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Line Bundles on Super Riemann Surfaces

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Abstract. We give the elements of a theory of line bundles, their classification, and their connections on super Riemann surfaces. There are several salient departures from the classical case. For example, the dimension of the Picard group is not constant, and there is no natural hermitian form on Pic. Furthermore, the bundles with vanishing Chern number aren't necessarily flat, nor can every such bundle be represented by an antiholomorphic connection on the trivial bundle. Nevertheless the latter representation is still useful in investigating questions of holomorphic factorization. We also define a subclass of all connections, those which are compatible with the superconformal structure. The compatibility conditions turn out to be constraints on the curvature 2-form.

1. Introduction

This paper is a sequel to [1, 2]. In those papers we described the theory of super Riemann surfaces (SRS) in differential-geometric terms.[1] In particular we defined a SRS $\hat{X}$ as a supermanifold of real dimension $2|2$ equipped with an additional structure. This "superconformal structure" amounts to an integrable reduction of the structure group of $\hat{X}$. $\hat{X}$ then has a canonical holomorphic line bundle $\hat{\omega}$, so we can define holomorphic $\frac{p}{2}$-differentials as sections of $\hat{\omega}^p$. We also get an analog of the Cauchy-Riemann operator, $\hat{\bar{\partial}}$, which can be used to define both the string action and actions for generalized first-order systems [3]. All in all, SRS show a remarkable formal similarity to ordinary Riemann surfaces, despite the fact that they cannot be thought of as having just one complex dimension.

In this paper we will carry the discussion further, turning to other structures on Riemann surfaces and their SRS analogs. We begin by reviewing the basic

[1] See the references in [2]

properties of the $\hat{\partial}$ operator and by discussing the associated cohomology groups. Next we define line bundles on complex supermanifolds and on SRS, and describe their classification. Following this, we describe in greater detail the relation between the super Cauchy Riemann operator $\hat{\partial}$ and the exterior derivative operator ∂, briefly mentioned in [2]. We then introduce both arbitrary connections and those compatible with the superconformal structure. The latter turn out to be distinguished by the fact that their curvatures obey certain constraints, a $2d$ version of the curvature constraints of superfield gauge theory. The curvature of a line bundle can be used to compute its Chern class, much as in the classical case.

In the case of odd spin structures, however, we will see several important differences between the theory of line bundles on SRS and the classical case. For one thing, the dimension of the Picard group is not constant. Also the group Pic_0 of bundles with Chern number zero cannot be represented in general by flat connections. Both of these pathologies suggest that perhaps our definition of $\hat{\mathcal{M}}$ is not yet the most useful one. On the other hand, for the even spin structures the above pathologies generically don't arise. Thus the even case closely parallels the classical theory.

In any case we can still describe Pic_0 in terms of connections on the trivial bundle, as we show in the last section. This differential geometric representation of a holomorphic family is useful when one wants to investigate holomorphic factorization. Finally we conclude with some open questions.

When one is given a class of analytic spaces such as SRS it is mathematically very natural to ask about the most general bundles and connections one can write compatible with the given structure. The answers which emerge then usually find their way into physical constructions. For example, once one knows that Riemann surfaces are important for string theory then it quickly becomes clear that *holomorphic* bundles are important, not just complex bundles. Still one may ask about the utility of considering arbitrary line bundles, when it seems to be only the untwisted $\frac{p}{2}$-differentials which enter into string theory. One answer is that a number of results in ordinary conformal field theory emerge only when one interpolates between different spin structures, for example the theorem on the determinant of the Dirac operator and the ensuing bosonization results [4, 5]. Even if in the end one considers only spin bundles, the results obtained by admitting twists are still important. It is not yet clear to us whether an exact analog of the theorem in [4] can be given on SRS, but the formalism developed here is a step in that direction.

Superdifferentials were introduced in [3, 6]. We also draw the reader's attention to the papers [7, 8], where bundles and jacobians are also discussed, and where some of our results are independently given.

2. The $\hat{\partial}$ Operator and its Cohomology

Before introducing bundles we will review and extend some notions from [2]; we focus in particular on the $\hat{\partial}$ operator and some of its properties. We will also give the Dolbeault theorem for $\bar{\hat{\partial}}$.

Throughout this paper u, θ will denote a set of superconformal coordinates for a SRS[2] $\hat{X}$ has a canonical holomorphic line bundle $\hat{\omega}$ of half-volume forms. Given the coordinates u, θ we get a local trivializing section v of $\hat{\omega}$:

$$v = [du d\theta], \tag{2.1}$$

where the right side is the Berezin volume element. We will write a general section of $\hat{\omega}$ as $v \cdot \lambda_+$, where λ_+ is a function. Let $\bar{v}$ be the complex conjugate of v. We can define an analog of the Cauchy-Riemann operator as follows:

$$\hat{\partial} f \equiv v \cdot (Df). \tag{2.2}$$

$\hat{\partial}$ is intrinsically defined, and we get an exact sequence

$$0 \to \mathbf{C} \hookrightarrow \hat{\mathcal{O}} \xrightarrow{\hat{\partial}} \hat{\omega} \to 0. \tag{2.3}$$

Here $\hat{\mathcal{O}}$ is the sheaf of holomorphic super functions on $\hat{X}$, while $\mathbf{C}$ is the constant sheaf of complex numbers. Any constant sheaf like $\mathbf{C}$ knows only about the topology of the ordinary topological space underlying $\hat{X}$.

