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Isospectral Hamiltonian Flows in Finite and Infinite Dimensions

I. Generalized Moser Systems and Moment Maps into Loop Algebras*

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Abstract. A moment map $\tilde{J}_r: \mathcal{M}_A \to (\widetilde{gl(r)}^+)^*$ is constructed from the Poisson manifold $\mathcal{M}_A$ of rank-r perturbations of a fixed $N \times N$ matrix A to the dual $(\widetilde{gl(r)}^+)^*$ of the positive part of the formal loop algebra $\widetilde{gl(r)} = gl(r) \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$. The Adler-Kostant-Symes theorem is used to give hamiltonians which generate commutative isospectral flows on $(\widetilde{gl(r)}^+)^*$. The pull-back of these hamiltonians by the moment map gives rise to commutative isospectral hamiltonian flows in $\mathcal{M}_A$. The latter may be identified with flows on finite dimensional coadjoint orbits in $(\widetilde{gl(r)}^+)^*$ and linearized on the Jacobi variety of an invariant spectral curve X_r which, generically, is an r-sheeted Riemann surface. Reductions of $\mathcal{M}_A$ are derived, corresponding to subalgebras of $gl(r, \mathbb{C})$ and $sl(r, \mathbb{C})$, determined as the fixed point set of automorphism groupes generated by involutions (i.e., all the classical algebras), as well as reductions to twisted subalgebras of $\widetilde{sl(r, \mathbb{C})}$. The theory is illustrated by a number of examples of finite dimensional isospectral flows defining integrable hamiltonian systems and their embeddings as finite gap solutions to integrable systems of PDE's.

1. Introduction

In 1979 Moser [32] showed that a number of well-known completely integrable finite dimensional hamiltonian systems could be uniformly understood in the framework of certain rank 2 isospectral deformations of matrices. The problem he considered involved hamiltonian flow $(x(t), y(t))$ in $\mathbb{R}^{2n}$ which, for a fixed $n \times n$ matrix A and real constants, a, b, c, d, leaves the spectrum of the matrix

$$L = A + ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y$$

invariant. Among the results he obtained were:

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(1) The elementary symmetric invariants of L, regarded as hamiltonians on $\mathbb{R}^{2n}$, give rise to flows which are mutually commutative, and hence isospectral, obeying equations of Lax type:

$$\dot{L}=[B,L].$$

(2) Associated to these flows is a certain invariant hyperelliptic curve X. The quotient of an invariant dense submanifold of the isospectral manifold by a 1-parameter group (or, in certain cases that of a codimension 1 constrained submanifold) is identifiable with the Jacobi variety $\mathcal{J}(X)$ so that the flows are identified with linear flows in $\mathcal{J}(X)$.

(3) The systems are completely integrable.

In further related developments, Adler and van Moerbeke [6, 7], and others [13, 37], recast the Lax equations for finite dimensional systems in the framework of loop algebras, where the spectral curve and linearization of flow on its Jacobi variety is very natural. Reyman and Semenov-Tian-Shansky [38, 39] have shown the relevance of the Adler, Kostant, Symes (AKS) theorem [5, 24, 44] to the linearization of isospectral hamiltonian flows in loop algebras, but did not examine the relationship to Moser's isospectral flows. The algebraic approach to integrable systems of PDE's developed by Sato [41] and the Kyoto school [9] has also been related to isospectral flows in loop algebras [11, 14, 42, 49]. Through inverse spectral methods, integrable systems of PDE's may be interpreted in terms of constrained harmonic oscillators in infinite dimensions [10, 32, 33] and solutions of "finite gap" or multi-soliton type derived algebraically from linear flows in Grassmanians and Jacobi varieties [8, 9, 12, 15, 21, 25–28, 31, 34, 36, 45, 46, 50, 51]. The PDE's that arise may be interpreted as integrability conditions implied by the commutativity of pairs of isospectral flows in loop algebras [11, 14, 42, 49].

The principal purpose of the present work is to provide a systematic link between finite dimensional integrable systems, flows in loop algebras, and integrable systems of PDE's through the use of moment maps. The construction of such maps allows us to apply the results of the AKS theorem to deduce a large class of commuting flows of isospectral type generalizing Moser's results to perturbations of arbitrary rank. As a consequence of our construction, not only can a much wider class of integrable finite dimensional systems linearizable on Abelian varieties be derived, but the intrinsic finite dimensional structure of the "algebraic" solutions to integrable systems of PDE's admitting a zero-curvature formulation can be determined and expressed in terms of commuting flows of finite dimensional systems. In many cases these systems are of interest in themselves.

More specifically, we consider the *generalized Moser problem:*

Given an $n\times n$ matrix A, consider a general rank r perturbation

$$L_A=A+FG^T,$$

where F and G are maximal rank rectangular $n\times r$ matrices. In the space of such pairs (F,G), endowed with the symplectic structure $\omega=\mathrm{tr}(dF^T\wedge dG)$, derive a maximal Poisson commuting (i.e. completely integrable) set of hamiltonian flows

$(F,G)\to(F(t),G(t))$ which give rise to isospectral deformations of L, and prove that these flows linearize on the Jacobi variety of a suitable spectral curve.

The solution of this problem through the moment map construction together with the AKS theorem is the main content of Sects. 2–4. In Sect. 2 we consider various hamiltonian group actions on the space of (F,G)'s. We define moment maps for these actions and derive in particular a moment map into the space in which the relevant AKS flows reside, the dual of the positive component of the loop algebra $\widetilde{gl(r)}$. This construction is central for all the subsequent development. We also address the question of invertibility and show how quotienting by certain hamiltonian group actions gives rise to a Poisson manifold on which the moment map becomes an immersion. In Sect. 3 we prove the basic theorem (3.2) showing why hamiltonians generating isospectral flows of AKS-type in $(\widetilde{gl(r)}^+)^*$ pull back to hamiltonians which fulfill the requirement of the generalized Moser problem (i.e. that they be isospectral for L_A and that they Poisson commute), both in the form stated above and in a slightly generalized version (Theorem 3.6). In Sect. 4 we study the spectral curves, which in general are r-sheeted for the rank r case. We show how the results of Reyman and Semenov-Tian-Shansky [39], concerning complete integrability of flows in finite dimensional orbits of loop algebras, and methods of Krichever and Novikov [25, 26, 27], and van Moerbeke and Mumford [31], concerning linearization in Jacobi varieties, can be applied to our situation to deduce complete integrability and linearization when the affine part of the curve is nonsingular. In Sect. 5 we discuss the problem of reductions of such systems under finite groups of automorphisms, in particular, reductions of $gl(r,\mathbb{C})$ by involutive automorphisms to the classical Lie algebras and reductions to the twisted subalgebras of loop algebras. Finally, in Sect. 6 we illustrate these results with a number of detailed examples involving both finite and infinite dimensional systems and the links between the two. For the finite dimensional case with rank 2 perturbations, we show how Moser's results may be deduced as particular cases of our own and we give an analysis of a rank 2 system which fits into another real form generalizing Moser's framework; namely, the Rosochatius system. We then proceed to the relation with systems of PDE's, realized as integrability conditions for commuting flows, treating as examples of the rank 2 case the NLS equation and the modified KdV equation. We also give, as illustrations of the rank 3 case, the coupled NLS and the Boussinesq equation.

A sequel to this paper [4] will deal with the generalization of the results in Sect. 4 involving linearization of the flows and complete integrability for the case of singular curves as well as presenting a detailed computation of the flows arising from the examples in Sect. 6.

2. Moment Maps

We first summarize the necessary definitions regarding moment maps (see e.g. [3, 17, 47]). Let (M,ω) be a symplectic manifold. For $f\in C^\infty(M)$ the associated hamiltonian vector field X_f is defined by:

$$X_f \lrcorner\, \omega = df\,, \tag{2.1}$$

and the Poisson bracket in $C^\infty(M)$ is:

$$\{f, g\} = -X_f(g). \tag{2.2}$$

Then $C^\infty(M)$ may be regarded as a Lie algebra with respect to the Poisson Lie bracket and, denoting the Lie algebra of vector fields on M by $\chi(M)$, the map

$$\beta : C^\infty(M) \to \chi(M), \qquad \beta(f) = -X_f \tag{2.3}$$

is a homomorphism of Lie algebras.

Let $\phi : G \times M \to M$ be a smooth group action preserving ω [written, in short, $\phi(g, x) = gx$, $g \in G$, $x \in M$] and denote by $\mathfrak{g}$ the Lie algebra of G. The infinitesimal $\mathfrak{g}$ action is given by the homomorphism

$$\sigma : \mathfrak{g} \to \chi(M),$$

defined by:

$$\sigma(\xi)(x) = -\frac{d}{dt}\phi(\exp t\xi, x)|_{t=0}, \qquad \xi \in \mathfrak{g},\ x \in M. \tag{2.4}$$

The G action is called hamiltonian if there exists a *moment map*:

$$J : M \to \mathfrak{g}^*$$

such that the hamiltonian flow generated by $\langle J, \xi\rangle$ (where $\langle\ ,\ \rangle$ denotes the dual pairing) coincides with $x \to \phi(\exp t\xi, x)$, i.e.,

$$X_{\langle J, \xi\rangle} = \sigma(\xi). \tag{2.5}$$

The moment map is *equivariant* if

$$J(\phi(g, x)) = \mathrm{Ad}^*_g J(x), \qquad \forall g \in G. \tag{2.6}$$

Let $j : \mathfrak{g} \to C^\infty(M)$ denote the *linear dual* of J, defined by

$$j(\xi) = \langle J, \xi\rangle. \tag{2.7}$$

If we are only concerned with the infinitesimal action (2.4) (i.e., if we do not require the flows induced by J to be complete) we may take the following as alternative equivalent definitions of an equivariant moment map.

A map $J : M \to \mathfrak{g}^*$ with $J(gx) = \mathrm{Ad}^*_g J(x)$ and $\beta \circ j : \mathfrak{g} \to \chi_\omega(M)$ a Lie algebra homomorphism. (2.8a)

A map whose linear dual $j : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism. (2.8b)

(The representation of $\mathfrak{g}$ in terms of vector fields on M is understood to be defined by $\sigma = \beta \circ j$.)

There is also a third characterization that will be useful. Recall that the space $\mathfrak{g}^*$ has a natural Poisson structure, called the Lie-Poisson structure [48], given by

$$\{\phi, \psi\}_{\mathfrak{g}^*}(\mu) = \left\langle \mu, \left[\frac{\delta\phi}{\delta\mu}, \frac{\delta\psi}{\delta\mu}\right]\right\rangle, \qquad \phi, \psi \in C^\infty(\mathfrak{g}^*), \qquad \mu \in \mathfrak{g}^*,$$

where $\delta\phi/\delta\mu$ denotes $d\phi(\mu)$ thought of as an element of $\mathfrak{g}=\mathfrak{g}^{**}$. We can then characterize an equivariant moment map equivalently as

> A map $J: M \to \mathfrak{g}^*$ that preserves the Poisson bracket, i.e. is a Poisson map with respect to the Lie-Poisson structure; if $\phi, \psi \in C^\infty(\mathfrak{g}^*)$, then $\{J^*\phi, J^*\psi\} = J^*\{\phi, \psi\}_{\mathfrak{g}^*}$. (2.8c)

Definitions (2.8a–c) remain valid when M is any Poisson manifold.

Henceforth, the term "equivariant" will be dropped since all moment maps considered will satisfy this condition.

Let $C_G^\infty(M)$ denote the G invariant functions on M. Since a symplectic G action on M preserves the Poisson bracket, it follows that $C_G^\infty(M)$ forms a subalgebra of $C^\infty(M)$. If G acts freely and properly on M, then M/G is a manifold with a Poisson structure inherited from the one on M through the identification $C^\infty(M/G) \sim C_G^\infty(M)$. The symplectic leaves of M/G turn out to be the Marsden-Weinstein reduced manifolds [48], i.e. they have the form

$$M_\mu = J^{-1}(\mu)/G_\mu = J^{-1}(\mathcal{O}_\mu)/G, \tag{2.9}$$

where $\mu \in \mathfrak{g}^*$, G_μ is the isotropy group of μ in G and $\mathcal{O}_\mu$ is the G-orbit through μ. Recall that the reduced manifold, M_μ, has a natural symplectic structure ω_μ such that $i^*\omega = \pi^*\omega_\mu$, where $i: J^{-1}(\mu) \to M$ is inclusion and $\pi: J^{-1}(\mu) \to M_\mu$ is the natural projection taking points to their G_μ orbits.

Let $M_{N,r}$ be the space of complex $N \times r$ matrices, and identify $M_{N,r} \sim M_{N,r}^*$ through the pairing

$$(F, G) = \mathrm{tr}(F^T G), \qquad F, G \in M_{N,r}.$$

We shall consider several group actions on $M_{N,r} \times M_{N,r}$ which are hamiltonian with respect to the symplectic form

$$\omega = \mathrm{tr}(dF \wedge dG^T). \tag{2.10}$$

For $n \leqq N$ let

$$G_r^n = GL(r, \mathbb{C}) \times \ldots \times GL(r, \mathbb{C}) \text{ (n times)} \tag{2.11}$$

be the direct product Lie group and

$$\mathfrak{g}_r^n = gl(r, \mathbb{C}) \oplus \ldots \oplus gl(r, \mathbb{C}) \text{ (n times)} \tag{2.11'}$$

its Lie algebra. Let $k_1, \ldots, k_n$ be positive integers with $k_i < r$ and $\sum_{i=1}^n k_i = N$. For $F \in M_{N,r}$ let F_i denote the $k_i \times r$ block whose j^{th} row is the $(k_1 + \ldots + k_{i-1} + j)^{th}$ row of F; i.e. F has the block form

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_i \\ \vdots \\ F_n \end{pmatrix}.$$

Expressing similarly $G=\begin{pmatrix}G_1\\ \vdots\\ G_n\end{pmatrix}$ with each G_i a $k_i\times r$ block, define a hamiltonian G_r^n action on $M_{N,r}\times M_{N,r}$ by

$$(g(F,G))_i=(F_ig_i^{-1},G_ig_i^T),\qquad g=(g_1,\dots,g_n)\in G_r^n. \tag{2.12}$$

Let $\sigma_r^n:\mathfrak{g}_r^n\to\chi(M_{N,r}\times M_{N,r})$ denote the corresponding infinitesimal action. The associated moment map

$$J_r^n:M_{N,r}\times M_{N,r}\to(\mathfrak{g}_r^n)^*$$

is given by

$$J_r^n(F,G)(X_1,\dots,X_n)=-\sum_{j=1}^n \operatorname{tr}(F_jX_jG_j^T), \tag{2.13}$$

where the traces in the right-hand sum involve $k_j\times k_j$ matrices.

Identifying $gl(r,\mathbb{C})^*$ with $gl(r,\mathbb{C})$ through the trace of matrix products, and hence $(\mathfrak{g}_r^n)^*$ with $\mathfrak{g}_r^n$, we have

$$J_r^n(F,G)=-(G_1^TF_1,\dots,G_n^TF_n)\in\mathfrak{g}_r^n. \tag{2.14}$$

Restricting the G_r^n action to that of the diagonal subgroup $G_r=\{(g,\dots,g)\in G_r^n\}$ $\sim GL(r,\mathbb{C})$ gives the action $G_r:M_{N,r}\times M_{N,r}\to M_{N,r}\times M_{N,r}$, defined by

$$g(F,G)=(Fg^{-1},Gg^T),\qquad g\in GL(r,\mathbb{C}) \tag{2.15}$$

with moment map

$$J_r(F,G)(X)=-\operatorname{tr}(G^TFX),\qquad X\in\mathfrak{g}_r=gl(r,\mathbb{C}), \tag{2.16}$$

or, through the above identification of $gl(r,\mathbb{C})^*$ with $gl(r,\mathbb{C})$,

$$J_r(F,G)=-G^TF. \tag{2.17}$$

Now let $\widetilde{gl(r)}=gl(r,\mathbb{C})\otimes\mathbb{C}[\lambda,\lambda^{-1}]]$ be the loop algebra of semi-infinite formal Laurent series in λ with coefficients in $gl(r,\mathbb{C})$; i.e., with elements $X(\lambda)\in\widetilde{gl(r)}$ which are formal series of the form $X(\lambda)=\sum_{i=-\infty}^{m} a_i\lambda^i$, $a_i\in gl(r,\mathbb{C})$, and Lie bracket with $Y(\lambda)=\sum_{j=-\infty}^{l} b_j\lambda_j$ given by

$$[X(\lambda),Y(\lambda)]=\sum_{k=-\infty}^{m+l}\sum_{i+j=k}[a_i,b_j]\lambda^k, \tag{2.18}$$

(the inner sum being finite and the outer one formal).

The algebra $\widetilde{gl(r)}$ has a nondegenerate, ad invariant inner product given by

$$\langle X(\lambda),Y(\lambda)\rangle=\operatorname{tr}((X(\lambda)Y(\lambda))_0)=\operatorname*{res}_{\lambda=0}\operatorname{tr}(\lambda^{-1}X(\lambda)Y(\lambda)), \tag{2.19}$$

where $(X(\lambda)Y(\lambda))_0$ denotes the constant term in the formal series $X(\lambda)Y(\lambda)$, or equivalently the formal residue at $\lambda=0$ of $\lambda^{-1}X(\lambda)Y(\lambda)$.

Let $\widetilde{gl(r)}^+$ denote the subalgebra of $\widetilde{gl(r)}$ given by the polynomials in λ and $\widetilde{gl(r)}^-$ the subalgebra of strictly negative series; i.e., series of the form $\sum_{i=-\infty}^{-1} a_i\lambda^i$. We can write $\widetilde{gl(r)}$ as the vector space direct sum $\widetilde{gl(r)}=\widetilde{gl(r)}^+\oplus\widetilde{gl(r)}^-$. The inner product (2.19) then gives the identification

$$(\widetilde{gl(r)}^+)^* \sim (\widetilde{gl(r)}^-)^\perp = \widetilde{gl(r)}_0^- , \tag{2.20}$$

where $\widetilde{gl(r)}_0^-$ denotes the subalgebra of $\widetilde{gl(r)}$ given by $\lambda\widetilde{gl(r)}^-$.

Fix n distinct complex numbers, $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since $X(\lambda)\in\widetilde{gl(r)}^+$ is a polynomial we can evaluate at $\lambda=\alpha_i$ to obtain $X(\alpha_i)\in gl(r,\mathbb{C})$. This gives a Lie algebra homomorphism

$$\mathscr{A}:\widetilde{gl(r)}^+\to\mathfrak{g}_r^n$$

defined by

$$\mathscr{A}(X(\lambda))=(X(\alpha_1),\ldots,X(\alpha_n)). \tag{2.21}$$

The kernel of this map is $a(\lambda)\widetilde{gl(r)}^+$, where $a(\lambda)=\prod_{i=1}^{n}(\lambda-\alpha_i)$. Hence we have the exact sequence of Lie algebra homomorphisms

$$0\to a(\lambda)\widetilde{gl(r)}^+ \xrightarrow{\iota} \widetilde{gl(r)}^+ \xrightarrow{\mathscr{A}} \mathfrak{g}_r^n\to 0,$$

where ι is inclusion. The dual sequence is thus

$$0\to(\mathfrak{g}_r^n)^* \xrightarrow{\mathscr{A}^*} (\widetilde{gl(r)}^+)^* \xrightarrow{\iota^*} (a(\lambda)\widetilde{gl(r)}^+)^*\to 0, \tag{2.23}$$

where, if we identify $(\mathfrak{g}_r^n)^*$ with $\mathfrak{g}_r^n$ by using the trace componentwise, and $(\widetilde{gl(r)}^+)^*$ with $\widetilde{gl(r)}_0^-$ as in (2.17), we get

Lemma 2.1.

$$\mathscr{A}^*(Y_1,\ldots,Y_n)=\lambda\sum_{i=1}^{n}\frac{Y_i}{\lambda-\alpha_i}=\sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}Y_i\alpha_i^k\right)\lambda^{-k}. \tag{2.24}$$

Proof. This may be verified by applying both sides to an element $X(\lambda)\in\widetilde{gl(r)}^+$ and comparing coefficients. The easiest way to see the result, however, is by viewing the second expression as a meromorphic function and using the interpretation of the inner product (2.19) as a formal residue at $\lambda=0$. □

Since $\mathscr{A}$ is a Lie algebra homomorphism, it follows that $\mathscr{A}^*$ is a Poisson map with respect to the Lie-Poisson structures on $(\mathfrak{g}_r^n)^*$ and $(\widetilde{gl(r)}^+)^*$. Hence the map $\tilde{J}_r: M_{N,r}\times M_{N,r}\to(\widetilde{gl(r)}^+)^*$, defined by

$$\tilde{J}_r=\mathscr{A}^*\circ J_r^n, \tag{2.25}$$

is a moment map in the sense of (2.8c) with respect to the infinitesimal action of $\widetilde{gl(r)}^+$ on $M_{N,r}\times M_{N,r}$ defined by $\sigma_r:\widetilde{gl(r)}^+\to\chi(M_{M,r}\times M_{N,r})$, $\sigma_r=\sigma_r^n\circ\mathscr{A}$.

Remark. We use this generalized definition of a moment map because the algebra $\widetilde{gl(r)}^+$ does not have an easily described group. The construction we have given in terms of formal series and polynomial algebras is the simplest for our present purposes. However, it is easily modified to permit a genuine group action as follows (see also [35]). Let $\overline{GL(r)}$ denote the loop group $H^s(S^1, GL(r, \mathbb{C}))$ for some $s > 1/2$, where H^s is the L^2 Sobolev space with s derivatives. The Lie algebra $\overline{gl(r)}$ of $\overline{GL(r)}$ is the set of Fourier series $\sum_{i=-\infty}^{\infty} a_i\lambda^i$, $\lambda \in S^1$, with $a_i \in gl(r, \mathbb{C})$, which converge in H^s. Let $\overline{GL(r)}^+$ be the subgroup of $\overline{GL(r)}$ given by loops which are H^s boundary values of $GL(r, \mathbb{C})$-valued holomorphic functions on the interior of the unit circle. The Lie algebra $\overline{gl(r)}^+$ of $\overline{GL(r)}^+$ is given by Fourier series of the form $\sum_{i=0}^{\infty} a_i\lambda^i$, so $\widetilde{gl(r)}^+ \subset \overline{gl(r)}^+$. Finally, if we assume the α_i's satisfy $|\alpha_i| < 1$ (by rescaling, if necessary), then we can define a group homomorphism

$$\alpha : \overline{GL(r)}^+ \to G_r^n \quad \text{by} \quad \alpha(g(\lambda)) = (g(\alpha_1), \dots, g(\alpha_n)),$$

since $g(\lambda) \in \overline{GL(r)}^+$ converges inside the unit circle. The composition of α with the action of G_r^n on $M_{N,r} \times M_{N,r}$ gives a symplectic action of $\overline{GL(r)}^+$ on $M_{N,r} \times M_{N,r}$. This action has a moment map $\overline{J}_r : M_{N,r} \times M_{N,r} \to (\overline{gl(r)^+})^*$, which when composed with the projection $(\overline{gl(r)^+})^* \to (\widetilde{gl(r)}^+)^*$ is $\tilde{J}_r$. In what follows we continue to use the formal algebras since it is standard in the literature.

