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Finite Dimensional Representations of the Quantum Analog of the Enveloping Algebra of a Complex Simple Lie Algebra

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Abstract. Let $\mathscr{G}$ be a complex simple Lie algebra. We show that when t is not a root of 1 all finite dimensional representations of the quantum analog $U_t\mathscr{G}$ are completely reducible, and we classify the irreducible ones in terms of highest weights. In particular, they can be seen as deformations of the representations of the (classical) $U\mathscr{G}$.

I. Introduction

To each complex simple Lie algebra $\mathscr{G}$, Jimbo associates the quantum analog of its enveloping algebra, let $U_t\mathscr{G}$, where t is a non-zero parameter, as follows (see also Drinfeld [2, 3]):

Let $(a_{ij})_{1\leqq i,j\leqq N}$ be the Cartan matrix of $\mathscr{G}$ and $(\alpha_i)_{1\leqq i\leqq N}$ a basis of simple roots; $U_t\mathscr{G}$ is the C-algebra generated by $(k_i^{\pm 1}, e_i, f_i)_{1\leqq i\leqq N}$ with relations:

$$k_i \cdot k_i^{-1} = k_i^{-1} \cdot k_i = 1; \quad k_i k_j = k_j k_i,$$
$$k_i e_j k_i^{-1} = t_i^{a_{ij}} e_j; \quad k_i f_j k_i^{-1} = t_i^{-a_{ij}} f_j,$$
$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{t_i^2 - t_i^{-2}},$$
$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{t_i^2} e_i^{1-a_{ij}-\nu} e_j e_i^\nu = 0 \quad \text{for } i \neq j,$$
$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{t_i^2} f_i^{1-a_{ij}-\nu} f_j f_i^\nu = 0 \quad \text{for } i \neq j,$$

where $t_i = t^{(\alpha_i|\alpha_i)/2}$, $(\ |\ )$ being the invariant inner product on $\oplus \mathbf{C}\alpha_i$, with $(\alpha_i|\alpha_i) \in \mathbf{Z}$.

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \dfrac{(t^m - t^{-m})(t^{m-1} - t^{-(m-1)}) \cdots (t^{m-n+1} - t^{-(m-n+1)})}{(t - t^{-1})(t^2 - t^{-2}) \cdots (t^n - t^{-n})} & \text{for } m > n > 0, \\ 1 \quad \text{for } n = 0 \text{ or } m = n. \end{cases}$$

So $t_i^{a_{ij}} = t_j^{a_{ji}} = t^{(\alpha_i|\alpha_j)}$. There is a coproduct: $\Delta\colon U_t\mathcal{G} \to U_t\mathcal{G} \otimes U_t\mathcal{G}$ defined by:

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$

$$\Delta(e_i) = e_i \otimes k_i^{-1} + k_i \otimes e_i; \quad \Delta(f_i) = f_i \otimes k_i^{-1} + k_i \otimes f_i,$$

and $U_t\mathcal{G}$ is a Hopf algebra with antipode S and augmentation ε respectively defined by:

$$S(k_i) = k_i^{-1}, \quad S(e_i) = -t_i^{-2} e_i, \quad S(f_i) = -t_i^2 f_i,$$
$$1 = \varepsilon(k_i) = \varepsilon(k_i^{-1}); \quad \varepsilon(e_i) = \varepsilon(f_i) = 0.$$

From now on, we shall assume that t is not a root of 1 and we shall study the finite dimensional representations of $U_t\mathcal{G}$.

In [4], Jimbo has shown that, for $\mathcal{G} = \mathrm{sl}(N+1)$, any irreducible finite dimensional representation can be deformed in an irreducible representation of $U_t\mathcal{G}$. We shall show, using analogs of highest weight modules, that all finite dimensional representations are essentially obtained in this way (after possibly tensoring by a 1-dimensional representation) and that all finite dimensional representations are completely reducible.

The paper is organised as follows: in sect. II, we give some lemmas on the general structure of $U_t\mathcal{G}$, in particular showing a triangular decomposition: $U_t\mathcal{G} = U_t n_- \otimes \mathbf{C}[T] \otimes U_t n_+$ as vector spaces (see notations below). In sect. III, we give general remarks on finite dimensional representations of $U_t\mathcal{G}$, which lead us to highest weights. In Sect. IV we treat the case of $U_t\,\mathrm{sl}(2)$, which is used in sect. V to get the result for any $U_t\mathcal{G}$.

Notations

· T is the subgroup of the group of invertible elements of $U_t\mathcal{G}$, generated by the k_i's, and $\mathbf{C}[T]$ is its group algebra.

· $U_t n_+$ (respectively $U_t n_-$) is the subalgebra of $U_t\mathcal{G}$ generated by the e_i's (respectively by the f_i's).

· $U_t b_+$ (respectively $U_t b_-$) is the subalgebra of $U_t\mathcal{G}$ generated by the e_i's and $k_i^{\pm 1}$'s). (respectively by e_i's and $k_i^{\pm 1}$'s).

· $A = \bigoplus_{i=1}^{N} \mathbf{Z}\alpha_i$ is the root lattice, and $Q_+ = \bigoplus_{i=1}^{N} \mathbf{N}\alpha_i$.

II. About the Structure of $U_t\mathcal{G}$

1. *Q-Gradation*

Proposition 1. *The action of k_i's by conjugation gives a Q-gradation on $U_t\mathcal{G}$, $U_t b_\pm$, $U_t n_\pm$ as follows: a monomial ξ in the generators e_i, f_i, k_i, is said to be the degree $\alpha = \sum_1^N n_i\alpha_i$, $n_i \in \mathbf{Z}$ iff:*

$$\forall i = 1, \ldots, N \quad k_i \xi k_i^{-1} = t_i^{(\alpha_i|\alpha)} \xi.$$

Proof. Let us note first that the $t_i^{(\alpha_i|\alpha)}$, $1 \leqq i \leqq N$, completely determine α: as

$(\alpha_i|\alpha)\in\mathbf{Z}$ and t is not a root of 1, the $t_i^{(\alpha_i|\alpha)}$ determine the integers $(\alpha_i|\alpha)$ which in turn determine α as $(\ |\ )$ is non-degenerate.

As each polynomial ξ where e_i appears n_i times and f_i m_i times is clearly of degree $\alpha=\sum_1^N (n_i-m_i)\alpha_i$, we see that $U_t\mathcal{G}$, $U_tb_\pm$, $U_tn_\pm$ are sums of their subspaces of degree.

Remark. $U_t\mathcal{G}\otimes U_t\mathcal{G}$ is then $Q\times Q$-graded, and also Q-graded via the total gradation.

Δ: $U_t\mathcal{G}\to U_t\mathcal{G}\otimes U_t\mathcal{G}$ is a morphism of Q-graded algebras.