The sequence (2.3) deals entirely with sheaves of holomorphic (or constant) functions. It will be important to generalize everything to spaces of *smooth* sections, analogous to the (p, q)-forms on Riemann surfaces. Defining differential forms as in [2], we can split the r-forms into spaces of smooth (p, q)-forms, where $p + q = r$:

$$\hat{\Omega}^{p,q} = \{\text{smooth } (p, q)\text{-forms on } \hat{X}\}. \tag{2.4}$$

We then have the usual exterior operator

$$\bar{\partial} : \hat{\Omega}^{p,q} \to \hat{\Omega}^{p,q+1}$$

and its conjugate, defined by $d = \partial + \bar{\partial}$. They define the exact sequences

$$0 \to \hat{\Omega}^p \hookrightarrow \hat{\Omega}^{p,0} \xrightarrow{\bar{\partial}} \hat{\Omega}^{p,1} \xrightarrow{\bar{\partial}} \hat{\Omega}^{p,2} \to \dots, \tag{2.5}$$

where $\hat{\Omega}^p$ are the holomorphic p-forms. For example when $p = 0$ the holomorphic functions are precisely those annihilated by $\bar{\partial}$, and so on.

The problem with (2.5) is that it does not terminate; there are no "top forms" in super geometry. Instead we need a complex based on $\hat{\bar{\partial}}$. Define[3]

$$\hat{\mathscr{A}}^{p,q} = \hat{\omega}^p \otimes \hat{\bar{\omega}}^q,$$

so that $\hat{\mathscr{A}}^{0,0} = \hat{\mathscr{A}}$ are the smooth functions, etc. In the definition of $\hat{\mathscr{A}}^{1,1}$ we add the condition that $v \otimes \bar{v} = \bar{v} \otimes v$, so that v behaves like $d\theta$. We will call sections of $\hat{\mathscr{A}}^{p,q}$ "$\left(\frac{p}{2}, \frac{q}{2}\right)$-differentials" to distinguish them from "(p, q)-forms", the sections of $\hat{\Omega}^{p,q}$.

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[2] Actually we must always consider *families* of SRS, as discussed for example in [2]. We will not make this explicit in the notation, but really u, θ are relative coordinates, the cohomology groups H^q below are $R^q p_*$, and so on (see [9])

[3] More precisely $\hat{\mathscr{A}}^{p,q} = (\hat{\omega}^p \otimes_{\hat{\mathcal{O}}} \hat{\mathscr{A}}) \otimes_{\hat{\bar{\mathcal{O}}}} \hat{\bar{\omega}}^q$

We can define an operator $\hat{\bar{\partial}}$ from $\hat{\mathscr{A}}^{p,0}$ to $\hat{\mathscr{A}}^{p,1}$ using (2.2) and

$$\hat{\bar{\partial}}(v\cdot\lambda_+)=v\bar{v}\cdot(\bar{D}\lambda_+), \tag{2.6}$$

and similarly for $\hat{\partial}$. This yields the sequences

$$0\to\hat{\omega}^p\hookrightarrow\hat{\mathscr{A}}^{p,0}\xrightarrow{\hat{\bar{\partial}}}\hat{\mathscr{A}}^{p,1}\to 0. \tag{2.7}$$

The sequences (2.7) are also exact. For example, a smooth function is holomorphic exactly when $\hat{\bar{\partial}}$ annihilates it. As explained in [2] this is possible because a single vector field in superspace can be nonintegrable. Moreover, just as in the classical case one finds that the sheaves $\hat{\mathscr{A}}^{p,q}$ are *fine*; that is, they admit a partition of unity subordinate to any cover of the base space X. To prove this we simply note that the smooth ordinary functions $\mathscr{A}$ on X admit partitions of unity. Furthermore $\hat{X}$ always admits a splitting as a smooth supermanifold [10]. Choose any such splitting and use it to pull the partition of unity from X up to $\hat{X}$. This is not canonical, but it does show that the $\hat{\mathscr{A}}^{p,q}$ are fine.

The resolutions (2.7) of $\hat{\mathcal{O}}$ and $\hat{\omega}$ by fine sheaves lead at once to a Dolbeault-type theorem[4]. Let

$$H_D^{p,1}=\Gamma(\hat{\mathscr{A}}^{p,1})/\hat{\bar{\partial}}\Gamma(\hat{\mathscr{A}}^{p,0}),$$

where $\Gamma(\cdot)=H^0(\cdot)$ is the space of sections of a sheaf. Then from the long sequences based on (2.7) we have

$$0\to H^0(\hat{\omega}^p)\to H^0(\hat{\mathscr{A}}^{p,0})\xrightarrow{\hat{\bar{\partial}}}H^0(\hat{\mathscr{A}}^{p,1})\to H^1(\hat{\omega}^p)\to H^1(\hat{\mathscr{A}}^{p,0})=0,$$

and

$$0=H^{q-1}(\hat{\mathscr{A}}^{p,1})\to H^q(\hat{\omega}^p)\to H^q(\hat{\mathscr{A}}^{p,0})=0 \quad \text{for} \quad q>1.$$

This shows that[5]

$$H^1(\hat{\omega}^p)\cong H_D^{p,1}, \qquad H^q(\hat{\omega}^p)=0, \qquad q>1. \tag{2.8}$$

One can also prove [11, 8] a Serre duality theorem:

$$H^1(\hat{\omega}^p)\cong H^0(\hat{\omega}^{1-p})^*. \tag{2.9}$$

3. Bundles

On a smooth supermanifold we can define complex line bundles as follows. Let $\hat{\mathscr{A}}^\times$ denote the smooth, invertible, even functions on $\hat{X}$. (These functions have an expansion in the anticommuting generators whose lowest term is nowhere vanishing.) A line bundle E on $\hat{X}$ is then defined by a collection of transition functions $\{g_{\alpha\beta}\}$ on the overlaps of a covering $\{U_\alpha\}$ of $\hat{X}$. The $g_{\alpha\beta}$ are in $\hat{\mathscr{A}}^\times(U_\alpha\cap U_\beta)$ and satisfy the usual cocycle condition; they are defined up to the usual coboundary. A collection of super functions related across patch overlaps by the $g_{\alpha\beta}$ is called a section of E; the sections constitute a sheaf which we will call $\mathscr{E}$.

