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Another Construction of the Central Extension of the Loop Group

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Abstract. By considering the geometry of the central extension of the loop group as a principal bundle it is shown that it must be the quotient of a larger group. This group is a central extension of the group of paths in the loop group and its cocycle is constructed as the holonomy around a certain path. Conversely it is shown that this definition of a cocycle gives a method of constructing the central extension. The Wess-Zumino term plays an important role in these constructions.

1. Introduction

Mickelsson (1987) [and see also Frenkel (1986)] gives a remarkably simple construction of the central extension $U(1) \to \hat{\Omega}G \to \Omega G$ of the loop group ΩG. It is well known that that the fibering $\hat{\Omega}G \to \Omega G$ is topologically non-trivial so that $\hat{\Omega}G$ cannot be constructed as $U(1) \oplus \Omega G$ with a new group multiplication $(g, \lambda)(h, \mu) = (gh, c(g, h)\lambda\mu)$ for some continuous cocycle $c(g, h)$. What Mickelsson does, however, is to consider a larger group DG which has ΩG as a quotient. He then shows that there is a cocycle which defines a central extension of DG and a normal subgroup of $DG \oplus U(1)$ such that the quotient group is $\hat{\Omega}G$.

By using the path fibration of the loop group another such construction is possible which gives rise to a different cocycle as the holonomy, or Wess-Zumino term, for a particular closed path in the loop group.

In this method the larger group DG arises as the group of paths and its central extension as a group of horizontal paths. It is easy to see then that it has the loop group central extension as a quotient and to identify the kernel as the loops in the loop group.

Conversely it is also readily shown that with this form of the cocycle DG has a central extension with a naturally defined normal subgroup and that the quotient gives a method of constructing the central extension of the loop group.

2. Invariant Connections

For the convenience of notation let $\mathcal{G}$ denote the loop group. In fact it could be any group with a central extension

$$U(1) \to \hat{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} .$$

From Koboyashi, and Nomizu we see that a $\hat{\mathscr{G}}$ right invariant connection on the $U(1)$ principal bundle $\hat{\mathscr{G}} \to \mathscr{G}$ is equivalent to a splitting of the exact sequence

$$T_1 U(1) \to T_1 \hat{\mathscr{G}} \to T_1 \mathscr{G} ,$$

where T_1 denotes the tangent space at the identity. The action of the group $\hat{\mathscr{G}}$ then transports this splitting to every other tangent space to define the horizontal subspaces that define the connection. This splitting is, of course, equivalent to a splitting of the exact sequence of a Lie algebras

$$LU(1) \to L\hat{\mathscr{G}} \to L\mathscr{G} ,$$

where L denotes the Lie algebra.

The facts we need to know about connections are as follows. They define the idea of a horizontal curve as one whose tangent vector is everywhere horizontal. For every choice of a curve on the base $\mathscr{G}$ and a point in $\hat{\mathscr{G}}$ above it there is a unique horizontal curve covering the curve on the base and going through the point. This is just a consequence of the uniqueness of solutions of first order differential equations with prescribed boundary conditions. When the connection is right invariant the group action sends horizontal curves to horizontal curves. If a horizontal curve begins and ends in the same fibre of the map π then there must be an element of $U(1)$ which maps the beginning point to the endpoint, this is called the holonomy of the curve. The image of such a curve under π is a loop and the holonomy is also the exponential of the integral of the curvature over any disk whose boundary is the loop or the parallel transport around the loop.

The curvature of such a connection is a right invariant two form whose value at the identity on X, $Y \in T_1 \mathscr{G}$ is the $LU(1)$ component of $[(0, X), (0, Y)]$, where $(0, X)$ and $(0, Y)$ are in $LU(1) \oplus L\mathscr{G}$. But the bracket of two such elements is $(C(X, Y), [X, Y])$, where C is the cocycle defined by the splitting of $L\hat{\Omega}G$ as $LU(1) \oplus L\mathscr{G}$.

To summarize: the splitting of the central extension of Lie algebras defines a cocycle and an invariant connection whose curvature, regarded as an antisymmetric pairing on the Lie algebra, is the cocycle. In the case of the loop group this is the well known cocycle

$$C(X, Y) = \frac{\varrho^2}{4\pi} \int_{S^1} \operatorname{tr}\left(X \frac{\partial Y}{\partial \theta} \right) d\theta ,$$

where ϱ is the length of the longest root of G.

