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Knots, Links, Braids and Exactly Solvable Models in Statistical Mechanics

Yasuhiro Akutsu[1] and Miki Wadati[2]

[1] Institute of Physics, Kanagawa University, Rokkakubashi, Kanagawa-ku, Yokohama 221, Japan
[2] Institute of Physics, College of Arts and Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

Abstract. We present a general method to construct the sequence of new link polynomials and its two variable extension from exactly solvable models in statistical mechanics. First, we find representations of the braid group from the Boltzmann weights of the exactly solvable models. Second, we give the Markov traces associated with new braid group representations and using them construct new link polynomials. Third, we extend the theory into a two-variable version of the new link polynomials. Throughout the paper, we emphasize the essential roles played by the exactly solvable models and the underlying Yang-Baxter relation.

1. Introduction

In physics we often deal with the configuration problem of one-dimensional objects, for instance, polymers, magnetic fluxes, dislocation lines and trajectories of particles. We generally call a one-dimensional object a string. A knot is a closed string which does not cross with itself. As a more generalized object, an assembly of knots with mutual entanglements is called a link. Classification of knots and links is known to be a longstanding problem in mathematics [1, 2]. In this paper we report an unexpected close connection between physics and mathematics. Namely, we present a general method to construct topological invariants for knots and links by using the theory of exactly solvable models in statistical mechanics.

We begin with the braid and the braid group. Braids are formed when n points on a horizontal line are connected by n strings to n points on another horizontal line directly below the first n points. A trivial n-braid is a configuration where no intersection between the strings is present. A general n-braid is constructed from the trivial n-braid by successive applications of the operation b_i, $i=1,2,\dots,n-1$. The operation b_i and its inverse b_i^{-1} are best understood by the graphs (Fig. 1). A the set of generators, $b_1, b_2, \dots, b_{n-1}$, define the braid group B_n [3]. By regarding the trivial n-braid as the identity operation in B_n, we can identify any element in B_n as an n-braid. To guarantee the topological equivalence between different

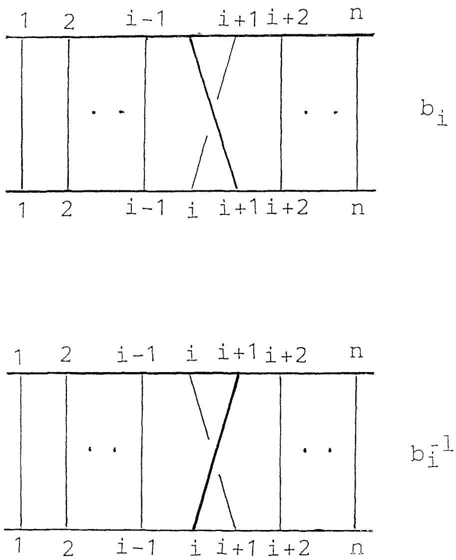

Fig. 1. Operations b_i and b_i^{-1}

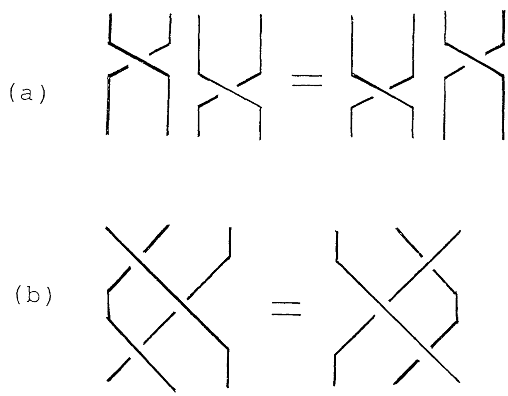

Fig. 2. Defining relations of the braid group

expressions of a braid in terms of braid group elements, Artin proved that the following conditions are necessary and sufficient (Fig. 2):

$$b_i b_j = b_j b_i, \qquad |i-j| \geqq 2, \tag{1.1a}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{1.1b}$$

We call them the defining relation of B_n. Then, each topologically equivalent class of the braids is identified with an element in B_n.

Given a braid, one may form a link by tying opposite ends. Conversely, according to Alexander's theorem [4], any link is represented by a closed braid. This fact gives the braid group a fundamental role in the knot theory. However, the representation of a link as a closed braid is highly non-unique. Therefore, the following theorem due to Markov [5] is important. The equivalent braids expressing the same link are mutually transformed by successive applications of two types of operations, type I and type II Markov moves:

$$\text{I. } AB \to BA\,(A, B \in B_n),$$
$$\text{II. } A \to Ab_n,\ A \to Ab_n^{-1},$$
$$(A \in B_n, b_n \in B_{n+1}). \tag{1.2}$$

Hence, to obtain a link polynomial which is a topological invariant, we first construct a suitable representation B'_n of the braid group B_n and then find a Markov move invariant quantity defined on B'_n.

Let us denote the representation of b_i by g_i and a link polynomial by $\alpha(\cdot)$. The link polynomial $\alpha(\cdot)$ must satisfy the conditions:

$$\text{I. } \alpha(AB) = \alpha(BA)\,(A, B \in B'_n), \tag{1.3a}$$
$$\text{II. } \alpha(Ag_n) = \alpha(Ag_n^{-1}) = \alpha(A)$$
$$(A \in B'_n, g_n \in B'_{n+1}). \tag{1.3b}$$

This quantity is readily obtained if we can find a linear functional $\phi(\cdot)$ on B'_n, called the Markov trace [6, 7], which have the following properties (the Markov properties):

$$\text{I. } \phi(AB) = \phi(BA)\,(A, B \in B'_n), \tag{1.4a}$$
$$\text{II. } \phi(Ag_n) = \tau\phi(A),\ \phi(Ag_n^{-1}) = \bar{\tau}\phi(A)$$
$$(A \in B'_n, g_n \in B'_{n+1}), \tag{1.4b}$$

with the parameters τ and $\bar{\tau}$ being given by

$$\tau = \phi(g_i), \qquad \bar{\tau} = \phi(g_i^{-1}) \quad \text{for all } i. \tag{1.5}$$

With the Markov trace $\phi(\cdot)$, the link polynomial $\alpha(\cdot)$ is given by

$$\alpha(A) = (\tau\bar{\tau})^{-(n-1)/2} \left(\frac{\bar{\tau}}{\tau}\right)^{e(A)/2} \phi(A) \qquad (A \in B'_n). \tag{1.6}$$

Here $e(A)$ is the exponent sum of g_i's appearing in the braid A. For instance, if $A = g_2^3 g_1^{-2}$, then $e(A) = 3 - 2 = 1$. The properties (1.3a) and (1.3b) are easily verified from (1.4a), (1.4b) and the definition of $e(A)$.