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[4] We thank M. Rothstein for a discussion on this point

[5] A similar result also follows in the case where $\hat{\omega}^p$ is replaced by an arbitrary holomorphic line bundle. Both versions were independently given in [8]

Since the transition functions of a line bundle are all even, it makes sense to assign a parity to a section of E. In fact we can divide line bundles into those of rank $1|0$ and those of rank $0|1$, depending on whether a local trivializing section s of E is even or odd, e.g. whether $\theta \cdot s = \pm s \cdot \theta$. This distinction is well-defined, since to change the parity of s requires that we multiply by an odd function, whereupon it vanishes when the nilpotents are set to zero, and hence is no longer trivializing. Given the transition functions of a bundle, its parity can be declared at will. However, we will adhere to the convention that $\hat{\omega}$ is of rank $0|1$, as implied by (2.1).

One can also define bundles of higher rank, but we will not do so here.

As in the classical case [12] one has the exponential sequences

$$0 \to \mathbf{Z} \hookrightarrow \hat{\mathscr{A}}_{\mathrm{ev}} \xrightarrow{e} \hat{\mathscr{A}}^{\times} \to 0,$$

where $e(f) = e^{2\pi i f}$ and $\hat{\mathscr{A}}_{\mathrm{ev}}$ are all the smooth even functions. The corresponding long sequence

$$\ldots \to H^1(\hat{\mathscr{A}}_{\mathrm{ev}}) \to H^1(\hat{\mathscr{A}}^{\times}) \to H^2(\mathbf{Z}) \to H^2(\hat{\mathscr{A}}_{\mathrm{ev}}) \to \ldots$$

then implies $H^1(\hat{\mathscr{A}}^{\times}) \cong H^2(\mathbf{Z})$, since $\hat{\mathscr{A}}$ is fine. Thus complex line bundles are completely classified by an element of $H^2(\mathbf{Z})$, the Chern class $c(E)$.

If we are given a family of complex manifolds, we simply define a family of line bundles as a single bundle over the total space, just as in the classical case. For example consider the family of manifolds $\hat{X} \times \mathbf{C}^{0|1}$ depending trivially on an odd parameter ζ. If $\{g_{\alpha\beta}\}$ is a class in $H^1(\hat{\mathscr{A}}^{\times}(\hat{X}))$, then it defines a bundle on $\hat{X}$ which also depends trivially on ζ. If however $\{\tilde{g}_{\alpha\beta}\}$ is a class in $H^1(\mathscr{A}_{\mathrm{od}}(\hat{X}))$, the *odd* super functions, then the functions $\{\bar{g}_{\alpha\beta} = 1 + \zeta \tilde{g}_{\alpha\beta}\}$ define an interesting family of bundles on $\hat{X}$. For this reason we will keep both parts of the cohomology, bearing in mind that the odd classes are associated to odd directions in the group of line bundles.

Similarly we can define the invertible even *holomorphic* functions $\hat{\mathcal{O}}^{\times}$. We again have an exponential sequence, but now $\hat{\mathcal{O}}$ is not fine. From (2.8) we have that $H^2(\hat{\mathcal{O}}) = 0$, however, and so we get the sequence

$$\ldots \to H^1(\mathbf{Z}) \to H^1(\hat{\mathcal{O}}) \xrightarrow{e} H^1(\hat{\mathcal{O}}^{\times}) \xrightarrow{c} H^2(\mathbf{Z}) \to 0.$$

This says that the Picard group $\mathrm{Pic} \equiv H^1(\hat{\mathcal{O}}^{\times})$ falls into disconnected components labeled by the Chern class $c(E)$. The zero component is

$$\mathrm{Pic}_0 = H^1(\hat{\mathcal{O}})/H^1(\mathbf{Z}). \tag{3.1}$$

Now suppose that the underlying supermanifold $\hat{X}$ is a SRS, or a family of SRS. As before we can define complex bundles on $\hat{X}$. Using the complex structure we can again define holomorphic bundles as well. One might think that we could go further and define a still more restrictive class of bundles using the full superconformal structure of $\hat{X}$, but this is not so, again essentially because the latter really adds no information to $\hat{X}$. $\hat{X}$ has the same structure sheaf $\hat{\mathcal{O}}$ regardless of whether we think of it as a complex manifold or as a SRS, and a line bundle is precisely a (locally free) sheaf of $\hat{\mathcal{O}}$-modules. Put differently, a "superconformal bundle" should have a $\hat{\bar{\partial}}$ operator. The condition for this operator to be well-defined, analogously to $\bar{\partial}$, is that all transition functions be annihilated by $\bar{D}$. But this simply says that the $g_{\alpha\beta}$ are holomorphic. There is thus no special class of bundles associated to a superconformal structure.

Using Serre duality, Eq. (3.1) tells us that the dimension of the Picard group equals that of $H^0(\hat{\omega})$. Consider first a "reduced" family of SRS, that is, a family with only commuting parameters, or none at all. For such a family we can expand a $\frac{1}{2}$-super-differential in components,

$$\omega = v \cdot (\omega_+ + \theta\omega_z),$$

where ω_+ and ω_z are respectively ordinary $\frac{1}{2}$- and 1-differentials on the corresponding Riemann surface X. Thus we have [9]

$$\hat{\omega} \cong \omega \mid \omega^{1/2}. \tag{3.2}$$

The bar means the direct sum, with the left element even and the right element odd[6]: ω is the ordinary canonical line on X. Thus for a reduced family the dimension of Pic is $g \mid q$, where g is the genus and $q = \dim H^0(\omega^{1/2})$. q is generically 0 or 1 on the even (respectively odd) component of $\hat{\mathscr{M}}$, but it can *jump* on sets of codimension one, a striking departure from the classical case. In any case the body of Pic is just the classical Picard group. This is an example of a general result about line bundles over a space with just one odd coordinate [13].

