Commun. Math. Phys. 115, 421–434 (1988)

Communications in
Mathematical
Physics

A General Relation Between Kink-Exchange and Kink-Rotation

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Abstract. The normal correlation between spin and statistics is shown to be valid for arbitrary kinks, among them the $SU(n)$ Skyrmions for $n \geqq 3$. It is assumed in the proof that no gauge-ambiguity attaches to the values of the underlying scalar field, and that conversely each configuration of this field is represented quantum mechanically by a Hilbert subspace of dimension precisely one.

I. Introduction

As has been known for several years, objects bearing fermionic *statistics* can occur in theories whose "elementary particles" are all bosons, for example, in field theories which impose only CCR's (Canonical Commutation Relations) but never CAR's (Canonical Anti-Commutation Relations) on their fundamental field operators. In the same way *spinorial* states – ones of angular momentum 1/2, 3/2, etc. – can occur in theories whose elementary particles are all tensorial (i.e. of integer spin), for example, field theories in which all the fundamental fields are scalars or vectors.

Such "emergent" fermionicity and spinoriality occurs in a large number of situations, the oldest example being a system of non-relativistic electrically *and* magnetically charged particles, interacting via the associated inverse-square forces (together with any short-range attractive forces that may be included in order to provide for the formation of bound states). In such a situation a bound state can be a spinorial fermion even if its elementary constituents are all (for example) bosons of spin zero [1]. Specifically, a "Saha dyon" formed from an electric monopole e and a magnetic monopole g will have "anomalous" spin *and* statistics whenever the product eg is an odd multiple of Planck's constant h.

Dyons with emergent spin and statistics can also occur in Lorentz invariant quantum field theories in flat space. This can occur, for example, in a theory

* Supported in part by NSF Grant No. PHY 83-18350

built on three basic bosonic fields: an $SU(2)$ vector-potential of "gauged isospin," an isovector Higgs scalar breaking this isospin to an "electromagnetic" $U(1)$, and an additional scalar field which transforms under $SU(2)$ as an iso-*spinor* (i.e., a field carrying spacetime spin 0 and isospin 1/2). If one of the minimal electric charges associated with this last field binds to one of the magnetically charged solitons of the theory ('t Hooft-Polyakov monopoles) then the resulting dyon will again be a spinorial fermion precisely when $eg/h = \text{odd integer}$. (See [2] and references therein.)

In curved spacetime, an analogous possibility occurs in a theory with only one fundamental field – the metric tensor of five-dimensional Kaluza-Klein gravity. Here, as well, there exists a magnetically charged[1] soliton (stable, at least classically) whose charge is minimal in the sense that it is related to the unit e of electric[1] charge by $eg/h = 1$. And once again any dyonic bound state uniting one of these solitons with one of the minimal electric charges of the theory (the so-called "pyrgons") would have to be a fermion of half-integer spin (see [3, 4]).

In the examples mentioned so far, magnetic monopoles enter in a crucial way, but there are also examples of a different type in which long-range, velocity-dependent forces play no role. One such case is that of topological geons occurring in "ordinary," $3+1$ dimensional, quantum gravity. Suppose, for illustration, that precisely N geons are present, all of the same type K, where K might be – among many other possibilities – the spherical manifold S^3/O^*. (Here O^* is the 48-element covering group in $SU(2)$ of the symmetry group of the octahedron, or equivalently, the cube. Some discussion of the space S^3/O^* may be found in [5].) The corresponding spacetime is topologically $R \times (R^3 \# K \# K \# \ldots \# K)$, where $\#$ is the connected sum; and we may suppose its metric to be globally hyperbolic and asymptotically flat. The formal rules of canonical quantum gravity in such a spacetime then imply that the quantum hilbert space possesses state-vectors for which all the geons (the N particles associated to the N topological structures K) are identical fermions of half-integer spin. Once again there is "emergence" because the only fundamental field is the metric $g_{\mu\nu}$, a tensor field quantized without the use of Grassmann numbers or their equivalent.

Another such example – the one of primary interest to us in this paper – is that of "kinks," which will be defined in detail below. Specifically, one may think of the longest known kink, the $SU(2)$ "Skyrmion" and of its more recently studied generalizations to $SU(n \geqq 3)$. In the simplest examples of this sort (pure nonlinear σ models) the only fundamental field is a spacetime scalar. Nonetheless, the quantum kinks may – and sometimes must – turn out to be spinorial fermions.

It is noteworthy that each of the gravitational examples mentioned above bears a close analogy to one of the flat-space examples. Thus, the Kaluza-Klein monopole, a topological geon in 5-dimensions, resembles a gauge-theory-monopole; and S^3/O^*, a topological geon in 4-dimensions, resembles an $SU(2)$-Skyrmion. There is also, however, a different sort of gravitational analog of the flat-space kinks, in which the *topology* is flat but the light-cones are twisted-up in a

[1] Whether one chooses to call this charge electric or magnetic is really a matter of convention

way formally identical to the twisting that characterizes an $SO(3)$ Skyrmion. In higher dimensions (though not in four) such kinks appear to afford a further example of "emergent" spin and statistics, albeit their role in quantum gravity is clouded by their unusual causal properties [6].

As a final example of "emergence" one may mention the possibility of spinorial states of the null string in $3+1$ dimensions [7]. Still further examples of a sort occur in $1+1$ and $2+1$ dimensions, of course, but it is doubtful whether they illustrate genuine emergence because there is no real meaning to spin or statistical types in less than three spatial dimensions [4].