In the recent discovery of a new polynomial invariant (Jones polynomial) for knots and links, Jones [7] utilized a C^*-algebra $A_{q,n}$ generated by $\{1, e_1, e_2, \ldots, e_n\}$. The generators $\{e_j\}$ are essentially identical with the Temperley-Lieb operators [8] and satisfy the relations:

$$e_i^* = e_i, \qquad e_i^2 = e_i,$$
$$e_i e_j = e_j e_i \quad \text{for} \quad |i-j| \geqq 2,$$
$$e_i e_{i\pm1} e_i = q^{-1} e_i. \tag{1.7}$$

The Temperley-Lieb algebra is known to describe the transfer matrices of the ferroelectric model (6-vertex model) and the self-dual Potts model (critical Potts model) in statistical mechanics [9, 10]. An ingenious idea by Jones is to use the algebra $A_{q,n}$ for a construction of a representation (Hecke algebra [11] representation) of the braid group. To observe this, we introduce a parameter t and a generator $\hat{g}_i$ by

$$q=2+t^{-1}+t, \tag{1.8}$$

$$\hat{g}_i=(t+1)e_i-1 \qquad (e_i\in A_{q,n-1}). \tag{1.9}$$

Then the Hecke algebra $H(t, n)$ is generated by $\{\hat{g}_i\}$. The generators satisfy both the defining relation (1.1) of the braid group and the quadratic relation (the reduction relation):

$$\hat{g}_i^2=(t-1)\hat{g}_i+t. \tag{1.10}$$

In most cases, the Jones polynomial is more powerful than the classic Alexander polynomial [12] in the sense that it detects properties of a link which could not be detected by the latter. The Jones polynomial and the Alexander polynomial are special cases of the two-variable extension [13] of the Jones polynomial. It is known that the Jones polynomial and its two-variable extension are still not complete. There exist infinitely many different links which have the same polynomial [14, 15]. The aim of the present paper is clear. We further pursue a close relation between the exactly solvable models and the link polynomials, and present a general method which leads to a sequence of new link polynomials and its two-variable extension.

The paper is organized as follows. In Sect. 2, we give a general prescription to have a braid group representation from the Boltzmann weights which satisfy the Yang-Baxter relation. In Sect. 3, we apply the prescription to the N-state vertex model and obtain a sequence of braid group representations. Further, defining the Markov traces associated with the new braid group representation, we construct a sequence of new link polynomials. In Sect. 4, we give an alternative method to obtain the braid group representation by introducing "composite" string with symmetrizers. In Sect. 5, we use the composite string representation of the braid group to obtain the two-variable extension of the new link polynomials. The last section is devoted to a summary of the paper.

2. Construction of Braid Group Representations from Solvable Models

Recent development in the theory of quantum completely integrable systems provides us a unified treatment of various exactly solvable models in $1+1$ dimensional field theory and in 2-dimensional classical statistical mechanics [16–20]. The central idea is that to each solvable model we can associate a family of commuting transfer matrices which are the generators of an infinite number of conserved quantities. The commutability condition is called the Yang-Baxter relation.

The Yang-Baxter relation takes different expressions depending on the types of models. For the $1+1$ dimensional field theory, the Yang-Baxter relation is the

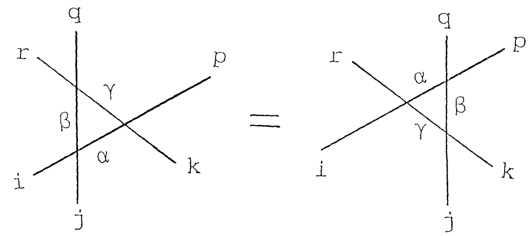

Fig. 3. The factorization equation

factorization condition [21] for the many-body S-matrices and is known as the factorization equation. We denote the scattering amplitude for the process $(i,j) \to (k,l)$ by $S^{ik}_{jl}(u)$, where u is the rapidity (we also refer to it as the spectral parameter). Then, the factorization equation reads as (Fig. 3)

$$\sum_{\alpha,\beta,\gamma} S^{\beta q}_{\gamma r}(v) S^{\alpha p}_{k\gamma}(u+v) S^{i\alpha}_{j\beta}(u) = \sum_{\alpha,\beta,\gamma} S^{\alpha p}_{\beta q}(u) S^{i\alpha}_{\gamma r}(u+v) S^{j\beta}_{k\gamma}(v). \tag{2.1}$$

In 2-dimensional statistical mechanics, we have two types of models, the vertex models and the IRF (interaction round a face) models [22]. Some of the IRF models are equivalent to the vertex models through the Wu-Kadanoff-Wegner transformation [23]. It is known that any factorized S-matrix can be interpreted as the Boltzmann weight of a solvable vertex model and that the factorization equation is the commutability condition of the transfer matrices of the vertex model [24]. For the IRF models, the Yang-Baxter relation is called the star-triangle relation which reads as

$$\sum_{c} w(b,d,c,a;u)\, w(a,c,f,g;u+v)\, w(c,d,e,f;v)$$
$$= \sum_{c} w(a,b,c,g;v)\, w(b,d,e,c;u+v)\, w(c,e,f,g;u). \tag{2.2}$$

Here $w(a,b,c,d;u)$ denotes the Boltzmann weight for the spin configuration (a,b,c,d) round a face.