Fix then a right invariant connection or equivalently a cocycle.

3. Lifting the Path Fibration

Mickelsson's group DG can be regarded as the group $\mathscr{P}\mathscr{G}$ of all paths $f : [0, 1] \to \mathscr{G}$ which begin at $1 \in \mathscr{G}$. A path in $\mathscr{G}$ is a function $f(r, \theta)$ and hence defines a map $f : D \to G$. We shall ignore the question of differentiability at the origin $r = 0$, $\theta = 0$.

There is an exact sequence of groups

$$\mathscr{L}\mathscr{G} \to \mathscr{P}\mathscr{G} \to \mathscr{G} ,$$

where the second map sends a path f to its endpoint $f(1)$ and the kernel is $\mathscr{L}\mathscr{G}$, the space of all based loops in $\mathscr{G}$. This is the path fibration of homotopy theory fame.

To "lift" this fibration consider the set $\mathscr{P}_h\hat{\mathscr{G}}$ of all horizontal paths in $\hat{\mathscr{G}}$ that begin in $U(1)$. The projection $\pi:\hat{\mathscr{G}}\to\mathscr{G}$ defines a projection π sending a path in $\hat{\mathscr{G}}$ to a path in $\mathscr{G}$ which begins at 1, that is, an element of $\mathscr{P}\mathscr{G}$. Given two such paths f and g their pointwise product fg is a path in $\hat{\mathscr{G}}$ beginning at $U(1)$, but it may not be horizontal. Define then the product $f*g$ to be that unique horizontal path which covers $\pi(f)\pi(g)$ and has endpoint $f(1)g(1)$. From the definition we have $\pi(f*g) = \pi(f)\pi(g)$, where juxtaposition denotes the pointwise product of paths in $\mathscr{G}$. It is straightforward to check that $(f*g)*h=f*(g*h)$ because $\pi((f*g)*h))=\pi(fgh) = \pi(f*(g*h))$ and $((f*g)h)(1)=f(1)g(1)h(1)=(f(g*h))(1)$, so the product is associative. The identity is the constant map 1 and the inverse of f is obtained by taking the pointwise inverse of $\pi(f)$ and lifting it horizontally to end at $f(1)^{-1}$.

This product makes $\mathscr{P}_h\hat{\mathscr{G}}$ into a group and the map that sends a path to its endpoint becomes a group homomorphism $\mathscr{P}_h\hat{\mathscr{G}}\to\hat{\mathscr{G}}$.

4. Defining the Central Extension

The map $\pi:\mathscr{P}_h\hat{\mathscr{G}}\to\hat{\mathscr{G}}$ is also a group homomorphism and, in fact there is a central extension

$$U(1)\to\mathscr{P}_h\hat{\mathscr{G}}\to\mathscr{P}\mathscr{G}\,.$$

This central extension has a canonical splitting or isomorphism

$$\mathscr{P}_h\hat{\mathscr{G}}\to U(1)\oplus\mathscr{P}\mathscr{G}\,,$$

which sends a path f to the pair $(f(0),\pi(f))$. The product on $\mathscr{P}_h\hat{\mathscr{G}}$ induces a product on $U(1)\oplus\mathscr{P}\mathscr{G}$ which has the form

$$(\mu,f)*(\lambda,g)=(\mu\lambda c(f,g),fg)$$

for some cocycle $c(f,g)$. The kernel of the homomorphism onto $\hat{\mathscr{G}}$ is the set of paths g in $\hat{\mathscr{G}}$ that begin at $U(1)$ and end at the identity 1. Because $\pi(g)$ is a loop this means the holonomy around $\pi(g)$ must be the inverse of $g(0)$. The image of the kernel in $U(1)\oplus\mathscr{P}\mathscr{G}$ consists therefore of all pairs $(\mathrm{hol}(g)^{-1},g)$ where $\mathrm{hol}(g)$ is the holonomy around $\pi(g)$. If any loop in $\mathscr{G}$ is contractible, then the holonomy around the loop is the exponential of the integral of the curvature over a disk whose boundary is the loop, and therefore the subgroup in question can be defined just by using the curvature which lives on $\mathscr{G}$. When $\mathscr{G}$ is the loop group, all loops are contractible and the holonomy is just the Wess-Zumino term as explained in Carey and Murray (1986) [see also Mickelsson (1986) where a similar result appears].