Lemma 1. *$\forall(m_1,\dots,m_n)\in\mathbf{N}^N$, $e_1^{m_1}\cdots e_N^{m_N}$ is non-zero in $U_t\mathcal{G}$.*

Proof.

a) There is always the fundamental representation of $U_t\mathcal{G}$ (given by the same formulas as the fundamental representation of $\mathcal{G}$, see Jimbo [6]) in which the e_i's are non-zero. (One can also mimic the proof in Humphreys [4] p. 97–99).

b) $\forall i\in\{1,\dots,N\}$, $\forall m\in\mathbf{N}$ $e_i^m\neq 0$.

As Δ, and also the $\Delta^{(m)}=(\Delta\otimes\mathrm{Id}^{\otimes(m-1)}\circ(\Delta\otimes\mathrm{Id}^{\otimes(m-2)}\circ\dots\circ\Delta$, are injective, it is enough to show that $\Delta^{(m)}(e_i^m)\neq 0$. Using the Q^m-gradation of $(U_t\mathcal{G})^{\otimes m}$, it is enough to check that the component of degree $(\alpha_i,\dots,\alpha_i)$ is non-zero.

Now, $\Delta^{(m)}(e_i)=u_1+\dots+u_m$, where $u_r=k_i\otimes\dots\otimes k_i\otimes e_i\otimes k_i^{-1}\otimes\dots\otimes k_i^{-1}$ (e_i at the r-th position) and $u_su_r=t_i^4u_ru_s$ for $r<s$. So, one computes $\Delta^{(m)}(e_i^{(m)})=[\Delta^{(m)}(e_i)]^m$ by the t_i^4-multinomial formula:

$$[\Delta^{(m)}(e_i)]^m=\sum_{n_1+\dots+n_m=m}\frac{\phi_m(t_i^4)}{\phi_{n_1}(t_i^4)\cdots\phi_{n_m}(t_i^4)}u_1^{n_1}\cdots u_m^{n_m},$$

and one gets the term of degree $(\alpha_i,\dots,\alpha_i)$ for $n_1=\cdot=n_m=1$. So, it is

$$\frac{\phi_m(t_i^4)}{[\phi(t_i^4)]^m}u_1\cdots u_m=\frac{(t_i^2-t_i^{-2})(t_i^4-t_i^{-4})\cdots(t_i^{2m}-t_i^{-2m})}{(t_i^2-t_i^{-2})^m}$$
$$e_ik_i^{m-1}\otimes e_ik_i^{m-3}\dots\otimes e_ik_i^{-(m-1)}.$$

Now, as k_i is invertible, we see that, $e_ik_i^{m-1}\otimes\dots\otimes e_ik_i^{-(m-1)}$ is non-zero.

c) Let $(m_1,\dots,m_N)\in\mathbf{N}$. In order to see that $e_1^{m_1}\cdots e_N^{m_n}\neq 0$, it is enough to consider the component of degree $(m_1\alpha_1,\dots,m_N\alpha_N)$ of $\Delta^{(N)}(e_1^{m_1}\cdots e_N^{m_N})$. But it is:

$$e_1^{m_1}k_2^{m_2}\cdots k_N^{m_N}\otimes k_1^{-m_1}e_2^{m_2}\cdots k_N^{m_N}\otimes\dots\otimes k_1^{-m_1}\cdots k_{N-1}^{-m_{N-1}}e_N^{m_N}$$

which is non-zero according to b).

3. A Basis for $\mathbf{C}[T]$

For $\alpha=\sum_1^N n_i\alpha_i\in Q$, let $k_\alpha=k_1^{n_1}\cdots k_N^{n_N}$.

Lemma 2. *The k_α's, $\alpha\in Q$, are linearly independent.*

Proof. Suppose $\sum_{\text{finite}}\lambda_\alpha k_\alpha=0$, $\lambda_\alpha\in\mathbf{C}^*$. As one can always multiply by a k_β (with a suitable β), one can assume that the α's in the finite sum belong to Q_+.

Then: $(\mathrm{Id}\otimes S)\circ\Delta(\sum\lambda_\alpha k_\alpha)=\sum\lambda_\alpha k_\alpha\otimes k_\alpha^{-1}=0$ in $U_t\mathscr{G}\otimes U_t\mathscr{G}$. Let L (respectively R) be the left (respectively right) regular representation of $U_t\mathscr{G}$.

So: $\sum\lambda_\alpha L(k_\alpha)\circ R(k_\alpha^{-1})=0$ in $\mathrm{End}(U_t\mathscr{G})$.

Evaluating on $e_1^{m_1}\cdots e_N^{m_N}$, $(m_1,\dots,m_N)\in\mathbf{N}^N$, one gets:

$$\sum\lambda_\alpha t^{(\alpha|\sum m_i\alpha_i)}=0\quad\forall(m_1,\dots,m_N)\in\mathbf{N}^N.$$

As we can also evaluate on $e_1^{km_1}\cdots e_N^{km_N}$ for each $k\in\mathbf{N}$, we see that t and all its power t^k are roots of a certain Laurent polynomial. As t is not a root of 1, its powers are 2 by 2 distincts so the Laurent polynomial must be 0. So

$$\sum_{\alpha/(\alpha|\sum m_i\alpha_i)=\text{fixedvalue}}\lambda_\alpha=0.$$

Using this remark, we shall give a proof by induction on the number p of terms in the sum (recall we have assumed $\lambda_\alpha\in\mathbf{C}^*$)

- the case $p=1$ is clear
- let us suppose the result true for p terms ($p\geqq 1$), and suppose there are $p+1$ terms: $\alpha^{(o)},\dots,\alpha^{(p)}$.

  It is enough to show that there exists $(m_1,\dots,m_N)\in\mathbf{N}^N$ such that:

$$(*)\qquad(\alpha^{(o)}|\textstyle\sum m_i\alpha_i)\notin\{(\alpha^{(k)}|\sum m_i\alpha_i)\,k=1,\dots,p\}$$

(because then the argument on Laurent polynomials gives $\lambda_{\alpha(0)}=0$, and we are back to a sum with p terms).

(*) reads: $\exists(m_1,m_N)\in\mathbf{N}^N$ such that: $\forall k=1,\dots,p\left(\alpha^{(o)}-\alpha^{(k)},\sum_1^N m_i\alpha_i\right)\neq 0$. But the $(\alpha^{(o)}-\alpha^{(k)},\cdot)$ are non-zero linear forms on h^*, which determine p hyperplanes in h^*. We have to see that there is a point of Q_+ outside the union of these hyperplanes. The proof is exactly the same as the classical one showing that any vector space on a field of characteristic 0 cannot be the union of a finite number of hyperplanes.

4. Basis for $U_tn_\pm$

As the vector space U_tn_+ is generated by monomial in the e_i's, there is a basis of U_tn_+ whose elements are some of these monomials; one can also assume that the monomials in this basis having a given Q-degree form a basis of the corresponding Q-component of U_tn_+.

Let $(E_r)_{r\in I}$ this basis.