The Boltzmann weights satisfying (2.1) or (2.2) define a set of operators [22], $\{X_i(u)\}$, which satisfy the relations:

$$X_i(u)X_j(v) = X_j(v)X_i(u) \quad \text{for} \quad |i-j| \geqq 2, \tag{2.3a}$$

$$X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u). \tag{2.3b}$$

In the vertex model we have the following expression of the operator $X_i(u)$:

$$X_i(u) = \sum_{klmn} S^{km}_{ln}(u) \cdot I^{(1)} \otimes I^{(2)} \otimes \dots \otimes E^{(i)}_{nk} \otimes E^{(i+1)}_{ml} \otimes I^{(i+2)} \otimes \dots, \tag{2.4a}$$

where $I^{(j)}$ is the identity acting on the j^{th} position and E_{nk} is a matrix whose elements are $(E_{nk})_{pq} = \delta_{np}\delta_{kq}$. In the IRF model, matrix elements of $X_i(u)$ are given by

$$[X_i(u)]_{\{l'\},\{l\}} = \delta_{l'_0 l_0}\delta_{l'_1 l_1}\dots\delta_{l'_{i-1} l_{i-1}}$$
$$\times\, w(l_i, l_{i+1}, l'_i, l_{i-1}; u) \cdot \delta_{l'_{i+1} l_{i+1}} \dots \delta_{l'_n l_n}. \tag{2.4b}$$

The operator $X_i(u)$ represents an element of the diagonal to diagonal transfer matrix [22].

By regarding the Yang-Baxter relation as a functional equation for the Boltzmann weights, we have a systematic method to find a new solvable model. We have shown that there exists at least an infinite number of solvable hierarchies (Grand Hierarchy) in 2-dimensional statistical mechanics [20, 25–29].

Now it is extremely interesting to notice the close similarity between (1.1) and (2.3). A graphical similarity between Figs. 2 and 3 is also intriguing. The only difference is that in (2.3) we have spectral parameters as the arguments of the operators. In other words, the Yang-Baxter relation contains superfluous information which might be important in future studies. For the present purpose, to have a braid group representation, we eliminate the spectral parameters. We have found [30, 31] that an interesting identification is, after suitable normalization if necessary, given by

$$b_i \to g_i = \lim_{u \to \infty} X_i(u). \tag{2.5}$$

Depending on the function of parametrization, all the known solutions to the Yang-Baxter relations are classified into three cases; (1) elliptic (2) trigonometric or hyperbolic (3) rational. Any model in the case (1) [respectively the case (2)] at criticality corresponds to a model in the case (2) [respectively case (3)]. Then, to obtain an interesting braid group representation as a well-defined limit (2.5), we should use a model at criticality which is parametrized by the trigonometric or hyperbolic functions. This explains an observation by Jones that the Temperley-Lieb algebra for the transfer matrices of the 6-vertex model (8-vertex model at criticality) and the self-dual (critical) Potts model appear in the construction of the braid group representation.

In what follows, we consider only the solvable models whose Boltzmann weights are parametrized by the hyperbolic (or equivalently trigonometric) functions. We assume that the unitarity condition holds:

$$\sum_{p,q} S_{pk}^{ql}(-u) S_{jq}^{ip}(u) = \delta_{ik}\delta_{jl}, \quad \text{for the vertex model}, \tag{2.6a}$$

$$\sum_{e} w(e,c,d,a;-u)\, w(b,c,e,a;u) = \delta_{bd}, \quad \text{for the IRF model}. \tag{2.6b}$$

Due to the unitarity (2.6) which amounts to

$$X_i(-u) = X_i(u)^{-1}, \tag{2.7}$$

we have the identification

$$g_i^{-1} = \lim_{u \to \infty} X_i(-u). \tag{2.8}$$

3. The N-State Vertex Model and New Link Polynomials

Among many solvable models with hyperbolic (or equivalently trigonometric) parametrization, we consider a series of vertex models (the N-state vertex model) proposed by Sogo, Akutsu and Abe [32]. This series includes the 6-vertex model as $N=2$ case and the 19-vertex model [33] as $N=3$ case.

The edge variables i, j, k, and l of the Boltzmann weights $\{S^{ik}_{jl}(u)\}$ for the N-state vertex model take the following values:

$$i, j, k, l = -s, -s+1, \ldots, s-1, s, \tag{3.1}$$

where "spin" s is related to the state number N by

$$N = 2s+1. \tag{3.2}$$

The model has the properties:

a) charge conservation condition;

$$S^{ik}_{jl}(u) = 0 \quad \text{unless} \quad i+j=k+l, \tag{3.3}$$

b) CPT invariances;

$$S^{ik}_{jl}(u) = S^{-i\,-k}_{-j\,-l}(u) = S^{jl}_{ik}(u) = S^{ki}_{lj}(u), \tag{3.4}$$

c) crossing symmetry;

$$S^{ik}_{jl}(u) = F(u) \cdot S_{jl}^{-k\,-i}(\lambda - u). \tag{3.5}$$

Here the parameter λ is called the crossing point of the spectral parameter and $F(u)$ is some function. For instance, parametrization of the weights which satisfy the unitarity condition (2.8) are shown below, for $N=2$, 3, and 4 cases.

1) $N=2$ $(s=1/2)$ case

$$S^{1/2\;1/2}_{1/2\;1/2}(u) = 1, \qquad S^{1/2\;1/2}_{-1/2\;-1/2}(u) = \frac{\sinh u}{\sinh(\lambda-u)},$$
$$S^{1/2\;-1/2}_{-1/2\;1/2}(u) = \frac{\sinh\lambda}{\sinh(\lambda-u)}. \tag{3.6}$$

2) $N=3$ $(s=1)$ case

$$S^{1\;1}_{1\;1}(u) = 1, \qquad S_{-1\;-1}^{\;\;1\;\;\;\;1}(u) = \frac{\sinh u \sinh(\lambda+u)}{\sinh(\lambda-u)\sinh(2\lambda-u)},$$
$$S_{-1\;\;\;1}^{\;\;1\;-1}(u) = \frac{\sinh\lambda \sinh 2\lambda}{\sinh(\lambda-u)\sinh(2\lambda-u)},$$
$$S^{1\;1}_{0\;0}(u) = \frac{\sinh u}{\sinh(2\lambda-u)}, \qquad S^{1\;0}_{0\;1}(u) = \frac{\sinh 2\lambda}{\sinh(2\lambda-u)},$$
$$S^{0\;\;\;\;1}_{0\;-1}(u) = \frac{\sinh 2\lambda \sinh u}{\sinh(\lambda-u)\sinh(2\lambda-u)},$$
$$S^{0\;0}_{0\;0}(u) = \frac{\sinh\lambda \sinh 2\lambda - \sinh u \sinh(\lambda-u)}{\sinh(\lambda-u)\sinh(2\lambda-u)}. \tag{3.7}$$