The point of this discussion is that if we knew the cocycle c then we could define the group $U(1)\oplus\mathscr{P}\mathscr{G}$ and a normal subgroup consisting of all pairs $(\mathrm{hol}(g)^{-1},g)$ where g is a loop in $\mathscr{G}$. The quotient group formed by factoring out this normal subgroup then fibres over $\mathscr{G}$ by the map that sends a path to its endpoint and defines the required central extension.

5. The Cocycle

The remarkable thing about the cocycle $c(f,g)$ defined above is that it is the holonomy around a path and therefore defined using only the curvature F of the

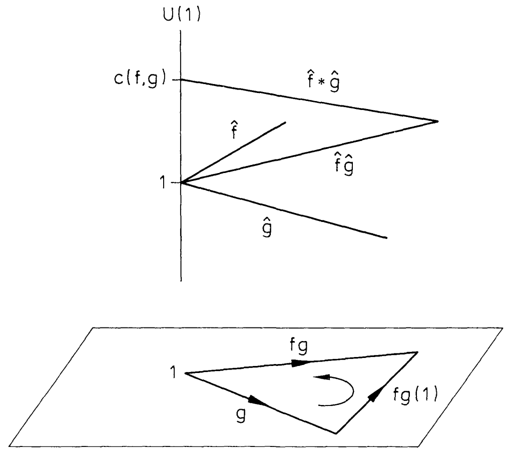

Fig. 1

right invariant connection. To see this note that the value of the cocycle can be calculated by lifting f and g to horizontal paths $\hat{f}$ and $\hat{g}$ in $\mathscr{G}$ which begin at 1. The beginning point of the product $\hat{f} * \hat{g}$ is then the value of the cocycle. To calculate this we need to join it to 1 by a horizontal path. The path $\hat{f} * \hat{g}$ joins $(\hat{f} * \hat{g})(0)$ to $\hat{f}(1)\hat{g}(1)$. What is more, the path $\hat{f}(0)\hat{g}$ is horizontal [in fact $\hat{f}(0) = 1$] and joins 1 to $\hat{f}(0)\hat{g}(1)$ and the path $\hat{f}\hat{g}(1)$ is horizontal (because the connection is right invariant) and joins $\hat{f}(0)\hat{g}(1)$ to the point $\hat{f}(1)\hat{g}(1)$. So the cocycle is the holonomy obtained by going along the paths $\hat{f}(0)\hat{g}$ and $\hat{f}\hat{g}(1)$, then backwards along $\hat{f} * \hat{g}$. The image of this path in $\mathscr{G}$ is the loop $f(0)g$, $fg(1)$, then fg backwards.

To calculate the cocycle we need to find a disk whose boundary is this loop. If $T = \{(t,s) \in \mathbb{R}^2 | 0 \leqq s \leqq t \leqq 1\}$ then the map $(t,s) \mapsto f(t)g(s)$ maps T onto $\mathscr{G}$ with boundary the given loop. Finally the cocycle is

$$c(f,g) = \exp\left(\frac{\varrho^2}{8\pi}\int_T F_{f(t)g(s)}\left(\frac{df}{dt}g(s), f(t)\frac{dg}{ds}\right)dsdt\right.$$

$$\left. = \exp\left(\frac{\varrho^2}{8\pi}\int_T F_1\left(\frac{df}{dt}f^{-1}(t), f(t)\frac{dg}{ds}g^{-1}(s)f^{-1}(t)\right)dsdt\right).$$

In the second integral we use the right invariance of F to evaluate it only at 1.

In the primary case of interest $\mathscr{G} = \Omega SU(n)$ $\varrho^2 = 4$, and we have

$$c(f,g) = \exp\frac{1}{2\pi}\int_{0 \leqq s \leqq t \leqq 1}\int_{0 \leqq \theta \leqq 2\pi} \mathrm{tr}\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial t}f^{-1}\right)f(t)\frac{\partial g}{\partial s}g^{-1}(s)f^{-1}(t)d\theta dsdt,$$

where we have suppressed the dependence of f and g on θ. The cocycle defined by Mickelsson is

$$c_M(f,g) = \exp\frac{1}{2\pi}\int_{0 \leqq r \leqq 1}\int_{0 \leqq \theta \leqq 2\pi} \mathrm{tr}\left(f^{-1}\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial r}g^{-1} - f^{-1}\frac{\partial f}{\partial r}\frac{\partial g}{\partial\theta}g^{-1}\right)drd\theta,$$

where we have suppressed the dependence on r and θ.

