq r_x$, then

$$|(u_x, v_x) - (u^z, v^z)| \leqq (1/2)\,[(m-\omega)^{1/2} - \delta], \quad \text{on} \quad (\varrho, \varrho + M), \tag{5.24}$$

where $z = u_x(\varrho)$. Note that, since J is bounded, r_x can be chosen to be bounded, uniformly in $x \in \Lambda$. Thus, if R is large enough, we have (5.24), for every $x \in \Lambda$. Therefore, from (5.21), (5.22), and (5.24), we get

$$\varrho_1 - \varrho \leqq M. \tag{5.25}$$

On the other hand, by (5.23), (5.24), and (5.25), we have

$$(u_x, v_x) \text{ is bounded in } C^1([\varrho, \varrho_1], \mathbf{R}^2), \text{ uniformly in } x \in \Lambda. \tag{5.26}$$

Let now σ be the area of the set $\{H(u,v) \leqq H(0,(m+\omega)^{1/2}, v \leqq -(m-\omega)^{1/2}\}$. It follows from (5.25), and (5.26), that if R is large enough, we have

$$\left| \int_{\varrho}^{\varrho_1} \left(1 - \frac{\varrho}{r}\right) u_x(r) v_x'(r)\, dr \right| \leqq \frac{\sigma}{2}, \quad \text{for every } x \in \Lambda. \tag{5.27}$$

Next, observe that the curve $\bigcup_{r \in [\varrho, \varrho_1]} \{u_x(r), v_x(r)\}$ has no multiple points, since on (ϱ, ϱ_1), we have $(v_x^2 - u_x^2)' > 0$, while $u_x > 0$, and $(v_x^2 - u_x^2)' < 0$, while $u_x < 0$. This comes from (2.6), assuming $R > 1/\omega$. Let Σ_x be the region contained between the curve, and the line $\{v = -(m-\omega)^{1/2}\}$, and let δ_x be its area. By (5.19), we have $\Sigma_x \supset \{H(u,v) \leqq H(0,(m+\omega)^{1/2}, v \leqq -(m-\omega)^{1/2}\}$; and so

$$\delta_x \geqq \sigma. \tag{5.28}$$

Now, applying Green's formula, we have

$$\delta_x = -\int_{\varrho}^{\varrho_1} u_x(r) v_x'(r)\, dr. \tag{5.29}$$

Putting together (5.27), (5.28), and (5.29), we get

$$\int_{\varrho}^{\varrho_1} \frac{1}{r} u_x(r) v_x'(r)\, dr \leqq -\frac{\sigma}{2\varrho}. \tag{5.30}$$

Integrating (2.5), between ϱ and ϱ_1, and using (1.5) and (5.30), we obtain the inequality

$$H_x(\varrho_1) - H_x(\varrho) \leqq -\sigma/\varrho < 0,$$

from which the result follows.

Corollary 5.5. *Let $n \geqq 0$ be such that $A_n \neq \emptyset$, and let $x_n = \operatorname{Sup} A_n$. Assume $x_n \in A_n$, and $x_n \geqq \operatorname{Sup} I_n$. Then, there exists $\varepsilon > 0$ such that $(x_n, x_n + \varepsilon) \subset I_{n+1}$.*

Proof. Let us set $(U, V) = (u_{x_n}, v_{x_n})$, and let R be the last zero of U. Assume for example that $U > 0$ on $(R, +\infty)$. By Lemma 4.6, there exists $\tau > R$, such that

$$0 < U(r) < V(r) \leqq (1/4)(m-\omega)^{1/2}, \quad \text{for} \quad r \geqq \tau. \tag{5.31}$$

Next, observe that for x close to x_n, we have the following. Both u_x and v_x have n zeroes on $(0, \tau)$, which are alternate, and

$$0 < u_x(\tau) < v_x(\tau) \leqq (1/2)(m-\omega)^{1/2}. \tag{5.32}$$

This follows from (2.4) and Lemma 4.6. Now, assume in addition that $x > x_n$. Let

$$\varrho_x = \operatorname{Sup}\{r \in (\tau, R_x),\ u_x > 0 \text{ and } v_x > 0 \text{ on } (\tau, r)\}.$$

Observe that, from (5.32) it follows easily that $u_x' < 0$ and $v_x' < 0$ on (τ, ϱ_x), and so

$$0 < u_x \leqq (1/2)(m-\omega)^{1/2}, \quad \text{and} \quad 0 < v_x \leqq (1/2)(m-\omega)^{1/2}, \quad \text{on} \quad (\tau, \varrho_x). \tag{5.33}$$

Next, $x \notin A_n$, and so, by Lemma 2.8, ϱ_x is finite. Therefore, $u_x(\varrho_x) = 0$, or $v_x(\varrho_x) = 0$. Furthermore, $x \notin I_n$, and so $u_x(\varrho_x) \neq 0$. Thus, $v_x(\varrho_x) = 0$. Note that, from (5.31), and (2.4), we have

$$\varrho_x \to +\infty, \quad \text{as} \quad x \to x_n. \tag{5.34}$$

Finally, from (1.4), (1.5), and (5.33), we get

$$(u_x + v_x)' \leqq -(1/2)(m-\omega)(u_x + v_x), \quad \text{on} \quad (\tau, \varrho_x). \tag{5.35}$$

Integrating (5.35), we obtain

$$u_x(\varrho_x) \leqq (1/2)(m-\omega)^{1/2} \exp((1/2)(m-\omega)\tau) \exp(-(1/2)(m-\omega)\varrho_x). \tag{5.36}$$

Putting together (5.34), (5.36), and applying Lemmas 5.3 and 5.4, we obtain that if x is close enough to x_n, then $u_x v_x$ has at least two zeroes after ϱ_x, and the first two zeroes are zeroes of u_x; and so $x \in I_{n+1}$. This completes the proof of Corollary 5.5.

End of the Proof of Theorem 1.1. Let $y_0 = \operatorname{Sup} I_0$. By Lemma 4.3, $y_0 \in A_0$. Let now $x_0 = \operatorname{Sup} A_0$. Applying Lemma 5.1, we get $x_0 \in A_0$. Therefore, by Corollary 5.5, there exists $\varepsilon_0 > 0$, such that $(x_0, x_0 + \varepsilon_0) \subset I_1$. Thus, $I_1 \neq \emptyset$. Let $y_1 = \operatorname{Sup} I_1$. We have $y_1 > x_0 \geqq y_0$; and so, by Lemma 5.2, $y_1 \in A_1$, then by Lemma 5.1, $x_1 = \operatorname{Sup} A_1 \in A_1$. Iterating this argument, we construct an increasing sequence $(x_n)_{n \geqq 0}$, with $x_n \in A_n$. The boundedness of $(x_n)_{n \geqq 0}$ follows from Lemma 4.8, and the exponential decay, from Lemma 4.6. Thus, the proof of Theorem 1.1 is complete.

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