Things get worse when we consider arbitrary families of SRS.[7] Consider a family of tori with one odd parameter ζ and superconformal patching conditions

$$\begin{cases} u \sim u + 1 - \zeta\theta \\ \theta \sim \zeta + \theta \end{cases} \quad \begin{cases} u \sim u + i \\ \theta \sim \theta \end{cases}.$$

The transition functions of $\hat{\omega}$ are all identically equal to one. The holomorphic sections of $\hat{\omega}$ are then spanned over $\mathbf{C}$ by

$$1; \quad \zeta, \zeta\theta.$$

Roughly speaking, the number of sections of $\hat{\omega}$ "jumps" as we leave the locus $\zeta = 0$. More precisely $H^0(\hat{\omega})$ fails to be free over the ring of functions $\wedge(\mathbf{C})$ on the parameter space; if it were free it would certainly be even-dimensional over $\mathbf{C}$. Again the problem arises only when there are spinor zero modes on X.

It is not clear to us how severely the above pathology affects SRS theory. We will proceed, but at times we will restrict to the case of split families in order to avoid it.

An even more restrictive class than the holomorphic bundles are the *flat* ones. A flat bundle is an equivalence class of *constant* transition $g_{\alpha\beta} \in \mathbf{C}$; those are classified by $H^1(\mathbf{C})$, a vector space of $2g$ complex dimensions. An important classical theorem states that every holomorphic bundle in Pic_0 has a flat representative [14]. We will now investigate the corresponding super statement.

As in the classical case we begin with (2.3), whence [using $H^2(\hat{\mathcal{O}}) = 0$]

$$0 \to H^0(\mathbf{C}) \to H^0(\hat{\mathcal{O}}) \xrightarrow{\hat{c}} H^0(\hat{\omega}) \to H^1(\mathbf{C}) \xrightarrow{\phi} H^1(\hat{\mathcal{O}}) \xrightarrow{\hat{c}} H^1(\hat{\omega}) \to H^2(\mathbf{C}) \to 0. \tag{3.3}$$

[6] This is sometimes written $\hat{\omega} \cong \omega^{1/2} \oplus \Pi\omega$

[7] We thank E. Witten for this observation

The Dolbeault theorem (2.8) says that $H^1(\hat{\omega})\cong H_D^{1,1}$. In the classical case this group is just $\mathbf{C}$, the isomorphism being integration of (1, 1)-forms. Thus the arrow labeled ϕ is onto, by exactness, and every bundle in Pic_0 has a flat representative [14]. In the super case, however, $H^2(\mathbf{C})$ is still $\mathbf{C}$ while $H^1(\hat{\omega})\cong H^0(\hat{\mathcal{O}})$ can be $\mathbf{C}^{1|q}$ with $q>0$, by arguments similar to those following (3.2). In this case ϕ cannot be onto. Bundles with vanishing Chern class are not necessarily flat if X has spinor zero modes.

4. Complexes

In this section we describe the relationship between the exterior and $\hat{\partial}$ sequences, Eqs. (2.5) and (2.7). Define

$$\begin{aligned}\mathscr{Z}^{0,1}&=\{\bar{\partial}\text{-closed }(0,1)\text{-forms}\}\subset\hat{\Omega}^{0,1},\\ \mathscr{Z}^{1,0}&=\{\partial\text{-closed }(1,0)\text{-forms}\}\subset\hat{\Omega}^{1,0},\\ \mathscr{Z}^{1,1}&=\{d\text{-closed }(1,1)\text{-forms}\}\subset\hat{\Omega}^{1,1}.\end{aligned}\tag{4.1}$$

Then we have

$$\hat{\mathscr{A}}^{p,q}\cong\mathscr{Z}^{p,q},\tag{4.2}$$

generalizing [2].[8] Equation (4.2) equates $\hat{\mathscr{A}}^{p,q}$ with a sheaf defined without reference to the superconformal structure. This is possible because a superconformal structure, when it exists, is unique [2].

Under the isomorphisms (4.2) the operators $\hat{\partial}$, $\hat{\bar{\partial}}$ correspond to ∂, $\bar{\partial}$. To show this, and to establish (4.2), choose superconformal coordinates u,θ. Dual to the basis $\left\{\frac{\partial}{\partial u}, D\right\}$, where $D=\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial u}$, we have the 1-forms $\{\eta, d\theta\}$, where

$$\eta\equiv du-d\theta\cdot\theta.\tag{4.3}$$

For the correspondence between $\hat{\mathscr{A}}^{1,0}$ and $\hat{\Omega}^{1,0}$ we then have [see (2.1)]

$$v\cdot\lambda_+\leftrightarrow d\theta\cdot\lambda_+ +\eta\cdot(D\lambda_+).\tag{4.4}$$

In different coordinates u',θ' we have

$$(\eta', d\theta)=(\eta, d\theta)\begin{pmatrix}\xi^2 & D\xi\\ 0 & \xi\end{pmatrix};\quad \begin{pmatrix}\partial_{u'}\\ D'\end{pmatrix}=\begin{pmatrix}\xi^{-2} & -\xi^{-3}D\xi\\ 0 & \xi^{-1}\end{pmatrix}\begin{pmatrix}\partial_u\\ D\end{pmatrix},$$

where $\xi\equiv D\theta'$, and hence $v'=v\cdot\xi$. From this one readily shows that (4.4) is intrinsically defined. It is also easy to see that the right side of (4.4) is the most general ∂-closed (1,0)-form, using the identity

$$\partial=d\theta\otimes D+\eta\otimes\partial_u.\tag{4.5}$$

This identity also makes it clear why under (4.4) the operators $\hat{\partial}$ and ∂ correspond: for any smooth f we have

$$\hat{\partial}f\leftrightarrow\partial f,$$

[8] These isomorphisms were given independently in [8]. They are isomorphisms of complex vector spaces – not of $\hat{\mathcal{O}}$-modules. Indeed the right side of (4.2) isn't an $\hat{\mathcal{O}}$-module at all

and ∂f is certainly in $\mathscr{L}^{1,0}$. Finally, complex conjugating the entire discussion shows that $\mathscr{L}^{0,1} \cong \hat{\mathscr{A}}^{0,1}$.