Combining (2.14) and (2.24) we obtain the following expression for the moment map $\tilde{J}_r$.

Proposition 2.2. *For $(F, G) \in M_{N,r} \times M_{N,r}$, we have*

$$\tilde{J}_r(F, G) = \sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\alpha_i - \lambda}, \tag{2.26}$$

where we identify $(\widetilde{gl(r)}^+)^*$ with $\widetilde{gl(r)}_0^-$.

Note that the restriction of $\tilde{J}_r$ to the subalgebra $gl(r) \subset \widetilde{gl(r)}^+$ reproduces the moment map J_r of (2.17).

We now wish to describe the image and fibres of the moment map

$$\tilde{J}_r : M_{N,r} \times M_{N,r} \to (\widetilde{gl(r)}^+)^* .$$

Since $\mathscr{A}^*$ is injective it is enough to describe the image and fibers of $J_r^n : M_{N,r} \times M_{N,r} \to (\mathfrak{g}_r^n)^* \sim \mathfrak{g}_r^n$. To describe the image of J_r^n first consider Eq. (2.14). Since the $r \times r$ matrix $G_i^T F_i$ has rank k_i or less, we may identify $\operatorname{Im} J_r^n \subset (\mathfrak{g}_r^n)^* \sim \mathfrak{g}_r^n$ as the set

$$\operatorname{Im} J_r^n = \{(X_1, \dots, X_n) \in \mathfrak{g}_r^n \mid X_i \text{ has rank } k_i \text{ or less}\} . \tag{2.27}$$

Now define H to be the direct product group

$$H = GL(k_1, \mathbb{C}) \times \dots \times GL(k_n, \mathbb{C}) \subset GL(N, \mathbb{C}), \tag{2.28}$$

where the inclusion is along the diagonal. Restricting the natural $GL(N, \mathbb{C})$ action on $M_{N,r}$ gives rise to the hamiltonian H action on $M_{N,r} \times M_{N,r}$, defined by

$$(h(F, G))_i = (h_i F_i, h_i^{-1T} G_i), \tag{2.29}$$

where

$$h = (h_1, \dots, h_n) \in H, \qquad h_i \in GL(k_i, \mathbb{C}) .$$

Let

$$\mathfrak{h} = gl(k_1, \mathbb{C}) \oplus \dots \oplus gl(k_n, \mathbb{C}) \tag{2.30}$$

denote the Lie algebra of H and $\mathfrak{h}^*$ its dual. If we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ by taking traces componentwise, the moment map for the H action (2.29) is given by

$$J_H(F, G) = (F_1 G_1^T, \ldots, F_n G_n^T). \tag{2.31}$$

Since the H action commutes with the G_r^n action, we conclude

Proposition 2.3. *$J_r^n(h(F, G)) = J_r^n(F, G)$ and $\tilde{J}_r(h(F, G)) = \tilde{J}_r(F, G)$ for all $h \in H$ and $(F, G) \in M_{N,r} \times M_{M,r}$.*

That is, the inverse image of any point under J_r^n or $\tilde{J}_r$ is invariant under H. In fact, the inverse image of any point is exactly the H orbit when we restrict to an open, dense set $\mathcal{M}^{\mathbf{k}} \subset M_{N,r} \times M_{N,r}$ labelled by the partition $\mathbf{k} = (k_1, \ldots, k_n)$ of N (with $k_i \leq r - 1$), and defined by

$$\mathcal{M}^{\mathbf{k}} = \{(F, G) \in M_{N,r} \times M_{N,r} \mid F_i, G_i \text{ have rank } k_i\}.$$

Clearly both H and G_r^n leave $\mathcal{M}^{\mathbf{k}}$ invariant and the two actions commute. Moreover, since (F_i, G_i) are of maximal rank, it follows that:

Proposition 2.4. *H acts freely on $\mathcal{M}^{\mathbf{k}}$ and*

$$(J_r^n)^{-1}(J_r^n(F, G)) = \{h(F, G) \mid h \in H\} = \tilde{J}_r^{-1}(\tilde{J}_r(F, G)) \quad \text{for} \quad (F, G) \in \mathcal{M}^{\mathbf{k}}. \tag{2.32}$$

Combining this with Proposition 2.3, we conclude

Corollary 2.5. *The infinitesimal $\widetilde{gl(r)}^+$ action on $\mathcal{M}^{\mathbf{k}}$ reduces to an infinitesimal action on the Poisson manifold $\mathcal{M}^{\mathbf{k}}/H$ with injective moment map $\tilde{J}_{r,0} : \mathcal{M}^{\mathbf{k}}/H \to (\widetilde{gl(r)}^+)^*$ given by*

$$\tilde{J}_{r,0}([(F, G)]_H) = \tilde{J}_r((F, G)), \tag{2.33}$$

where $[(F, G)]_H \in \mathcal{M}^{\mathbf{k}}/H$ denotes the H-orbit of $(F, G) \in \mathcal{M}^{\mathbf{k}}$.

Remark. Notice that the H action and the G_r^n action are not entirely distinct since the center of G_r^n,

$$\mathcal{D} = \{(d_1 I_r, \ldots, d_n I_r) \mid d_i \in \mathbb{C} \backslash \{0\}\} \subset G_r^n \tag{2.34}$$

and that of H,

$$\mathcal{D}' = \{(d_1 I_{k_1}, \ldots, d_n I_{k_n}) \mid d_i \in \mathbb{C} \backslash \{0\}\} \subset H, \tag{2.35}$$

(where I_l denotes the $l \times l$ identity) may be identified, together with their action on $M_{N,r} \times M_{N,r}$. The corresponding Lie algebras $\mathfrak{d}_n$ and $\mathfrak{d}'_n$ may be identified with $\mathbb{C}^n$, and the moment map

$$J_{\mathcal{D}} : M_{N,r} \times M_{N,r} \to \mathfrak{d}_n^* \sim \mathbb{C}^n$$

is given by

$$J_{\mathcal{D}}((F, G)) = (\operatorname{tr} G_1^T F_1, \ldots, \operatorname{tr} G_n^T F_n). \tag{2.36}$$

Moreover, $\mathfrak{d}_n$ is the image under $\mathcal{A}$ of the center $\tilde{\mathfrak{d}} \subset \widetilde{gl(r)}^+$, the latter consisting of polynomial multiples of the $r \times r$ identity matrix, I_r. Therefore, the part of $\tilde{J}_r$ which generates this algebra is also just the trace

$$\tilde{J}_{\mathfrak{d}}(F,G)=\lambda\left(\sum_{i=1}^{n}\frac{\operatorname{tr}(G_i^T F_i)}{\alpha_i-\lambda}\right)I_r. \tag{2.37}$$

Now, let

$$A=\operatorname{diag}(\alpha_1,\ldots,\alpha_1,\alpha_2,\ldots,\alpha_2,\ldots,\alpha_n,\ldots,\alpha_n) \tag{2.38}$$

be the complex diagonal $N\times N$ matrix with eigenvalues α_i repeated k_i times. Then Eq. (2.26) can be rewritten as

$$\tilde{J}_r(F,G)=\lambda G^T(A-\lambda I)^{-1}F=-\sum_{k=0}^{\infty}G^T A^k F\lambda^{-k}. \tag{2.39}$$

Let $\mathscr{M}_0\subset M_{N,r}\times M_{N,r}$ be the open, dense submanifold

$$\mathscr{M}_0=\{(F,G)\in M_{M,r}\times M_{N,r}\,|\,F \text{ and } G \text{ have rank } t\}, \tag{2.40}$$

and let

$$\mathscr{M}_A=\{A+FG^T\,|\,(F,G)\in\mathscr{M}_0\} \tag{2.41}$$

denote the space of rank r perturbations of A. The group $G_r=GL(r,\mathbb{C})$ acts freely and properly on $\mathscr{M}_0$ and therefore gives rise to a manifold structure on the quotient space of orbits, $\mathscr{M}_0/G_r$. Since the projection $\pi:\mathscr{M}_0\to\mathscr{M}_A$ given by $(F,G)\to L_A=A+FG^T$ has as its fibers precisely these G_r orbits, we may identify $\mathscr{M}_A$ with $\mathscr{M}_0/G_r$. Through this identification $\mathscr{M}_A$ has a natural Poisson structure with functions on $\mathscr{M}_A$ interpreted as $GL(r)$ invariant functions on $\mathscr{M}_0$. In the following section we shall study hamiltonian flows on $M_{N,r}\times M_{N,r}$ which derive from $GL(r)$ invariant hamiltonians, project within $\mathscr{M}_0$ to flows on $\mathscr{M}_A$ which are isospectral, and leave $\mathscr{M}^k$ invariant.

As a final remark, notice that the groups H and $GL(r,\mathbb{C})$ act freely and properly on the open dense submanifold $\mathscr{M}=\mathscr{M}^{\mathbf{k}}\cap\mathscr{M}_0\subset M_{N,r}\times M_{N,r}$. Hence the preceding analysis of the H action on $\mathscr{M}^k$ can be equally applied to $\mathscr{M}$. The G_r^n action on $\mathscr{M}^k$, however, only gives a local action on $\mathscr{M}$, and therefore the moment map (2.14) must be interpreted in the infinitesimal sense.

3. Isospectral Flows

Poisson Commutativity and the AKS *Theorem*

We now consider flows of rank r perturbations, $L_A\equiv A+FG^T$, of the matrix A defined in Sect. 2, which are isospectral and which arise as projections of $GL(r,\mathbb{C})$ invariant hamiltonian flows in $\mathscr{M}_0$. Define $L_k=L_k(F,G)$, the elementary symmetric invariants of L_A, regarded as functions of (F,G), by:

$$\det(A+FG^T-\lambda I)\equiv\sum_{k=0}^{N-1}L_k\lambda^k+(-\lambda)^N.$$

Since the functions L_k determine the spectrum of $A+FG^T$, we are looking for flows in $M_{N,r}\times M_{N,r}$ which leave invariant all the L_k's. We shall describe a large family of such flows in this section.

To begin with we need to recall the theorem of Adler [5], Kostant [24], and Symes [44] (see also [38, 39, 20]). Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate, ad invariant inner product $\langle , \rangle$ (i.e. $\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0$). Suppose $\mathfrak{g}$ is the vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, where $\mathfrak{k}$ and $\mathfrak{l}$ are subalgebras. Then $\mathfrak{g}^*$ can be identified with $\mathfrak{k}^* \oplus \mathfrak{l}^*$ by identifying $\mathfrak{k}^*$ with $\mathfrak{l}^0$, the annihilator in $\mathfrak{g}^*$ of $\mathfrak{l}$, and $\mathfrak{l}^*$ with $\mathfrak{k}^0$. Using the inner product we can also identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Correspondingly, $\mathfrak{k}^* \sim \mathfrak{l}^0$ is identified with the orthogonal annihilator $\mathfrak{l}^\perp \subset \mathfrak{g}$ and $\mathfrak{l}^*$ with $\mathfrak{k}^\perp \subset \mathfrak{g}$. If $X \in \mathfrak{g}$ we write $X = X_{\mathfrak{k}^\perp} + X_{\mathfrak{l}^\perp}$, where $X_{\mathfrak{k}^\perp} \in \mathfrak{k}^\perp$ and $X_{\mathfrak{l}^\perp} \in \mathfrak{l}^\perp$. Using this notation we can write the ad* maps for $\mathfrak{k}^*$ and $\mathfrak{l}^*$ in terms of the bracket in $\mathfrak{g}$. Namely, for $X \in \mathfrak{k}$ and $Y \in \mathfrak{l}^\perp \sim \mathfrak{k}^*$, we get

$$\mathrm{ad}_{\mathfrak{k}}^*(X)(Y) = [X, Y]_{\mathfrak{l}^\perp}, \tag{3.2a}$$

and for $U \in \mathfrak{l}$ and $V \in \mathfrak{k}^\perp \sim \mathfrak{l}^*$, we get

$$\mathrm{ad}_{\mathfrak{l}}^*(U)(V) = [U, V]_{\mathfrak{k}^\perp}. \tag{3.2b}$$

Let $I(\mathfrak{g}^*)$ denote the ring of infinitesimally Ad* invariant functions on $\mathfrak{g}^*$, i.e. $f \in I(\mathfrak{g}^*)$ iff $\langle \mu, [\delta f/\delta\mu, X]\rangle = 0$ for all $X \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, where $\delta f/\delta\mu$ denotes $df(\mu)$ thought of as an element of $\mathfrak{g} \sim \mathfrak{g}^{**}$.

Theorem 3.1 (AKS). (1) *For $\hat{f}$ and $\hat{g} \in I(\mathfrak{g}^*)$, let f and g denote their restrictions to $\mathfrak{l}^* \sim \mathfrak{k}^0$. Then $\{f, g\}_{\mathfrak{l}^*} = 0$, where $\{ , \}_{\mathfrak{l}^*}$ is the Lie-Poisson bracket on $\mathfrak{l}^*$.*

(2) *Let $\hat{f} \in I(\mathfrak{g}^*)$ and let f be its restriction to $\mathfrak{l}^*$. Using the identifications $\mathfrak{g}^* \sim \mathfrak{g}$ and $\mathfrak{l}^* \sim \mathfrak{k}^\perp$, Hamilton's equation for the hamiltonian f on $\mathfrak{k}^\perp$ is given by*

$$\dot{X} = [d\hat{f}(X^\flat)_+, X] = -[d\hat{f}(X^\flat)_-, X], \qquad X \in \mathfrak{k}^\perp, \tag{3.3}$$

where, if $\xi \in \mathfrak{g}$, ξ_+, and ξ_- are respectively the $\mathfrak{l}$ and $\mathfrak{k}$ components of ξ, $X^\flat \in \mathfrak{g}^*$ is the point in $\mathfrak{g}^*$ corresponding to $X \in \mathfrak{g}$ under the identification $\mathfrak{g} \sim \mathfrak{g}^*$, and $d\hat{f}(X^\flat) \in \mathfrak{g}$ is the differential of $\hat{f}$ at $X^\flat$ considered as an element of $\mathfrak{g} = \mathfrak{g}^{**}$.

Now identify $\mathfrak{g} = \widetilde{gl(r)}$, $\mathfrak{k} = \widetilde{gl(r)}^-$, and $\mathfrak{l} = \widetilde{gl(r)}^+$. Part (1) of this theorem then states that elements of the ring of functions

$$\mathscr{F}_+ = \{\phi \in C^\infty((\widetilde{gl(r)}^+)^*) \mid \phi = \hat{\phi}\,|_{(\widetilde{gl(r)}^+)^*},\ \hat{\phi} \in I(\widetilde{gl(r)}^*)\} \tag{3.4}$$

commute in the Lie-Poisson structure of $(\widetilde{gl(r)}^+)^*$. Since the moment map $\tilde{J}_r : M_{N,r} \times M_{N,r} \to (\widetilde{gl(r)}^+)^*$ is a Poisson map, it follows that elements of the ring of functions

$$\mathscr{F} = \{\tilde{J}_r^*\phi \mid \phi \in \mathscr{F}_+\} \subset C^\infty(M_{N,r} \times M_{N,r}) \tag{3.5}$$

Poisson commute on $M_{N,r} \times M_{N,r}$. This leads us to our first main result.

Theorem 3.2. *$L_k \in \mathscr{F}$ for $k = 0, 1, \ldots, N-1$. Hence the hamiltonian flow for any $f \in \mathscr{F}$ (in particular, for the L_k's themselves) preserves the L_k's,* i.e. *it is isospectral.*

Proof. Identify $\widetilde{gl(r)}^*$ with $\widetilde{gl(r)}$ by the ad invariant inner product (2.19). For $X(\lambda) \in \widetilde{gl(r)}$, let $X^\flat(\lambda)$ denote the corresponding element of $\widetilde{gl(r)}^*$. Define $\hat{\phi}_k \in I(\widetilde{gl(r)}^*)$ by

$$\hat{\phi}_k(X^\flat(\lambda)) \equiv \left(p(\lambda)\det\left(I + \frac{1}{\lambda}X(\lambda)\right)\right)_k, \qquad k = 0, \ldots, N-1, \tag{3.6}$$

where $p(\lambda)=\det(A-\lambda I)$ and the right-hand side denotes the coefficient of λ^k in the formal series $p(\lambda)\det\left(I+\frac{1}{\lambda}X(\lambda)\right)$. Let $\phi_k=\hat{\phi}_k|_{(\widetilde{gl(r)}^+)^*}$. We then have,

$$L_k=\tilde{J}_r^*\phi_k\in\mathscr{F}\,. \tag{3.1}$$

This follows directly from Eq. (2.39) and the identity

$$\det(A+FG^T-\lambda I)=\det(A-\lambda I)\det(I+G^T(A-\lambda I)^{-1}F)\,. \quad \square \tag{3.8}$$

Although we defined the L_k's as functions on $M_{N,r}\times M_{N,r}$ they may also be regarded as functions on the manifold $\mathscr{M}_A=\{A+FG^T\,|\,(F,G)\in\mathscr{M}_0\}$ [recall that $\mathscr{M}_0\subset M_{N,r}\times M_{N,r}$ is the open dense submanifold consisting of (F,G)'s that have maximal rank]. In fact this may be done for all functions in $\mathscr{F}$.

Proposition 3.3. *If $f\in\mathscr{F}$, then f is $GL(r,\mathbb{C})$ invariant. Hence, $\mathscr{F}|_{\mathscr{M}_0}$ is projectable to a ring of functions $\mathscr{F}_A$ on $\mathscr{M}_A$ whose elements commute in the Poisson structure for $\mathscr{M}_A$.*

Proof. Let $GL(r,\mathbb{C})$ act in the natural way on $\widetilde{gl(r)}^*\sim\widetilde{gl(r)}$, i.e.

$$g\left(\sum_{i=-\infty}^{m}X_i\lambda^i\right)=\sum_{i=-\infty}^{m}(\mathrm{Ad}_gX_i)\lambda^i\,,\qquad g\in GL(r)\,.$$

Notice this action leaves $(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-$ invariant. If $\hat{\phi}\in I(gl(r)^*)$ it is invariant under this action, hence its restriction $\phi=\hat{\phi}|_{(\widetilde{gl(r)}^+)^*}$ is also invariant under the action. It is clear from (2.26) that $\tilde{J}_r$ intertwines the diagonal $G_r=GL(r,\mathbb{C})$ action (2.15) on $M_{N,r}\times M_{N,r}$ with the restriction of the above action to $(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-$, i.e.

$$\tilde{J}_r\cdot g=g\cdot\tilde{J}_r\,. \tag{3.9}$$

Thus $\tilde{J}_r^*\phi$ must be invariant under the G_r action. $\square$

It also follows from Proposition 2.4 that $\mathscr{F}|_{\mathscr{M}^{\mathbf{k}}}$ projects to a ring of functions $\tilde{\mathscr{F}}$ on $\mathscr{M}^{\mathbf{k}}/H$ which still Poisson commute. In fact, since we have an injective moment map $\tilde{J}_{r,0}:\mathscr{M}^{\mathbf{k}}/H\to(\widetilde{gl(r)}^+)^*$ we can use part (2) of the AKS Theorem 3.1 to get the equations of motion on $\mathscr{M}^{\mathbf{k}}/H$.

Proposition 3.4. *Let $\phi=\hat{\phi}|_{(\widetilde{gl(r)}^+)^*}$ for some $\hat{\phi}\in I(\widetilde{gl(r)}^*)$ and let $f=\tilde{J}_{r,0}^*\phi$. Let X_f be the hamiltonian vector field for f on $\mathscr{M}^{\mathbf{k}}/H$. Then X_f at the point $[(F,G)]_H\in\mathscr{M}^{\mathbf{k}}/H$ is defined by*

$$(\tilde{J}_{r,0*}X_f)(\mathscr{N})=[(d\hat{\phi}(\mathscr{N}))_+,\mathscr{N}]=-[(d\hat{\phi}(\mathscr{N}))_-,\mathscr{N}]\,, \tag{3.10}$$

where $d\hat{\phi}(\mathscr{N})=(d\hat{\phi}(\mathscr{N}))_+-(d\hat{\phi}(\mathscr{N}))_-$ is the splitting of $d\hat{\phi}(\mathscr{N})\in\widetilde{gl(r)}$ into its $\widetilde{gl(r)}^+$ and $\widetilde{gl(r)}^-$ pieces, and

$$N=N(\lambda)=\tilde{J}_r(F,G)\in(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-\,. \tag{3.11}$$

We remark that it is also straightforward to write the equations of motion for $\tilde{J}_r^*\phi$ on all of $M_{N,r}\times M_{N,r}$ by the methods used to describe collective motion in reference [18].

Shifted Hamiltonians

A slight generalization of the above formulation is possible which allows us to give isospectral deformations of matrices of the form

$$L_A(a) \equiv A + FaG^T \tag{3.12}$$

for some fixed $a \in GL(r, \mathbb{C})$. The functions $\mathscr{F}$ given by the AKS theorem generally do not preserve the spectrum of this matrix. We can however describe a similar collection of flows which are isospectral for (3.12) by using a generalization of the AKS theorem due to Reyman et al. [40].

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{l}$ be as in the AKS theorem. An element Y of $\mathfrak{k}^* \sim \mathfrak{l}^\perp$ is called an (infinitesimal) character of $\mathfrak{k}$ if $\mathrm{ad}_{\mathfrak{k}}^*(X)(Y) = [X, Y]_{\mathfrak{l}^\perp} = 0$ for all $X \in \mathfrak{k}$. For $\hat{\phi} \in I(\mathfrak{g}^*)$ let $\hat{\phi}_Y$ be defined by

$$\hat{\phi}_Y(\mu) = \hat{\phi}(\mu + Y), \qquad \mu \in \mathfrak{g}^* \tag{3.13}$$

and let $\phi_Y = \hat{\phi}|_{\mathfrak{l}^*}$.

Theorem 3.5 (RSTSF). (1) *Let $\hat{\phi}$ and $\hat{\psi}$ be in $I(\mathfrak{g}^*)$, then $\{\phi_Y, \psi_Y\}_{\mathfrak{l}^*} = 0$, where $\{\,,\}_{\mathfrak{l}^*}$ is the Lie-Poisson bracket on $\mathfrak{l}^*$.*

(2) *If $\hat{\phi} \in I(\mathfrak{g}^*)$ Hamilton's equations for ϕ_Y through $X \in \mathfrak{l}^* \sim \mathfrak{k}^\perp$ are given by*

$$\dot{X} = [d\hat{\phi}((X+Y)^\flat)_+, X+Y] = -[d\hat{\phi}((X+Y)^\flat)_-, X+Y], \tag{3.14}$$

where $(X+Y)^\flat$ is the point in $\mathfrak{g}^*$ corresponding to $X + Y \in \mathfrak{g}$.