Lemma 3. *$(E_t\cdot k_\alpha)_{r\in I,\alpha\in Q}$ is a basis of U_tb_+. So $U_tb_+\simeq U_tn_+\otimes\mathbf{C}[T]$ as vector spaces.*

Proof. According to the defining relations of $U_t\mathscr{G}$, these elements generate U_tb_+. Let us show they are linearly independent.

Suppose $\sum\lambda_rE_rk_{\alpha_r}=0$, $\lambda_r\in\mathbf{C}^*$. One can assume that all the terms have the same Q-degree β. The term of degree $(\beta,0)$ in $\Delta(\sum\lambda_rE_rk_{\alpha_r})$ must be 0, so:

$$\sum\lambda_rE_rk_r\otimes k_\beta k_{\alpha_r}=0,$$

$$\sum_\alpha\left(\sum_{\{r/\alpha_r=\alpha\}}\lambda_rE_rk_{\alpha_r}\right)\otimes k_\alpha k_\beta=0.$$

As k_α's, for distinct α's, are independent:

$$\sum_{\{r/\alpha_r=\alpha\}} \lambda_r E_r k_\alpha = 0, \quad \text{so } \sum \lambda_r E_r = 0 \text{ and } \forall r\ \lambda_r = 0.$$

Remark. Let θ the algebra automorphism given by $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(k_i) = k_i^{-1}$.

Let $F_r = \theta(E_r)$. Then $(F_r)_{r\in I}$ is a basis of $U_t n_-$ having the same properties as $(E_r)_{r\in I}$.

5. The Triangular Decomposition of $U_t\mathcal{G}$

Proposition 2. *$(E_r \cdot F_{r'} \cdot k_\alpha)_{(r,r',\alpha)\in I\times I\times Q}$ is a basis of $U_t\mathcal{G}$. So $U_t\mathcal{G} \simeq U_t n_- \otimes \mathbf{C}[T] \otimes U_t n_+$ as vector spaces and $U_t\mathcal{G}$ is a free $U_t b_+$-module.*

Proof. It is enough to show the linear independence. Suppose $\sum \lambda_{r,r',\alpha} E_r.\ F_{r'} k_\alpha = 0$, $\lambda_{r,r',\alpha} \in \mathbf{C}^*$. For $r\in I$, let α_r the Q-degree of E_r (and-$\alpha_{r'}$ for $F_{r'}$). Then, the Q-degree of $E_r \cdot F_{r'} \cdot k_\alpha$ is $\alpha_r - \alpha_{r'}$ and we can assume that the couples (r,r') in the sum are such that $\alpha_r - \alpha_{r'} = $ constant.

We shall use an order relation $\leqq$ on Q, defined as follows:

for $\alpha = \sum n_i \alpha_i \in Q$, let $m_i(\alpha) = n_i$, $l(\alpha) = \sum_1^N m_i(\alpha) \in \mathbf{Z}$. For $\alpha \neq \alpha'$, we say that $\alpha < \alpha'$ if:

a) $l(\alpha) < l(\alpha')$ or

b) $l(\alpha) = l(\alpha')$ and the smallest index i such that $m_i(\alpha) \neq m_i(\alpha')$ verifies: $m_i(\alpha) < m_i(\alpha')$. This order is total, and compatible with the addition.

Now, consider $I_0 = \{r\in I /$ the degree α_r of E_r is maximal for $\leqq\}$. Then, in $\Delta(\sum \lambda_{r,r',\alpha} E_r F_{r'} k_\alpha) = 0$, the component of $Q\times Q$-degree (maximal, minimal) must be 0:

$$\sum_{r\in I_0} \lambda_{r,r',\alpha}(E_r k_{\alpha_{r'}} \otimes k_{\alpha_r}^{-1} F_{r'}) k_\alpha \otimes k_\alpha = 0.$$

Here α_r is fixed, so $\alpha_{r'}$ also,

$$\sum_{r\in I_0} \lambda_{r,r',\alpha}(E_r k_\alpha \otimes F_{r'} k_\alpha) = 0,$$

$$\sum_{(r',\alpha)\ 2\text{ by }2\text{ distinct}} \left(\sum_{r\in I_0, (r',\alpha)\text{ fixed}} \lambda_{r,r',\alpha} E_r k_\alpha \right) \otimes F_{r'} k_\alpha = 0.$$

As the $F_{r'} k_\alpha$ are independent, $\forall (r',\alpha)$ fixed $\sum_{r\in I_0} \lambda_{r,r',\alpha} E_r k_\alpha = 0$, so $\lambda_{r,r',\alpha} = 0$.

III. General Remarks on the Finite Dimensional Representations

Let ρ a representation of $U_t\mathcal{G}$ in the finite dimensional vector space V.

Lemma 4. 1. *The operators $\rho(e_i)$, $\rho(f_i)$ $(1 \leqq i \leqq N)$ are nilpotent.*

2. *If ρ is irreducible, the $\rho(k_i)$'s are simultaneously diagonalisable and $V = \oplus V_\mu$, where, for $\mu = (\mu_1, \ldots, \mu_N)$,*

$$V_\mu = \{v\in V / \forall i \quad \rho(k_i)v = \mu_i v.\}$$

Remark. Such a μ defines a character μ: $T \to \mathbb{C}^*$, this allows us to speak about weights of the representation.

Proof. 1. For $1 \leqq i \leqq N$, the relation $\rho(k_i)\rho(e_i)\rho(k_i)^{-1} = t^{(\alpha_i|\alpha_i)}\rho(e)$ shows that if the spectrum of $\rho(e_i)$ contains a non-zero element, it contains an infinity of elements. So, this spectrum is $\{0\}$ and $\rho(e_i)$ is nilpotent. Same proof for $\rho(f_i)$.

2. As the $\rho(k_i)$ commute, they have a common eigenvector v and we have to see that each is diagonalisable. Let $E = \{W$ subspace of V, $\dim W \geqslant 1/\forall i$, $\rho(k_i)|_W$ diagonalisable$\}$ $E \neq \emptyset$ as $\mathbb{C}\cdot v \in E$. Let $W \in E$ of maximal dimension and suppose $\dim W < \dim V$:

a) if W is invariant under $\rho(e_i)$ and $\rho(f_i)$, we must have $W = V$ due to the irreducibility of V.

b) assume there exists $w \in W$ and $j \in \{1, \ldots, N\}$ such that $\rho(e_j)w \notin W$. (The case $\rho(f_j)w \notin W$ is similar.) As $W = \oplus W_\mu$, where $W_\mu = \{w/\rho(k_i)w = \mu_i w\}$, we can assume that $w \in W_\mu$ for a certain μ. Then $\rho(k_i)\rho(e_j)w = t_i^{a_{ij}}\rho(e_j)\rho(k_i)w = \mu_i t_i^{a_{ij}}\rho(e_j)w$. So $w' = \rho(e_j)w$ is a common eigenvector of all $\rho(k_i)$'s and $W' = W \oplus \mathbb{C}w'$ belongs to E, with $\dim W' > \dim W$. Contradiction.