We now turn to $\hat{\mathscr{A}}^{1,1}$. Here we let

$$|v|^2 \cdot \varphi_{+-} \leftrightarrow d\bar{\theta} \wedge d\theta \cdot \varphi_{+-} + d\bar{\theta} \wedge \eta \cdot D\varphi_{+-} - \bar{\eta} \wedge d\theta \cdot \bar{D}\varphi_{+-} + \bar{\eta}\eta \cdot \bar{D}D\varphi_{+-}. \tag{4.6}$$

Again one verifies that the right side is the most general closed (1, 1)-form, and that the correspondence is natural.

To summarize, on a SRS we have the equivalent complexes

$$\begin{array}{ccc} \hat{\mathscr{A}} & \xrightarrow{\hat{\partial}} & \hat{\mathscr{A}}^{1,0} \\ \downarrow \hat{\bar{\partial}} & & \downarrow \hat{\bar{\partial}} \\ \hat{\mathscr{A}}^{0,1} & \xrightarrow{\hat{\partial}} & \hat{\mathscr{A}}^{1,1} \end{array} \qquad \begin{array}{ccc} \hat{\mathscr{A}} & \xrightarrow{\partial} & \mathscr{L}^{1,0} \\ \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\ \mathscr{L}^{0,1} & \xrightarrow{\partial} & \mathscr{L}^{1,1} \end{array}.$$

The differential forms which correspond to super differentials have only their top component independent: (4.4) and (4.6) give the lower components in terms of the top one.

The differentials in $\hat{\mathscr{A}}^{1,1}$ are by definition volume forms on $\hat{X}$, so one can integrate them if $\hat{X}$ is compact. If we expand φ_{+-} in powers of θ as $\varphi_{+-} = \ldots + \bar{\theta}\theta\varphi_{u\bar{u}}$, then the integral gives

$$\int_{\hat{X}} \varphi = \int_{X} du d\bar{u} \varphi_{u\bar{u}}.$$

In particular the integral is zero if φ is of the form $\hat{\partial}\bar{\lambda}$ or $\hat{\bar{\partial}}\lambda$, a total derivative.

The relationship between $\bar{\partial}$ and $\hat{\bar{\partial}}$ can also be generalized to the case where these operators are coupled to a holomorphic line bundle. Given a holomorphic line bundle E we can generalize $\bar{\partial}$ from $\hat{\Omega}^{p,q}$ to $\hat{\Omega}^{p,q}_E$, the smooth E-valued (p, q)-forms. Simply choose a local holomorphic trivializing section s of E and let $\bar{\partial}(\lambda \cdot s) \equiv (\bar{\partial}\lambda) \cdot s$, where λ is a (p, q)-form. This definition clearly factors through a change in the section s, so we always have

$$\Omega^{p,q}_E \xrightarrow{\bar{\partial}} \Omega^{p,q+1}_E.$$

We can then discuss E-valued differentials, letting $\mathscr{E}^{p,q} = \hat{\mathscr{A}}^{p,q} \otimes \mathscr{E}$. As before we have

$$\mathscr{E}^{0,1} \cong \mathscr{L}^{0,1}_E,$$

and the evident correspondence between $\hat{\bar{\partial}}$ and $\bar{\partial}$ acting on sections of E. Before we set up the rest of (4.1), however, we must first introduce a connection.

5. Connections

On a complex line bundle we define a connection in the usual way, as a linear map

$$\nabla : \mathscr{E} \to \hat{\Omega}^{1,0}_E; \qquad \bar{\nabla} : \mathscr{E} \to \hat{\Omega}^{0,1}_E,$$

satisfying

$$\nabla(f\psi) = \partial f \cdot \psi + f \cdot \nabla\psi. \tag{5.1}$$

Given a local trivializing section s for E, define even 1-forms a,

$$\nabla s = a \cdot s ; \qquad \bar{\nabla} s = \bar{a} \cdot s .$$

Under a change of s the connection forms transform in the usual way. For an arbitrary section $\psi = \psi_{\cdot} \cdot s$ we have

$$\nabla(\psi_{\cdot} \cdot s) = (\partial + a)\psi_{\cdot} \cdot s .$$

The dot reminds that $\psi_{\cdot}$ is not "gauge"-invariant.

If E is a holomorphic bundle it makes sense to require that $\bar{\nabla} = \bar{\partial}$, or $\bar{a} \equiv 0$ with respect to any holomorphic trivialization. In this case we say that $(\nabla, \hat{\bar{\partial}})$ is a holomorphic connection.

Given a connection we can at once extend ∇, $\bar{\nabla}$ to act on the E-valued forms. This works because the exterior derivative satisfies (see e.g. [2])

$$d(\lambda \cdot f) = (d\lambda) \cdot f + \lambda \wedge df ,$$

while the connection satisfies (5.1). Thus we let

$$\nabla(\lambda_{\cdot} \cdot s) = (\partial \lambda_{\cdot} + \lambda_{\cdot} \wedge a) \cdot s . \tag{5.2}$$

This prescription usually does not lead to a complex, however: $(\nabla + \bar{\nabla})^2 \neq 0$ in general. Instead we have that

$$(\nabla + \bar{\nabla})^2 \psi = \mathscr{F} \cdot \psi ,$$

where $\mathscr{F}$ is a 2-form called the curvature:

$$\mathscr{F} = \partial a + (\partial \bar{a} + \bar{\partial} a) + \bar{\partial} \bar{a} .$$

The curvature is always a closed 2-form: $d\mathscr{F} = 0$, the Bianchi identity.