To apply this theorem to our situation, first notice that any $Y(\lambda) \in (\widetilde{gl(r)}^-)^* = \lambda \widetilde{gl(r)}^+$ of the form $Y(\lambda) \equiv \lambda Y$, $Y \in gl(r, \mathbb{C})$, is a character of $\widetilde{gl(r)}^-$. Thus for $Y \in gl(r, \mathbb{C})$ and $\hat{\phi} \in I(\widetilde{gl(r)}^*)$, we define

$$\hat{\phi}_Y(X(\lambda)) = \hat{\phi}(X(\lambda) + \lambda Y) \tag{3.15}$$

and set $\phi_Y = \hat{\phi}_Y|_{(\widetilde{gl(r)}^+)^*}$. If $\hat{\phi}$ and $\hat{\psi}$ are in $I(\widetilde{gl(r)}^*)$ we see that, since $\tilde{J}_r$ is a Poisson map, part (1) of Theorem 3.5 implies that $\tilde{J}_r^* \phi_Y$ and $\tilde{J}_r^* \psi_Y$ Poisson commute on $M_{N,r} \times M_{N,r}$. Define the Poisson commutative rings,

$$\mathscr{F}_+^Y \equiv \{\phi_Y \in C^\infty((\widetilde{gl(r)}^+)^*) \mid \phi_Y = \hat{\phi}_Y|_{(\widetilde{gl(r)}^+)^\perp},\ \hat{\phi} \in I(\widetilde{gl(r)}^*)\} \tag{3.16a}$$

and

$$\mathscr{F}^Y = \{\tilde{J}_r^* \phi_Y \mid \hat{\phi} \in \mathscr{F}_+^Y\} \subset C^\infty(M_{N,r} \times M_{N,r}). \tag{3.16b}$$

Theorem 3.6. *Let $Y \in gl(r, \mathbb{C})$ be such that $I + Y$ is invertible. Let $f \in \mathscr{F}^Y$. Then the hamiltonian flow of f on $M_{N,r} \times M_{N,r}$ preserves the spectrum of*

$$L_A(a) = A + FaG^T,$$

where $a = (I+Y)^{-1}$.

Proof. This follows as in Theorem 3.2, by noting that the elementary symmetric invariants $L_k(a)$, defined by

$$\det(A + FaG^T - \lambda I) \equiv \sum_{k=0}^{N-1} L_k(a)(F, G)\lambda^k + (-\lambda)^N$$

may be expressed:

$$L_k(a) = \tilde{J}_r^* \psi_{k,Y} \in F^y,$$

$$\text{where } \hat{\psi}_k(X^\flat(\lambda)) = a(\lambda)\hat{\phi}_k(X^\flat(\lambda)), \qquad X^\flat(\lambda) \in gl(r)^*.$$

To apply part (2) of Theorem 3.5 we use the one-to-one moment map $\tilde{J}_{r,0}: \mathcal{M}^{\mathbf{k}}/H \to (gl(r)^+)^*$ to get a result corresponding to Proposition 3.4; namely,

Proposition 3.7. *Let $f_Y = \tilde{J}_{r,0}^* \phi_Y$ for some $\bar{\phi} \in I(\widetilde{gl(r)}^*)$. Let X_{f_Y} be the hamiltonian vector field for f_Y on $\mathcal{M}^{\mathbf{k}}/H$. Then X_{f_Y} is defined by*

$$\tilde{J}_{r,0*} X_{f_Y}(N) = [(d\hat{\phi}(\mathcal{N} + \lambda Y))_+, \mathcal{N} + \lambda Y] = -[(d\hat{\phi}(\mathcal{N} + \lambda Y))_-, \mathcal{N} + \lambda Y], \tag{3.17}$$

where $\mathcal{N} = \mathcal{N}(\lambda)$ is $\tilde{J}_r(F, G)$ and $(F, G) \in \mathcal{M}^{\mathbf{k}}$.

Proposition 3.3 does not generalize to this case so easily. What turns out to be the most useful generalization of Proposition 3.3 is the following.

Proposition 3.8. *For $a = (I + Y)^{-1}$ let $GL(r, \mathbb{C})_a = \{g \in GL(r, \mathbb{C}) \,|\, gag^{-1} = a\}$ denote the stabilizer of $a \in GL(r, \mathbb{C})$ under conjugation. Then the functions $\mathcal{F}^Y$ are $GL(r, \mathbb{C})_a$ invariant.*

Proof. Notice that for $g \in GL(r, \mathbb{C})_a$ we have $gYg^{-1} = Y$, and hence for $\phi_Y \in \mathcal{F}_+^Y$, $g \in GL(r, \mathbb{C})_a$ we see

$$\phi_Y(g\mathcal{N}(\lambda)g^{-1}) = \phi_Y(\mathcal{N}(\lambda)).$$

Since $\tilde{J}_r$ is $GL(r, \mathbb{C})$ equivariant, it follows that $\tilde{J}_r^* \phi_Y$ is $GL(r, \mathbb{C})_a$ invariant. □

4. Integrability, Spectral Curves, and Linearization

We say that a hamiltonian function h on a symplectic manifold S, ω is *completely integrable* if there exists a ring of functions $\mathcal{F}$ containing h such that

1) $\{f, g\} = 0$ for all $f, g \in \mathcal{F}$.

2) For any $p \in S$ the simultaneous level set $L_p = \{x \in S \,|\, f(x) = f(p) \text{ for all } f \in \mathcal{F}\}$ is a submanifold of S.

3) For all $p \in S$ the hamiltonian vector fields $X_f(p)$, $f \in \mathcal{F}$, span the tangent space of L_p.

Such a ring $\mathcal{F}$ will be referred to as a completely integrable ring of functions.

Remark. Notice that 1)–3) are equivalent to assuming that the submanifolds L_p are Lagrangian. When S is finite dimensional this agrees with the usual definition of complete integrability because under the assumptions 1, 2, 3 we may always choose a set of $1/2 \dim S$ locally independent generators $\{f_i\}$ of $\mathcal{F}$ whose domain of independence extends to an open set in S consisting of a union of L_p's.

In Sect. 3 we described a ring of Poisson commuting functions $\mathcal{F}$ on $M_{N,r} \times M_{N,r}$ whose hamiltonian flows through the point (F, G) leave invariant the spectrum of the matrix $A + FG^T$. The functions in $\mathcal{F}$ are invariant under the action of H and $GL(r, \mathbb{C})$ on $M_{N,r} \times M_{N,r}$. Since the $GL(r, \mathbb{C})$ action commutes with the H action, and maps $\mathcal{M}^{\mathbf{k}} \subset M_{N,r} \times M_{N,r}$ to itself, it reduces to a hamiltonian $GL(r, \mathbb{C})$

action on the Poisson manifold $\mathcal{M}^{\mathbf{k}}/H$ with moment map

$$J_{r,0}:\mathcal{M}^{\mathbf{k}}/H \to (gl(r,\mathbb{C}))^*$$

given by

$$J_{r,0}([(F,G)]_H) = J_r((F,G)) = -G^T F, \tag{4.1}$$

where $(F,G)\in\mathcal{M}^{\mathbf{k}}$ and $[(F,G)]_H\in\mathcal{M}^{\mathbf{k}}/H$ denotes the H orbit through (F,G). The ring $\mathcal{F}$ reduces to a Poisson commuting ring $\tilde{\mathcal{F}}$ of $GL(r,\mathbb{C})$ invariant functions on $\mathcal{M}^{\mathbf{k}}/H$. Let $S_{[(F,G)]_H}\subset\mathcal{M}^{\mathbf{k}}/H$ denote the symplectic leaf in the Poisson manifold $\mathcal{M}^{\mathbf{k}}/H$ through $[(F,G)]_H$ and let $\bar{S}_{[(F,G)]_H}$ denote the Marsden-Weinstein reduction of $S_{[(F,G)]_H}$ through $[(F,G)]_H$ by the $GL(r,\mathbb{C})$ action, i.e.

$$\bar{S}_{[(F,G)]_H} = [(J_{r,0})^{-1}(J_{r,0}([(F,G)]_H))\cap S_{[(F,G)]_H}]/G_{J_r(F,G)},$$

where $G_{J_r(F,G)}\subset GL(r,\mathbb{C})$ is the isotropy group of $J_r(F,G)=J_{r,0}([(F,G)]_H)$. This can be interpreted simply as the Marsden-Weinstein reduction under H of the symplectic leaf of $\mathcal{M}^k/GL(r,\mathbb{C})$ through $G^T F$.

The main result of this section is contained in the following theorem.

Theorem 4.1. *Suppose that k_i, the multiplicity of the eigenvalue α_i for the matrix A, is equal to $r-1$ for all i. Then there exists an open dense submanifold $\tilde{\mathcal{M}}\subset\mathcal{M}^{\mathbf{k}}$ such that for $(F,G)\in\tilde{\mathcal{M}}$ the ring $\tilde{\mathcal{F}}_{[(F,G)]_H}$ is completely integrable on $\bar{S}_{[(F,G)]_H}$. The flows of the ring $\tilde{\mathcal{F}}_{[(F,G)]_H}$ linearize on the Jacobi variety of an $r-1$ sheeted algebraic curve.*

Remark. This theorem is valid in greater generality (involving all values $1\leqq k_i\leqq r-1$). The proof for the more general case, which involves desingularization of singular curves, will be left to the sequel [4, 23].

Before giving the proof of Theorem 4.1 we note that as a corollary we can construct a ring of Poisson commuting functions $\mathcal{G}$ on $\tilde{\mathcal{M}}$, containing $\mathcal{F}$, which is completely integrable on an open dense subset of $\tilde{\mathcal{M}}$. To do this we need the following generalization of a theorem of Mischenko and Fomenko [29]. A related theorem may also be found in [19].

Theorem 4.2. *Let S,ω be a symplectic manifold with a hamiltonian action of the semisimple Lie group G, with moment map $J:S\to\mathfrak{g}^*$. Assume that J is a submersion and S/G is a manifold. Let $\mathcal{F}$ be a G invariant ring of functions on S which projects to a completely integrable ring of functions on the symplectic leaves of S/G. Then there exists a ring of functions $\mathcal{G}$ containing $\mathcal{F}$ which is completely integrable on an open dense set of S.*

Proof. For a semisimple Lie algebra of rank k and dimension $2n+k$ Mischenko and Fomenko [30] have shown that there are n functions on $\mathfrak{g}^*$ which Poisson commute and are independent on generic orbits in $\mathfrak{g}^*$. Let $\mathcal{H}$ be the ring of functions on $\mathfrak{g}^*$ generated by these n functions. Let $\mathcal{G}$ be the ring of functions on S generated by $J^*\mathcal{H}\cup\mathcal{F}\cup J^*(I(\mathfrak{g}^*))$. Since the hamiltonian vector fields generated by functions in $J^*\mathcal{H}\cup J^*(I(\mathfrak{g}^*))$ are all tangential to the G orbits the functions in $\mathcal{G}$ all Poisson commute. Since J is a submersion, the G action is locally free so the generic symplectic leaves of S/G have dimension equal to $\dim S-\dim G-\operatorname{rank}G$.

Hence $\mathcal{F}$ locally generates $1/2(\dim S-\dim G-\operatorname{rank}G)$ independent hamiltonian vector fields on S/G. Since $\operatorname{rank}G$ independent elements of $J^*(I(\mathfrak{g}^*))$ give trivial flows on S/G it follows that $\mathscr{F} \cup J^*(I(\mathfrak{g}^*))$ is locally generated by $1/2(\dim S - \dim G + \operatorname{rank} G)$ independent functions. $\mathscr{H}$ is generated by $1/2(\dim G - \operatorname{rank} G)$ independent functions. Since J is a submersion and the generic orbits in $\mathfrak{g}^*$ fill out an open dense set of $\mathfrak{g}^*$, there must be an open dense set of S on which $J^*\mathscr{H}$ is still generated by $1/2(\dim G - \operatorname{rank} G)$ independent functions. Since the projection to S/G of a hamiltonian flow generated by an element of $J^*\mathscr{H}$ is trivial, and $\mathscr{H}$ is independent from $I(\mathfrak{g}^*)$, it follows that $J^*\mathscr{H}$ is independent from $\mathscr{F} \cup J^*(I(\mathfrak{g}^*))$. We conclude that $\mathscr{G}$ is locally generated by $1/2 \dim S$ independent functions on an open dense subset of S. □

This theorem may be applied in conjunction with Theorem 4.1 to conclude that the hamiltonian functions in $\mathscr{F}$ generate completely integrable flows in $\mathscr{M}^{\mathbf{k}}$ itself. To see this, note that the reduced spaces $\bar{S}_{[(F,G)]_H}$ which arise by two sequential Marsden-Weinstein reductions [first by H and then by $GL(r, \mathbb{C})$] may be obtained in a single step by reducing under the action of the product $H \times GL(r, \mathbb{C})$. Since this group does not act effectively, it is sufficient to use $H \times SL(r, \mathbb{C})$ instead. There is an H-invariant open dense submanifold $\tilde{\mathscr{M}} \subset \mathscr{M}^{\mathbf{k}}$ in which this action is locally free, and hence the corresponding moment map:

$$J_{H \times SL(r,\mathbb{C})} : \tilde{\mathscr{M}} \to \mathfrak{h}^* \oplus sl(r, \mathbb{C})^*,$$

$$J_{H \times SL(r,\mathbb{C})} : (F, G) \to (J_H(F, G), J_r(F, G) - 1/r \operatorname{tr} J_r(F, G) I)$$

is a submersion. The symplectic leaves of $\tilde{\mathscr{M}}/H$ are just $\{\bar{S}_{[F,G]_H}\}$ and hence the conditions of Theorem 4.2 are satisfied. □

We now turn to the proof of Theorem 4.1. This is accomplished by reducing to an equivalent theorem in $(\widetilde{gl(r)}^+)^*$. Recall that the moment map $\tilde{J}_{r,0} : \mathscr{M}^{\mathbf{k}}/H \to (\widetilde{gl(r)}^+)^*$ is injective. This can only happen if $\tilde{J}_{r,0}$ maps symplectic leaves in $\mathscr{M}^{\mathbf{k}}/H$ to symplectic leaves in $(\widetilde{gl(r)}^+)^*$ (with the Lie-Poisson structure).

Proposition 4.3. *The map* $\tilde{J}_{r,0} : \mathscr{M}^{\mathbf{k}}/H \to (\widetilde{gl(r)}^+)^*$ *has its image in the finite dimensional subspace*

$$(\widetilde{gl(r)}_A^+)^* = \left\{ \sum_{i=1}^{n} \frac{\lambda \mu_i}{\lambda - \alpha_i} \right\}.$$

The image consists of the Poisson submanifold

$$(\widetilde{gl(r)}^+)_{\mathbf{k}}^* = \left\{ \sum \frac{\lambda \mu_i}{\lambda - \alpha_i}, \ \operatorname{rank} \mu_i = k_i \right\},$$

and the map $\tilde{J}_{r,0} : \mathscr{M}^{\mathbf{k}}/H \to (\widetilde{gl(r)}^+)_{\mathbf{k}}^*$ *is a diffeomorphism which preserves the Poisson bracket. Thus, the restriction of* $\tilde{J}_{r,0}$ *to any symplectic leaf of* $\mathscr{M}^{\mathbf{k}}/H$ *is a symplectomorphism to the corresponding symplectic leaf in* $(\widetilde{gl(r)}^+)_{\mathbf{k}}^* \subset (\widetilde{gl(r)}^+)^*$. *The hamiltonian flow of a function in* $\mathscr{F}$ *leaves* $\mathscr{M}^{\mathbf{k}}$ *invariant.*

Remark. The tangent bundle to the symplectic leaves in $(\widetilde{gl(r)}^+)_A^*$ is generated by the $\operatorname{ad}_{gl(r)}^*$ vector fields, and therefore it is reasonable to refer to these leaves as Ad* orbits, even though the corresponding transformation group is only well defined on these orbits.

Proof. $\tilde{J}_{r,0}(F,G)=\sum_{i=1}^{n}\lambda\frac{G_i^T F_i}{\alpha_i-\lambda}\in(\widetilde{gl(r)}_A^+)^*$, since $G_i^T F_i$ has rank k_i. Any rank k_i matrix μ_i can be expressed $\mu_i=G_i^T F_i$, where the pair of rank k_i matrices (F_i, G_i) are determined up to the equivalence

$$(F_i, G_i)\sim(h_i F_i, (h_i^T)^{-1}G_i), \qquad h_i\in GL(k_i).$$

Thus the map $\tilde{J}_{r,0}$ is invertible on $(\widetilde{gl(r)}^+)^*_{\mathbf{k}}$ and hence a diffeomorphism. The equivariance of $\tilde{J}_{r,0}$ implies it is a Poisson map and therefore carries symplectic leaves in $\mathcal{M}^{\mathbf{k}}/H$ to those in $(\widetilde{gl(r)}^+)^*_{\mathbf{k}}$. Since the symplectic leaves in $\mathcal{M}^{\mathbf{k}}/H$ are just the $G_r^n(r)$ orbits, the corresponding orbits in $(gl(r)^+)^*_{\mathbf{k}}$ are of the form $\left\{\sum_{i=1}^{n}\frac{g_i\mu_i g_i^{-1}}{\alpha_i-\lambda}, g_i\in GL(r)\right\}$. Since any function $f\in\mathcal{F}$ is the pull-back of one in $(\widetilde{gl(r)}^+)^*_{\mathbf{k}}$, its hamiltonian vector field lies in the module generated by the infinitesimal $\mathfrak{g}_r^n$ action, and hence is tangential to $\mathcal{M}^{\mathbf{k}}$. □

Now let $\mathcal{N}(\lambda)=\tilde{J}_r(F,G)$. By Proposition 4.3 we can identify

$$S_{[(F,G)]_H}\sim\mathcal{O}_{\mathcal{N}(\lambda)}, \tag{4.2}$$

where $\mathcal{O}_{\mathcal{N}(\lambda)}$ denotes the ad* orbit in $(\widetilde{gl(r)}^+)^*$ through $\mathcal{N}(\lambda)$. The map $\tilde{J}_{r,0}$ intertwines the $GL(r,\mathbb{C})$ action on $\mathcal{M}^k/H$ with the natural $GL(r,\mathbb{C})$ action on $(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-$ given by

$$g:\sum_{i=-\infty}^{0}X_i\lambda^i\to\sum_{i=-\infty}^{0}(\mathrm{Ad}_g X_i)\lambda^i, \qquad g\in GL(r,\mathbb{C}).$$

This action on $(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-$ is hamiltonian with moment map

$$J:\widetilde{gl(r)}_0^-\to gl(r,\mathbb{C})^*\sim gl(r,\mathbb{C})$$

defined by

$$J\left(\sum_{i=-\infty}^{0}X_i\lambda^i\right)=X_0. \tag{4.3}$$

We thus have $J_{r,0}=J\circ\tilde{J}_{r,0}$. Letting $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$ denote the Marsden-Weinstein reduction of $\mathcal{O}_{\mathcal{N}(\lambda)}$ through $\mathcal{N}(\lambda)$ by this $GL(r,\mathbb{C})$ action, we can identify

$$\bar{S}_{[(F,G)]_H}\sim\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}.$$

Since the ring of functions $\tilde{F}$ is given by the pullback by $\tilde{J}_{r,0}$ of the ring of functions $\mathcal{F}_+$ on $(\widetilde{gl(r)}^+)^*$ we see that Theorem 4.1 is equivalent to

Theorem 4.4. *Suppose $k_i=r-1$ for all i. For $(F,G)\in\tilde{\mathcal{M}}$ let $\mathcal{N}(\lambda)=\tilde{J}_r(F,G)$. The ring of functions F_+, reduced to $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$, is completely integrable on $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$.*

A theorem equivalent to this is proved by Reyman and Semenov-Tian-Shansky in [39] using the theory of Krichever [25, 26] (cf. also van Moerbeke and Mumford [31] concerning linearization of flows in Jacobi varieties). For the sake of completeness we give a brief summary of this theory here.

Let $\mathcal{L}(\lambda)\in\widetilde{gl(r)}_0^-$ be a polynomial in λ^{-1}, i.e.

$$\mathcal{L}(\lambda)=\mathcal{L}_0+\mathcal{L}_1\lambda^{-1}+\ldots+\mathcal{L}_{n-1}\lambda^{-n+1}.$$

Assume $\mathscr{L}(\lambda)$ satisfies the following genericity conditions:

1) The spectrum of $\mathscr{L}(\lambda)$ is simple for all but finitely many λ.
2) $\mathscr{L}_0$ and $\mathscr{L}_{n-1}$ have simple spectrum.
3) The affine curve in $\mathbb{C}^2$ described by $\det(\lambda^{n-1}\mathscr{L}(\lambda)-z)=0$ is nonsingular and irreducible. (4.4)

Let X be the smooth compactification of the affine curve $\det(\lambda^{n-1}\mathscr{L}(\lambda)-z)=0$. X is called the *spectral curve* of $\mathscr{L}$. If $x\in X$ is not a branch point there is a unique one dimensional eigenspace $E_{\mathscr{L}}(x)\subset\mathbb{C}^r$ of $\lambda^{n-1}(x)\mathscr{L}(\lambda(x))$ with eigenvalue $z(x)$, i.e. $\lambda^{n-1}(x)\mathscr{L}(\lambda(x))v(x)=z(x)v(x)$, where $v(x)\in E_L(x)$. Since X is smooth this extends across the branch points to give a holomorphic line bundle $E_{\mathscr{L}}\to X$.

Proposition 4.5 [39]. a) *X has genus $g=\frac{1}{2}r(r-1)(n-1)-r+1$,*
b) *$E_{\mathscr{L}}$ has degree $-g-r+2$.*

Now let $\mathcal{O}_{\mathscr{L}}\subset\widetilde{gl(r)}^-\sim(\widetilde{gl(r)}^+)^*$ be the symplectic leaf through $\mathscr{L}$ with respect to the Lie-Poisson structure of $(\widetilde{gl(r)}^+)^*$. The elements of $\mathcal{O}_{\mathscr{L}}$ are polynomials in λ^{-1} of degree $n-1$. Let $T_{\mathscr{L}}\subset\mathcal{O}_{\mathscr{L}}$ denote the elements of $\mathcal{O}_{\mathscr{L}}$ which are isospectral with $\mathscr{L}$, i.e. $\mathscr{M}\in\mathcal{O}_{\mathscr{L}}$ is in $T_{\mathscr{L}}$ if and only if $\hat{\phi}(\mathscr{L})=\hat{\phi}(\mathscr{M})$ for all $\hat{\phi}\in I(\widetilde{gl(r)}^*)$. Thus $\mathscr{M}\in T_{\mathscr{L}}$ has the same spectral curve as $\mathscr{L}$. The construction above gives a line bundle $E_{\mathscr{M}}\to X$ of degree $-g-n+2$. The degree zero line bundles over X are isomorphic to the Jacobian, $\mathscr{J}_X$, of X. Thus we can define a map

$$I: T_{\mathscr{L}}\to\mathscr{J}_X$$

by

$$I(\mathscr{M})=E^*_{\mathscr{L}}\otimes E_{\mathscr{M}},$$

where $E^*_{\mathscr{L}}$ is the dual bundle of $E_{\mathscr{L}}$.

Now take $\phi\in\mathscr{F}_+$ and let $\mathscr{L}_t(\lambda)$ denote the integral curve of the hamiltonian ϕ with initial point $\mathscr{L}_0(\lambda)=\mathscr{L}(\lambda)$. Notice $\mathscr{L}_t(\lambda)\in T_{\mathscr{L}}$ for all t since the ring of functions $\mathscr{F}_+$ Poisson commutes.