Definition. A vector $v \in V \backslash \{0\}$ is said a highest weight vector if there exists $\lambda = (\lambda_1, \ldots, \lambda_N) \in (\mathbf{C}^*)^N$ such that: $\rho(k_i)v = \lambda_i v \, \forall i = 1, \ldots, N$,

$$\rho(e_i)v = 0 \, \forall i = 1, \ldots, N.$$

Proposition 3. *For each finite dimensional representation (ρ, V), there is at least a highest weight vector in V.*

Proof. a) As the $\rho(k_i)$'s are simultaneously trigonalisable, the set of weights P is non-empty; The subvectorspace $V' = \oplus V_\mu$ of V is non-zero and invariant under $U_t\mathcal{G}$. We consider the subrepresentation of $U_t\mathcal{G}$ in V' and look for a highest weight vector in V'.

b) In V', we only have to show that $V_0 = \bigcap_1^N \operatorname{Ker} \rho(e_j)$ is not zero (as it is invariant under the $\rho(k_i)$'s, they have a common eigenvector in it). This follows classically from the lemma:

Lemma 5. *There exists an integer M such that:* $\forall j_1, \ldots, j_p \in \{1, \ldots, N\}$, $\rho(e_{j_1}) \cdots \rho(e_{j_p}) = 0$ *in* $\operatorname{End} V'$ *as soon as* $p \geqq M$.

Proof. It is enough to check that: $\forall \mu \in P$, $\forall v \in V'_\mu$, $\rho(e_{j_1}) \cdots \rho(e_{j_p})v = 0$ for p big enough. Let us fix $\mu \in P$; Then $v' = \rho(e_{j_1}) \cdots \rho(e_{j_p})v \in V'_\mu$, with $\mu'_i = \mu_i t_i^{\sum_k a_{ij_k}}$. Let n_k be the number of times e_k appears in $\{e_{j_1}, \ldots, e_{j_p}\}$; $\mu'_i = \mu_i t_i^{\sum n_k a_{ik}}$. As V' is finite dimensional, there is only a finite number of weights $\mu, \mu^{(1)}, \ldots, \mu^{(2)}$, and it is enough to see that for $p \geqslant M$, μ' is not in this list for $i \in \{1, \ldots, N\}$, let $x_i^{(s)} = \mu_i^{(s)}/\mu_i$; we have to find $i_0 \in \{1, \ldots, N\}$ such that:

$$t_{i_0}^{\sum n_k a_{i_0 k}} \notin \{1, x_{i_0}^{(1)}, \ldots, x_{i_0}^{(r)}\}.$$

As t is non-zero, let us fix $\tau \in \mathbb{C}$ such that $t = \exp(2i\pi\tau)$. As t is not a root of 1, $\tau \notin \mathbb{Q}$. As each $x_i^{(s)}$ is not zero, we fix $y_i^{(s)}$ such that $x_i^{(s)} = \exp(2i\pi y_i^{(s)})$. Then, an equality $t_i^{\sum n_k a_{ik}} = x_i^{(s)}$ gives:

$$\frac{(\alpha_i|\alpha_i)}{2} \sum_1^N n_k a_{ik} = y_i^{(s)} + \frac{m}{\tau} \quad \text{for a certain } m,$$

$$\sum_{1}^{N} n_k(\alpha_i|\alpha_k) = y_i^{(s)} + \frac{m}{\tau}.$$

As the left-hand side belongs to $\mathbf{Z}$, the right-hand side must also, and as there is at most an integer m such that $y_i^{(s)} + m/\tau \in \mathbf{Z}$. Let us put $z_i^{(s)} = y_i^{(s)} + m/\tau$. Suppose that for each $i \in \{1,\dots,N\}$, there exists $s \in \{0,\dots,r\}$ such that

$$\sum_{k=1}^{N} n_k(\alpha_i|\alpha_k) = z_i^{(s)}.$$

We have a linear system, with unknowns $(n_1,\dots,n_N)$ and matrix $((\alpha_i|\alpha_k))$ which is invertible. So, given $(z_1^{(s_1)},\dots,z_N^{(s_N)})$, there is at most an integral solution to the system. As we can form only a finite number of N-uples $(z_1^{(s_1)},\dots,z_N^{(s_N)})$, we see that if $(n_1,\dots,n_N)$ is not in a certain finite set, there is always an index i_0 such that: $t_{i_0}^{\sum n_k a_{i_0 k}} \notin \{1, x_{i_0}^{(1)},\dots,x_{i_0}^{(s)}\}$. Let $M = \sup(|n_1| + |n_2| + \cdots + |n_N|) + 1$, where $(n_1,\dots,n_N)$ belongs to the excluded finite set, and we get the lemma.

Proposition 4. *Let V be a cyclic $U_t\mathscr{G}$-module generated by a highest weight vector v_+, with weight $\lambda = (\lambda_1,\dots,\lambda_N)$.*

1) *V is spanned by v_+ and the $\rho(f_{i_1})\cdots\rho(f_{i_p})v_+$, $i_1,\dots,i_p \in \{1,\dots,N\}$, and such a vector, if non-zero, is a vector of weight $\mu = (\mu_1,\dots,\mu_N)$ with $\mu_k = \lambda_k \cdot t_k^{-\sum_j a_{kij}}$.*

2) *All the weights of V are of this form.*

3) *For each weight μ, dim $V_\mu < \infty$ and dim $V_\lambda = 1$.*

4) *V is an indecomposable $U_t\mathscr{G}$-module, with a unique maximal proper submodule.*

Proof. (Compare Humphreys [4]). Quite analogous to the classical one, using the decomposition $U_t\mathscr{G} = U_t n_- \otimes \mathbb{C}[T] \otimes U_t n_+$. (For 3), use the same argument as in Lemma 5 to prove that $\rho(f_{j_1})\cdots\rho(f_{j_r})v_+$ and $\rho(f_{i_1})\cdots\rho(f_{i_p})v_+$ have the same weight iff $\forall i$, f_i appears the same number of times in $\{f_{j_1},\dots,f_{j_r}\}$ and in $\{f_{i_1},\dots,f_{i_p}\}$.

Proposition 5. *If ρ and ρ' are irreducible representations with the same highest weight, they are equivalent.*

Now, given an irreducible finite dimensional representation, we know that it has highest weight, necessarily unique. In order to determine the possible values of $\lambda = (\lambda_1,\dots,\lambda_N)$, we shall consider, for each $i = 1,\dots,N$, the restriction of the representation to the subalgebra generated by $k_i^{\pm 1}, e_i, f_i$ (which is isomorphic to $U_t\mathrm{sl}(2)$).

IV. Finite Dimensional Representations of $U_t\mathrm{sl}(2)$

We shall call $k^{\pm 1}$, e, f the generators.