3) $N=4$ ($s=3/2$) case

$$S^{3/2\ 3/2}_{3/2\ 3/2}(u)=1\,,$$

$$S^{3/2\ 3/2}_{-3/2\ -3/2}(u)=\frac{\sinh u\sinh(\lambda+u)\sinh(2\lambda+u)}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ -3/2}_{-3/2\ 3/2}(u)=\frac{\sinh\lambda\sinh 2\lambda\sinh 3\lambda}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ 3/2}_{1/2\ 1/2}(u)=\frac{\sinh u}{\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ 3/2}_{-1/2\ -1/2}(u)=\frac{\sinh u\sinh(\lambda+u)}{\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ 1/2}_{1/2\ 3/2}(u)=\frac{\sinh 3\lambda}{\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ 1/2}_{-1/2\ 1/2}(u)=\frac{2\sqrt{\sinh\lambda\sinh 3\lambda}\cosh\lambda\sinh u}{\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ 1/2}_{-3/2\ -1/2}(u)=\frac{\sinh 3\lambda\sinh u\sinh(\lambda+u)}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ -1/2}_{-1/2\ 3/2}(u)=\frac{\sinh 2\lambda\sinh 3\lambda}{\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{3/2\ -1/2}_{-3/2\ 1/2}(u)=\frac{\sinh 2\lambda\sinh 3\lambda\sinh u}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{1/2\ 1/2}_{1/2\ 1/2}(u)=\frac{\sinh 2\lambda\sinh 3\lambda-\sinh u\sinh(\lambda-u)}{\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{1/2\ 1/2}_{-1/2\ -1/2}(u)=\frac{\sinh u[\sinh 2\lambda\sinh 3\lambda-\sinh u\sinh(\lambda-u)]}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,,$$

$$S^{1/2\ -1/2}_{-1/2\ 1/2}(u)=\frac{\sinh 2\lambda[\sinh\lambda\sinh 3\lambda-2\cosh\lambda\sinh u\sinh(\lambda-u)]}{\sinh(\lambda-u)\sinh(2\lambda-u)\sinh(3\lambda-u)}\,. \tag{3.8}$$

The S-matrices for general N have been given in [32].

In order to construct an interesting braid group representation, we further introduce the symmetry-breaking transformation [32],

$$\tilde{S}^{ik}_{jl}(u)=\exp[(k-i-l+j)u/2]\cdot S^{ik}_{jl}(u)\,. \tag{3.9}$$

Due to the charge conservation condition (3.3), the resulting asymmetrized Boltzmann weights $\{\tilde{S}^{ik}_{jl}(u)\}$ also satisfy the factorization equation. We note that the transformation does not spoil the unitarity condition. Substituting $\tilde{S}^{ik}_{jl}(u)$ into (2.4a), we obtain the generator g_i of the braid group representation. We find that g_i satisfies the reduction relation

$$(g_i-c_1)(g_i-c_2)\dots(g_i-c_N)=0\,, \tag{3.10}$$

where

$$c_i = (-1)^{i+1} t^{[N(N-1)-(N-i+1)(N-i)]/2}, \qquad (3.11a)$$

$$t = e^{2\lambda}. \qquad (3.11b)$$

The reduction relation (3.10) for $N=2$ case is, by an identification $g_i \to -\hat{g}_i$, that of the the Hecke algebra (1.10).

A remaining task is to introduce the Markov trace which should satisfy the property (1.4). We have found that the Markov trace $\phi(\cdot)$ associated with our braid group representation is given in a form [30, 31]

$$\phi(A) = \mathrm{Tr}(HA). \qquad (3.12)$$

Here Tr stands for the ordinary trace, that is, the sum of diagonal elements of the matrix. And, the matrix H is given by a tensor product of a diagonal matrix h:

$$H = h \otimes h \otimes \ldots \otimes h \otimes \ldots, \qquad (3.13)$$

with

$$(h)_{pq} = t^{-p} \delta_{pq} \Bigg/ \left(\sum_{k=-s}^{s} t^{-k} \right). \qquad (3.14)$$

The denominator in (3.14) gives the proper normalization for the identity I_n in B'_n:

$$\phi(I_n) = 1. \qquad (3.15)$$

The Markov trace defined by (3.12)–(3.15) is a generalization of the Powers state [34, 35]. The parameters τ and $\bar{\tau}$ in (1.5) are given by

$$\tau(t) = 1/(1 + t + t^2 + \ldots + t^{N-1}), \qquad (3.16a)$$

$$\bar{\tau}(t) = t^{N-1}/(1 + t + t^2 + \ldots + t^{N-1}) = \tau(1/t). \qquad (3.16b)$$

Thus, we arrive at a conclusion that the link polynomial $\alpha(A)$ for elements A in B'_n is given by

$$\alpha(A) = [t^{-(N-1)/2}(t + t + t^2 + \ldots + t^{N-1})]^{n-1} [t^{(N-1)/2}]^{e(A)} \phi(A), \qquad (3.17)$$

where $e(A)$ is the exponent sum of g_i's appearing in the braid A.

From the reduction relation (3.10), the Alexander-Conway relation for the Jones polynomial (the $N=2$ case) [7, 12, 36] is derived:

$$\alpha(L_+) = (1-t)t^{1/2}\alpha(L_0) + t^2 \alpha(L_-). \qquad (3.18)$$

In the above, by L_+, L_0, and L_-, we have denoted links which have the configurations of g_i, g_i^0, and g_i^{-1}, respectively, at an intersection. Similarly, we can show that

$$\alpha(L_{2+}) = t(1 - t^2 + t^3)\alpha(L_+) + t^2(t^2 - t^3 + t^5)\alpha(L_0) - t^8 \alpha(L_-) \qquad (3.19)$$

for the $N=3$ case and

$$\alpha(L_{3+}) = t^{3/2}(1 - t^3 + t^5 - t^6)\alpha(L_{2+}) + t^6(1 - t^2 + t^3 + t^5 - t^6 + t^8)\alpha(L_+)$$
$$- t^{9/2} t^8 (1 - t + t^3 - t^6)\alpha(L_0) - t^{20}\alpha(L_-), \qquad (3.20)$$

for the $N=4$ case. In the above expressions, the meanings of L_{2+} and L_{3+} should be clear.