When the underlying manifold $\hat{X}$ is a SRS we can generalize $\hat{\partial}$ and $\hat{\bar{\partial}}$ to $\hat{\nabla}$ and $\hat{\bar{\nabla}}$ in a similar way. A "superconformal connection" is a derivation

$$\hat{\nabla} : \mathscr{E} \to \mathscr{E}^{1,0} ,$$

and similarly $\hat{\bar{\nabla}}$.

We can now generalize a part of (4.2), the relation between $\mathscr{E}^{1,0}$ and $\hat{\Omega}_E^{1,0}$. We let $\mathscr{Z}_E^{1,0}$ be the ∇-closed E-valued (1,0)-forms. Then $\mathscr{Z}_E^{1,0}$ corresponds to $\mathscr{E}^{1,0}$ via

$$v \cdot \lambda_{+ \cdot} \cdot s \leftrightarrow [d\theta \lambda_{+ \cdot} + \eta(D\lambda_{+ \cdot} + A_+ \lambda_{+ \cdot})] \cdot s , \tag{5.3}$$

where A_+ is the odd function defined by

$$\hat{\nabla} s = A \cdot s = (v \cdot A_+) \cdot s ,$$

so that

$$\hat{\nabla}(\psi_{\cdot} \cdot s) = (\hat{\partial} + A)\psi_{\cdot} \cdot s .$$

The correspondence (5.3) is easily seen to be gauge- and superconformally invariant.

Moreover given a superconformal connection we obtain a connection by letting $\nabla \psi$ be the form corresponding to $\hat{\nabla} \psi$, or in other words by letting a correspond to A under (4.4). Thus a superconformal connection is just a special case of a connection, one whose connection form a is ∂-closed in any trivialization (and similarly $\bar{a}$).

We can again define curvature by

$$(\hat{\nabla}+\hat{\bar{\nabla}})^2\psi = F\cdot\psi\,; \qquad F=\hat{\partial}\bar{A}+\hat{\bar{\partial}}A\,.$$

Under (4.6) F corresponds to the curvature $\mathscr{F}$ of the associated connection on forms. Thus superconformal connections are distinguished by the fact that their curvature forms $\mathscr{F}$ are not merely closed, but closed (1, 1)-forms:

$$\mathscr{F}_{\theta\theta}=\mathscr{F}_{\theta u}=\mathscr{F}_{\bar{\theta}\bar{\theta}}=\mathscr{F}_{\bar{\theta}\bar{u}}=0 \quad \text{for superconformal connections}. \tag{5.4}$$

Conversely, if (5.4) holds for a connection then by definition $a\in\mathscr{Z}^{1,0}$, $\bar{a}\in\mathscr{Z}^{0,1}$ and the connection is superconformal. Thus the superconformal connections are precisely those which obey the curvature "constraints" (5.4). These constraints are all "conventional" in the sense of [15]: given an arbitrary connection we can force it to be superconformal by discarding part of it (the coefficient a_z of η) and replacing it (take $a_z = Da_+$). The constraints express the compatibility of a connection with the superconformal structure, much as the torsion constraints express the consistency of the superconformal structure itself.

If the curvature vanishes we call the connection *flat*. It is easy to see that every flat bundle admits a flat connection, namely $A=\bar{A}\equiv 0$. Again not every bundle in Pic_0 has this property.

Since $\hat{\partial}$ and $\hat{\bar{\partial}}$ anticommute, F is gauge-invariant. Moreover, two connections ∇, ∇' differ by a *global* differential δA, so that under a change of connection F changes by a total derivative. Thus the net curvature $\int F$ depends only upon the bundle E itself. We will now relate the total curvature to the Chern class.

To calculate the Chern number of a bundle[9] $[g]\in H^1(\hat{\mathcal{O}}^\times)$, recall that each class in $H^1(\hat{\mathcal{O}}^\times)$ determines a unique element $c[g]$ of $H^2(\mathbf{Z})\cong\mathbf{Z}$. The Chern number of $[g]$ is its image in $\mathbf{Z}$; we will show how to compute the Chern number via the curvature.

The integral class $c[g]$ determines an element of $H^2(\mathbf{C})$. From (3.3)

$$H^2(\mathbf{C})\cong H^1(\hat{\omega})/\hat{\partial}H^1(\hat{\mathcal{O}})\,, \tag{5.5}$$

so one can represent this element by a class $[B]\in H^1(\hat{\omega})$. More explicitly, given the representative $\{g_{\alpha\beta}\}$ of $[g]$, let

$$B_{\alpha\beta}=\hat{\partial}\log g_{\alpha\beta}\,. \tag{5.6}$$

We now recall that from the Dolbeault theorem (2.8)

$$H^1(\hat{\omega})\cong\Gamma(\hat{\mathscr{A}}^{1,1})/\bar{\partial}\Gamma(\hat{\mathscr{A}}^{1,0})\,. \tag{5.7}$$

To make this explicit, we take $B=\delta A$, where δ is the Cech coboundary operator and $\{A_\alpha\}$ is a collection of smooth (1, 0)-differentials on the patches of X. By (5.6) this means that $\{A_\alpha\}$ defines a holomorphic connection for the bundle given by $g_{\alpha\beta}$. Thus the choice of $\{A_\alpha\}$ is ambiguous by the addition a global (1, 0)-differential C, $A_\alpha\to A_\alpha+C$. As usual we define $F\in H^0(\hat{\mathscr{A}}^{1,1})$ by $F_\alpha=\hat{\bar{\partial}}A_\alpha$. Clearly F is global: $\delta F=0$.