Theorem 1. 1. *If $\lambda \in \mathbf{C}^*$ is the highest weight of a finite dimensional representation of $U_t\mathrm{sl}(2)$, then $\lambda = \omega \cdot t^m$, where $\omega \in \{1, -1, i, -i\}$, $m \in \mathbf{N}$.*

2. *For each $m \in \mathbf{N}$ and $\omega \in \{1, -1, i, -i\}$, $\lambda = \omega \cdot t^m$ is the highest weight of an irreducible representation of dimension $(m+1)$, and the weights of this representation are exactly: $\omega t^m, \omega t^{m-2},\dots,\omega t^{-m}$.*

3. *Every finite dimensional representation of $U_t\mathrm{sl}(2)$ is completely reducible.*

Proof. 1. Let v be a vector with highest weight λ and put, for $p \in \mathbf{N}$, $v_p = (1/p!)\rho(f)^p \cdot v$. Then:

i) $\rho(f)v_p = (p+1)v_{p+1}$

ii) $\rho(k)v_p = \lambda t^{-2p} v_p$

and the formula $[e, f^p] = f^{p-1}((t^{2p} - t^{-2p})/(t^2 - t^{-2}))\cdot((k^2 t^{-2(p-1)} - k^{-2} t^{2(p-1)})/(t^2 - t^{-2}))$ and the fact that $\rho(e)\cdot v = 0$, show that we have:

iii) $\rho(e)v_p = \dfrac{t^{2p} - t^{-2p}}{(t^2 - t^{-2})} \cdot \dfrac{t^{-2(p-1)}\lambda^2 - t^{2(p-1)}\lambda^{-2}}{(t^2 - t^{-2})} v_{p-1}, p \geqq 1.$

As V is finite dimensional, there is a first integer m such that $v_m = 0$. Then, as t is not a root of 1, $\lambda^4 = t^{4(m-1)}$, so $\lambda = \omega t^{m-1}$, $\omega \in \{1, -1, i, -i\}$.

2. Let V be a $\mathbf{C}$-vector space with basis $(v_0, \ldots, v_m)$, on which k, e, f act by the same formulas i), ii), iii) with $\lambda = \omega t^m$. Then $\rho(k), \rho(e), \rho(f)$ verify the defining relations of $U_t\mathrm{sl}(2)$: so (ρ, v) is a representation of $U_t\mathrm{sl}(2)$ and it is irreducible since the v_p's are the only weight vectors possible (up to scalar).

3. We have to check that if V is a finite dimensional $U_t\mathrm{sl}(2)$-module and V' an invariant subspace of V, then there is an invariant subspace V'' such that $V = V' \oplus V$.

a) Case where V' is of Codimension 1. By using induction on the dimension of V', one classically reduces to the case where V' is also irreducible; so, it is a highest weight module. Let us call $\omega \cdot t^m$ its highest weight.

Lemma 6. 1. *$C = ((kt - k^{-1}t^{-1})^2/(t^2 - t^{-2})^2) + fe$ is in the center of $U_t\mathrm{sl}(2)$ and it acts in every finite dimensional irreducible representation, by a non-zero scalar. (Compare Jimbo [3]).*

2. *For $\omega' \in \{1, -1, i, -i\}$, let $C' = C - (\omega' t - \omega'^{-1} t^{-1})^2/(t^2 - t^{-2})^2$. It acts in every finite dimensional irreducible representation by a non-zero scalar if the dimension of the representation is greater than 2.*

Proof. One checks immediately that C and C' commute with e, f, k. So, they are in the center of $U_t\mathrm{sl}(2)$ and act by a scalar in every irreducible representation. This scalar is obtained by evaluating on the highest weight vector v_0. For C, one gets $((\omega t^{m+1} - \omega^{-1} t^{-(m+1)})/(t^2 - t^{-2}))^2$ which is non-zero as t is not a root of 1.

For C', one gets: $((\omega^2 t^{2(m+1)} + \omega^{-2} t^{-2(m+1)} - \omega'^2 t^2 - \omega'^{-2} t^{-2})/(t^2 - t^{-2})^2)$.

But $\omega^2 = \omega^{-2}$ and $\omega'^2 = \omega'^{-2}$.

It is zero if and only if $\omega^2(t^{2(m+1)} + t^{-2(m+1)}) = \omega'^2(t^2 + t^{-2})$,

$$\frac{t^{2(m+1)} + t^{-2(m+1)}}{t^2 + t^{-2}} = \left(\frac{\omega'}{\omega}\right)^2 \in \{1, -1\}.$$

But

$$\frac{t^{2(m+1)} + t^{-2(m+1)}}{t^2 + t^{-2}} = 1 \Leftrightarrow t^2(t^{2m} - 1) = t^{-2(m+1)}(t^{2m} - 1)$$

impossible if t is not a root of 1 ($m \geqq 1$ as the dimension of the representation is $m+1$)

$$\frac{t^{2(m+1)} + t^{-2(m+1)}}{t^2 + t^{-2}} = -1 \Leftrightarrow t^2(t^{2m} + 1) = t^{-2(m+1)}(t^{2m} + 1)$$

impossible if t is not a root of 1.

Proof of a). Suppose first that $\dim V' \geqq 2$.

Consider the representation of $U_t\mathrm{sl}(2)$ in V/V', which is 1-dimensional: e_i, f_i act by 0 and k_i by a scalar $\omega' \in \{1, -1, i, -i\}$. Define C' as in the lemma and let it act in V: it takes V' into V', where it acts by a non-zero scalar according to Lemma 6, and in fact it takes V into V' as it acts by 0 in V/V' (by choice of ω'). So $V_2 = \ker C'$ is 1-dimensional and $V = V' \oplus V_2$. Furthermore, V_2 is invariant under $U_t\mathrm{sl}(2)$ as C' belongs to the center.

Suppose now $\dim V' = 1$ and $\dim V = 2$. The only non-trivial case is the one where ω, the weight of the representation in V' is equal to ω', the weight of the representation in V/V'. So, there exists a basis (v_1, v_2) in V in which $\rho(k)$ has matrix $\begin{pmatrix} \omega & \alpha \\ 0 & \omega \end{pmatrix}$, $\alpha \in \mathbf{C}$.

Then $\rho(k)[\rho(e)v_1] = t^2\omega\rho(e)v_1$, so $\rho(e)v_1 = 0$.

Then $\rho(k)[\rho(e)v_2] = t^2\rho(e)[\omega v_2 + \alpha v_1] = t^2\omega\rho(e)v_2$, so $\rho(e)v_2 = 0$ and $\rho(e) = 0$. Similarly, $\rho(f) = 0$.