Since we have the explicit forms of the braid group representations and the Markov traces, the evaluation of link polynomials, $\alpha(A)$ defined by (3.17), is automatic. In fact, we have given new link polynomials up to closed 3-braids by applying the $N=3$ theory [37].

There exists a link polynomial corresponding to the exactly solvable model. Our theory can be extended into any solvable vertex models and IRF models. The braid group representation and the Markov trace are readily found for the IRF models.

4. Composite String Representations

The N-state vertex model can be considered to describe the scattering for spin $(N-1)/2$ particles which has the factorization property. In particular, the 6-vertex model ($N=2$) corresponds to the spin 1/2 factorized S-matrix. We recall that a multiplet of spin 1/2 particles contains higher spin particles. For example, from a pair of spin 1/2 particles, we can make two "composite particles"; one with spin 1 and the other with spin 0. Kulish and Sklyanin [18] pointed out that the $N=3$ vertex model can be interpreted as the factorized S-matrix for the composite spin 1 particles. The spin 1 factorized S-matrix can be constructed through the following two steps:

Step 1: prepare the four-body S-matrix expressing the collision of two pairs of spin 1/2 particles.

Step 2: multiply projectors (symmetrizers) to extract the spin 1 component for each pair.

This process of making higher spin S-matrices from lower spin S-matrices is sometimes termed as the fusion procedure. By applying the formula (2.5) to the resulting S-matrix, the fusion procedure is directly translated into the braid group representations.

Let us denote the spin s representation of B_n by $B_n^{[s]}$. For instance, $B_n^{[1/2]}$ is equivalent to $H(t,n)$ by an identification $g_i \to -\hat{g}_i$. Note that we always regard the $B_n^{[s]}$ as a group algebra. For definiteness, let us explain the $N=3$ case. To have a spin 1 representation of B_n, we prepare n pairs of strings (totally $2n$ strings). From the generators $\{g_j;\, j=1, 2, \ldots, 2n-1\}$ of $B_n^{[1/2]}$, we define the following operator $G_i (i=1,2,\ldots,n-1)$:

$$G_i = P_i^{(+)} P_{i+1}^{(+)} g_{2i} g_{2i-1} g_{2i+1} g_{2i} P_i^{(+)} P_{i+1}^{(+)}, \tag{4.1}$$

where the symmetrizer $P_i^{(+)}([P_i^{(+)}]^2 = P_i^{(+)})$ acting the i^{th} pair is explicitly given by

$$P_i^{(+)} = \frac{1}{1+t}(t + g_{2i-1}). \tag{4.2}$$

Noting that the projector can be identified as a sum of elements in B_2 (or $B_2^{[1/2]}$ as a group algebra), we immediately see that the generators $\{G_i;\ i=1,2,\ldots,n-1\}$ satisfy the defining relations of B_n. We call this representation of B_n the composite string representation.

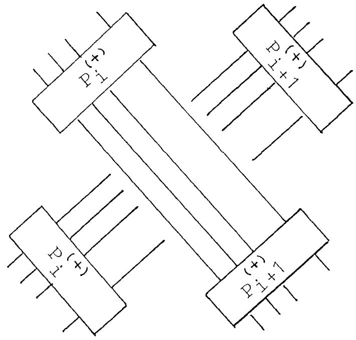

Fig. 4. Generator G_i defined in (4.4)

The above procedure is readily extended into higher spin representations. The general prescription of the composite string representation of B_n for the N-state case may be summarized as follows. For notational simplicity, we write k for $N-1$. Prepare n sets of k strings (totally $k \times n$ strings). Define an operator $g_j^{(p)}$ (p: positive integer) by

$$g_j^{(p)} = \prod_{m=1}^{p} g_{j-p-1+2m}\,. \tag{4.3}$$

In the above, we are free from the operator ordering problem due to the property (1.1). Then we have the expression of the generator G_i for the N-state case (Fig. 4):

$$G_i = P_i^{(+)} P_{i+1}^{(+)} \bar{G}_i P_i^{(+)} P_{i+1}^{(+)} \tag{4.4a}$$

with

$$\bar{G}_i = g_{ki}^{(1)} g_{ki}^{(2)} \dots g_{ki}^{(k-1)} g_{ki}^{(k)} g_{ki}^{(k-1)} \dots g_{ki}^{(2)} g_{ki}^{(1)}, \tag{4.4b}$$

where the symmetrizer $P_i^{(+)}$ acts on the ith set of k strings. We can verify that the symmetrizer $P^{(+)}$ is an element in $B_k^{[1/2]}$, which guarantees $\{G_i;\, i=1,2,\dots,n\}$ to satisfy the defining relation of B_n. For example, $P_i^{(+)}$ for $N=4$ case is given by

$$P_i^{(+)} = \frac{1}{(1+t)(1+t+t^2)} [t^3 + t^2(g_{3i-2} + g_{3i-1}) + t(g_{3i-2}g_{3i-1} + g_{3i-1}g_{3i-2}) + g_{3i-2}g_{3i-1}g_{3i-2}]\,. \tag{4.5}$$

The projector $P_i^{(+)}$ for general N can be constructed, for instance, recursively. Thus we have constructed the representation of braid group generator b_i. Further, we have the following identifications of the remaining operators:

$$I \to P_1^{(+)} P_2^{(+)} \dots P_n^{(+)}, \tag{4.6}$$

$$G_i^{-1} \to P_i^{(+)} P_{i+1}^{(+)} (\bar{G}_i)^{-1} P_i^{(+)} P_{i+1}^{(+)}\,. \tag{4.7}$$

The appearance of $P_i^{(+)}$'s in (4.6) and (4.7) means that we are considering only the highest spin space for each set of strings. We have confirmed that the expression (4.4) indeed reproduces the results in previous sections for any $N>2$. The reduction relations can also be derived by using (4.4), (4.6), and (4.7). For instance, we have a cubic relation for $N=3$:

$$G_i^3=(1-t^2+t^3)G_i^2+(t^2-t^3+t^5)G_i-t^5P_i^{(+)}P_{i+1}^{(+)}, \tag{4.8}$$

which should be compared with (3.10).