These cocycles are not the same but they do agree on the Lie algebra of $U(1)\oplus\mathscr{P}\mathscr{G}$. Freed (1986) has shown that the loop group is a Hilbert Lie group and therefore, because the Lie group structures defined by the different cocycles have the same Lie algebra, they will be isomorphic Lie groups. Alternatively Omori et al. (1986) have shown that the same is true if the loop group is regarded as a Fréchet Lie group. As the cocycles define isomorphic Lie group structures they must be cohomologous. It would be interesting to see an explicit coboundary relating the cocycles; this would be a function $\chi:\mathscr{P}\mathscr{G}\to U(1)$ such that

$$\chi(f)\chi(g)c(f,g)=c_M(f,g)\chi(fg)$$

for all f and g.

6. Constructing the Kac-Moody Central Extension

If we start with the curvature definition of the cocycle, then to construct the central extension of the loop group we have to show that it is a cocycle, that is it defines an associative product on $U(1)\oplus\mathscr{G}$, and that the set $\{(\mathrm{hol}^{-1}(g),g)\,|\,g\in\mathscr{L}\mathscr{G}\}$ is a subgroup. These calculations are readily performed by drawing the paths which are being integrated over and remembering that the curvature is right invariant, so right translating a region of integration leaves the integral of the curvature unchanged.

Firstly to show that c is a cocycle we want to show that

$$c(f,g)c(fg,h)=c(f,gh)c(g,h)\,.$$

This follows by integrating the curvature over the faces of the tetrahedron in Fig. 2 with the orientation given and noting that this must be a 2π multiple of an integer because of the integrality of the curvature. Notice that the face not joining 1 is the translate by $h(1)$ of the region used to define $c(f,g)$. Because of the invariance of the curvature under right translation the exponential of the integral over this face also defines $c(f,g)$.

To show that we have a subgroup we want to show that

$$(\mathrm{hol}^{-1}(g),g)*(\mathrm{hol}^{-1}(h),h)=(\mathrm{hol}^{-1}(gh),gh)$$

or

$$\mathrm{hol}^{-1}(g)\,\mathrm{hol}^{-1}(h)c(g,h)=\mathrm{hol}^{-1}(gh)\,.$$

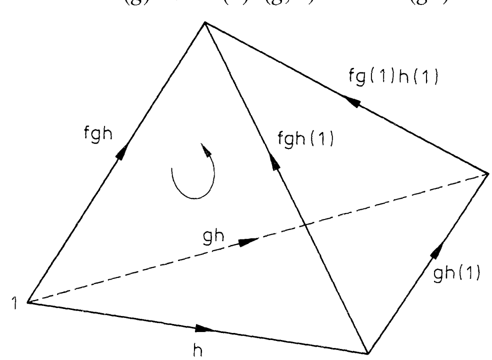

Fig. 2

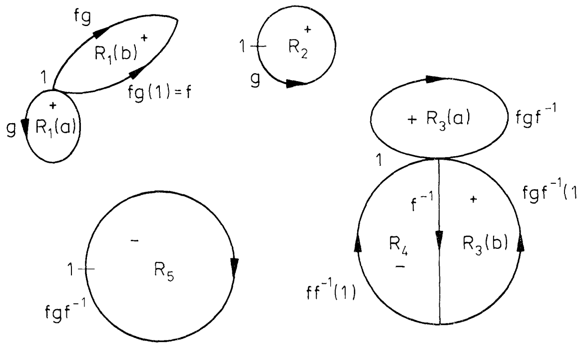

Fig. 3

But g and h are loops so the cocycle is calculated by going around the (loop) h then the (loop) $gh(1)=g$, and then the (loop) gh backwards; but parallel transport around a loop is holonomy, so we must have

$$c(g,h)=\mathrm{hol}(g)\,\mathrm{hol}(h)\,\mathrm{hol}^{-1}(gh),$$

which gives the requires result.