Then the relation $[e,f] = (k^2 - k^{-2})/(t^2 - t^{-2})$ implies $\rho(k)^2 = \rho(k^{-1})^2$, so $\alpha = 0$.

b) General Case. V' of any codimension. Let

$$\mathscr{V} = \{f \in \mathscr{L}(V, V')/f_{|V'} \text{ is a scalar operator}\},$$
$$\mathscr{V}' = \{f \in \mathscr{L}(V, V')/f_{|V'} = 0\}.$$

Then $\mathscr{V}'$ is a subspace of codimension 1 in $\mathscr{V}$.

One makes $U_t\mathrm{sl}(2)$ act in $\mathscr{L}(V, V')$ after identifying $\mathscr{L}(V, V')$ with $V' \otimes V^*$ and putting: $\bar{\rho} = (\rho \otimes \tilde{\rho}) \circ \Delta$, where $\tilde{\rho} = {}^t\rho \circ S$ is the contragradient representation in V^*. If one fixes a basis $(y_1, \ldots, y_p)$ of V', one can write any $\varphi \in \mathscr{L}(V, V')$ uniquely as $\varphi = \sum y_i \otimes x_i^*$ for some $x_i^* \in V^*$.

One then checks without difficulty that $\mathscr{V}$ and $\mathscr{V}'$ are invariant under $\bar{\rho}$. Applying a), there exists an invariant subspace $\mathscr{V}''$ such that $\mathscr{V} = \mathscr{V}' \oplus \mathscr{V}''$. Let $\varphi = \sum y_i \otimes x_i^*$ a non-zero element in $\mathscr{V}''$: it acts in V' by a non-zero scalar and $\operatorname{Ker}\varphi = \cap_i \operatorname{Ker} x_i^*$ verifies $V = \operatorname{Ker}\varphi \oplus V'$. Furthermore, $\operatorname{Ker}\varphi$ is invariant under $U_t\mathscr{G}$ (because $\mathscr{V}''$ was) and $\operatorname{Ker}\varphi$ is the sought for space.

Corollary. *If $\lambda = (\lambda_1, \ldots, \lambda_N)$ is the highest weight of a finite dimensional irreducible representation of $U_t\mathscr{G}$, then, necessarily, λ_k is of the form $\lambda_k = \omega_k t_k^{m_k}$. $\omega_k \in \{1, -1, i, -i\}$, $m_k \in \mathbf{N}$.*

V. Finite Dimensional Representations of $U_t\mathscr{G}$

1. Any 1-dimensional representation is irreducible, with highest weight $\omega = (\omega_1, \ldots, \omega_N) \in \{1, -1, i, -i\}^N$. Let us denote it by $(\rho_\omega . \mathbb{C}_\omega)$. If (ρ, V) is any finite dimensional irreducible representation, with highest weight λ, then $(\rho \otimes \rho_\omega) \circ \Delta$ gives an irreducible representation in $V \otimes \mathbb{C}_\omega$, with highest weight $\omega . \lambda = (\omega_1 \lambda_1, \ldots, \omega_N \lambda_N)$.

2. Let $\tilde{\lambda}$ a dominant weight of $\mathscr{G}$ (with the basis of roots (α_i)). One can associate to it a character of T, noted $t^{\tilde{\lambda}}$, by: $t^{\tilde{\lambda}}(k_i) = t_i^{\tilde{\lambda}(H_i)}$, where $(H_1, \ldots, H_N)$ is the coroot system associated with $(\alpha_1, \ldots, \alpha_N)$.

The corollary shows that to each highest weight λ, one can associate a 1-dimensional representation $(\rho_\omega, \mathbb{C}_\omega)$ and a dominant weight $\tilde{\lambda}$ defined by $\tilde{\lambda}(H_i) = \langle \tilde{\lambda}, \alpha_i \rangle = m_i \in \mathbb{N}$.

This is the first point of the following theorem:

Theorem 2.

1. *If (ρ, V) is a finite dimensional irreducible representation with highest weight λ, then $\lambda = \omega . t^{\tilde{\lambda}}$, where $\omega \in \{1, -1, i, -i\}^N$ and $\tilde{\lambda}$ is a dominant weight of $\mathscr{G}$.*
2. *Any character of T of this form is the highest weight of a finite dimensional irreducible representation.*
3. *Any finite dimensional representation of $U_t\mathscr{G}$ is completely reducible.*

Proof.

2. According to the remarks in 1., we only have to consider the case where $\lambda = t^{\tilde{\lambda}}$. But, for each $\lambda \in (\mathbf{C}^*)^N$, one can construct the universal standard cyclic module with highest weight λ, call it $Z(\lambda)$, by an induced module construction: consider the 1-dimensional space D_λ, with basis v_+, on which $U_t b_+$ acts as follows:

$$e_i . v_+ = 0 \ \forall i$$
$$k_i . v_+ = \lambda_i v_+ \ \forall i.$$

Put $Z(\lambda) = U_t\mathscr{G} \bigotimes_{U_t b_+} D_\lambda$: it is a left $U_t\mathscr{G}$-module in which $1 \otimes v_+$ is not zero because $U_t\mathscr{G}$ is a free right $U_t b_+$-module, and $1 \otimes v_+$ generates $Z(\lambda)$. Taking the quotient by the maximal proper submodule (see Prop. 4), we get an irreducible module with highest weight λ: $V(\lambda)$. The fact that, when $\tilde{\lambda}$ is dominant, $V(t^{\tilde{\lambda}})$ is finite dimensional will follow from:

Proposition 6. *Let $V(t^{\tilde{\lambda}})$ the irreducible module as above, where the dominant weight $\tilde{\lambda}$ is defined by the positive integers $m_i = \tilde{\lambda}(H_i)$. Then:*

1. $f_i^{m_i+1} \cdot v_+ = 0 \quad \forall i = 1, \ldots, N.$
2. *For each $1 \leqq i \leqq N$, $V(t^{\tilde{\lambda}})$ contains a non-zero finite dimensional L_i-module (L_i is the subalgebra generated by $e_i, f_i, k_i^{\pm 1}$).*
3. *$V(t^{\tilde{\lambda}})$ is the sum of the finite dimensional L_i-submodules.*
4. *The Weyl group W acts on the set P of weights. Each weight subspace V_μ is finite dimensional and* $\dim V_{\sigma\mu} = \dim V_\mu \ \forall \sigma \in W$.
5. *The set of weights P is finite.*

Then, $V(t^{\tilde{\lambda}})$ being irreducible, it equals the sum of its weight subspaces and 4. *and* 5. *show that it is finite dimensional.*

Proof of proposition (Compare Humphreys [4]).