Combination of our general formula (2.9) and the fusion procedure known in the factorized S-matrix theory has naturally led us to the composite string representation of the braid group. We remark here that the above construction procedure does not utilize any specific representation of $B_n^{[1/2]}$ we started from. The only requirement is that the projector $P^{(+)}$ should be an element (or a sum of elements) of the starting representation. In this sense, the projector itself should not necessarily be the symmetrizer. In fact, for the $N=3$ case, we can construct a different braid group representation from that in (4.2) by using the antisymmetrizer

$$P_i^{(-)}=\frac{1}{1+t}(1-g_{2i-1}). \tag{4.9}$$

5. Two-Variable Extension

In a previous section we have introduced the composite string representation of the braid group. We use the composite string representation to extend our new link polynomials to those with two variables. In the case of the Jones polynomial, the two-variable extension [13] has been made both in a combinatorial way and in a C^*-algebraic way. As the latter, Ocneanu introduced a trace functional $\psi(\cdot)$ defined on $B_n^{[1/2]}$ which has the proper normalization

$$\psi(I)=1, \tag{5.1}$$

and has the Markov properties

$$\psi(AB)=\psi(BA)\,(A,B\in B_n^{[1/2]}), \tag{5.2}$$

$$\psi(Ag_n)=z\cdot\psi(A)\,(A\in B_n^{[1/2]},g_n\in B_{n+1}^{[1/2]}), \tag{5.3a}$$

$$\psi(Ag_n^{-1})=\bar{z}\cdot\psi(A)\,(A\in B_n^{[1/2]},g_n\in B_{n+1}^{[1/2]}), \tag{5.3b}$$

with

$$z=\psi(g_n), \tag{5.4a}$$

$$\bar{z}=\psi(g_n^{-1}). \tag{5.4b}$$

An important point is that the quantity

$$z=\psi(g_i)\quad\text{for all } i \tag{5.5}$$

is independent of the parameter t and is a free parameter. This contrasts to the Jones's trace $\phi(\cdot)$ [see (3.17), (3.18), and (3.19) with $N=2s+1=2$]. The pair of variables (t,z) enters into the two-variable Jones polynomial. Hence, we may attain the two-variable extension of the link polynomials $N\geqq 3$ by generalizing the Ocneanu's trace. Interpretation of our higher spin representation of B_n in terms of composite strings presented in the previous section is helpful for this purpose.

Let us explain the simplest case $N=3$. Since the spin 1 representation $B_n^{[1]}$ can be interpreted as a sub group-algebra in $B_{2n}^{[1/2]}$, the Ocneanu's trace $\psi(\cdot)$ naturally acts on $B_n^{[1]}$. Then the induced functional $\psi^{[1]}(\cdot)$ defined by

$$\psi^{[1]}(A)=\psi(A)/[\psi(P_i^{(+)})]^n \qquad (A\in B_n^{[1]}) \tag{5.6}$$

is a natural candidate of the generalized Ocneanu's trace. The quantity appearing in the denominator is given by

$$\psi(P_i^{(+)})=\frac{1}{1+t}(t+z), \tag{5.7}$$

where we have combined (4.2) and (5.3). The denominator in (5.6) is important to guarantee the proper normalization. A task left for us is to check the Markov properties

$$\psi^{[1]}(AB)=\psi^{[1]}(BA)\,(A,B\in B_n^{[1]}), \tag{5.8}$$

$$\psi^{[1]}(AG_n)=Z\cdot\psi^{[1]}(A)\,(A\in B_n^{[1]},G_n\in B_{n+1}^{[1]}), \tag{5.9a}$$

$$\psi^{[1]}(AG_n^{-1})=\bar{Z}\cdot\psi^{[1]}(A)\,(A\in B_n^{[1]},G_n\in B_{n+1}^{[1]}), \tag{5.9b}$$

with

$$Z=\psi^{[1]}(G_i)=\frac{(1+t)z^2}{t+z}, \tag{5.10a}$$

$$\bar{Z}=\psi^{[1]}(G_i^{-1})=\frac{(1+t)\bar{z}^2}{1+t\bar{z}}. \tag{5.10b}$$

Using (5.2)–(5.4) in the definition (5.6), we can confirm that (5.8) and (5.9) hold. Hence the functional $\psi^{[1]}(\cdot)$ is indeed the generalized Ocneanu's trace.

We apply the formula (5.6) to obtain the two-variable link polynomial. Before doing this, we introduce a variable ω by

$$\omega=\bar{z}/z=\psi(g_i^{-1})/\psi(g_i)=\frac{z+t-1}{tz} \tag{5.11}$$

and change variables from (t,z) to (t,ω). We then have

$$Z=\frac{1-t}{1-\omega t}\cdot\frac{1-t^2}{1-\omega t^2}, \tag{5.12a}$$

$$\bar{Z}=\omega^2 Z. \tag{5.12b}$$

Therefore we obtain the two-variable extension $\alpha_\omega^{[1]}(\cdot)$ of the $N=3$ polynomial as

$$\alpha_\omega^{[1]}(A)=\left(\frac{1-t}{1-\omega t}\cdot\frac{1-t^2}{1-\omega t^2}\cdot\omega\right)^{-(n-1)}\omega^{e(A)}\psi^{[1]}(A) \qquad (A\in B_n^{[1]}). \tag{5.13}$$

It is interesting to compare this expression with that of the $N=2$ case,

$$\alpha_\omega^{[1/2]}(A)=\left(\frac{1-t}{1-\omega t}\cdot\sqrt{\omega}\right)^{-(n-1)}(\sqrt{\omega})^{e(A)}\psi^{[1/2]}(A). \tag{5.14}$$

The one variable polynomials are reproduced by setting

$$\omega = t \,. \tag{5.15}$$

We know that for general N the projector $P_i^{(+)}$ is expressed as sum of elements in $B_n^{[1/2]}$. Then the construction of two-variable link polynomials for general N can be done in a similar fashion [38].