[9] Here $[g]$ denotes the cohomology class of $\{g_{\alpha\beta}\}$

From (5.5) and (5.7) we thus find that a bundle $[g]\in H^1(\hat{\mathcal{O}}^\times)$ determines an element of $\Gamma(\hat{\mathscr{A}}^{1,1})$, namely the curvature form F, up to

$$F\to F+\hat{\bar{\partial}}C\,. \tag{5.8}$$

The Chern number of the bundle is then given by

$$c[g]=\frac{i}{2\pi}\int_{\hat{X}}F\,. \tag{5.9}$$

We have already seen that this expression is unaffected by a change in the choice of connection on E, Eq. (5.8). Moreover it is clearly additive under tensor products and zero whenever $[g]\in \mathrm{Pic}_0$. The last point follows since in that case F takes the form $\hat{\partial}\tilde{C}$ for a global (0, 1)-differential $\tilde{C}$, again a total divergence. Finally, we should check that $c[g]$ is an integer. This is done in the appendix.

6. Families of Bundles

One often wants to study the behavior of the determinants of a family of operators, for example $\hat{\bar{\partial}}^\dagger\hat{\bar{\partial}}$ coupled to a family of line bundles. Since the Picard group is disconnected, we can study the variation of such a determinant by coupling it to $E_0\otimes E$, where E_0 is fixed and E lies in Pic_0. Rather than parametrizing E by a family of transition functions, in the classical case it is equivalent and computationally simpler to represent it by a family of connections on the *trivial* bundle [16, 4]. We can realize every bundle once by choosing the connections to be antiabelian differentials. In this section we will treat the corresponding super case, extending the discussion of [7, 8].

Since E is in Pic_0 we can take the logarithms of its transition functions, representing them by the cocycle $\{\lambda_{\alpha\beta}\}\in Z^1(\hat{\mathcal{O}})$ [see (3.1)]. Regarding $\{\lambda_{\alpha\beta}\}$ as a cocycle in $\hat{\mathscr{A}}$, it is trivial since $\hat{\mathscr{A}}$ is fine. Thus we can write

$$\lambda_{\alpha\beta}=\tau_\alpha-\tau_\beta\,. \tag{6.1}$$

where $\{\tau_\alpha\}$ define a 0-cochain: $\{\tau_\alpha\}\in C^0(\hat{\mathscr{A}})$. The $\{\tau_\alpha\}$ define a trivialization of E which is not holomorphic: given a section $\{\psi_\alpha\}$ of E, the corresponding global function is $f=e^{-2\pi i\tau_\alpha}\psi_\alpha$. Under this correspondence the operator $\hat{\bar{\partial}}$ on $\mathscr{E}$ is unitarily equivalent to $\hat{\bar{\partial}}+\bar{A}$ on $\hat{\mathscr{A}}$, where

$$\bar{A}=2\pi i\hat{\bar{\partial}}\tau_\alpha \quad \text{on} \quad U_\alpha \tag{6.2}$$

defines a global (0, 1)-differential.

The representation (6.2) of a bundle by the (0, 1) part of a connection is redundant; after all, $\hat{\mathscr{A}}^{0,1}(\hat{X})$ is a function space, while Pic_0 is finite-dimensional. In fact replacing τ_α by $\tau_\alpha+\tau'$ for a global smooth function τ does not affect (6.1), but it changes $\bar{A}$ by $\hat{\bar{\partial}}\tau'$. Thus all that matters is the Dolbeault class of $\bar{A}$; by the Dolbeault theorem (2.8) we then have a faithful representation of the tangent to Pic_0 by vector potentials. Serre duality (2.9) then says that $T(\mathrm{Pic})\cong H_D^{0,1}\cong H^0(\hat{\omega})^*$.

Unlike the classical case, however, we cannot find a unique representative for each class of $H_D^{0,1}$ as a global section of $\hat{\bar{\omega}}$, an antiholomorphic differential.

Consider for example a reduced family of SRS, with odd spin structure. We have

$$\bar{A}_- = \alpha_- + \theta\alpha_{+-} + \bar{\theta}\alpha_{\bar{u}} + \bar{\theta}\theta\alpha_{\bar{u}+},$$

$$\bar{D}\tau' = \beta_- + \theta\beta_{+-} + \bar{\theta}\partial_{\bar{u}}\beta + \bar{\theta}\theta\partial_{\bar{u}}\beta_+ .$$

We see that there are Dolbeault groups obstructing the removal of the terms $\bar{\theta}\alpha_{\bar{u}}$ and $\bar{\theta}\theta\alpha_{\bar{u}+}$. The latter is not antiholomorphic, since it depends on θ, so we do not get a nice slice for $H_D^{0,1}$. This problem is just another manifestation of the fact that bundles in Pic_0 aren't necessarily flat.

Nevertheless we can use the representation of bundles near the identity of Pic given by (6.2) to address holomorphic factorization [7]. While we have no nice slice for $\Gamma(\hat{\mathscr{A}}^{0,1})/\hat{\bar{\partial}}\Gamma(\hat{\mathscr{A}})$, still this is a complex vector space modulo a complex subspace, so we do get the complex structure on Pic in this way. This means that a function on Pic is holomorphic precisely when its variation with respect to any (0, 1) variation A is zero.