1. Let $w = f_i^{m_i+1} \cdot v_+$ and let us show that, if $w \neq 0$, it is a highest weight vector, with highest weight different from $t^{\tilde{\lambda}}$ (such a vector cannot exist as $V(t^{\tilde{\lambda}})$ is irreducible). First, $k_j \cdot w = t_j^{-a_{ji}(m_i+1)} f_i^{m_i+1} k_j v_+ = t_j^{-a_{ji}(m_i+1)} t_j^{\tilde{\lambda}(H_j)} w$. So, if $w \neq 0$, it is a weight vector with weight $t^{\tilde{\lambda}-(m_i+1)\alpha_i} \neq t^{\tilde{\lambda}}$. Then, as for $i \neq j$, e_j and f_i commute, $e_j \cdot w = 0$. For $i = j$, the relation

$$[e_i, f_i^{m_i+1}] = f_i^{m_i} . \frac{t_i^{2(m_i+1)} - t_i^{-2(m_i+1)}}{t^2 - t^{-2}} . \frac{k_i^2 t_i^{-2m_i} - k_i^{-2} t_i^{2m_i}}{t^2 - t^{-2}}$$

and the fact that $k_i \cdot v_+ = t_i^{m_i} v_+$ shows that $e_i \cdot w = 0$. So w would be a highest weight vector.

2. For $1 \leqq i \leqq N$, consider the subvectorspace spanned by $v_+, f_i \cdot v_+, \ldots, f_i^{m_i+1} \cdot v_+$. Commutation rules between e_i, f_i and k_i show that it is invariant under L_i.

3. Let V' the sum of the finite dimensional L_i-submodules. According to 2), $V' \neq \{0\}$. To check that $V' = V(t^{\tilde{\lambda}})$, it is enough to see that it is invariant under all e_j, f_j, k_j.

Remark. $1 - a_{ij} \in \{1, \ldots, 4\}$. If $1 - a_{ij} = 1$, then $e_i e_j = e_j e_i$. For $1 - a_{ij} \geqq 2$, put $e_{i,j} = e_i e_j - t_i^{2a_{ij}} e_j e_i$. Then, if $1 - a_{ij} = 2$, one defining relation gives $e_i e_{i,j} - t_i^{4+2a_{ij}} e_{i,j} e_i = 0$. If $1 - a_{ij} = 3$, put $e_{i,i,j} = e_i e_{i,j} - t_i^{4+2a_{ij}} e_{i,j} e_i$, and we have $e_i e_{i,i,j} = t_i^{8+2a_{ij}} e_{i,i,j} e_i$. For $1 - a_{ij} = 4$, put $e_{i,i,i,j} = e_i e_{i,i,j} - t_i^{8+2a_{ij}} e_{i,i,j} e_i$ and then: $e_i e_{i,i,i,j} = t^{12+2a_{ij}} e_{i,i,i,j} e_i$. Same remark with the f_i's. Now, the invariance of V' will result from the following fact: if W is an invariant finite dimensional L_i-submodule, then the vector space spanned by $e_j W, f_i W, k_j W, \quad e_{i,j} W, f_{i,j} W, \ldots, e_{i,i,i,j} W$ and $f_{i,i,i,j} W$ (where $j \in \{1, \ldots, N\} \backslash \{i\}$) is finite dimensional and invariant under L_i according to the remark. So, $U_t \mathscr{G}(W) \subset V'$.

4. The finite dimensionality of each V is proved as in Proposition 4. Let $\mu = t^{\tilde{\mu}} \in P$ and $\sigma_i \in W$ associated with the simple root α_i. Let us show that $\sigma_i(t^{\tilde{\mu}})$, defined as $t^{\sigma_i(\tilde{\mu})}$, *belongs to* P. *But the subspace* $\bigoplus_{k \in \mathbf{Z}} V_{t^{\tilde{\mu}+k\alpha_i}}$ *is invariant under* L_i; *let us fix* $v_\mu \in V_\mu \backslash \{0\}$. According to 3), there is a non-trivial finite dimensional subspace V'' of $\bigoplus V_{t^{\tilde{\mu}+k\alpha_i}}$, invariant under L_i and containing v_μ. According to the complete reducibility theorem for $U_{t_i}\mathrm{sl}(2)$, V'' is a direct sum of irreducible L_i-modules. As $\mu = t^{\tilde{\mu}}$ is a weight for the representation in V'', $\mu_i = t_i^{\tilde{\mu}(H_i)}$ appears as a weight of one of the irreducible summands. According to Theorem 1, $t_i^{-\tilde{\mu}(H_i)}$ is also a weight for this irreducible L_i-module. But, as the possible weights are restrictions of those of V'', there is k in $\mathbf{Z}$ such that:

$$t_i^{-\tilde{\mu}(H_i)} = t_i^{\tilde{\mu}(H_i) + k\alpha_i(H_i)}, \quad \text{that is} \quad 2\tilde{\mu}(H_i) = -k\alpha_i(H_i).$$

But

$$\sigma_i(\tilde{\mu}) = \tilde{\mu} - \frac{2\tilde{\mu}(H_i)}{(\alpha_i, \alpha_i)} \alpha_i = \tilde{\mu} + k\alpha_i.$$

So, $t^{\sigma_i(\tilde{\mu})} \in P$.

5. Using 4, the proof is exactly the same as the classical one.

Proof of Point 3) in Theorem 2. (Complete reducibility) We shall use a result due to Professor A. Borel, which he has obtained as a generalisation of an argument allowing him to prove the complete reducibility theorem for complex semi-simple Lie algebras without using the Casimir operator.

His result is the following:

Theorem (A Borel): *Let A be an algebra, M an additive category of A-modules and $\mathscr{S}$ the set of classes of simple A-modules in M. Assume*:

1. *M is closed under the formation of subquotients. Every element of M has a finite Jordan–Holder series.*

2. *There is an involutive functor $V \to V^*$ on M, reversing the arrows, preserving $\mathscr{S}$, direct sums and short exact sequences.*

3. *There is a partial order $\leqq$ in $\mathscr{S}$ such that $V \leqq W \Rightarrow V^* \leqq W^*$. (In the sequel, write $<$ for $\lneqq$).*

4. *Let* $0 \to U \to V \to W \to 0$ *be a short exact sequence in* M, *with* U *and* W *in* $\mathscr{S}$. *If* V *is indecomposable, then* $U < W$.

Then under those conditions, every element of M *is a direct sum of elements in* $\mathscr{S}$.

In fact, Borel's proof remains true if one replaces (2) by the little more general hypothesis:

(2′) There are two functors F_1 and F_2 on M, reversing arrows, preserving $\mathscr{S}$, direct sums and exact sequences, and which are inverse one of the other. Then, (3) must be true for F_1 and F_2.

It is under this form that we shall apply the result to $A = U_t\mathscr{G}$ and to the additive category of finite dimensional $U_t\mathscr{G}$-modules.

Let us check that the four conditions are satisfied:

1. is clear.

2. Let F_1 the functor contragredient representation $(\rho, V) \to (\rho_1, V^*)$, where $\rho_1 = {}^t\rho \circ S$ (S is the antipode), and F_2 the functor skew contragredient representation: $(\rho, V) \to (\rho_2, V^*)$, where $\rho_2 = {}^t\rho \circ S'$ (S' is the skew antipode: $S': U_t\mathscr{G} \to U_t\mathscr{G}$ is linear, antimultiplicative and inverse of S).