Theorem 4.6. [39]. a) *$I(\mathscr{L}_t(\lambda))$ is a one parameter subgroup of $\mathscr{J}_X$.*
b) *Every one parameter subgroup of $\mathscr{J}_X$ can be realized this way.*

From this theorem it would follow that $\mathscr{F}_+$ is completely integrable if we knew that $I: T_{\mathscr{L}}\to\mathscr{J}_X$ were bijective. However, this is not the case. Recall that the functions in $\mathscr{F}_+$ are invariant under the $GL(r,\mathbb{C})$ action on $(\widetilde{gl(r)}^+)^*\sim\widetilde{gl(r)}_0^-$. Thus this action maps $T_{\mathscr{L}}$ to itself.

Proposition 4.7 [39]. *The fibers of I are the $GL(r,\mathbb{C})$ orbits in $T_{\mathscr{L}}$.*

Let $\bar{\mathcal{O}}_{\mathscr{L}}=J^{-1}(\mathscr{L}_0)/G_{\mathscr{L}_0}$ denote the Marsden-Weinstein reduction of the orbit $\mathcal{O}_{\mathscr{L}}$ through $\mathscr{L}(\lambda)$, where $G_{\mathscr{L}_0}$ is the isotropy group of $\mathscr{L}_0$ in $GL(r,\mathbb{C})$.

Corollary 4.8 [39]. *The ring of functions $\mathscr{F}_+$, reduced to $\bar{\mathcal{O}}_{\mathscr{L}}$, is completely integrable on $\bar{\mathcal{O}}_{\mathscr{L}}$.*

We now apply this theorem to prove Theorem 4.4. Because we assume $k=r-1$, i.e. each of the eigenvalues α_i is repeated $r-1$ times, for $(F,G)\in\mathscr{M}^k$,

$\mathcal{N}(\lambda)=\tilde{J}_r(F,G)$ has the form

$$\mathcal{N}(\lambda)=\lambda\sum_{i=1}^{n}\frac{\mu_i}{\lambda-\alpha_i}, \tag{4.5}$$

where $\mu_i\in gl(r,\mathbb{C})$ has rank $r-1$. Let

$$\mathcal{L}(\lambda)=\lambda^{-n}a(\lambda)\mathcal{N}(\lambda), \tag{4.6}$$

where $a(\lambda)=\prod_{i=1}^{n}(\lambda-\alpha_i)$. Then $\mathcal{L}(\lambda)$ is a polynomial in λ^{-1} of the form

$$\mathcal{L}(\lambda)=\mathcal{L}_0+\ldots+\mathcal{L}_{n-1}\lambda^{-n+1}$$

with $\det(\lambda^{n-1}\mathcal{L}(\lambda))$ having simple zeros at $\lambda=\alpha_i$, $i=1,\ldots,n$. If we let (F,G) vary in $\mathcal{M}^{\mathbf{k}}$ the μ_i's can be arbitrary rank $r-1$ matrices so we are able to get arbitrary polynomials $\mathcal{L}(\lambda)=\mathcal{L}_0+\ldots+\mathcal{L}_{n-1}\lambda^{-n+1}$ with $\det(\lambda^{n-1}\mathcal{L}(\lambda))$ having simple zeros at $\lambda=\alpha_i$, $i=1,\ldots,n$. Since the conditions (4.4) on $\mathcal{L}(\lambda)$ are generic we conclude that there must be an open dense set $\tilde{\mathcal{M}}\subset\mathcal{M}^{\mathbf{k}}$ such that $\mathcal{L}(\lambda)$ satisfies (4.3) as long as $(F,G)\in\tilde{\mathcal{M}}$.

Let $\mathcal{O}_{\mathcal{N}(\lambda)}$ and $\mathcal{O}_{\mathcal{L}(\lambda)}$ be the symplectic leaves of $(\widetilde{gl(r)}^+)^*$ through $\mathcal{N}(\lambda)$ and $\mathcal{L}(\lambda)$ respectively. Let $T_{\mathcal{N}(\lambda)}$ and $T_{\mathcal{L}(\lambda)}$ be the intersections of $\mathcal{O}_{\mathcal{N}(\lambda)}$ and $\mathcal{O}_{\mathcal{L}(\lambda)}$ with the set

$$\{X(\lambda)\in(\widetilde{gl(r)}^+)^*\mid\phi(X(\lambda))=\phi(\mathcal{N}(\lambda))\text{ for all }\phi\in\mathcal{F}_+\},$$

and let $\bar{T}_{\mathcal{N}(\lambda)}$ and $\bar{T}_{\mathcal{L}(\lambda)}$ denote the reductions of $T_{\mathcal{N}(\lambda)}$ and $T_{\mathcal{L}(\lambda)}$ by the $GL(r,\mathbb{C})$ action on $(\widetilde{gl(r)}^+)^*$ at the point $J(\mathcal{N}(\lambda))=\mathcal{N}_0=\mathcal{L}_0=J(\mathcal{L}(\lambda))$. Corollary 4.8 says that the reduced flows of $\mathcal{F}_+$ through $\mathcal{L}(\lambda)$ span a neighborhood in $\bar{T}_{\mathcal{L}(\lambda)}$.

Proposition 4.9. *Take $\phi\in\mathcal{F}_+$ and consider the hamiltonian flow $\mathcal{L}_t(\lambda)$ for ϕ with $\mathcal{L}_0(\lambda)=\mathcal{L}(\lambda)$. There is a $\psi\in\mathcal{F}_+$ whose hamiltonian flow $\mathcal{N}_t(\lambda)$ with $\mathcal{N}_0(\lambda)=\mathcal{N}(\lambda)$ satisfies*

$$\lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda)=\mathcal{L}_t(\lambda),\quad \textit{all } t.$$

Proof. First take ϕ to be of the form $\hat{\phi}_{jk}|_{(\widetilde{gl(r)}^+)^*}$, where

$$\hat{\phi}_{jk}(X(\lambda))=\frac{1}{k}\operatorname{tr}((\lambda^jX^k(\lambda))_0),\qquad X(\lambda)\in\widetilde{gl(r)}.$$

Then $d\hat{\phi}_{jk}(X(\lambda))=\lambda^jX^{k-1}(\lambda)$, so $\mathcal{L}_t(\lambda)$ is determined by

$$\frac{d}{dt}\mathcal{L}_t(\lambda)=[(\lambda^j\mathcal{L}_t^{k-1}(\lambda))_+,\mathcal{L}_t(\lambda)],\qquad \mathcal{L}_0(\lambda)=\mathcal{L}(\lambda).$$

Define

$$\hat{\psi}_{jk}(X(\lambda))=\frac{1}{k}\operatorname{tr}((\lambda^j(\lambda^{-n}a(\lambda))^{k-1}X^k(\lambda))_0)$$

then

$$d\hat{\psi}_{jk}(X(\lambda))=\lambda^j(\lambda^{-n}a(\lambda))^{k-1}X^{k-1}(\lambda)$$

so $\mathcal{N}_t(\lambda)$ satisfies

$$\frac{d}{dt}\mathcal{N}_t(\lambda)=[(\lambda^j(\lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda))^{k-1})_+,\mathcal{N}_t(\lambda)].$$

Multiplying by $\lambda^{-n}a(\lambda)$ gives

$$\frac{d}{dt}(\lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda)) = [(\lambda^j(\lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda))^{k-1})_+, \lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda)],$$

hence $\lambda^{-n}a(\lambda)\mathcal{N}_t(\lambda) = \mathcal{L}_t(\lambda)$.

To prove the proposition for general $\phi \in \mathcal{F}_+$ we notice that the ϕ_{jk}'s determine the spectrum of $\mathcal{L}(\lambda)$, and hence they generate the ring $\mathcal{F}_+$ when restricted to $T_{\mathcal{L}(\lambda)}$. □

Since multiplication by $\lambda^{-n}a(\lambda)$ is injective on $\widetilde{gl(r)}_0^-$, we conclude that the reduced flows of $\mathcal{F}_+$ through the point $\mathcal{N}(\lambda)$ span a space of dimension $\frac{1}{2}r(r-1)(n-1)-r+1$ in $\bar{T}_{\mathcal{N}(\lambda)}$, i.e. there are at least $\frac{1}{2}r(r-1)(n-1)-r+1$ functions in $\mathcal{F}_+$ which are independent on $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$. If we show $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$ has dimension $r(r-1)(n-1)-2r+2$ we can conclude $\mathcal{F}_+$ is completely integrable on $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)}$. Recall that Eq. (4.1) gives

$$\mathcal{O}_{\mathcal{N}(\lambda)} = \left\{ \sum_{i=1}^{n} \lambda \frac{g_i \mu_i g_i^{-1}}{\alpha_i - \lambda} \,\middle|\, g_i \in GL(r, \mathbb{C}),\ i = 1, \ldots, n \right\}.$$

Since the orbit of a generic $\mu \in gl(r, \mathbb{C})$ of rank $r-1$ has dimension $r(r-1)$ we conclude that for $(F, G) \in \tilde{\mathcal{M}}$, $\dim \mathcal{O}_{\mathcal{N}(\lambda)} = nr(r-1)$.

Finally, to compute the dimension of $\bar{\mathcal{O}}_{\mathcal{N}(\lambda)} = J^{-1}(\mathcal{N}_0)/G_{\mathcal{N}_0}$ first notice that for

$$\mathcal{M} = \mathcal{M}_0 + \ldots + \mathcal{M}_{n-1}\lambda^{-n+1} \in \mathcal{O}_{\mathcal{N}(\lambda)}$$

the value $J(\mathcal{M}) = \mathcal{M}_0$ is arbitrary in $gl(r, \mathbb{C})$ except that

$$\begin{aligned} \operatorname{tr} \mathcal{M}_0 &= \operatorname{tr} \sum_{i=1}^{n} g_i \mu_i g_i^{-1} \qquad (\text{some } g_i \in GL(r, \mathbb{C})) \\ &= \sum_{i=1}^{n} \operatorname{tr}(g_i \mu_i g_i^{-1}) = \sum_{i=1}^{n} \operatorname{tr}(\mu_i) = \operatorname{tr} \mathcal{N}_0 . \end{aligned}$$

Hence, assuming $\mathcal{N}_0$ is generic, $J^{-1}(\mathcal{N}_0) \subset \mathcal{O}_{\mathcal{N}(\lambda)}$ has dimension $nr(r-1)-(r^2-1)$. Furthermore, since $\mathcal{N}_0$ is generic, $G_{\mathcal{N}_0}$ is an abelian group of dimension r in $GL(r, \mathbb{C})$. However, the action of the one dimensional group $\{cI_r \mid c \in \mathbb{C}\backslash\{0\}\}$ is trivial in $(\widetilde{gl(r)}^+)^*$, so we conclude

$$\dim \bar{\mathcal{O}}_{\mathcal{N}(\lambda)} = nr(r-1)-(r^2-1)-(r-1) = (n-1)r(r-1)-2r+2 .$$

This completes the proof of Theorem 4.4. □

Remark. To this point we have considered the flows of $\mathcal{F}_+$ on $(\widetilde{gl(r)}^+)^*$ mainly as a tool in understanding isospectral flows in $\mathcal{M}_A$. Of course it is also useful to go in the other direction. Namely, let $\mathcal{L}(\lambda) = \mathcal{L}_0 + \ldots + \mathcal{L}_{n-1}\lambda^{-n+1}$ be a polynomial element of $\widetilde{gl(r)}_0^-$. Let $\alpha_1, \ldots, \alpha_n$ be n zeros of $\det(\lambda^{n-1}\mathcal{L}(\lambda))$. Then $\mathcal{N}(\lambda) = \frac{1}{a(\lambda)}\lambda^n \mathcal{L}(\lambda)$ $\left(\text{with } a(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i)\right)$ is in the image of $\tilde{J}_r$, where the k_i's are determined by $k_i = \operatorname{rank}(\mathcal{L}(\alpha_i))$. We can now consider a symplectic leaf in $\mathcal{M}^{\mathbf{k}}/H$ as a model for the symplectic leaf in $\widetilde{gl(r)}_0^-$ through $\mathcal{N}(\lambda)$ and the flows of $\mathcal{N}(\lambda)$ determined by hamiltonians in $\mathcal{F}_+$ can be interpreted as hamiltonian flows in $\mathcal{M}^{\mathbf{k}}/H$.

There is some ambiguity here in terms of the choices of the α_i's. Usually the most convenient choice is to take the α_i's to be the highest order zeros of $\det(\lambda^{n-1}\mathscr{L}(\lambda))$. Generic flows in $\mathscr{M}^{\mathbf{k}}/H$ are of course given only when all the other zeros of $\det(\lambda^{n-1}\mathscr{L}(\lambda))$ have order one.

5. Reduction to Subalgebras

In the previous sections we have shown a correspondence between complex rank r isospectral perturbations of the matrix $L_A = A + FG^T$ and the hamiltonian flows of elements in $\mathscr{F}_+$ on finite dimensional orbits in $(\widetilde{gl(r)}^+)^*$. Since there is an injective moment map $\tilde{J}_{r,0}: \mathscr{M}^{\mathbf{k}}/H \to (\widetilde{gl(r)}^+)^*$, we can consider the symplectic leaves of $\mathscr{M}^{\mathbf{k}}/H$ as models for the finite dimensional orbits of $(\widetilde{gl(r)}^+)^*$. In this section we show how to modify our spaces to provide appropriate models for the finite dimensional orbits in the duals of various subalgebras of $\widetilde{gl(r)}^+$.

The first class of subalgebras we consider are given as follows. Let $\mathfrak{k} \subset gl(r, \mathbb{C})$ be a subalgebra and $\tilde{\mathfrak{k}} \subset \widetilde{gl(r)}$ the corresponding formal loop algebra. The decomposition $\widetilde{gl(r)} = \widetilde{gl(r)}^+ \oplus \widetilde{gl(r)}^-$, restricted to $\tilde{\mathfrak{k}}$, gives the corresponding decomposition $\tilde{\mathfrak{k}} = \tilde{\mathfrak{k}}^+ \oplus \tilde{\mathfrak{k}}^-$, where $\tilde{\mathfrak{k}}^+ = \widetilde{gl(r)}^+ \cap \tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{k}}^- = \widetilde{gl(r)}^- \cap \tilde{\mathfrak{k}}$ are subalgebras. The moment map $\tilde{J}_r: M_{N,r} \times M_{N,r} \to (\widetilde{gl(r)}^+)^*$ corresponds to the infinitesimal action

$$\sigma_r: \widetilde{gl(r)}^+ \to \chi(M_{N,r} \times M_{N,r})$$

given in block form by

$$(\sigma_r(X(\lambda))(F,G))_i = (F_i X(\alpha_i), -G_i X^T(\alpha_i)), \quad i = 1, \ldots, n, \tag{5.1}$$

where we have made the usual identification $T_{(F,G)}(M_{N,r} \times M_{N,r}) \sim M_{N,r} \times M_{N,r}$. We can restrict this to an infinitesimal action of $\tilde{\mathfrak{k}}^+$,

$$\sigma_{\mathfrak{k}}: \tilde{\mathfrak{k}}^+ \to \chi(M_{N,r} \times M_{N,r}),$$

with corresponding moment map

$$\tilde{J}_{\mathfrak{k}}: M_{N,r} \times M_{N,r} \to (\tilde{\mathfrak{k}}^+)^*$$

given by

$$\tilde{J}_{\mathfrak{k}} = \pi \circ \tilde{J}_r, \tag{5.2}$$

where $\pi: (\widetilde{gl(r)}^+)^* \to (\tilde{\mathfrak{k}}^+)^*$ is the dual of the inclusion map $\tilde{\mathfrak{k}}^+ \to \widetilde{gl(r)}^+$.

The subalgebras $\mathfrak{k}$ we consider will be such that there exists a reductive decomposition $gl(r, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{l}$ with $\mathfrak{l}$ an $\mathrm{ad}_{\mathfrak{k}}$-invariant complementary subspace to $\mathfrak{k} \subset gl(r, \mathbb{C})$. This determines a decomposition $(\widetilde{gl(r)}^+)^* = (\tilde{\mathfrak{k}}^+)^* \oplus (\tilde{\mathfrak{l}}^+)^*$, where we identify $(\tilde{\mathfrak{k}}^+)^* \sim (\tilde{\mathfrak{l}}^+)^0$, $(\tilde{\mathfrak{l}}^+)^* \sim (\tilde{\mathfrak{k}}^+)^0$. The map π is given by projection to $(\tilde{\mathfrak{k}}^+)^*$ along $(\tilde{\mathfrak{l}}^+)^*$. [More generally, we consider a nested sequence of subalgebras $\mathfrak{k}_0 = gl(r, \mathbf{C}) \supset \mathfrak{k}_1 \supset \mathfrak{k}_2 \supset \ldots$ such that each $\mathfrak{k}_i$ admits a reductive decomposition $\mathfrak{k}_i = \mathfrak{k}_{i+1} \oplus \mathfrak{l}_{i+1}$. The arguments that follow apply equally to each subalgebra of the sequence.] Let

$$\mathscr{M}_{\mathfrak{k}} = \tilde{J}_r^{-1}((\tilde{\mathfrak{k}}^+)^*) = \{(F,G) \mid (\tilde{J}_r(F,G), \xi) = 0 \ \ \forall \xi \in \tilde{\mathfrak{l}}^+\},$$

and let $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}} = \mathscr{M}_{\mathfrak{k}} \cap \mathscr{M}^{\mathbf{k}}$ so that $\tilde{J}_r$ maps $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}$ into $(\tilde{\mathfrak{k}}^+)^*$ with $\tilde{J}_r^{-1}(\tilde{J}_r(F,G))$ given by the H orbit of (F,G) for $(F,G) \in \mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}$. Since $\mathfrak{l}$ is $\mathrm{ad}_{\mathfrak{k}}$ invariant, the $\mathrm{ad}^*_{\tilde{\mathfrak{k}}^+}$ action on $(\tilde{\mathfrak{k}}^+)^*$

$\subset (\widetilde{gl(r)}^+)^*$ is the restriction of the $\mathrm{ad}^*_{\widetilde{gl(r)}^+}$ action to $(\tilde{\mathfrak{k}}^+)^*$. Furthermore, $\tilde{J}_r$ is equivariant for the $\widetilde{gl(r)}^+$ actions on $M_{N,r} \times M_{N,r}$ and $(\widetilde{gl(r)}^+)^*$. It follows that $\mathcal{M}_{\mathfrak{k}}$ is invariant under the $\tilde{\mathfrak{k}}^+$ action on $M_{N,r} \times M_{N,r}$ (i.e. the vector fields in $\mathrm{Im}\,\sigma_{\mathfrak{k}}$ are tangent to $\mathcal{M}_{\mathfrak{k}}$). Since this action commutes with the H action we also have a $\tilde{\mathfrak{k}}^+$ action on $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H \subset \mathcal{M}^{\mathbf{k}}/H$. If $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ is a Poisson submanifold of $\mathcal{M}^{\mathbf{k}}/H$ (i.e., a union of symplectic leaves), this action is hamiltonian with moment map $\tilde{J}_{\mathfrak{k},0} : \mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H \to (\tilde{\mathfrak{k}}^+)^*$ given by

$$\tilde{J}_{\mathfrak{k},0} = \tilde{J}_{r,0}|_{\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H}\,.$$

Hence, in this case the symplectic leaves of $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ are the appropriate models for the symplectic leaves in $(\tilde{\mathfrak{k}}^+)^*$. This situation occurs only if $\mathfrak{l}$ is the center of $gl(r)$. More generally, for subalgebras $gl(r) \supset \mathfrak{k}_1 \supset \mathfrak{k}_2 \supset \ldots$, it occurs if $\mathfrak{l}_{i+1} \subset \mathfrak{k}_i$ is central.

In general, when $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ is not a Poisson submanifold of $\mathcal{M}^{\mathbf{k}}/H$, we proceed as follows. By restricting the α_i's if necessary, we find a $\tilde{\mathfrak{k}}^+$ invariant symplectic submanifold $W \subset \mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}$ along with a subgroup $H_{\mathfrak{k}} \subset H$ leaving W invariant such that $W/H_{\mathfrak{k}} = \mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$. In other words, we reduce the H-bundle $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}} \to \mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ to an $H_{\mathfrak{k}}$-bundle $W \to W/H_{\mathfrak{k}}$ so that the total space is $\tilde{\mathfrak{k}}^+$ invariant and symplectic. (Actually we shall often have different connected components of $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$. That is, $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ will decompose into the union of a finite number of components

$$\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H = \bigcup_i W_i/H^i_{\mathfrak{k}}$$

with W_i the components of W and $H^i_{\mathfrak{k}}$ a subgroup of H leaving W_i invariant. In this case the following arguments must be applied to each component separately.) It follows that $W/H_{\mathfrak{k}}$ is a Poisson manifold with a hamiltonian $\tilde{\mathfrak{k}}^+$ action whose moment map

$$\tilde{J}_{\mathfrak{k},0} : W/H_{\mathfrak{k}} \to (\tilde{\mathfrak{k}}^+)^*$$

is given by $\tilde{J}_{r,0}|_{\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H}$ under the identification of $\mathcal{M}^{\mathbf{k}}_{\mathfrak{k}}/H$ with $W/H_{\mathfrak{k}}$ and $(\tilde{\mathfrak{k}}^+)^*$ with $(\tilde{\mathfrak{l}}^+)^0$. Hence the symplectic leaves of $W/H_{\mathfrak{k}}$ are the appropriate models for finite dimensional symplectic leaves in $(\tilde{\mathfrak{k}}^+)^*$.

Poisson Submanifold Reduction: $gl(r, \mathbb{C}) \supset sl(r, \mathbb{C})$

The first case we consider is $\mathfrak{k} = sl(r, \mathbb{C})$. If we split $gl(r, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{l}$, where $\mathfrak{l} = \{cI_r | c \in \mathbb{C}\}$ and use the pairing (2.19) to identify $(\widetilde{gl(r)}^+)^*$ with $\widetilde{gl(r)}_0^-$, we get an identification of $(\widetilde{sl(r, \mathbb{C})}^+)^*$ with $\widetilde{sl(r, \mathbb{C})}_0^- = \widetilde{sl(r, \mathbb{C})} \cap \widetilde{gl(r)}_0^-$. The projection $\pi : (\widetilde{gl(r)}^+)^* \to (\widetilde{sl(r, \mathbb{C})}^+)^*$ is then

$$\pi(X(\lambda)) = X(\lambda) - \frac{1}{r}\mathrm{tr}(X(\lambda))I_r, \qquad X(\lambda) \in \widetilde{gl(r)}_0^-\,.$$

Thus, in this case (5.2) becomes

$$\tilde{J}_{\mathfrak{k}}(F, G) = \tilde{J}_r(F, G) - \frac{1}{r}\mathrm{tr}(\tilde{J}_r(F, G))I_r\,. \tag{5.3}$$

The space $\mathcal{M}_{\mathfrak{k}}$ is given by $\{(F, G) \,|\, \mathrm{tr}(\tilde{J}_r(F, G)) = 0\}$, i.e.

$$\mathrm{tr}\left(\lambda \sum_{i=1}^{n} \frac{G_i^T F_i}{\lambda - \alpha_i}\right) = 0\,.$$

Since this must hold for all λ, it is equivalent to requiring

$$\operatorname{tr}(G_i^T F_i)=0 \quad \text{or} \quad \operatorname{tr}(F_i G_i^T)=0.$$

Since functions of the form $\phi \circ J_H$, where $\phi:\mathfrak{h}^*\to\mathbb{C}$ is Ad_H^* invariant, project to central elements in the Poisson algebra on $\mathcal{M}^{\mathbf{k}}/H$, their simultaneous level sets are Poisson submanifolds. But $\{\phi_i(\mu)=\operatorname{tr}\mu_i$, where $\mu=(\mu_1,\dots,\mu_n)\in\mathfrak{h}^*\}$ are such Ad_H^* invariant functions. Hence $\mathcal{M}_{\mathfrak{t}}^{\mathbf{k}}/H\subset\mathcal{M}^{\mathbf{k}}/H$ is a Poisson submanifold.