Lastly to show that this is a normal subgroup we need

$$(\lambda, f)*(\mathrm{hol}^{-1}(g), g)*(\lambda^{-1}c(f,f^{-1})^{-1}, f^{-1})=(\mathrm{hol}^{-1}(fgf^{-1}), fgf^{-1}).$$

Equivalently it is enough to show that

$$c(f,g)\,\mathrm{hol}^{-1}(g)c(fg,f^{-1})c(f,f^{-1})^{-1}=\mathrm{hol}(fgf^{-1})^{-1}.$$

From Fig. 3 this means that

$$\int_{R_1(a)} F+\int_{R_1(b)} F-\int_{R_2} F+\int_{R_3(a)} F+\int_{R_3(b)} F-\int_{R_4} F=-\int_{R_5} F.$$

This follows from Fig. 3, noting that $R_1(b)f^{-1}(1)=-(R_3(b)\cup(-R_4))$, where the sign indicates the opposite orientation. In Fig. 3 the regions marked with $a+$ are given the usual orientation and the regions marked with $a-$ the opposite.

Finally we want to show that the central extension constructed in this way has the correct Lie algebra cocycle. Let $f(t)=\exp(\tau X(t))$ and $g(t)=\exp(\sigma Y(t))$ be elements in $\mathscr{P}\mathscr{G}$ near the identity where exp is the exponential map for the group $\mathscr{G}$. The Lie algebra cocycle is then

$$C(X,Y)=\frac{d^2}{d\sigma d\tau}(c(f,g)-c(g,f))_{\tau=\sigma=0}.$$

It is straightforward to calculate that this is

$$\begin{aligned}C(X,Y)&=\iint_{0\le s\le t\le 1} F_1\left(\frac{dX}{dt},\frac{dY}{ds}\right)dtds+\iint_{0\le t\le s\le 1} F_1\left(\frac{dX}{dt},\frac{dY}{ds}\right)dtds\\&=\iint_{0\le s,t\le 1} F_1\left(\frac{dX}{dt},\frac{dY}{ds}\right)dtds=F_1(X(1),Y(1)).\end{aligned}$$

For the loop group the curvature on the tangent space to the identity of DG is

$$F_1(X, Y) = \frac{\varrho^2}{4\pi} \int_{S^1} \operatorname{tr}\left(X(1,\theta) \frac{\partial Y}{\partial \theta}(1,\theta) \right) d\theta .$$

So the cocycle on $L\mathcal{G}$ is just F.

7. The Central Extension as an Associated Bundle

The group $U(1) \oplus \mathcal{P}\mathcal{G}$ with the $*$ product fibres over $\mathcal{G}$ by the map sending a path to its endpoint. The kernel of this map is $U(1) \oplus \mathcal{L}\mathcal{G}$ with the $*$ product. This means that $U(1) \oplus \mathcal{P}\mathcal{G}$ is a principal bundle over $\mathcal{G}$ and furthermore that $\hat{\mathcal{G}}$ is an associated bundle induced by the group homomorphism

$$U(1) \oplus \mathcal{L}\mathcal{G} \to U(1),$$
$$(z, g) \mapsto z \operatorname{hol}(g) .$$

This is a homomorphism because we have already seen that if g and h are two loops, then

$$c(g, h) = \operatorname{hol}(g)\operatorname{hol}(h)\operatorname{hol}(gh)^{-1},$$

so $(z, g) * (w, h) = (zwc(g, h), gh)$ maps to $(zw \operatorname{hol}(g)\operatorname{hol}(h)\operatorname{hol}(gh)^{-1})\operatorname{hol}(gh) = (z \operatorname{hol}(g))\,(w \operatorname{hol}(h))$.

This means that sections of the line bundle over $\mathcal{G}$ associated to $\hat{\mathcal{G}}$ can be considered as maps $\phi : U(1) \oplus \mathcal{P}\mathcal{G} \to C$ satisfying $\phi((w, f) * (z, g)) = z^{-1} \operatorname{hol}(g)^{-1} \phi(w, f)$. Equivalently they can be defined as functions $\varphi : \mathcal{P}\mathcal{G} \to C$ by letting $\varphi(f) = \phi(1, f)$. Then φ must satisfy $\varphi(fg) = \phi(1, fg) = \phi((1, f) * (c(f, g)^{-1}, g)) = c(f, g)\operatorname{hol}(g)^{-1}\varphi(f)$. This is the definition that is often used in the physics literature (Mickelsson (1987)).

8. Conclusion

The path fibration of the loop group gives a definition of a cocycle on the group of paths in the loop group which can be used to define the central extension of the loop group. This appears to be a different cocycle to that defined by Mickelsson; however, they both give rise to the same Lie algebra so must be cohomologous.

It is interesting the important role that the Wess-Zumino term, in the guise of holonomy, plays in all these constructions.

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