To close this section, we want to mention the relationship between our $N=3$ polynomial and the Kauffman polynomial [39]. Kauffman constructed a two-variable link polynomial which has a combinatorial definition and is different from the two-variable Jones polynomial. Very recently, Birman and Wenzl [40], and independently Murakami [41] devised an algebra corresponding to the Kauffman polynomial, which we call Birman-Wenzl-Murakami (BWM) algebra. The BWM algebra denoted by $C_n(l, m)$ is generated by two kinds of operators: one is $\{\tilde{G}_i\}_{i=1}^{n-1}$ satisfying the defining relation of the braid group and the other is $\{\tilde{E}_i\}_{i=1}^{n-1}$. They satisfy the following relations

$$\tilde{G}_i + \tilde{G}_i^{-1} = m(1 + \tilde{E}_i) \,, \tag{5.16a}$$

$$\tilde{E}_i \tilde{E}_{i\pm1} \tilde{E}_i = \tilde{E}_i \,, \tag{5.16b}$$

$$\tilde{G}_{i\pm1} \tilde{G}_i \tilde{E}_{i\pm1} = \tilde{E}_i \tilde{G}_{i\pm1} \tilde{G}_i = \tilde{E}_i \tilde{E}_{i\pm1} \,, \tag{5.16c}$$

$$\tilde{G}_{i\pm1} \tilde{E}_i \tilde{G}_{i\pm1} = \tilde{G}_i^{-1} \tilde{E}_{i\pm1} \tilde{G}_i^{-1} \,, \tag{5.16d}$$

$$\tilde{G}_{i\pm1} \tilde{E}_i \tilde{E}_{i\pm1} = \tilde{G}_i^{-1} \tilde{E}_{i\pm1} \,, \tag{5.16e}$$

$$\tilde{E}_{i\pm1} \tilde{E}_i \tilde{G}_{i\pm1} = \tilde{E}_{i\pm1} \tilde{G}_i^{-1} \,, \tag{5.16f}$$

$$\tilde{G}_i \tilde{E}_i = \tilde{E}_i \tilde{G}_i = l^{-1} \tilde{E}_i \,, \tag{5.16g}$$

$$\tilde{E}_i \tilde{G}_{i\pm1} \tilde{E}_i = l \tilde{E}_i \,, \tag{5.16h}$$

$$\tilde{E}_i \tilde{E}_j = \tilde{E}_j \tilde{E}_i \quad \text{if } |i-j| \geqq 2 \,, \tag{5.16i}$$

$$\tilde{E}_i^2 = [m^{-1}(l + l^{-1}) - 1] \tilde{E}_i \,, \tag{5.16j}$$

$$\tilde{G}_i^2 = m(\tilde{G}_i + l^{-1} \tilde{E}_i) - 1 \,. \tag{5.16k}$$

Combining (5.16a) and (5.16k) we see that the generator $\tilde{G}_i$ satisfies a cubic relation

$$\tilde{G}_i^3 = \left(m + \frac{1}{l}\right) \tilde{G}_i^2 - \left(1 + \frac{m}{l}\right) \tilde{G}_i + \frac{1}{l} \,. \tag{5.17}$$

Associated with the algebra $C_n(l, m)$, Birman and Wenzl defined a Markov trace by using the Kauffman polynomial. Hence the Kauffman polynomial is a trace type invariant. Since ours is also a trace type invariant with a braid group generator having a cubic relation, the Kauffman polynomial should be related to our $N=3$ polynomial. In fact, by renormalizing our braid group generator G_i as

$$\tilde{G}_i = \frac{G_i}{t\sqrt{-1}} \tag{5.18}$$

and *defining* the operator $\tilde{E}_i$ through the relation

$$\tilde{G}_i + \tilde{G}_i^{-1} = \frac{1-t^2}{t\sqrt{-1}}(1+\tilde{E}_i), \tag{5.19}$$

we can directly verify that the operators $\{\tilde{G}_i\}$ and $\{\tilde{E}_i\}$ indeed satisfy the relation (5.16) with

$$m = \frac{1-t^2}{t\sqrt{-1}}, \tag{5.20}$$

$$l = \frac{\sqrt{-1}}{t^2}. \tag{5.21}$$

Hence we see that our $N=3$ braid group representation $B_n^{[1]}$ corresponds to a special case of $C_n(l,m)$. But this does not mean that our $N=3$ polynomial is a special case of the Kauffman polynomial. The reason is the following. Both of the two variables in the Kauffman polynomial enter into the algebra. In our $N=3$ two-variable polynomial, roles of two variables are different; one for the algebra and the other for the trace functional. Then our two-variable polynomial is different from the Kauffman polynomial. It is quite interesting to observe that our two-variable polynomial at the point $\omega = t$ coincides with the Kauffman polynomial with (5.20) and (5.21).

We expect that the analog of BWM algebra structure should also exist for $N \geqq 4$ polynomials [38]. This will give us combinatorial or "graphical" interpretation of our hierarchy of link polynomials.

6. Summary

In this paper, we have discussed a close relation between the knot theory and the exactly solvable models in statistical mechanics. We have presented a general prescription to construct a braid group representation from the Boltzmann weights of a solvable model satisfying the Yang-Baxter relation (factorization equation, star-triangle relation). Through the discussion, we have shown that the starting solvable model must be critical. This gives a natural explanation for the appearance of the Temperley-Lieb algebra in the Jones polynomial. We have applied the formula to a sequence of solvable vertex models to have a sequence of new braid group representations. With the Markov traces associated with these new representations, we have successfully constructed new link polynomials. Further, by utilizing the "fusion" method which is known to give higher spin factorized S-matrices, we have been led to the composite string representation of our hierarchy of braid group representations. This representation enables us to construct the two-variable extensions of our new polynomials.

Recently Murakami [42] presented braid group representations also by using multiple strings. His representation is seemingly similar, but it is different from ours in that the symmetrizer is not used. At present, we have not found an interpretation of his result from a viewpoint of the factorized S-matrix theory. Detailed account on this subject will be given in a separate paper [38].

In conclusion, we like to emphasize that the sequence of new link polynomials in this paper is a specialization of our theory. The method of constructing the braid group representation is quite general and can be used for any vertex model and IRF model. For example, we can apply our method to an IRF model called the eight-vertex SOS model [25]. Then, an important question is how many independent link polynomials can be made from solvable models. This problem will be the subject of our future study.