Reductions of $gl(r,\mathbb{C})$ and $sl(r,\mathbb{C})$ by Automorphisms

Besides $sl(r,\mathbb{C})$, we consider the other classical complex algebras ($so(r,\mathbb{C})$, $sp(n,\mathbb{C})$) and the real forms ($gl(r,\mathbb{R})$, $sl(r,\mathbb{R})$, $u(p,q)$, $su(p,q)$, $o(p,q)$, $so(p,q)$, $sp(p,q)$, $sp(n,\mathbb{R})$, $su^*(2n)$, $so^*(2n)$). These algebras can all be described as the fixed points in $gl(r,\mathbb{C})$ or $sl(r,\mathbb{C})$ of a finite group of automorphisms generated by one or two linear or antilinear involutions.

The involutive automorphisms required are of the three forms (see e.g. [22])

$$\begin{aligned} &\text{(a)} && \sigma(X)=-tX^Tt^{-1},\\ &\text{(b)} && \sigma(X)=t\bar{X}t^{-1}, \\ &\text{(c)} && \sigma(X)=-t\bar{X}^Tt^{-1}, \end{aligned} \tag{5.4}$$

where t is one of the three matrices

$$I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix},\qquad p+q=r,$$

$$K_{p,q}=\begin{pmatrix} I_p & 0 & 0 & 0\\ 0 & -I_q & 0 & 0\\ 0 & 0 & I_p & 0\\ 0 & 0 & 0 & -I_q\end{pmatrix},\qquad 2(p+q)=r,$$

$$\gamma_p=\begin{pmatrix} 0 & I_p\\ -I_p & 0\end{pmatrix},\qquad 2p=r$$

(where I_p denotes the $p\times p$ identity matrix).

To keep the notation simple we shall only show how to reduce $gl(r,\mathbb{C})$ by any one of the σ's. The same procedure may be applied sequentially to reduce any of the subalgebras of $gl(r,\mathbb{C})$.

Since $\sigma^2=\mathrm{Id}$ we can split $gl(r,\mathbb{C})$ into the ± 1 eigenspaces of σ. The subalgebra $\mathfrak{k}$ is the $+1$ eigenspace and we let $\mathfrak{l}$ denote the -1 eigenspace. Thus σ gives a natural splitting $gl(r,\mathbb{C})=\mathfrak{k}\oplus\mathfrak{l}$, and we can identify $\mathfrak{k}^*$ with $\mathfrak{l}^0\subset(gl(r,\mathbb{C}))^*$. A bit of caution should be taken here in terms of real or complex duals. Until now we have considered $gl(r,\mathbb{C})$ as a complex vector space and $(gl(r,\mathbb{C}))^*$ as the space of complex linear functionals into $\mathbb{C}$. If $\mathfrak{k}\subset gl(r,\mathbb{C})$ is a real subalgebra, then by $\mathfrak{k}^*$ we mean the real linear functionals (into $\mathbb{R}$). In this case, the embedding $\mathfrak{k}^*\sim\mathfrak{l}^0\subset(gl(r,\mathbb{C}))^*$ must be understood with respect to the real dual on the right-hand side. To identify $\mathfrak{k}^*$ with $\mathfrak{l}^\perp\subset gl(r,\mathbb{C})$ we use the real inner product, $\mathrm{Re}(\operatorname{tr}(XY))$, instead of the complex one, $\mathrm{tr}(XY)$, to give a pairing between $gl(r,\mathbb{C})$ and $gl(r,\mathbb{C})^* \sim gl(r,\mathbb{C})$. This situation will arise for subalgebras defined by the antilinear automorphisms; i.e. types (b) and (c).

With respect to the appropriate (real or complex) pairing, σ preserves the appropriate inner product and hence $\mathfrak{k}$ and $\mathfrak{l}$ are orthogonal. Therefore, when we identify $(\widetilde{gl(r)}^+)^*$ with $\widetilde{gl(r)}^-$ [using either the inner product (2.19) for the complex case or the inner product

$$\langle X(\lambda), Y(\lambda)\rangle \equiv \mathrm{Re}\,\mathrm{tr}((X(\lambda)Y(\lambda))_0) \tag{5.5}$$

for the real case], we obtain a corresponding identification of $(\tilde{\mathfrak{k}}^+)^*$ with $\tilde{\mathfrak{k}}_0^- = \tilde{\mathfrak{k}} \cap \widetilde{gl(r)}_0^-$. Thus, for each σ the corresponding subspace $\mathcal{M}_{\mathfrak{k}} \subset M_{N,r} \times M_{N,r}$ can be described by

$$\mathcal{M}_{\mathfrak{k}} = \{(F,G) \mid \tilde{\sigma}(\tilde{J}_r(F,G)) = \tilde{J}_r(F,G)\},$$

where $\tilde{\sigma}: \widetilde{gl(r)} \to \widetilde{gl(r)}$ is defined by the action of σ on each component of the formal power series, i.e.

$$\tilde{\sigma}\left(\sum X_i \lambda^i\right) = \sum \sigma(X_i)\lambda^i .$$

Case (a). For (F,G) to be in $\mathcal{M}_{\mathfrak{k}}$ we require

$$\sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\lambda - \alpha_i} = -\sum_{i=1}^{n} \frac{\lambda t F_i^T G_i t^{-1}}{\lambda - \alpha_i}. \tag{5.6}$$

If this is to hold for all λ it is necessary that

$$G_i^T F_i = -t F_i^T G_i t^{-1}, \qquad i = 1, \ldots, n .$$

We can identify the space $\mathcal{M}^{\mathbf{k}}/H$ with the set $\{(G_1^T F_1, \ldots, G_n^T F_n) | (F,G) \in \mathcal{M}^{\mathbf{k}}\} \subset \mathfrak{g}_r^n$ since the projection $(F,G) \to [(F,G)]_H = (G_1^T F_1, \ldots, G_n^T F_n)$ is a principal H fibration. Thus $\mathcal{M}_{\mathfrak{k}}^{\mathbf{k}}/H$ is identified with the subspace

$$\{(G_1^T F_1, \ldots, G_n^T F_n) \mid G_i^T F_i = -t(G_i^T F_i)^T t^{-1}\} .$$

Suppose $(F,G) \in \mathcal{M}_{\mathfrak{k}}^{\mathbf{k}}$. Since F_i and G_i have maximal rank, there exists an $m_i \in GL(k_i, \mathbb{C})$ such that

$$t F_i^T = -G_i^T m_i, \qquad G_i t^{-1} = m_i^{-1} F_i .$$

Combining these two equations yields

$$F_i = -m_i (m_i^T)^{-1} F_i t^T t^{-1}, \qquad G_i = -(m_i^T)^{-1} m_i G_i t^{-1} t^T .$$

If $t = I_{p,q}$ or $K_{p,q}$, then $t^T = t$, so we have

$$m_i = -m_i^T .$$

This is possible in the case that k_i is even for all i. Assume this to be the case and set $\kappa_i = 1/2 k_i$.

Now if $(\tilde{F}, \tilde{G}) = (hF, (h^{-1})^T G)$, $h \in H$ is in the H orbit of (F,G), then $(\tilde{F}, \tilde{G})$ satisfies

$$t \tilde{F}_i^T = -\tilde{G}_i^T \tilde{m}_i, \qquad \tilde{G}_i t^{-1} = \tilde{m}_i^{-1} \tilde{F}_i ,$$

where $\tilde{m}_i = h_i m_i h_i^T$, $h=(h_1, \ldots, h_n)$. Since m_i is antisymmetric it is possible to normalize m_i to equal γ_{κ_i} by means of this action. That is, each H orbit in $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}$ has at least one element (F, G) satisfying

$$F_i = \gamma_{\kappa_i} G_i t .$$

Let

$$W = \{(F, G) \in \mathscr{M}^{\mathbf{k}} \mid F_i = \gamma_{\kappa_i} G_i t\} . \tag{5.7}$$

It is straightforward to check that this is a symplectic submanifold of $\mathscr{M}^{\mathbf{k}}$. The H-bundle $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}} \to \mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}/H$ reduces to the $H_{\mathfrak{k}}$-subbundle $W \to \mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}/H$,

$$H_{\mathfrak{k}} = \{(h_1, \ldots, h_n) \in H \mid \gamma_{\kappa_i} = h_i \gamma_{\kappa_i} h_i^T\} = \{(h_1, \ldots, h_n) \in H \mid h_i \in Sp(\kappa_i, \mathbf{C})\} . \tag{5.8}$$

Finally a direct computation shows that the $\tilde{\mathfrak{k}}^+$ action leaves W invariant, i.e. that

$$\sigma_r(X(\lambda))(F, G) \in T_{(F,G)} W$$

when $X(\lambda) \in \tilde{\mathfrak{k}}^+$ and $(F, G) \in W$.

On the other hand, if $t = \gamma_p$ (so $r = 2p$), then $t^T = -t$ and we have

$$m_i = m_i^T .$$

By means of the H action, $m_i \to h_i m_i h_i^T$, it is possible to normalize to $m_i = I_{k_i}$; i.e., each H orbit in $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}$ has at least one element (F, G) satisfying $G = F\gamma_p$. Let

$$W = \{(F, G) \in \mathscr{M}^{\mathbf{k}} \mid G = F\gamma_p\} . \tag{5.9}$$

Again, W is a $\tilde{\mathfrak{k}}$-invariant symplectic submanifold of $\mathscr{M}^{\mathbf{k}}$ which is principal bundle over $\mathscr{M}_{\mathfrak{k}}^{\mathbf{k}}/H$, where

$$H_{\mathfrak{k}} = \{(h_1, \ldots, h_n) \in H \mid h_i^T h_i = I_{k_i}\} = \{(h_1, \ldots, h_n) \in H \mid h_i \in O(k_i, \mathbf{C})\} . \tag{5.10}$$

Remark. Reductions of $gl(r, \mathbf{C})$ or $sl(r, \mathbf{C})$ by σ of type (a) alone yield the following algebras

$\mathfrak{k} = so(r, \mathbf{C})$: Reduce $sl(r, \mathbf{C})$ with $t = I_r$.

$\mathfrak{k} = sp(p, \mathbf{C})$: Reduce $gl(r, \mathbf{C})$ with $t = \gamma_p, \quad r = 2p$.

Although $t = I_{p,q}$ or $K_{p,q}$ leads to a reduction equivalent to $t = I_r$, when combined with further reductions under anti-linear automorphisms the result may be inequivalent.

Case (b). For (F, G) to be in $\mathscr{M}_{\mathfrak{k}}$ we require

$$\sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\lambda - \alpha_i} = \sum_{i=1}^{n} \frac{\lambda t \bar{G}_i^T \bar{F}_i t^{-1}}{\lambda - \bar{\alpha}_i} . \tag{5.11}$$

For this to hold for all λ it is necessary that the α_i's either be real or come in complex conjugate pairs. Ordering the α_i's so that $\alpha_{2i} = \bar{\alpha}_{2i-1}$, $i = 1, \ldots, m$ and $\alpha_j = \bar{\alpha}_j$, $j = 2m+1, \ldots, n$, Eq. (5.11) implies

$$G_{2i}^T F_{2i} = t \bar{G}_{2i-1}^T \bar{F}_{2i-1} t^{-1} , \qquad i = 1, \ldots, m$$

(hence $k_{2i}=k_{2i-1}$), and

$$G_j^T F_j = t\bar{G}_j^T \bar{F}_j t^{-1}, \quad j=2m+1, \ldots, n.$$

If $t=I_{p,q}$ or $K_{p,q}$, then by an analysis similar to case (a) we can take

$$W=\{(F,G)\in \mathcal{M}^{\mathbf{k}} \mid F_{2i}=\bar{F}_{2i-1}t,\ G_{2i}=\bar{G}_{2i-1}t, \text{ for } i=1,\ldots,m, \\ \text{and } F_j=\bar{F}_j t,\ G_j=\bar{G}_j t,\ j=2m+1,\ldots,n\}. \tag{5.12}$$

This is a $\tilde{\mathfrak{k}}^+$-invariant real symplectic submanifold of $\mathcal{M}^{\mathbf{k}}$ which is a principal $H_{\mathfrak{k}}$ bundle over $M_{\mathfrak{k}}^{\mathbf{k}}/H$, where

$$H_{\mathfrak{k}}=\{(h_1,\ldots,h_n)\in H \mid h_{2i}=\bar{h}_{2i-1},\ i=1,\ldots,m \text{ and } h_j=\bar{h}_j,\ j=2m+1,\ldots,n\}. \tag{5.13}$$

On the other hand, if $t=\gamma_p$, $r=2p$, we can take

$$W=\{(F,G)\in \mathcal{M}^{\mathbf{k}} \mid F_{2i}=\bar{F}_{2i-1}\gamma_p,\ G_{2i}=\bar{G}_{2i-1}\gamma_p,\ i=1,\ldots,m \\ \text{and } F_j=\gamma_{\kappa_j}\bar{F}_j\gamma_p^{-1},\ G_j=\gamma_{\kappa_j}\bar{G}_j\gamma_p^{-1}\}, \tag{5.14}$$

which again is a $\tilde{\mathfrak{k}}^+$-invariant real symplectic submanifold of $\mathcal{M}^{\mathbf{k}}$ fibering over $W/H_{\mathfrak{k}} \sim \mathcal{M}_{\mathfrak{k}}^{\mathbf{k}}/H$ with

$$H_{\mathfrak{k}}=\{(h_1,\ldots,h_n)\in H \mid h_{2i}=\bar{h}_{2i-1},\ i=1,\ldots,m, \\ \text{and } h_j=\gamma_{\kappa_j}\bar{h}_j\gamma_{\kappa_j}^{-1},\ j=2m+1,\ldots,n\}. \tag{5.15}$$

Remarks. (1) Reductions of $gl(r,\mathbb{C})$ or $sl(r,\mathbb{C})$ by σ of type (b) yield the following classical algebras:

$gl(r,\mathbb{R})$: Reduce $gl(r,\mathbb{C})$ with $t=I$.

$sl(r,\mathbb{R})$: Reduce $sl(r,\mathbb{C})$ with $t=I$.

$su^*(2p)$: Reduce $sl(r,\mathbb{C})$ with $t=\gamma_p$, $r=2p$.

(2) If we first reduce $gl(r,\mathbb{C})$ [or $sl(r,\mathbb{C})$] to $gl(r,\mathbb{R})$ [respectively $sl(r,\mathbb{R})$] by means of a σ of type (b), then further reduce by a σ of type (a), we get

$so(p,q)$: Reduce $sl(r,\mathbb{R})$ with $t=I_{p,q}$, $p+q=r$.

$sp(p,\mathbb{R})$: Reduce $gl(r,\mathbb{R})$ with $t=\gamma_p$, $r=2p$.

When reducing $gl(r,\mathbb{R})$ to $sp(p,\mathbb{R})$ a slightly more complicated situation arises. Taking all α_i's real, the space $W\subset M_{N,r}\times M_{N,r}$ which corresponds to $gl(r,\mathbb{R})$ is $W=M_{N,r}(\mathbb{R})\times M_{N,r}(\mathbb{R})$, where $M_{N,r}(\mathbb{R})$ denotes the real $N\times r$ matrices. The subgroup of H which leaves W invariant is

$$H(\mathbb{R})=GL(k_1,\mathbb{R})\times \ldots \times GL(k_n,\mathbb{R}).$$

The complication that arises in reduction to $sp(p,\mathbb{R})$ is that if $m_i\in GL(k_i,\mathbb{R})$ with $m_i=m_i^T$, we are only able to normalize m_i to some I_{p_i,q_i}, $p_i+q_i=k_i$, by means of the $H(\mathbb{R})$ action. In this case, for each choice of (p_i,q_i) we get a different W and a different $H_{\mathfrak{k}}$, each such W projecting to a different component of $\mathcal{M}_{\mathfrak{k}}^{\mathbf{k}}/H$; i.e., there is a finite stratification of $\mathcal{M}_{\mathfrak{k}}^{\mathbf{k}}/H$ based on the signatures (p_i,q_i).

Case (c). For (F, G) to be in $\mathscr{M}_{\mathfrak{t}}$ we have

$$\sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\lambda-\alpha_i} = -\sum_{i=1}^{n} \frac{\lambda t \bar{F}_i^T G_i t^{-1}}{\lambda-\bar{\alpha}_i}. \tag{5.16}$$

Again, we can order the α_i's such that $\alpha_{2i}=\bar{\alpha}_{2i-1}$, $i=1,\ldots,m$, and $\bar{\alpha}_j=\alpha_j$, $j=2m+1,\ldots,n$. Then (5.16) implies

$$G_{2i}^T F_{2i} = -t\bar{F}_{2i-1}^T \bar{G}_{2i-1} t^{-1}, \qquad i=1,\ldots,m$$

and

$$G_j^T F_j = -t\bar{F}_j^T \bar{G}_j t^{-1}, \qquad j=2m+1,\ldots,n.$$

In the case $t=I_{p,q}$ or $K_{p,q}$ we find that for each fixed choice of $p_j, q_j, j=2m+1,\ldots,n$, with $p_j+q_j=k_j$ we can take

$$\begin{aligned} W=\{(F,G)\in\mathscr{M}^{\mathbf{k}} \mid F_{2i-1}=\bar{G}_{2i}t^{-1},\ F_{2i}=-\bar{G}_{2i-1}t^{-1},\ i=1,\ldots,m \\ \text{and } F_j=\sqrt{-1}I_{p_j,q_j}\bar{G}_j t^{-1},\ j=2m+1,\ldots,n\}, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} H_{\mathfrak{t}}=\{(h_1,\ldots,h_n)\in H \mid h_{2i}=(\bar{h}_{2i-1}^{-1})^T,\ i=1,\ldots,m, \\ \text{and } h_j=I_{p_jq_j}(\bar{h}_j^{-1})^T I_{p_jq_j},\ j=2m+1,\ldots,n\}. \end{aligned} \tag{5.18}$$

Again $W\subset\mathscr{M}^{\mathbf{k}}$ is a real symplectic submanifold and $\mathscr{M}_{\mathfrak{t}}^{\mathbf{k}}/H$ is the disjoint union of the submanifolds $W/H_{\mathfrak{t}}$ corresponding to the various choices of signature (p_j, q_j).

In the case $t=\gamma_p$ we find that for each fixed choice of integers (p_j, q_j), $j=2m+1,\ldots,n$, with $p_j+q_j=k_j$, we can take

$$\begin{aligned} W=\{(F,G)\in\mathscr{M}^{\mathbf{k}} \mid F_{2i-1}=-\bar{G}_{2i}\gamma_p,\ F_{2i}=-\bar{G}_{2i-1}\gamma_p,\ i=1,\ldots,m, \\ \text{and } F_j=-I_{p_j,q_j}\bar{G}_j\gamma_p,\ j=2m+1,\ldots,n\} \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} H_{\mathfrak{t}}=\{(h_1,\ldots,h_n)\in H \mid h_{2i}=(\bar{h}_{2i-1}^{-1})^T,\ i=1,\ldots,m \\ \text{and } h_j=I_{p_j,q_j}(\bar{h}_j^{-1})^T I_{p_j,q_j},\ j=2m+1,\ldots,n\}. \end{aligned} \tag{5.20}$$

Again, $\mathscr{M}_{\mathfrak{t}}^{\mathbf{k}}/H$ is the disjoint union of the submanifolds $W/H_{\mathfrak{t}}$ corresponding to the various choices of signatures (p_j, q_j).

Remark. (1) Reductions of $gl(r,\mathbb{C})$ or $sl(r,\mathbb{C})$ by σ of type (c) yield the following classical algebras:

$u(p,q)$: Reduce $gl(r,\mathbb{C})$ with $t=I_{p,q}$, $p+q=r$.

$su(p,q)$: Reduce $sl(r,\mathbb{C})$ with $t=I_{p,q}$, $p+q=r$.

(2) The algebra $so^*(2p)$ is obtained by reducing $so(2p,\mathbb{C})$ by a σ of type (c) with $t=\gamma_p$ and the algebra $sp(p,q)$ is obtained by reducing $sp(p+q,\mathbb{C})$ by a σ of type (c) with $t=K_{p,q}$, $2(p+q)=r$.

The reduction conditions corresponding to involutive automorphisms are summarized in Table 5.1.

Table 5.1. Reductions by involutive automorphisms

$\sigma(x)$	t	$H=\{h_i\}_{i=1,\dots,n}$	W	$\{\alpha_i\}$, $i=1,\dots,n$
a) $-tX^Tt^{-1}$	$I_{p,q}$ or $K_{p,q}$	$h_i\in Sp(\kappa_i,\mathbb{C})$	$F_i=\gamma_{\kappa_i}G_it$	$\alpha_i\in\mathbb{C}$
	γ_p	$h_i\in O(k_i,\mathbb{C})$	$G=F\gamma_p$	
b) $t\bar{X}t^{-1}$	$I_{p,q}$ or $K_{p,q}$	$h_{2i}=\bar{h}_{2i-1}\in GL(k_i,\mathbb{C})$ $h_j\in GL(k_j,\mathbb{R})$	$F_{2i}=\bar{F}_{2i-1}t$, $G_{2i}=\bar{G}_{2i-1}t$ $F_j=\bar{F}_jt$, $G_j=\bar{G}_jt$	$\alpha_{2i}=\bar{\alpha}_{2i-1}\in\mathbb{C}$, $i=1,\dots,m$ and
	γ_p	$h_{2i}=\bar{h}_{2i-1}\in GL(k_i,\mathbb{C})$ $h_j\in SU^*(2\kappa_j)$	$F_{2i}=\bar{F}_{2i-1}t$, $G_{2i}=\bar{G}_{2i-1}t$ $F_j=\gamma_{\kappa_j}\bar{F}_jt^{-1}$, $G_j=\gamma_{\kappa_j}\bar{G}_jt^{-1}$	$\alpha_j\in\mathbb{R}$, $j=2m+1,\dots,n$
c) $-t\bar{X}^Tt^{-1}$	$I_{p,q}$ or $K_{p,q}$	$h_{2i}=(\bar{h}_{2i-1}^{-1})^T\in GL(k_i,\mathbb{C})$ $h_j\in U(p_j,q_j)$ $p_j+q_j=k_j$	$F_{2i-1}=\bar{G}_{2i}t^{-1}$, $F_{2i}=\bar{G}_{2i-1}t^{-1}$ $F_j=\sqrt{-1}I_{p_jq_j}\bar{G}t^{-1}$	$\alpha_{2i}=\bar{\alpha}_{2i-1}\in\mathbb{C}$, $i=1,\dots,m$ and $\alpha_j\in\mathbb{R}$,
	γ_p	$h_{2i}=(\bar{h}_{2i}^{-1})^T\in GL(k_i,\mathbb{C})$ $h_j\in U(p_j,q_j)$ $p_j+q_j=k_j$	$F_{2i-1}=-\bar{G}_{2i}t$ $F_{2i}=-\bar{G}_{2i-1}t$ $F_j=-I_{p_jq_j}\bar{G}_jt$	$j=2m+1,\dots,n$

Twisted Algebras

To conclude this section we discuss a class of subalgebras of $\widetilde{gl(r)}^+$ which are not of the form $\tilde{\mathfrak{k}}^+$ for some $\mathfrak{k}\subset gl(r,\mathbb{C})$; namely, the twisted subalgebras of $\widetilde{gl(r)}^+$ [or $\widetilde{sl(r,\mathbb{C})}^+$].