3. An element in S is characterised by its highest weight $\omega.t^{\tilde{\lambda}}$, where $\omega \in \{1, -1, i, -i\}$ and $\tilde{\lambda}$ is a dominant weight in $\mathscr{G}$.

Let $\preccurlyeq$ the usual partial order on the weights in $\mathscr{G}$. Define the order on $\mathscr{S}$ by:

$$\omega.t^{\tilde{\lambda}} \leqq \omega'.t^{\tilde{\lambda}'} \Leftrightarrow \omega = \omega' \quad \text{and} \quad \tilde{\lambda} \preccurlyeq \tilde{\lambda}'.$$

As $S(k_i) = S'(k_i) = k_i^{-1}, \omega.t^{\tilde{\mu}}$ is a weight in $(\rho, V) \Leftrightarrow \omega^{-1}.t^{-\tilde{\mu}}$ is a weight in (ρ_1, V^*) (or (ρ_i, V^*)).

So, to prove $V \leqq W \Rightarrow V^* \leqq W^*$, we are back to the classical case: if w_0 is the longest element of the Weyl group W, then $i = -w_0$ defines an involution in $\mathfrak{h}^*$ preserving the weight lattice and the order on it. On the set of characters of T of the form $\omega.t^{\tilde{\mu}}$, where $\omega \in \{1, -1, i, -i\}^N$ and $\tilde{\mu}$ is a weight of $\mathscr{G}$, one has an involution I given by: $I(\omega.t^{\tilde{\mu}}) = \omega^{-1}.t^{i(\tilde{\mu})}$, which preserves the order. Now, it is easy to see that if V is an irreducible $U_t\mathscr{G}$-module with highest weight $\omega.t^{\tilde{\lambda}}$, then $F_1(V)$ and $F_2(V)$ are irreducible with highest weight $\omega^{-1}.t^{i(\tilde{\lambda})}$.

4. We shall follow Borel's proof for the classical case.

Let $O \to V(\omega.t^{\tilde{\lambda}}) \to V \to V(\omega'.t^{\tilde{\mu}}) \to 0$ be a short exact sequence with V indecomposable. Then V is cyclic with respect to any vector v not contained in $V(\omega.t^{\tilde{\lambda}})$. Put $\lambda = \omega.t^{\tilde{\lambda}}$, $\mu = \omega.t^{\tilde{\mu}}$.

Let us show that $\lambda \neq \mu$. If not, V_λ is 2-dimensional and there is no weight $v > \lambda$ in V. So V_λ is killed by $U_t n_+$. So, any $v \in V_\lambda \backslash \{0\}$ is a highest weight vector and generates an irreducible submodule whose intersection with V_λ is 1-dimensional. As $\dim V(\lambda)_\lambda = 1$, taking $v \in V_\lambda \backslash V(\lambda)_\lambda$, we see that the cyclic module generated by v should be V. Contradiction.

So, $\lambda \neq \mu$. We shall prove that there is in $V \backslash V(\lambda)$ a vector with weight μ killed by $U_t n_+$. As such a vector must generate V, it will follow that $\lambda < \mu$. Let us note that the space $V^{U_t n_+}$ of vectors killed by $U_t n_+$ is 2-dimensional: $\dim V^{U_t n_+} \leqq 2$ because $\dim V(\lambda)^{U_t n_+} = \dim V(\mu)^{U_t n_+} = 1$, and $\dim V^{U_t n_+} \geqq 1$ because $V(\lambda)^{U_t n_+} \subset V^{U_t n_+}$. One can suppose that μ is not $< \lambda$, because, if $\mu < \lambda$, taking the dual exact sequence $0 \to V(\mu)^* \to V^* \to V(\lambda)^* \to 0$, one has $I(\mu) < I(\lambda)$ and in particular $I(\lambda)$ is not $< I(\mu)$. So one gets $\dim (V^*)^{U_t n_+} = 2$, but as $(V^*)^{U_t n_+}$ is the dual of $V/U_t n_+.V$, it has the same dimensions as $V^{U_t n_+}$.

So suppose that μ is not $<\lambda$. Then V cannot have weights $v<\mu$ because such a weight would be necessarily a weight of $V(\lambda)$ and we should have $\mu<v\leqq\lambda$. Now, there is $x\in V$ whose image in $V(\mu)$ generates $V(\mu)_\mu$. So, for $i=1,\ldots,N$, $\rho(k_i)x-\mu_i x\in V(\lambda)$ and $\rho(e_i)x\in V(\lambda)$. Put $y_i=\rho(k_i)x-\mu_i x\in V(\lambda)$, and note that $(\rho(k_j)-\mu_j)y_i=(\rho(k_i)-\mu_i)y_j$. As μ is not $<\lambda$ it cannot be a weight in $V(\lambda)$; so, there is $i\in\{1,\ldots,N\}$ such that: $\rho(k_i)-\mu_i|_{V(\lambda)}$ is invertible. Put $z=(\rho(k_i)-\mu_i)^{-1}(y_i)\in V(\lambda)$. Then $(\rho(k_j)-\mu_j)z=y_j=(\rho(k_j)-\mu_j)x$. So $x'=x-z$ is such that: $\forall i$, $(\rho(k_i)-\mu_i)x'=0$, $\rho(e_i)x'\in V(\lambda)$ and has the same image as x in $V(\lambda)$. Let us show that in fact $\rho(e_i)x'=0\ \forall i$.

If not, let i such that $\rho(e_i)x'\in V(\lambda)\setminus\{0\}$. Then $\forall j\in\{1,\ldots,N\}$,

$$\rho(k_j)\rho(e_i)x'=t_j^{a_{ji}}\rho(e_i)\rho(k_j)x'=t_j^{a_{ij}}\mu_j\rho(e_i)x'=\omega_j' t_j^{(\tilde{\mu}+\alpha_i)(H_j)}\rho(e_i)x'.$$

So $\rho(e_i)x'$ should be a vector with weight $\omega'.t^{\tilde{\mu}+\alpha_i}>\omega'.t^{\tilde{\mu}}=\mu$. Impossible. So, x' is the sought for vector and we have also proved that $\dim V^{U_t n_+}=2$.

The only remaining case is the one where $\mu<\lambda$, with $\dim V^{U_t n_+}=2$. As $\dim V(\lambda)^{U_t n_+}=1$, there is an $x\in V^{U_t n_+}\setminus V(\lambda)^{U_t n_+}$. Its image in $V(\mu)$ is not zero and is killed by $U_t n_+$. So each of its components $\bar{x}_v$ in the decomposition $V(\mu)=\bigoplus V(\mu)_v$ is a highest weight vector if $\bar{x}_v\neq 0$. So, as $V(\mu)$ is irreducible, only $\bar{x}_\mu\neq 0$. So the μ-component x_μ of x is not zero and, as it is also killed by $U_t n_+$, it is the sought for vector.

The theorem is now completely proved.

These results have been announced in [7].

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