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References

1. Birman, J.S.: Braids, links, and mapping class groups. Princeton, NJ: Princeton University Press 1974
2. Rolfsen, D.: Knots and links. Berkeley, CA: Publish or Perish 1976
3. Artin, E.: Ann. Math. **48**, 101 (1947)
4. Alexander, J.W.: Proc. Natl. Acad. Sci. USA **9**, 93 (1923)
5. Markov, A.A.: Recueil Math. Moscov 73 (1935)
6. Jones, V.F.R.: Invent. Math. **72**, 1 (1983)
7. Jones, V.F.R.: Bull. Am. Math. Soc. **12**, 103 (1985)
8. Temperley, H.N.V., Lieb, E.H.: Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: Some exact results for the "percolation" problem. Proc. Roy. Soc. Lond. A **322**, 251 (1971)
9. Lieb, E.H., Wu, F.Y.: In: Phase transitions and critical phenomena, Vol. 1, p. 331. Domb, C., Green, M.S. (eds.). London: Academic Press 1972
10. Baxter, R.J., Kelland, S.B., Wu, F.Y.: Equivalence of the Potts' model or Whitney polynomial with an ice-type model. J. Phys. A **9**, 397 (1976)
11. Bourbaki, N.: Groupes et algebres de Lie. Paris: Hermann 1968, Chap. 4
12. Alexander, J.W.: Trans. Am. Math. Soc. **30**, 275 (1928)
13. Freyd, P., Yetter, D., Hoste, J., Lickorish, W.B.R., Millett, K., Ocneanu, A.: Bull. Am. Math. Soc. **12**, 239 (1985)
14. Birman, J.S.: Invent. Math. **81**, 287 (1985)
15. Kanenobu, T.: Math. Ann. **275**, 555 (1986)
16. Faddeev, L.D.: Sov. Sci. Rev. Math. Phys. C **1**, 107 (1981)
17. Thacker, H.B.: Exact integrability in quantum field theory and statistical mechanics. Rev. Mod. Phys. **53**, 253 (1981)
18. Kulish, P.P., Sklyanin, E.K.: Lecture Notes in Physics, Vol. 151, p. 61. Berlin, Heidelberg, New York: Springer 1982
19. Wadati, M.: In: Dynamical problems in soliton systems, p. 68. Takeno, S. (ed.). Berlin, Heidelberg, New York: Springer 1985
20. Wadati, M., Akutsu, Y.: Exactly solvable models in statistical mechanics. In: Springer Series in Nonlinear Dynamics. Lakshmanan, M. (ed.). Berlin, Heidelberg, New York: Springer 1988
21. Karowski, M., Thun, H.J., Truong, T.T., Weisz, P.H.: On the uniqueness of a purely elastic S-matrix in $(1+1)$ dimensions. Phys. Lett. **67** B, 321 (1977)
Zamolodchikov, A.B., Zamolodchikov, A.B.: Factorized S-Matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. Ann. Phys. (NY) **120**, 253 (1979)
Sogo, K., Uchinami, M., Nakamura, A., Wadati, M.: Nonrelativistic theory of factorized S-matrix. Prog. Theor. Phys. **66**, 1284 (1981)

Sogo, K., Uchinami, M., Akutsu, Y., Wadati, M.: Classification of exactly solvable two-component models. Prog. Theor. Phys. **68**, 508 (1981)
22. Baxter, R.J.: Exactly solved models in statistical mechanics. London: Academic Press 1982
23. Wu, F.Y.: Ising model with four-spin interactions. Phys. Rev. B **4**, 2312 (1971)
Kadanoff, L.P., Wegner, F.J.: Some critical properties of the eight vertex model. Phys. Rev. B **4**, 3989 (1981)
24. Zamolodchikov, A.B.: Z_4-symmetric factorized S-matrix in two space-time dimensions. Commun. Math. Phys. **69**, 165 (1979)
25. Andrews, G.E., Baxter, R.J., Forrester, P.J.: Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities. J. Stat. Phys. **35**, 193 (1984)
26. Kuniba, A., Akutsu, Y., Wadati, M.: J. Phys. Soc. Jpn **55**, 1092, 2170, and 3338 (1986)
Kuniba, A., Akutsu, Y., Wadati, M.: An exactly solvable 4-state IRF model. Phys. Lett. **116**A, 382 (1986) and An exactly solvable 5-state IRF model. **117**A, 358 (1986)
Baxter, R.J., Andrews, G.E.: Lattice gas generalization of the hard hexagon model. I. Star-triangle relation and local densities. J. Stat. Phys. **44**, 249 (1986)
Andrews, G.E., Baxter, R.J.: Lattice gas generalization of the hard hexagon model. II. The local densities as elliptic functions. J. Stat. Phys. **44**, 713 (1986)
27. Akutsu, Y., Kuniba, A., Wadati, M.: J. Phys. Soc. Jpn. **55**, 1466 and 1880 (1986)
28. Kuniba, A., Akutsu, Y., Wadati, M.: J. Phys. Soc. Jpn. **55**, 2605 (1986)
29. Akutsu, Y., Kuniba, A., Wadati, M.: J. Phys. Soc. Jpn. **55**, 290 (1986)
Date, E., Jimbo, M., Miwa, T., Okado, M.: Fusion of the eight vertex SOS model. Lett. Math. Phys. **12**, 209 (1986)
30. Akutsu, Y., Wadati, M.: J. Phys. Soc. Jpn. **56**, 839 (1987)
31. Akutsu, Y., Wadati, M.: J. Phys. Soc. Jpn. **56**, 3039 (1987)
32. Sogo, K., Akutsu, Y., Abe, T.: New factorized S-matrix and its application to exactly solvable q-state model. I and II. Theor. Phys. **70**, 730 and 739 (1983)
33. Zamolodchikov, A.B., Fateev, V.A.: A model of facterized S-matrix and an integrable spin-1 Heisenberg chain. Sov. J. Nucl. Phys. **32**, 298 (1980)
34. Powers, R.T.: Ann. Math. **86**, 138 (1967)
35. Pimsner, M., Popa, S.: Preprint
36. Conway, J.H.: In: Computational problems in abstract algebra, p. 329. Leach, J. (ed.). London: Pergamon Press 1969
37. Akutsu, Y., Deguchi, T., Wadati, M.: J. Phys. Soc. Jpn. **56**, 3464 (1987)
38. Deguchi, T., Akutsu, Y., Wadati, M.: J. Phys. Soc. Jpn. **57**, No. 3 (1988)
39. Kauffman, L.H.: Preprint
40. Birman, J.S., Wenzl, H.: Preprint
41. Murakami, J.: Preprint
42. Murakami, J.: Preprint

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