Let σ be an automorphism of $gl(r,\mathbb{C})$ [or $sl(r,\mathbb{C})$] of order k and let $q=e^{2\pi i/k}$. Define an automorphism $\tilde{\sigma}$ on $\widetilde{gl(r)}$ by

$$\tilde{\sigma}\left(\sum_{j=-\infty}^{p} X_j\lambda^j\right)=\sum_{j=-\infty}^{p}\sigma(X_j)q^{-j}\lambda^j, \tag{5.21}$$

and let $\widehat{gl(r)}\subset\widetilde{gl(r)}$ be the subalgebra defined by the fixed points of $\tilde{\sigma}$, i.e.

$$\widehat{gl(r)}=\{X(\lambda)\in\widetilde{gl(r)}\mid\tilde{\sigma}(X(\lambda))=X(\lambda)\}. \tag{5.22}$$

If we decompose $gl(r,\mathbb{C})$ into k eigenspaces $\mathfrak{g}_j$, $j=0,\dots,k-1$, of σ with eigenvalues $\sigma|_{\mathfrak{g}_j}=q^j$, then

$$\widehat{gl(r)}=\left\{X(\lambda)=\sum_{l=-\infty}^{p}X_l\lambda^l\mid X_l\in\mathfrak{g}_j,\ \text{where } l\equiv j \bmod k\right\}. \tag{5.23}$$

Since $\tilde{\sigma}$ is an automorphism of order k, we can also split $\widetilde{gl(r)}$ into k eigenspaces, $\tilde{\mathfrak{g}}_j$, $j=0,\dots,k-1$, with $\tilde{\sigma}|_{\tilde{\mathfrak{g}}_j}=q^j$ so

$$\widetilde{gl(r)}=\bigoplus_{j=0}^{k-1}\tilde{\mathfrak{g}}_j \tag{5.24}$$

with $\tilde{\mathfrak{g}}_0=\widehat{gl(r)}$. Relative to this splitting we may identify

$$\widetilde{gl(r)}^*=\bigoplus_{j=0}^{k-1}\tilde{\mathfrak{g}}_j^*, \tag{5.25}$$

where $\tilde{\mathfrak{g}}_j^*$ is identified with the annihilator of $\bigoplus_{l\neq j}\tilde{\mathfrak{g}}_l$.

Since σ is an automorphism it preserves the (real or complex) inner product on $gl(r, \mathbb{C})$. Hence $\tilde{\sigma}$ preserves the inner product (2.19) when σ is linear, and (5.5) when σ is antilinear. This gives the identification

$$\widetilde{gl(r)}^* \sim \left(\bigoplus_{j=1}^{k-1} \tilde{\mathfrak{g}}_j\right)^{\perp} = \widehat{gl(r)}. \tag{5.26}$$

Letting $\widehat{gl(r)}^{\pm} = \widetilde{gl(r)}^{\pm} \cap \widehat{gl(r)}$, we also get the identification

$$(\widehat{gl(r)}^{+})^* \sim (\widehat{gl(r)}^{-})^{\perp} = \widehat{gl(r)}_0^{-}, \tag{5.27}$$

where $\widehat{gl(r)}_0^{-} = \widetilde{gl(r)}_0^{-} \cap \widehat{gl(r)}$.

Let $\pi: \widetilde{gl(r)}^* \to \widehat{gl(r)}^*$ denote the projection dual to the inclusion $i: \widehat{gl(r)} \to \widetilde{gl(r)}$. Under the above identifications π is the projection of $\widetilde{gl(r)}$ to $\widehat{gl(r)}$ relative to the decomposition (5.24).

Lemma 5.1. *The projection* $\pi: \widetilde{gl(r)} \to \widehat{gl(r)}$ *along* $\bigoplus_{j=1}^{k-1} \tilde{\mathfrak{g}}_j$ *is given by*

$$\pi(X(\lambda)) = \frac{1}{k} \sum_{j=0}^{k-1} \tilde{\sigma}^j(X(\lambda)). \tag{5.28}$$

Proof. Direct computation. □

Notice $\pi(\widetilde{gl(r)}_0^{-}) = \widehat{gl(r)}_0^{-}$. The infinitesimal $\widetilde{gl(r)}^{+}$ action on $M_{N,r} \times M_{N,r}$ restricts to an infinitesimal action of the subalgebra $\widehat{gl(r)}^{+}$. This action has a moment map $\hat{J}_r: M_{N,r} \times M_{N,r} \to (\widehat{gl(r)}^{+})^*$ given by

$$\hat{J}_r = \pi \circ \tilde{J}_r.$$

Proposition 5.2. *Assume* $q^l \alpha_i \neq \alpha_j$ *for all* i, j, l. *For* $(F, G) \in \mathcal{M}_0$, $\hat{J}_r^{-1}(\hat{J}_r(F, G))$ *is the* H *orbit through* (F, G). *Thus, since the* H *action on* $\mathcal{M}^{\mathbf{k}}$ *commutes with the* $\widehat{gl(r)}^{+}$ *action,* $\hat{J}_r$ *reduces to an injective moment map* $\hat{J}_{r,0}: \mathcal{M}^{\mathbf{k}}/H \to (\widehat{gl(r)}^{+})^*$.

Proof. It is enough to show that π is injective on the image of $A^*: (\mathfrak{g}_r^n)^* \to (\widetilde{gl(r)}^{+})^*$.

Suppose $\pi\left(\lambda \sum_{i=1}^{n} \frac{Y_i}{\lambda - \alpha_i}\right) = 0$. By Lemma 5.1 this implies

$$0 = \sum_{j=0}^{k-1} \tilde{\sigma}^j\left(\lambda \sum_{i=1}^{n} \frac{Y_i}{\lambda - \alpha_i}\right) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} \lambda \frac{\sigma^j(Y_i)}{\lambda - q^j \alpha_i}. \tag{5.29}$$

The assumption $q^l \alpha_i \neq \alpha_j$ assures that all of the poles $q^j \alpha_i$ are distinct. Thus (5.29) implies $Y_i = 0$, $i = 1, \ldots, n$. □

From Proposition 5.2 we conclude that as long as the $q^j \alpha_i$ are distinct the symplectic leaves of $\mathcal{M}^{\mathbf{k}}/H$ give appropriate models for finite dimensional orbits in $(\widehat{gl(r)}^{+})^*$.

6. Examples

To illustrate the results derived in the previous sections, we now consider in detail a number of integrable hamiltonian systems. In particular, we show how Moser's [32] examples are recovered in the $r = 2$ framework and also consider the

Rosochatius system [16] which, though briefly mentioned in [32], requires our extended framework for full details. We then also show how the analysis is related to completely integrable PDE's, treating the nonlinear Schrödinger (NLS) and modified Korteweg-de Vries (mKdV) equation as illustrations of the $r=2$ case and the coupled nonlinear Schrödinger (CNLS) and Boussinesq equations as examples for $r=3$. The relationship of these PDE's to our framework resides in the fact that commutativity of any pair of hamiltonian flows in the loop algebra framework entails integrability conditions which can be expressed in terms of PDE's for the matrix coefficients.

A. Moser's Examples
(Finite Dimensional $\widetilde{sl(2,\mathbb{R})}$ Flows with Shifted Hamiltonians)

For these examples we take $r=2$ (so $k_i=1$, $i=1,\ldots,n$, and

$$H=\mathscr{D}=\{(d_1 I_2,\ldots,d_n I_2)\,|\,d_i\in\mathbb{C}\backslash 0\}),$$

the α_i's are taken to be real, and we reduce $\widetilde{gl(2)}$ to $\widetilde{sl(2,\mathbb{R})}$. This reduction is done in two steps. First the reduction of $\widetilde{gl(2)}$ to $\widetilde{gl(2,\mathbb{R})}$ is given by the reality conditions

$$F=\bar{F},\qquad G=\bar{G}, \tag{6.1}$$

where $F,G\in M_{n,2}$.

The group H reduces to $\{(d_1 I_2,\ldots,d_n I_2)\,|\,d_i\in\mathbb{R}\backslash 0\}$. Secondly, the reduction of $\widetilde{gl(2,\mathbb{R})}$ to $\widetilde{sl(2,\mathbb{R})}$ is given by the condition $J_{\mathscr{D}}=0$ which in this case reads

$$G_i\cdot F_i=0,\qquad i=1,\ldots,n. \tag{6.2}$$

Since $r=2$ we can write

$$G_i=a_i F_i+b_i F_i\gamma_1,$$

where

$$\gamma_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

(i.e. G_i is the sum of a vector parallel to F_i and a vector perpendicular to F_i). The condition (6.2) implies $a_i=0$. Furthermore, the H action transforms

$$F_i\to d_i F_i,\qquad G_i\to d_i^{-1}G_i,\qquad d_i\in\mathbb{R}\backslash 0,$$

hence, choosing $d_i=(\sqrt{|b_i|})^{-1}$, we can reduce the fibration $J_{\mathscr{D}}^{-1}(0)\to J_{\mathscr{D}}^{-1}(0)/H$ to a fibration $W\to J_{\mathscr{D}}^{-1}(0)/H$, where W is given by

$$W=\{(F,G)\in\mathscr{M}^{\mathbf{k}}\,|\,F=\bar{F},\ G_i=\pm F_i\gamma_1\},$$

and the fibers are given by the orbits of the finite subgroup of H, $\{(\pm 1,\ldots,\pm 1)\}$.

The different components of W, determined by a choice of $+$ or $-$ signs for each i, all project onto $J_{\mathscr{D}}^{-1}(0)/H$. Henceforth we restrict our attention to the component

$$W_0=\{(F,G)\in\mathscr{M}^{\mathbf{k}}\,|\,F=\bar{F},\ G_i=-F_i\gamma_1\}.$$

If we let $\left(\frac{1}{\sqrt{2}}\vec{x}, \frac{1}{\sqrt{2}}\vec{y}\right)$ denote the column vectors of F, then G has column vectors $\left(\frac{1}{\sqrt{2}}\vec{y}, -\frac{1}{\sqrt{2}}\vec{x}\right)$ and the symplectic structure on $\mathcal{M}^{\mathbf{k}}$ becomes

$$\omega = d\vec{x} \wedge d\vec{y}$$

when restricted to W_0.

Now, for $Y \in sl(2,\mathbb{R})$, consider the class, $\mathcal{F}^Y$, of translated hamiltonians

$$\phi_Y(\mu) = \phi(\mu + \lambda Y),$$

where $\phi \in I(\widetilde{sl(2,\mathbb{R})}^*)|_{(\widetilde{sl(2,\mathbb{R})}^+)^*}$. By Theorem 3.6 and Proposition 3.8, such functions pulled back under

$$\tilde{J}_2 : W \to (\widetilde{sl(2,\mathbb{R})}^+)^*$$

generate commuting isospectral flows for

$$L = A + F\alpha G^T = A + ax\otimes x + bx\otimes y + cy\otimes x + dy\otimes y,$$

where

$$2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\gamma_1 = (I+Y)^{-1}\gamma_1. \tag{6.3}$$

In particular, the elementary symmetric invariants L_k of L belong to this class. There are n of these and the space W_0 has dimension $2n$, hence they must generate the entire ring $\mathcal{F}^Y$ and comprise a completely integrable system.

From (6.3) the matrix Y is given by

$$Y = \frac{1}{2\Delta}\begin{pmatrix} -c-2\Delta & a \\ -d & b-2\Delta \end{pmatrix},$$

where $\Delta = ad - bc$.

For Y to be in $sl(2,\mathbb{R})$ we must have $b-c=4$. On the other hand, the spectral curve X is given by

$$\det\left(\sum_{i=1}^{n} \frac{\lambda G_i^T F_i}{\alpha_i - \lambda} + \lambda Y - z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0. \tag{6.4}$$

Thus by translating the spectral parameter z by $\lambda\left(\frac{b-c}{4\Delta} - 1\right)$ we can replace Y by Y' with

$$Y' = \tfrac{1}{2}\begin{pmatrix} -b' & a' \\ -d' & b' \end{pmatrix},$$

where $b' = \frac{b+c}{2\Delta}$, $a' = \frac{a}{\Delta}$, $d' = \frac{d}{\Delta}$. If we next replace z by $\lambda z/2$, we can write (6.4) as

$$\det\begin{pmatrix} -Q_\lambda(\vec{x},\vec{y}) - b' - z & -Q_\lambda(\vec{y},\vec{y}) + a \\ Q_\lambda(\vec{x},\vec{x}) - d & Q_\lambda(\vec{x},\vec{y}) + b' - z \end{pmatrix} = 0,$$

where, following Moser's notation,

$$Q_\lambda(\vec{\xi},\vec{\eta})=\sum_{i=1}^{n}\frac{\xi_i\eta_i}{\lambda-\alpha_i} \tag{6.5}$$

for $\vec{\xi}=(\xi_1,\ldots,\xi_n)$, $\vec{\eta}=(\eta_1,\ldots,\eta_n)\in\mathbb{R}^n$.

The moment map for the $SL(2,\mathbb{R})$ action on W_0 is given by

$$J_{SL(2,\mathbb{R})}(\vec{x},\vec{y})=\frac{1}{2}\begin{pmatrix}\vec{x}\cdot\vec{y} & \vec{y}\cdot\vec{y}\\ -\vec{x}\cdot\vec{x} & -\vec{x}\cdot\vec{y}\end{pmatrix}.$$

Let $SL(2,\mathbb{R})_{Y'}$ denote the stabilizer group of Y' in $SL(2,\mathbb{R})$. According to the general construction of Sect. 4 the Jacobi variety $\mathscr{J}(X)$ on which the flows linearize is given by the Marsden-Weinstein reduction of the joint level sets of the invariants L_i by the action of $SL(2,\mathbb{R})_{Y'}$. In particular if $Y'=0$, $SL(2,\mathbb{R})_{Y'}=SL(2,\mathbb{R})$, so we reduce by the whole $SL(2,\mathbb{R})$ action. Setting

$$\vec{x}\cdot\vec{y}=\alpha,\qquad \|\vec{y}\|^2=\beta,\qquad \|\vec{x}\|^2=\gamma,$$

where α, β, and γ are constants, $\mathscr{J}(X)$ is given by quotienting the intersection of these joint level sets with those of the L_i's by the one parameter subgroup generated by

$$\frac{1}{2}\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\in sl(2,\mathbb{R}),$$

i.e. the hamiltonian flow for

$$F(\vec{x},\vec{y})=\alpha\vec{x}\cdot\vec{y}-\tfrac{1}{2}\beta\|\vec{x}\|^2-\tfrac{1}{2}\gamma\|\vec{y}\|^2.$$

If, however, $Y'\neq 0$ the group $SL(2,\mathbb{R})_{Y'}$ is one dimensional. The conserved quantities are given by the restriction of $J_{SL(2,\mathbb{R})}(\vec{x},\vec{y})$ to $sl(2,\mathbb{R})_{Y'}$, i.e.

$$F(\vec{x},\vec{y})=\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\vec{x}\cdot\vec{y} & \vec{y}\cdot\vec{y}\\ -\vec{x}\cdot\vec{x} & -\vec{x}\cdot\vec{y}\end{pmatrix}\begin{pmatrix}-b' & a\\ -d & b'\end{pmatrix}\right]=-\tfrac{1}{2}a\|\vec{x}\|^2-\tfrac{1}{2}d\|\vec{y}\|^2-b'\vec{x}\cdot\vec{y}.$$

The quotient of the joint level sets of $F(\vec{x},\vec{y})$ and the L_k's by $SL(2,\mathbb{R})_{Y'}$ may be identified with any conveniently chosen section transversal to the fibers, giving rise to a second constraint, $G(\vec{x},\vec{y})=0$. This constraint need not be invariant, but transversality requires that $\{F,G\}\neq 0$.

For example, for the Neumann oscillator we take

$$Y=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix},$$

giving $F(\vec{x},\vec{y})=-\|\vec{x}\|^2$, and take the level set $\|\vec{x}\|^2=1$ together with the section $\vec{x}\cdot\vec{y}=0$, defining T^*S^{n-1}. (This may be regarded as a special case of the Rosochatius system, cf. part B of this section.) The hamiltonian is $H=L_1=\frac{1}{2}\sum y_i^2+\sum\alpha_i x_i^2$, where

$$2\det\left(\frac{1}{\lambda}\tilde{J}_2(\vec{x},\vec{y})+\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}\right)=\frac{\mathscr{L}_0\lambda^{n-1}+\mathscr{L}_1\lambda^{n-2}+\ldots+\mathscr{L}_{n-1}}{\Pi(\lambda-\alpha_i)}.$$

More generally, Moser's constrained systems, which are given by a pair of relations

$$F(\vec{x},\vec{y})=0, \qquad G(\vec{x},\vec{y})=0,$$

may best be understood in the sense of Marsden-Weinstein reduction as follows. Quotienting by the 1-parameter invariance group H_F generated by $F(\vec{x},\vec{y})$, and taking a zero level set for its moment map J_F (i.e. $F=0$), we choose a transversal section to the fibration $J_F^{-1}(0)\to J_F^{-1}(0)/H_F$ whose image in $J_F^{-1}(0)$ is defined by $G(\vec{x},\vec{y})=0$. This is applicable whether or not $F(\vec{x},\vec{y})$ happens to be a part of the $J_{SL(2,\mathbb{R})}$ moment map, as long as it is a conserved quantity.

To see the relationship between the two formulations, note that Moser's constrained hamiltonian is of the form: $h_c=h+\mu F$, with μ chosen so that $\{h,G\}=0$ on $J_F^{-1}(0)$. Since the projection $J_F^{-1}(0)\to J_F^{-1}(0)/H_F$ restricted to the section

$$\sigma_G: J_F^{-1}(0)/H_F\to J_F^{-1}(0)$$

defined by $G(\vec{x},\vec{y})=0$ is a symplectomorphism, the flow of h_c in $J_F^{-1}(0)$ will correspond to that of h in $J_F^{-1}(0)/H_F$, provided X_{h_c} [which is tangential to $\mathrm{im}(\sigma_G)$] and X_h (which is not) project to the same vector field on $J_F^{-1}(0)/H_F$. Since $F=0$ on $J_F^{-1}(0)$, we have $X_{h_c}=X_h+\mu X_F$, and hence this is the case. It follows that the *constrained* system satisfies the same isospectral equations and linearizes on the same Jacobi variety as the unconstrained one, provided we determine the flow in $J_F^{-1}(0)/H_F$ and map it to $J_F^{-1}(0)$ by σ_G.

B. Rosochatius System [Finite Dimensional $\widetilde{u(1,1)}$ Flows] [16, 32]

For this example the phase space is still T^*S^{n-1} as for the Neumann oscillator, but an additional inverse square potential is added to give the hamiltonian

$$H^{\varepsilon}=\tfrac{1}{2}\sum y_i^2+\sum\frac{\mu_i^2}{x_i^2}+\varepsilon\sum\alpha_i x_i^2. \tag{6.6}$$

We again use $r=2$ but now reduce $\widetilde{gl(2)}$ to $\widetilde{u(1,1)}$ by the involution $\sigma(X)=\gamma_1\bar{X}^T\gamma_1$. This corresponds to a definition of $u(1,1)$ taken with respect to the off diagonal hermitian form

$$h=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},$$

chosen to simplify the reduction of the resulting system to the Neumann oscillator. Assuming the α_i's are real this gives the reality condition

$$W=\{(F,G)\in\mathcal{M}^{\mathbf{k}}\mid G=\bar{F}\gamma_1\}.$$

If we denote the columns of F by $\frac{1}{\sqrt{2}}(\vec{z},\vec{p})$, then the columns of G are $\frac{1}{\sqrt{2}}(-\vec{\bar{p}},\vec{\bar{z}})$. The reduced group H_k is given by $U(1)\times\ldots\times U(1)$ (n times) and has moment map

$$J_H(\vec{z},\vec{p})=\tfrac{1}{2}(p_1\bar{z}_1-z_1\bar{p}_1,\ldots,p_n\bar{z}_n-z_n\bar{p}_n).$$

We consider the Marsden-Weinstein reduction of W at

$$J_H(\vec{z},\vec{p})=\sqrt{-2}(\mu_1,\ldots,\mu_n),$$

i.e. $J_H^{-1}(\sqrt{-2}(\mu_1,\ldots,\mu_n))/H$. The quotient has a transverse section given by requiring $z_i=x_i$ to be real. Since $\operatorname{Im}(z_i p_i)=\sqrt{-2}\mu_i$ we can write

$$p_i=-y_i+\sqrt{-2}\frac{\mu_i}{x_i}, \qquad y_i\in\mathbb{R}$$

and the constrained symplectic form is

$$\omega=d\vec{x}\wedge d\vec{y}.$$

On the constrained space, the moment map

$$\tilde{J}_2(F,G)=-\sum_{i=1}^{n}\frac{\lambda G_i^T F_i}{\lambda-\alpha_i}\equiv\mathcal{N}$$

takes the form

$$\mathcal{N}(\vec{x},\vec{y};\lambda)=-\tfrac{1}{2}\sum_{i=1}^{n}\left(\frac{\lambda}{\lambda-\alpha_i}\right)\left\{\begin{pmatrix} x_i y_i & -y_i^2\\ x_i^2 & -x_i y_i\end{pmatrix}+\sqrt{-2}\mu_i\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\left(\frac{\mu_i^2}{x_i^2}\right)\begin{pmatrix}0&1\\0&0\end{pmatrix}\right\}.$$

To obtain the hamiltonian consider the translated invariant

$$\phi\equiv 2\det\left(\frac{\mathcal{N}}{\lambda}+\varepsilon\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\equiv\frac{\mathcal{L}_0\lambda^{n-1}+\mathcal{L}_1\lambda^{n-2}+\ldots+\mathcal{L}_{n-1}}{a(\lambda)},$$

where

$$\mathcal{L}_0=\varepsilon\sum x_i^2,$$

$$\begin{aligned}\mathcal{L}_1&=\tfrac{1}{2}(\textstyle\sum x_i^2)(\sum y_i^2)-\tfrac{1}{2}(\sum x_i y_i)^2+(\sum x_i^2)\left(\sum\left(\frac{\mu_i}{x_i}\right)\right)^2+\varepsilon\sum\alpha_i x_i^2-(\sum\mu_i)^2\\ &\quad-\varepsilon(\textstyle\sum\alpha_i)(\sum x_i^2)\\ &\equiv\tilde{H}^\varepsilon-(\textstyle\sum\mu_i)^2-\varepsilon\sum\alpha_i.\end{aligned}$$

For $\varepsilon\neq 0$ the stabilizer in $U(1,1)$ of $\varepsilon\begin{pmatrix}0&1\\0&0\end{pmatrix}$ is generated by $\begin{pmatrix}0&1\\0&0\end{pmatrix}$. The projection of $J_{u(1,1)}(\vec{x},\vec{y})$ onto this subalgebra is

$$F(\vec{x},\vec{y})=\tfrac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\vec{x}\cdot\vec{y}+\sqrt{-2}\vec{\mu} & -\|y\|^2-2\left(\sum\frac{\mu_i^2}{x_i^2}\right)\\ \|\vec{x}\|^2 & -\vec{x}\cdot\vec{y}+\sqrt{-2}\vec{\mu}\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]=\tfrac{1}{2}\|\vec{x}\|^2.$$

This function generates the flow

$$(\vec{x},\vec{y})\to(\vec{x},\vec{y}+t\vec{x}) \quad \text{on } \mathbb{R}^{2n}.$$

We reduce $\mathbb{R}^{2n}$ by this $\mathbb{R}$ action at $F(\vec{x},\vec{y})=\frac{1}{2}$. This gives an $\mathbb{R}$-fibration $\pi:F^{-1}(1)\to F^{-1}(1)/\mathbb{R}$ of which a cross section is given by

$$G(\vec{x},\vec{y})\equiv\vec{x}\cdot\vec{y}=0.$$

Hence, the constrained space we consider is

$$F^{-1}(1)/\mathbb{R} \sim T^*S^{n-1}$$

with projected hamiltonian H^ε (such that $\pi^* H^\varepsilon = \tilde{H}^\varepsilon|_{F^{-1}(1)}$) given by (6.6). Note that choosing $\vec{\mu}=0$ corresponds to a further reduction to the subgroup $su(1,1) \sim sl(2,\mathbb{R})$ giving back the Neumann oscillator of the previous section.