Finally, in the classical case the representation (6.2) provides a natural hermitian norm on Pic. Given a holomorphic tangent we represent it uniquely by $\bar{A}$ and let its norm be

$$\|\bar{A}\|^2 = \int_X A \wedge \bar{A} . \tag{6.3}$$

This norm plays an important role in the theory of theta functions and in Quillen's theorem. Unfortunately, in the super case things are again not so nice. The definition analogous to (6.3) is

$$(A, \bar{A}') = \int_{\hat{X}} A\bar{A}' , \tag{6.4}$$

where $A \in H^0(\hat{\omega})$ and $\bar{A}' \in H_D^{0,1}$. This formula exhibits the isomorphism $H^0(\hat{\omega})^* \cong H_D^{0,1}$. We have seen, however, that when there are spinor zero modes $H_D^{0,1}$ is not naturally represented by $H^0(\hat{\bar{\omega}})$. Consequently (6.4) does not give a norm on Pic. Thus in this case it's not clear what a super analog of the Quillen theorem should say.

7. Conclusion

We have described a number of features of line bundles on super Riemann surfaces. There are some close parallels between bundles on SRS and their classical counterparts, especially for even spin structures. These include basic results of classification: the Picard group Pic falls into components labelled by the Chern number. Each connected component is isomorphic to Pic_0, which in the even case has dimension $g|0$ except at the theta divisor in spin moduli space.

Various pathologies appear, however, when the underlying Riemann surface X has spinor zero modes, and in particular in the case of odd spin structures. Bundles in Pic_0 aren't necessarily flat, and cannot necessarily be represented by flat connections. In addition the dimension of Pic can jump, both as the even and the odd moduli are varied. All of these annoyances point out the fact that SRS theory is *not* an automatic generalization of the classical theory, and they cast some doubt on the utility of arbitrary bundles on SRS in applications to string theory. It may be that some of the basic constructions need to be modified to ameliorate the problems mentioned above.

Appendix

Given a family of complex bundles and a family of SRS we want to show that $c[g]$ defined in (5.9) is an integer. To do this, we first show how to construct (non-canonically) from $[g]$ a bundle $[g^0]$ on the body of $\hat{X}$. We will then show that $c[g]$ is the ordinary Chern number of the associated $[g^0]$.

To define $g^0_{\alpha\beta}$, first recall that all smooth supermanifolds are split. For the family of SRS in question, let v, $\bar{v}$, ψ, $\bar{\psi}$ be relative coordinates respecting the splitting (in particular v, $\bar{v}$ can be taken to be complex coordinates for X). We can then expand out each $g_{\alpha\beta}$ in terms of all the nilpotents: $g_{\alpha\beta}=g^0_{\alpha\beta}+g^{\mathcal{N}}_{\alpha\beta}$, where $g^0_{\alpha\beta}$ is a function only of even coordinates and even parameters. Thus $g^0_{\alpha\beta}$ defines a bundle $[g^0]$ on X. Let a^0_α be an admissible connection for $[g^0]$, that is $a^0_\alpha-a^0_\beta=\partial\log g_{\alpha\beta}$. The projection from $\hat{X}$ to X then enables us to pull this connection back to $\hat{X}$.

While we cannot consistently define $\log g^0_{\alpha\beta}$ if the Chern class is non-zero, we can define

$$\lambda_{\alpha\beta}=\log\left(1+\frac{g^{\mathcal{N}}_{\alpha\beta}}{g^0_{\alpha\beta}}\right)$$

by means of its power series expansion. Now define $[\tau]\in C^0(\hat{\mathscr{A}})$ by $\delta\tau=\lambda$. Then the connection $a_\alpha=a^0_\alpha+\partial\tau_\alpha$ is an admissible connection for $[g]$. Its curvature is $F=\bar{\partial}a_\alpha+\partial\bar{a}_\alpha$; clearly the only nonvanishing component of F is $F_{v\bar{v}}$.

In superconformal coordinates u, θ we have

$$c[g]=\frac{1}{2\pi}\int d^2u d^2\theta F_{+-};$$

if we write this in arbitrary coordinates, for example v, ψ, it becomes

$$c[g]=\frac{1}{2\pi}\int d^2v d^2\psi \det(E^A_M)E^M_+E^N_-F_{NM}\,. \tag{A.1}$$

where E_A is any frame in the given superconformal class on $\hat{X}$. In principle we could just work out the Jacobian for the change of coordinates $(u,\theta)\to(v,\psi)$, but this is somewhat complicated in detail. Instead, and without loss of generality, assume that there are no even parameters: the parameter space $\hat{Y}=\mathbf{C}^{0|q}$. Following [17] we can begin with a Riemann surface X and construct a holomorphic odd family of SRS by choosing $2g-2$ "gravitino" fields, $(-\frac{1}{2},1)$-differentials χ^i on X. Let v be a holomorphic coordinate for X and $\psi=\sqrt{dv}$ a holomorphic coordinate on the split surface $\hat{X}=(X,\wedge\omega^{1/2})$. Then $\hat{X}\times\hat{Y}$ becomes a family of SRS with frame

$$E_+=\psi\partial_v+\tfrac{1}{2}\bar{\chi}|\psi|^2\partial_{\bar{v}}+(1-\tfrac{1}{4}|\psi|^2|\chi|^2)\partial_\psi+\tfrac{1}{2}\bar{\chi}\psi\partial_{\bar{\psi}}\,, \tag{A.2}$$

where $\chi=\chi^i\zeta^i$ and ζ^i are odd coordinates for $\hat{Y}$. The coordinates v, ψ are holomorphic only when $\zeta^i=0$.

We now put the frame (A.2) in (A.1) and use the fact $\det E^M_A=1+\mathcal{O}(|\psi|^2)$ from [17]; this immediately gives

$$c[g]=\frac{i}{2\pi}\int_X \bar{\partial}a^0+\partial\bar{a}^0=c[g^0]\,,$$

the classical expression for the Chern number of $[g^0]$. In particular, we can easily see from this result that $c[\hat{\omega}]=g-1$.

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Note added in proof. The problem of non locally free sheaves discussed in Sect. 3 has recently been developed in detail by Hodgkin [18].