The spectral curve X for this system is defined by

$$P(\lambda, z) \equiv \det\left(\frac{\mathcal{N}(\vec{x},\vec{y};\lambda)}{\lambda} + \varepsilon\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + z\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = 0.$$

Defining a new parameter w by

$$-\frac{w^2}{a^2(\lambda)} \equiv 2\left(z - \sqrt{-2}\sum \frac{\mu_i}{\lambda-\alpha_i}\right)^2,$$

X is given by $w^2 = S^\varepsilon(\lambda)$, where

$$\begin{aligned}\frac{S^\varepsilon(\lambda)}{a^2(\lambda)} &\equiv \tfrac{1}{2}\left(\sum\frac{x_i^2}{\lambda-\alpha_i}\right)\left(\sum\frac{y_i^2}{\lambda-\alpha_i}\right) - \tfrac{1}{2}\left(\sum\frac{x_i y_i}{\lambda-\alpha_i}\right)^2 \\ &\quad + \left(\sum\frac{x_i^2}{\lambda-\alpha_i}\right)\left(\sum\frac{\mu_i^2/x_i^2}{\lambda-\alpha_i}\right) + \varepsilon\left(\sum\frac{x_i^2}{\lambda-\alpha_i}\right). \end{aligned} \tag{6.7}$$

From this it is evident that if $\varepsilon \neq 0$ the genus of X is $n-1$ and if $\varepsilon = 0$ the genus is $n-2$.

One can rewrite (6.7) as

$$\frac{S^\varepsilon(\lambda)}{a^2(\lambda)} = \tfrac{1}{2}\sum\frac{F_i}{\lambda-\alpha_i} + \sum\frac{\mu_i^2}{(\lambda-\alpha_i)^2},$$

where

$$F_i \equiv \tfrac{1}{2}\sum_{j\neq 1}\left[\frac{(x_i y_j - y_i x_j)^2 + 2(x_i^2/x_j^2)\mu_j^2 + 2(x_j^2/x_i^2)\mu_i^2}{\alpha_i - \alpha_j}\right] + \varepsilon x_i^2 \tag{6.8}$$

are n invariants with one relation $\sum F_i = \varepsilon$.

Remark. For the degenerate case $\varepsilon = 0$, the stabilizer is the entire $SU(1,1)$ generated by $\sum x_i^2$, $\sum x_i y_i$, and $\sum(y_i^2/2 + \mu_i^2/x_i^2) = H^0$. Thus these are all constants of motion. Setting $\sum x_i^2 = 1$ and $\sum x_i y_i = 0$ gives an invariant symplectic submanifold on which the flow linearizes on the $\mathbb{C}^*$-extended Jacobi variety of the genus $n-2$ curve. Fixing $H^0 = E = \text{const}$ and quotienting by the stabilizer of $\begin{pmatrix}0 & E\\ 1 & 0\end{pmatrix}$ gives a constant (in the ordinary Jacobi variety), i.e. the flow is entirely vertical in the extension [and, in fact, strictly periodic, i.e. generated by a $U(1)$ action]. This corresponds to geodesic flow on S^{n-1}. For full details regarding the flow for the general case, expressed in abelian integrals, see [16].

C. Nonlinear Schrödinger Equation
(Finite Gap Solutions from $\widetilde{sl(2,\mathbb{C})}$ and Reductions)

The NLS equation is a PDE determined as the integrability condition for a commutative pair of flows on the dual of a reduced loop algebra $\tilde{\mathfrak{k}}^+ \subset \widetilde{sl(2,\mathbb{C})}^+$, (see e.g. [14]). The "finite gap" solutions are determined by flows on finite dimensional symplectic leaves in $(\tilde{\mathfrak{k}}^+)^*$. Using the moment map $\tilde{J}_k$ these flows can be pulled back to isospectral flows of matrices which can be interpreted as finite dimensional hamiltonian systems (cf. [36]).

The NLS equation has two distinct forms

$$u_{xx}+\sqrt{-1}u_t=2|u|^2u, \tag{6.9a}$$

$$u_{xx}+\sqrt{-1}u_t=-2|u|^2u. \tag{6.9b}$$

These arise from the complex form of NLS:

$$u_{xx}+\sqrt{-1}u_t=2vu^2, \qquad v_{xx}-\sqrt{-1}v_t=2uv^2. \tag{6.10}$$

by setting $v=\bar{u}$ or $v=-\bar{u}$.

The reduction of $\widetilde{gl(2,\mathbb{C})}$ to $\widetilde{sl(2,\mathbb{C})}$ is given by the condition $J_{\mathscr{D}}=0$; i.e., $J_{\mathscr{D}}^{-1}(0)/H$ gives the appropriate model for finite dimensional symplectic leaves in $(\widetilde{sl(2,\mathbb{C})}^+)^*$. By an argument similar to that in part A of this section, the fibration $J_{\mathscr{D}}^{-1}(0)\to J_{\mathscr{D}}^{-1}(0)/H$ can be reduced to a fibration $W\to J_{\mathscr{D}}^{-1}/H$, where

$$W=\{(F,G)\in\mathscr{M}^{\mathbf{k}}\mid G=F\gamma_1\}$$

is an $\widetilde{sl(2,\mathbb{C})}^+$ invariant symplectic submanifold of $J_{\mathscr{D}}^{-1}(0)$ and the fibers of $W\to J_{\mathscr{D}}^{-1}(0)/H$ are given by the orbits of the finite subgroup $\{(\pm1,\pm1,\ldots,\pm1)\}\subset H$.

Let the column vectors of F be denoted by $\frac{1}{\sqrt{2}}(\vec{x},\vec{y})$, $\vec{x},\vec{y}\in\mathbb{C}^n$. Those of G are then $\frac{1}{\sqrt{2}}(-\vec{y},\vec{x})$ and the symplectic form is given by $\omega=d\vec{x}\wedge d\vec{y}$ when restricted to W. The moment map for the $\widetilde{SL(2,\mathbb{C})}$ action on W is

$$\tilde{J}_2(\vec{x},\vec{y})=\frac{\lambda}{2}\begin{pmatrix} Q_\lambda(\vec{x},\vec{y}) & Q_\lambda(\vec{y},\vec{y}) \\ -Q_\lambda(\vec{x},\vec{y}) & -Q_\lambda(\vec{x},\vec{y})\end{pmatrix},$$

where Q_λ is given in (6.5).

Let $\mathscr{N}(\lambda)\equiv\tilde{J}_2(\vec{x},\vec{y})$ and $\mathscr{L}(\lambda)=\frac{a(\lambda)}{\lambda^n}\mathscr{N}(\lambda)=\mathscr{L}_0+\mathscr{L}_1\lambda^{-1}+\ldots+\mathscr{L}_{n-1}\lambda^{-n+1}$. Consider the ad* invariant functions on $(\widetilde{sl(2,\mathbb{C})})^*$ given by

$$\phi_k(X(\lambda))=\tfrac{1}{2}\operatorname{tr}\left(\left(\frac{a(\lambda)\lambda^k}{\lambda^n}X(\lambda)^2\right)_0\right), \qquad k=1,2,\ldots.$$

If we let t_k denote the time parameter for the hamiltonian flow of $\hat{\phi}_k$ we have as usual

$$\frac{d}{dt_k}\mathscr{N}(\lambda)=[(d\phi_k(\mathscr{N}(\lambda)))_+,\mathscr{N}(\lambda)]$$

or equivalently,

$$\frac{d}{dt_k}\mathscr{L}(\lambda)=[(\lambda^k\mathscr{L}(\lambda))_+,\ \mathscr{L}(\lambda)]. \qquad (6.11)$$

In particular

$$\frac{d}{dt_1}\mathscr{L}(\lambda)=[\lambda\mathscr{L}_0+\mathscr{L}_1,\ \mathscr{L}(\lambda)], \qquad (6.12a)$$

$$\frac{d}{dt_2}\mathscr{L}(\lambda)=[\lambda^2\mathscr{L}_0+\lambda\mathscr{L}_1+\mathscr{L}_2,\ \mathscr{L}(\lambda)]. \qquad (6.12b)$$

The leading term $\mathscr{L}_0$ is given by the value of the $SL(2,\mathbb{C})$ moment map, i.e.

$$\mathscr{L}_0=\tfrac{1}{2}\begin{pmatrix}\sum x_iy_i & \sum y_i^2\\ -\sum x_i^2 & -\sum x_iy_i\end{pmatrix},$$

and is thus an invariant of the flows. To get NLS we choose the level set

$$\sum x_i^2=0,\quad \sum y_i^2=0,\quad \text{and}\quad \sum x_iy_i=\sqrt{-1} \qquad (6.13a)$$

and define

$$\mathscr{L}_1=\begin{pmatrix}s & v\\ u & -s\end{pmatrix}\quad\text{and}\quad \mathscr{L}_2=\begin{pmatrix}S & V\\ U & -S\end{pmatrix},$$

where

$$u=-\sum\alpha_ix_i^2,\quad v=\sum\alpha_iy_i^2,\quad\text{and}\quad s=\tfrac{1}{2}\sum\alpha_ix_iy_i-\tfrac{1}{2}\sqrt{-1}\sum\alpha_i. \qquad (6.13b)$$

Because of our choice of $\mathscr{L}_0$ it follows that

$$s=-\frac{\sqrt{-1}}{2}(\operatorname{tr}\lambda L^2)_0\quad\text{and}\quad uv-\sqrt{-1}S=\tfrac{1}{2}\operatorname{tr}(\lambda^2L^2(\lambda))_0,$$

and therefore we can choose $s=0$, i.e.

$$\sum\alpha_ix_iy_i=\sqrt{-1}\sum\alpha_i,\quad\text{and}\quad S=\sqrt{-1}uv. \qquad (6.13c)$$

Letting x denote t_1 and t denote t_2, the commutativity of the two flows imply Eq. (6.10).

The real form (6.9a) comes from a reduction to $\mathfrak{k}_1=su(2)$, i.e.

$$\mathfrak{k}_1=\{X\in sl(2,\mathbb{C})\mid X=-\bar{X}^T\}.$$

Applying the results of Sect. 5, case c, requires the α_i's to appear in complex conjugate pairs and the following relation,

$$x_{2i-1}=-\bar{y}_{2i},\quad x_{2i}=\bar{y}_{2i-1}. \qquad (6.14a)$$

The real form (6.9b) comes from a reduction to $\mathfrak{k}_2=su(1,1)$, i.e.

$$\mathfrak{k}_2=\{X\in sl(2,\mathbb{C})\mid X=-t\bar{X}^Tt\},$$

where $t=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Again applying the results of Sect. 5 we may for this case choose the α_i's to be real, together with the reality conditions

$$x_i=\sqrt{-1}\,\bar{y}_i\,. \tag{6.14b}$$

An analysis similar to that in part A of this section gives the curve X and the Jacobi variety $\mathscr{J}(X)$ on which the flows linearize. Furthermore, a study of the reality conditions (6.14a) and (6.14b) leads to the linearization of the flows for (6.9a) and (6.9b). These computations have already been done in [36]. Note that to reconstruct u, v explicitly it is necessary to do a further integration of the hamiltonian equations corresponding to the restriction of the moment map to the stabilizer of $\mathscr{N}_0=\mathscr{L}_0$, i.e. ϕ_0. The nonreduced isospectral manifold then becomes a $\mathbb{C}^*$ extension of $\mathscr{J}(X)$ as discussed in [36].

D. Modified Korteweg-de Vries Equation [The Twisted Loop Algebra $\widetilde{sl(2,\mathbb{R})}$]

The mKdV equation

$$u_t-6u^2u_x+u_{xxx}=0 \tag{6.15}$$

is determined as the integrability condition for a pair of commutative flows on the dual of a subalgebra of $\widetilde{gl(2)}^+$. In this case the subalgebra is a twisted loop algebra [14, 49].

We begin with the reduced algebra $\widetilde{sl(2,\mathbb{R})}$ as in part A of this section and thus restrict attention to the $\widetilde{sl(2,\mathbb{R})}^+$ invariant symplectic subspace $W\to\mathscr{M}^{\mathbf{k}}$ given by

$$W=\{(F,G)\in\mathscr{M}^{\mathbf{k}}\mid F=\bar{F} \text{ and } G=-F\gamma_1\}\,.$$

As above, we write $F=\frac{1}{\sqrt{2}}(\vec{x},\vec{y})$ with $\vec{x},\vec{y}\in\mathbb{R}^n$ so that $G=\frac{1}{\sqrt{2}}(-\vec{y},\vec{x})$ and the symplectic form of $\mathscr{M}^{\mathbf{k}}$ becomes $\omega=d\vec{x}\wedge d\vec{y}$ when restricted to W.

Now define the twisted loop algebra $\widehat{sl(2,\mathbb{R})}$ by

$$\widehat{sl(2,\mathbb{R})}=\{X(\lambda)\in\widetilde{sl(2,\mathbb{R})}\mid tX(\lambda)t=X(-\lambda)\}\,,$$

where $t=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. An element $\xi(\lambda)\in\widehat{sl(2,\mathbb{R})}$ has the form

$$\xi(\lambda)=\sum_{i=-\infty}^{p}\xi_i\lambda^i\,,\qquad \xi_i\in sl(2,\mathbb{R})\,,$$

where ξ_i is a multiple of $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ if i is even and ξ_i is off diagonal if i is odd.

We identify $\widehat{sl(2,\mathbb{R})}^*$ with $\widehat{sl(2,\mathbb{R})}$ using the inner product (2.16). This gives an identification

$$(\widehat{sl(2,\mathbb{R})}^+)^*\sim(\widehat{sl(2,\mathbb{R})}^-)^\perp=\widehat{sl(2,\mathbb{R})}_0^-\,,$$

where $\widehat{sl(2,\mathbb{R})}_0^-$ is the subalgebra of $\widehat{sl(2,\mathbb{R})}$ consisting of elements of the form

$$X(\lambda)=\sum_{j=-\infty}^{0}X_j\lambda^j\,.$$

Under this identification the moment map

$$\hat{J}_2 : W \to (\widehat{sl(2,\mathbb{R})}^+)^* \sim \widehat{sl(2,\mathbb{R})}_0^-$$

for the $\widehat{sl(2,\mathbb{R})}^+$ action on W is given by

$$\hat{J}_2(F,G) = \frac{\lambda}{2}\sum_{i=1}^{n}\left(\frac{G_i^T F_i}{\alpha_i - \lambda} - \frac{tG_i^T F_i t^{-1}}{\alpha_i + \lambda}\right) = \frac{\lambda}{2}\begin{pmatrix} \sum_{i=1}^{n} \frac{\lambda x_i y_i}{\lambda^2 - \alpha_i^2} & \sum_{i=1}^{n} \frac{\alpha_i y_i^2}{\lambda^2 - \alpha_i^2} \\ -\sum_{i=1}^{n} \frac{\alpha_i x_i^2}{\lambda^2 - \alpha_i^2} & -\sum_{i=1}^{n} \frac{\lambda x_i y_i}{\lambda^2 - \alpha_i^2} \end{pmatrix}.$$

Let $\mathcal{N}(\lambda)$ denote $\hat{J}_2(F,G)$ and

$$\mathcal{L}(\lambda) = \frac{a(\lambda)}{\lambda^{2n}}\mathcal{N}(\lambda) = \mathcal{L}_0 + \frac{1}{\lambda}\mathcal{L}_1 + \ldots + \frac{1}{\lambda^{2n-1}}\mathcal{L}_{2n-1},$$

where $a(\lambda) = \prod_{i=1}^{n}(\lambda^2 - \alpha_i^2)$. Notice $\mathcal{L}(\lambda) \in \widehat{sl(2,\mathbb{R})}_0^-$ because $a(\lambda)$ is a polynomial in λ^2.

Define $\phi_k \in I(\widehat{sl(2,\mathbb{R})}^*)$ by

$$\phi_k(X(\lambda)) = \tfrac{1}{2}\operatorname{tr}\left(\left(\frac{\lambda^{2k}a(\lambda)}{\lambda^{2n}}X(\lambda)^2\right)_0\right), \qquad k = 1, \ldots, n-1.$$

Let t_k denote the time parameter for the hamiltonian flow of $\phi_k|_{(\widehat{sl(2,\mathbb{R})}^+)^*}$, then

$$\frac{d}{dt_k}\mathcal{L}(\lambda) = [(\lambda^{2k}\mathcal{L}(\lambda))_+, \mathcal{L}(\lambda)]. \tag{6.16}$$

The flows are isospectral for both $\mathcal{N}(\lambda)$ and $\mathcal{L}(\lambda)$, and hence $\det(\mathcal{N}(\lambda))$ and $\det(\mathcal{L}(\lambda))$ are invariant. The spectral curve X is hyperelliptic with affine part given by

$$z^2 = \det(\lambda^{2n}\mathcal{L}(\lambda)).$$

The leading term

$$\mathcal{L}_0 = \tfrac{1}{2}\begin{pmatrix} \sum x_i y_i & 0 \\ 0 & -\sum x_i y_i \end{pmatrix} \tag{6.17a}$$

is an invariant of the flows which we choose to be zero.

It follows that

$$\mathcal{L}_1 = \tfrac{1}{2}\begin{pmatrix} 0 & \sum \alpha_i y_i^2 \\ -\sum \alpha_i x_i^2 & 0 \end{pmatrix} \tag{6.17b}$$

is also an invariant, which we fix by

$$\sum \alpha_i y_i^2 = 1, \qquad \sum \alpha_i x_i^2 = 1. \tag{6.18}$$

Define

$$\mathcal{L}_2 = u\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{L}_3 = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \quad \mathcal{L}_4 = w\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.19}$$

where

$$u=\tfrac{1}{2}\sum \alpha_i^2 x_i y_i, \quad p=\tfrac{1}{2}\sum \alpha_i^3 y_i^2-\tfrac{1}{2}\sum \alpha_i^2,$$
$$q=-\tfrac{1}{2}\sum \alpha_i^3 x_i^2+\tfrac{1}{2}\sum \alpha_i^2, \quad w=\sum \alpha_i^4 x_i^2 y_i^2-2u\sum \alpha_i^2. \tag{6.20}$$

The function

$$\tfrac{1}{2}\operatorname{tr}(\mathscr{L}_2^2+\mathscr{L}_1\mathscr{L}_3+\mathscr{L}_3\mathscr{L}_1)=u^2+\tfrac{1}{2}(q-p)$$

is also a constant which we set equal to zero.

Letting x denote the variable t_1 and t denote t_2, it follows from the commutativity of the flows for ϕ_1 and ϕ_2 that u satisfies the mKdV equation (6.15).

E. Boussinesq Equation [Rank $r=3$ Deformations; Constrained $\widetilde{sl(3,\mathbb{R})}$ Flows] Here we consider the Boussinesq equation [2, p. 232]

$$3u_{tt}+u_{xxxx}+12(uu_x)_x=0 \tag{6.21}$$

as an example of rank $r=3$ perturbations (cf. also [43]).

The x and t Boussinesq flows are given by Lax pairs

$$\frac{\partial}{\partial x}\mathscr{L}=[A,\mathscr{L}], \quad \frac{\partial}{\partial t}\mathscr{L}=[B,\mathscr{L}], \tag{6.22}$$

where

$$A=\lambda\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\-3u_x-3v&-3u&0\end{pmatrix}, \tag{6.23a}$$

$$B=\lambda\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}+\begin{pmatrix}2u&0&1\\-u_x-3v&-u&0\\-u_{xx}-3v_x&-2u_x-3v&-u\end{pmatrix}, \tag{6.23b}$$

and $\mathscr{L}$ is a matricial polynomial in λ^{-1}, $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1\lambda^{-1}+\ldots+\mathscr{L}_n\lambda^{-n}$.

The commutation relation

$$\frac{\partial}{\partial t}A-\frac{\partial}{\partial x}B=[B,A]$$

yields the equations

$$u_{xx}+2v_x=u_t, \quad u_{xxx}+3v_{xx}-6uu_x=3u_{xt}+3v_t,$$

which, upon elimination of v, yield (6.21).

The Lax pairs (6.22) are obtained by imposing symplectic constraints on AKS flows in $(\widetilde{sl(3,\mathbb{R})}^+)^*$ as follows.

Consider the hamiltonians

$$\Phi_k(\mathcal{N}(\lambda))=\tfrac{1}{2}\operatorname{tr}\left(\frac{a(\lambda)}{\lambda^n}\lambda^k\mathcal{N}^2(\lambda)\right)_0, \tag{6.24a}$$

$$\Psi_k(\mathcal{N}(\lambda))=\tfrac{1}{3}\operatorname{tr}\left(\left(\frac{a(\lambda)}{\lambda^n}\right)^2\lambda^k\mathcal{N}^3(\lambda)\right)_0. \tag{6.24b}$$

Let

$$H_0=\Phi_1=\operatorname{tr}(\mathcal{N}_0\mathcal{N}_1)-\tfrac{1}{2}(\textstyle\sum\alpha_i)\operatorname{tr}(\mathcal{N}_0^2), \tag{6.25a}$$

$$H_1=\Psi_2=\operatorname{tr}(\mathcal{N}_0^2\mathcal{N}_2+\mathcal{N}_0\mathcal{N}_1^2)-2(\textstyle\sum\alpha_i)\operatorname{tr}(\mathcal{N}_1\mathcal{N}_0^2)+[\tfrac{2}{3}(\sum\alpha_i)^2-\tfrac{1}{3}(\sum\alpha_i^2)]\operatorname{tr}(\mathcal{N}_0^3) \tag{6.25b}$$

generate the x and t flows, respectively, where

$$\mathcal{N}(\lambda)=\mathcal{N}_0+\mathcal{N}_1\lambda^{-1}+\mathcal{N}_2\lambda^{-2}+\dots. \tag{2.26}$$

With $\mathcal{L}(\lambda)=\dfrac{a(\lambda)}{\lambda^n}\mathcal{N}(\lambda)=\mathcal{L}_0+\mathcal{L}_1\lambda^{-1}+\dots+\mathcal{L}\lambda_n^{-n}$, we have

$$\frac{\partial}{\partial x}\mathcal{L}=[(dH_0)_+,\mathcal{L}],$$

and

$$\frac{\partial}{\partial t}\mathcal{L}=[(dH_1)_+,\mathcal{L}],$$

where

$$(dH_0)_+=\lambda\mathcal{L}_0+\mathcal{L}_1,$$

and

$$(dH_1)_+=\lambda^2\mathcal{L}_0^2+\lambda(\mathcal{L}_0\mathcal{L}_1+\mathcal{L}_1\mathcal{L}_0)+\mathcal{L}_1^2+\mathcal{L}_0\mathcal{L}_2+\mathcal{L}_2\mathcal{L}_0.$$

As usual, $\mathcal{L}_0$ is preserved by all of the flows so we can choose

$$\mathcal{L}_0=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}. \tag{6.27}$$

In general, letting t_k denote the flow for Φ_k and s_k the flow for Ψ_k, we have

$$\frac{d}{dt_k}\mathcal{L}=[(\lambda^k\mathcal{L})_+,\mathcal{L}]=-[(\lambda^k\mathcal{L})_-,\mathcal{L}], \tag{6.28a}$$

$$\frac{d}{ds_k}\mathcal{L}=[(\lambda^k\mathcal{L}^2)_+,\mathcal{L}]=-[(\lambda^k\mathcal{L}^2)_-,\mathcal{L}]. \tag{6.28b}$$

From this it follows that

$$\frac{d}{dt_k}\mathcal{L}_1=[\mathcal{L}_0,\mathcal{L}_{k+1}] \tag{6.29a}$$

and

$$\frac{d}{ds_k}\mathscr{L}_1=\left[\mathscr{L}_0, \sum_{i+j=k+1}\mathscr{L}_i\mathscr{L}_j\right]. \tag{6.29b}$$

If we take

$$\mathscr{L}_1=\begin{pmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix},$$

it follows from (6.29a and b) that a_2, a_3, b_2, b_3 are constants of motion. We choose $a_2=b_3=1$ and $a_3=b_2=0$, so $\mathscr{L}_1$ has the form

$$\mathscr{L}_1=\begin{pmatrix} a_1 & 1 & 0\\ b_1 & 0 & 1\\ c_1 & c_2 & -a_1\end{pmatrix}.$$

Setting

$$\mathscr{L}_2=\begin{pmatrix} A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{pmatrix},$$

we have

$$(dH_0)_+=\lambda\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}+\begin{pmatrix} a_1 & 1 & 0\\ b_1 & 0 & 1\\ c_1 & c_2 & -a_1\end{pmatrix}$$

and

$$(dH_1)_+=\lambda\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}+\begin{pmatrix} a_1^2+b_1+A_3 & a_1 & 1\\ a_1b_1+c_1+B_3 & b_1+c_2 & -a_1\\ b_1c_2+A_1+C_3 & c_1+A_2-c_2a_1 & c_2-a_1+A_3\end{pmatrix}.$$

To put these in the form (6.23a) and (6.23b) we need to add constraints. Let

$$F_1(\mathscr{N}(\lambda))=\Phi_0(\mathscr{N}(\lambda))=\tfrac{1}{2}\operatorname{tr}\left(\frac{a(\lambda)}{\lambda^n}\mathscr{N}(\lambda)^2\right)_0,$$

$$F_2(\mathscr{N}(\lambda))=\Psi_1(\mathscr{N}(\lambda))=\tfrac{1}{3}\operatorname{tr}\left(\left(\frac{a(\lambda)}{\lambda^n}\right)^2\lambda\mathscr{N}(\lambda)^3\right)_0,$$

$$G_1(\mathscr{N}(\lambda)=b_1-A_3, \quad\text{and}\quad G_2(\mathscr{N}(\lambda))=a_1.$$

Computing Poisson brackets we find

$$\{F_1,G_1\}=1, \qquad \{F_1,G_2\}=0,$$

$$\{F_2,G_1\}=-G_2, \qquad \{F_2,G_2\}=1.$$

It follows that the constraints $F_1=F_2=G_1=G_2=0$ define symplectic submanifolds of the symplectic leaves in $(\widetilde{sl(3,\mathbb{R})^+})^*$. Since F_1 and F_2 are commuting constants of motion, the constrained hamiltonians $\tilde{H}_0$ and $\tilde{H}_1$ are given by

$$\tilde{H}_0=H_0-\{H_0,G_1\}F_1-\{H_0,G_2\}F_2$$

and

$$\tilde{H}_1=H_1-\{H_1,G_1\}F_1-\{H_1,G_2\}F_2.$$

Since the F's and G's vanish on the constrained space, the constrained hamiltonian vector fields are given by

$$(d\tilde{H}_0)_+=(dH_0)_+-\{H_0,G_1\}(dF_1)_+-\{H_0,G_2\}(dF_2)_+$$

and

$$(d\tilde{H}_1)_+=(dH_1)_+-\{H_1,G_1\}(dF_1)_+-\{H_1,G_2\}(dF_2)_+.$$

Computing

$$\{H_0,G_1\}=-\frac{d}{dx}(b_1-A_3)=-A_2+2B_3,$$

$$\{H_1,G_1\}=-\frac{d}{dt}(b_1-A_3)=2A_1+2C_3+2b_1^2-b_1c_2,$$

$$\{H_0,G_2\}=-\frac{d}{dx}a_1=A_3,$$

$$\{H_1,G_2\}=-\frac{d}{dt}a_1=A_2+B_3,$$

we find

$$(d\tilde{H}_0)_+=\lambda\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\c_1+A_2-2B_3&c_2-A_3&0\end{pmatrix}$$

and

$$(d\tilde{H}_1)_+=\lambda\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}+\begin{pmatrix}2b_1&0&1\\c_1-A_2&b_1+c_2&0\\2b_1c_2-2b_1^2-A_1-C_3&c_1-B_3&b_1+c_2\end{pmatrix}.$$

Now $\mathscr{L}(\lambda)$ is traceless so $A_1+B_2+C_3=0$, i.e., $2A_1-B_2-C_3=3A_1$. Furthermore $\Phi_2(\mathcal{N}(\lambda))=4b_1+2c_2$ is a constant of motion which we choose to be zero. Thus $(d\tilde{H}_1)_+$ takes the form

$$(d\tilde{H}_1)_+=\lambda\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}+\begin{pmatrix}2b_1&0&1\\c_1-A_2&b_1&0\\-6b_1^2+B_2&c_1-B_3&-b_1\end{pmatrix}.$$

Taking

$$b_1 = A_3 = -\tfrac{1}{2}c_2 = u\,, \qquad c_1 - A_2 = -u_x - 3v\,,$$
$$c_1 - B_3 = -2u_x - 3v\,, \qquad B_2 = 6u^2 - u_{xx} - 3v_x\,,$$

$(d\tilde{H}_0)_+$ and $(d\tilde{H}_1)_+$ take the form (6.23a) and (6.23b) respectively.

Remark. Since the constraints F_1 and F_2 are from the set of commuting functions Φ_j, Ψ_k it follows that the Φ's and Ψ's still Poisson commute after applying the full set of constraints $F_1 = F_2 = G_1 = G_2 = 0$. In fact these constraints may be interpreted as a part of the Marsden-Weinstein reduction of the orbits in $(\widetilde{sl(3,\mathbb{R})}^+)^*$ by the $sl(3,\mathbb{R})$ action. F_1 and F_2 give parts of the $sl(3,\mathbb{R})$ moment map and $G_1 = 0$, $G_2 = 0$ defines a section of the Marsden Weinstein reduction by the abelian subgroup of $sl(3,\mathbb{R})$ generated by the hamiltonian flows of F_1 and F_2.

We now use the moment map

$$\tilde{J}_3 : \mathcal{M}^{\mathbf{k}} \to (\widetilde{gl(3)}^+)^*$$

to interpret these flows as isospectral rank 3 perturbations. For simplicity, let us assume that the eigenvalues α_i are real and distinct; thus $k_i = 1$ for each i. We write the matrices F and G, $(F, G) \in \mathcal{M}^{\mathbf{k}}$, in the form

$$F = \begin{pmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_n \end{pmatrix}, \qquad G = \begin{pmatrix} \vec{p}_1 \\ \vdots \\ \vec{p}_n \end{pmatrix}, \tag{6.30}$$

where $\vec{q}_i$, $\vec{p}_i \in \mathbb{C}^3$. The reduction of $(\widetilde{gl(3)}^+)^*$ to $(\widetilde{sl(3,\mathbb{R})}^+)^*$ requires that $\vec{q}_i$, $\vec{p}_i \in \mathbb{R}^3$ and

$$I_{\mathcal{D}}(F, G) = (\vec{q}_1 \cdot \vec{p}_1, \ldots, \vec{q}_n \cdot \vec{p}_n) = 0\,. \tag{6.31}$$

The groups H and $\mathcal{D}$ are the same, and the action is given by $\vec{q}_i \to \alpha_i \vec{q}_i$, $\vec{p}_i \to \alpha_i^{-1} \vec{p}_i$, $\alpha_i \in \mathbb{R} \backslash 0$. Thus we can identify the reduced space $J_{\mathcal{D}}^{-1}(0)/H$ with the space

$$\begin{aligned} W = &\{(F, G) \in \mathcal{M}^{\mathbf{k}} \mid \vec{q}_i,\ \vec{p}_i \in \mathbb{R}^3,\ \|\vec{q}_i\| = 1,\ \vec{q}_i \cdot \vec{p}_i = 0\} \\ &\sim \mathrm{T}^*\mathrm{S}^2 \times \ldots \times \mathrm{T}^*S^2 \quad (n \text{ times})\,. \end{aligned} \tag{6.32}$$

The symplectic structure induced on W is such that the latter identification is symplectic.

The moment map, restricted to W, now gives

$$\tilde{J}_3 = \mathcal{N}(\lambda) = \lambda \sum_{i=1}^{n} \frac{\vec{p}_i^T \vec{q}_i}{\alpha_i - \lambda}\,.$$

Writing the columns of F and G as $(\vec{x}, \vec{y}, \vec{z}) \in \mathbb{R}^{3n}$ and $(\vec{u}, \vec{v}, \vec{w}) \in \mathbb{R}^{3n}$ respectively [so $\vec{q}_i = (x_i, y_i, z_i)$ and $\vec{p}_i = (u_i, v_i, w_i)$], we have

$$\mathcal{N}(\lambda) = -\lambda \begin{pmatrix} Q_\lambda(\vec{x}, \vec{u}) & Q_\lambda(\vec{y}, \vec{u}) & Q_\lambda(\vec{z}, \vec{u}) \\ Q_\lambda(\vec{x}, \vec{v}) & Q_\lambda(\vec{y}, \vec{v}) & Q_\lambda(\vec{z}, \vec{v}) \\ Q_\lambda(\vec{x}, \vec{w}) & Q_\lambda(\vec{y}, \vec{w}) & Q_\lambda(\vec{z}, \vec{w}) \end{pmatrix} \tag{6.33}$$

where Q_λ is given in (6.5). Substitution of this into (6.24a and b) gives the hamiltonians H_0 and H_1 on W in terms of $(\vec{x}, \vec{y}, \vec{z}, \vec{u}, \vec{v}, \vec{w})$. Similarly, the relations $0=F_1=F_2=G_1=G_2$ may be regarded as defining constrained hamiltonians $\tilde{H}_0$ and $\tilde{H}_1$ in W whose flows furthermore leave invariant the submanifold determined by the relations

$$N_0=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \quad N_1=\begin{pmatrix}0&1&0\\u&0&1\\c_1+\sum\alpha_i&-\frac{1}{2}u&0\end{pmatrix}, \tag{6.34}$$

where

$$c_1+\sum\alpha_i=-\sum x_i w_i \alpha_i, \tag{6.35}$$

and

$$u=-\sum x_i v_i \alpha_i \tag{6.36}$$

is the solution of the Boussinesq equation (6.21).

F. Coupled Nonlinear Schrödinger Equation ($\widetilde{su(1,2)}$ Flows)

The CNLS equation is ([2], p. 97)

$$\begin{aligned}\sqrt{-1}u_t+u_{xx}&=2u(|u|^2+|v|^2),\\ \sqrt{-1}v_t+v_{xx}&=2v(|u|^2+|v|^2).\end{aligned} \tag{6.37}$$

This is obtained from the complex form

$$\begin{aligned}\sqrt{-1}u_t+u_{xx}&=2u(uU+vV),\\ \sqrt{-1}v_t+v_{xx}&=2v(uU+vV),\\ -\sqrt{-1}U_t+U_{xx}&=2U(uU+vV),\\ -\sqrt{-1}V_t+V_{xx}&=2V(uU+vV),\end{aligned} \tag{6.38}$$

with the reality condition $U=\bar{u}$, $V=\bar{v}$. Other real forms may also be obtained as for the CNLS equation by choosing differing signs $U=\pm\bar{u}$, $V=\pm\bar{v}$. As in the preceding example, Eq. (6.38) is obtained as the integrability condition for a pair of commutative flows given by Lax equations of the form (6.22), with:

$$A=\frac{\sqrt{-1}}{3}\lambda\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}+\begin{pmatrix}0&\bar{u}&\bar{v}\\u&0&0\\v&0&0\end{pmatrix}, \tag{6.39a}$$

$$\begin{aligned}B=&\frac{\sqrt{-1}}{3}\lambda^2\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}+\lambda\begin{pmatrix}0&\bar{u}&\bar{v}\\u&0&0\\v&0&0\end{pmatrix}\\ &+\sqrt{-1}\begin{pmatrix}|u|^2+|v|^2&-\bar{u}_x&-\bar{v}_x\\u_x&-|u|^2&-\bar{v}u\\v_x&-\bar{u}v&-|v|^2\end{pmatrix}.\end{aligned} \tag{6.39b}$$

These may again be interpreted as flows on $(\widetilde{sl(3,\mathbb{C})}^+)^*$ generated by AKS hamiltonians of the class (6.24a, b), with a reduction to the subalgebra $\tilde{\mathfrak{k}}\equiv\widetilde{su(1,2)}\subset\widetilde{sl(3,\mathbb{C})}$. The moment map $\tilde{J}_3$ will then give an interpretation of the flows on finite dimensional symplectic leaves in $(\widetilde{su(1,2)}^+)^*$ as isospectral flows of matrices.

Using the same notation as in the preceding example, the reduction $\widetilde{gl(3,\mathbb{C})}\supset\widetilde{sl(3,\mathbb{C})}$ is given by the condition $J_{\mathscr{D}}=0$, i.e.

$$\operatorname{tr}\mathscr{L}_i=0\,. \tag{6.40}$$

The reality conditions $\bar{u}=U$, $\bar{v}=V$ giving (6.38) follows from the reduction of $sl(3,\mathbb{C})$ to $su(1,2)$ by

$$-t\bar{\mathscr{L}}_j^T t^{-1}=\mathscr{L}_j \tag{6.41}$$

with

$$t=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}.$$

Taking $(F,G)\in M_{n,3}\times M_{n,3}$ with rows $(F_i=(x_i,y_i,z_i),\,G_i)$, (so $k_i=1$) and real eigenvalues α_i, the reduction procedure of Sect. 5 gives:

$$G_i=\sqrt{-1}(\overline{x_i},-\overline{y_i},-\overline{z_i}) \tag{6.42}$$

from (6.41). Here x_i, y_i, $z_i\in\mathbb{C}$ provide complex Darboux coordinates on the reduced space $W=(\mathscr{C}^3)^n$ with symplectic structure

$$\omega=\sqrt{-1}\sum(dx_i\wedge d\bar{x}_i-dy_i\wedge d\bar{y}_i-dz_i\wedge d\bar{z}_i)\,.$$

To implement (6.40) we take the constraint

$$|x_i|^2-|y_i|^2-|z_i|^2=0\,, \tag{6.43}$$

and quotient by the S^1 action

$$g\cdot(x_i,y_i,z_i)=(gx_i,gy_i,gz_i)\,,\qquad g\in S^1$$

on each of the n copies of $\mathbb{C}^3$.

This quotient space may be identified with $\mathbb{C}^2$, where Darboux coordinates are given by $\eta_i=e^{-\sqrt{-1}\theta_i}y_i$ and $\zeta_i=e^{-\sqrt{-1}\theta_i}z_i$, and the phase θ_i is given by $x_i=e^{\sqrt{-1}\theta_i}|x_i|$. Since the moment map is independent of this phase we may write it (and its components) in terms of η_i and ζ_i,

$$\begin{aligned}\tilde{J}_{su(1,2)}(\zeta,\eta)=N(\lambda)&=\sum_{i=1}^{n}\frac{\sqrt{-1}\lambda}{\alpha_i-\lambda}\begin{pmatrix}\varrho_i^2&\eta_i\varrho_i&\zeta_i\varrho_i\\-\bar{\eta}_i\varrho_i&-|\eta_i|^2&-\bar{\eta}_i\zeta_i\\-\bar{\zeta}_i\varrho_i&-\bar{\zeta}_i\eta_i&-|\zeta_i|^2\end{pmatrix}\\&=-\frac{\lambda^n}{a(\lambda)}(\mathscr{L}_0+\mathscr{L}_1\lambda^{-1}+\mathscr{L}_2\lambda^{-2}+\ldots)\,,\end{aligned} \tag{6.44}$$

where

$$\varrho_i=\sqrt{|\eta_i|^2+|\zeta_i|^2}\,. \tag{6.45}$$

The leading term $\mathscr{L}_0$ is an invariant of the flows since it is given by $J_{su(1,2)}$. We choose the level set

$$\mathscr{L}_0 = \frac{\sqrt{-1}}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{6.46}$$

i.e.

$$\sum|\eta_i|^2 = 1/3, \quad \sum|\zeta_i|^2 = 1/3, \quad \sum \eta_i\sqrt{|\eta_i|^2+|\zeta_i|^2} = 0,$$
$$\text{and} \quad \sum \eta_i\bar{\zeta}_i = 0, \quad \sum \zeta_i\sqrt{|\eta_i|^2+|\zeta_i|^2} = 0. \tag{6.47}$$

Furthermore $\mathscr{L}_1$ satisfies (6.29a) and (6.29b). Thus, if we write

$$\mathscr{L}_1 = \begin{pmatrix} a & \bar{u} & \bar{v} \\ u & b & \bar{d} \\ v & d & c \end{pmatrix},$$

we see that with our choice of $\mathscr{L}_0$ the entries a, b, c, d are invariants of all the flows. We choose $a=b=c=d=0$ where

$$a = \sqrt{-1}\sum(\alpha_i(|\eta_i|^2+|\zeta_i|^2) - \tfrac{2}{3}\alpha_i), \quad b = -\sqrt{-1}\sum(\alpha_i|\eta_i|^2 - \tfrac{1}{3}\alpha_i),$$
$$c = -\sqrt{-1}\sum(\alpha_i|\zeta_i|^2 - \tfrac{1}{3}\alpha_i), \quad \text{and} \quad d = -\sqrt{-1}\sum \alpha_i\eta_i\bar{\zeta}_i. \tag{6.48}$$

Thus

$$\mathscr{L}_1 = \begin{pmatrix} 0 & \bar{u} & \bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \tag{6.49}$$

where

$$u = -\sqrt{-1}\sum \alpha_i\bar{\eta}_i\sqrt{|\eta_i|^2+|\zeta|^2}, \quad v = -\sqrt{-1}\sum \alpha_i\bar{\zeta}_i\sqrt{|\eta_i|^2+|\zeta_i|^2}. \tag{6.50}$$

Let $\mathscr{L}_2$ be denoted by

$$\mathscr{L}_2 = \sqrt{-1}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

and let x denote t_1 and t denote t_2. The commutativity of the Φ_1 and Φ_2 flows then implies

$$\mathscr{L}_2 = \sqrt{-1}\begin{pmatrix} a_1 & -\bar{u}_x & -\bar{v}_x \\ u_x & b_2 & c_2 \\ v_x & b_3 & c_3 \end{pmatrix}.$$

with $(a_1)_x = (|u|^2+|v|^2)_x$, $(b_2)_x = (-|u|^2)_x$, $(c_3)_x = (-|v|^2)_x$, $(c_2)_x = (-\bar{v}u)_x$, and $(b_3)_x = (-\bar{u}v)_x$.

Setting $\Phi_2(\mathcal{N}(\lambda))=0$, it follows that $a_1=|u|^2+|v|^2$. To get

$$\begin{aligned} A&=(d\phi_1(N(\lambda)))_+=\lambda L_0+L_1\,,\\ B&=(d\phi_2(N(\lambda)))_+=\lambda^2 L_0+\lambda L_1+L_2 \end{aligned}\tag{6.51}$$

in the form (6.39a, b), we must set

$$b_2+|u|^2=c_3+|v|^2=b_3+v\bar{u}=c_2+u\bar{v}=0\,.\tag{6.52}$$

These quantities are invariants of all the flows in $\mathscr{F}_+$, and therefore this does not require constraining the hamiltonians. To see this, consider the matrix

$$S=\mathscr{L}_2+i\mathscr{L}_1^2\,.$$

Let $\phi\in\mathscr{F}_+$ and τ denote the parameter for its hamiltonian flow. Then

$$\frac{d}{d\tau}\mathscr{L}_2=[\mathscr{L}_0, d\phi(\mathscr{L})_{-2}]+[\mathscr{L}_1, d\phi(\mathscr{L})_{-1}]\,,$$

and $\frac{d}{d\tau}\mathscr{L}_1^2=\mathscr{L}_1[\mathscr{L}_0, d\phi(\mathscr{L})_{-1}]+[\mathscr{L}_0, d\phi(\mathscr{L})_{-1}]\mathscr{L}_1$. Allowing arbitrary matrices for $d\phi(\mathscr{L})_{-1}$ and $d\phi(\mathscr{L})_{-2}$ leaves the lower right-hand 2×2 block zero. Since S has

$$\begin{pmatrix} i(b_2+|u|^2) & i(c_2+\bar{v}u)\\ i(b_3+\bar{u}v) & i(c_3+|v|^2)\end{pmatrix}$$

as its lower right-hand 2×2 block we see that these quantities are invariants of the flows.

The detailed computation of the linearized flows on Jacobi varieties of spectral curves for these and other examples will be presented in the sequel [4].

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