Having called to mind the above array of examples in which spin type and statistical type both behave, in a sense, anomalously, it is natural to inquire whether the normal *correlation* between spin and statistics might not also show an "anomaly" in some of these cases. In this connection the well-known spin-statistics Theorem of Axiomatic Field Theory [8] is powerless because the axioms on which it rests are not all obeyed in these examples. Indeed none of the particles we have discussed is obviously created by any local field operator. Moreover, in the case of gauge theories the fundamental local operators which do exist fail to transform simply under the Lorentz group; in string theories such operators either don't exist at all or have an infinite number of components. Further, certain of the above examples lack Lorentz invariance, while others involve a spacetime which is not even *topologically* flat, and on which, therefore, an action of the Poincaré group could not even be defined.

Despite these difficulties, a spin-statistics correlation has been established for many of the above cases, and the results so far suggest that such a correlation will exist whenever – and to the extent that – the underlying theory incorporates the possibility of pair creation and annihilation (cf. [9–11, 2, 12, 13]).

This principle would imply, in particular, that *all* Skyrmions – not just the $SU(2)$ ones – should manifest a spin-statistics correlation, since pair creation is certainly available to them. In the sequel we will establish such a correlation for arbitrary kinks, including of course the $SU(n \geqq 3)$ Skyrmions as special cases. A consequence is that spin-1/2-baryons modelled as Skyrmions are correctly predicted to be fermionic independently of whether one assumes two, three, or more flavors for the underlying chiral field.

The proof per se is presented in Sect. V, but three preliminary sections are taken up with introducing the concepts and notation which will underlie our argument. Specifically Sect. II is a discussion in general of the "mechanism" which in particular makes emergent spinoriality and fermionicity possible for kinks; Sect. III *defines* spin-type and statistical-type in correspondingly general language, introducing the maps f_{stat} and f_{spin}; and Sect. IV specializes to kink theories the key definitions of Sects. II and III, notably the definitions of the configuration space Q and of the two maps just mentioned. The proof itself, as given in Sect. V, then amounts to demonstrating that the mapping f_{stat} can be deformed continuously into a certain restriction of f_{spin}. To conclude the paper, Sect. VI discusses possible extensions of the present results, and intimates that the present derivation might be the root of a general argument that would comprehend all other instances of spin-statistics correlation as special cases.

II. Concerning the Bundle of Configuration Eigenvectors

Earlier investigations [4, 9] have made it clear that distinctions like that between bosons and fermions are most adequately expressed in terms of a certain $U(1)$ bundle which occurs as a subset of the quantum hilbert space H. Let us recall the definition of this bundle, which in the next section will afford us a convenient formulation of the spin-statistics correlation that we are aiming to prove.

To begin with, let Q be the joint spectrum of a complete set of (mutually commuting) "position observables" of the system in question. For example, if the system is a "free top" then the points of Q can be parametrized by three cartesian coordinates locating the top's center of mass together with three Euler angles specifying its orientation. Thus Q in this case is the six-dimensional manifold $R^3 \times SO(3)$.

Or, to take another example, if the system is a scalar field $\hat{\phi}$, then a single configuration point $q \in Q$ will be a mapping ϕ from physical space R^3 to the manifold, Φ, in which the field takes its values: $Q = \mathrm{Map}(R^3, \Phi)$. This example is, of course, the one of primary interest to us; and we will assume additionally that the mappings $\phi \in Q$ are continuous and approach a fixed constant $e \in \Phi$ as the spatial radius $r \to \infty$.

We must also assume in general that Q is given to us, not just as a set, but as a topological space. For example, the free top's configuration space $R^3 \times SO(3)$ would be endowed with its product topology. For the scalar field (nonlinear sigma model) we can endow Q with any convenient topology induced by the manifold topology of Φ, e.g. the restriction to Q of the compact-open topology [14] on continuous functions from R^3 to Φ.

Now by assumption there exists for each $q \in Q$ a state-vector $|\psi\rangle \in H$ which corresponds to q in the sense that if $\{\hat{Q}_k\}$ is a complete set of position observables and $\{q_k\}$ are the corresponding coordinates of q, then $|\psi\rangle$ is a joint eigenvector of the $\hat{Q}_k$ such that for all k, $\hat{Q}_k|\psi\rangle = q_k|\psi\rangle$. This condition, however, does not determine $|\psi\rangle$ uniquely. Even if supplemented by the requirement that $\langle\psi|\psi\rangle = 1$, it leaves free *at least* the phase of $|\psi\rangle$. In general, much more than a phase will be left free because the system will have additional degrees of freedom beyond those corresponding to changes of the configuration point $q \in Q$. For systems of point particles these additional degrees of freedom might of course be spin, isospin, color, etc., but they might also be the non-local degrees of freedom which characterize exchange relations of the parastatistical sort. (Some amplification of this statement can be found in [12].) In what follows, however, we will ignore all such possibilities and assume that there corresponds to each $q \in Q$ an eigenvector $|\psi\rangle$ which is unique up to phase.

[We are also ignoring here the fact that normalizable eigenvectors of the $\hat{Q}_k$ can exist only in the trivial case that Q is a collection of isolated points. In all other cases $|\psi\rangle$ is really a slightly smeared eigenvector, representing not just the single point $q \in Q$, but a small neighborhood thereof. This tacit use of smeared eigenvectors will not affect our considerations below, but it does have the interesting consequence that the bundle $\bar{Q}$ which we will define will *not* inherit a connection from its natural embedding in H.]

Now let us associate to each $q \in Q$ the "fiber" $\bar{Q}(q)$ comprising all $|\psi\rangle \in H$ which correspond to q in the sense just described. For us this fiber will be a circle (since it is parameterized by a phase) and the union of all such fibers will be a principal $U(1)$ bundle over Q. The bundle is principal because the rule $|\psi\rangle \rightarrow e^{i\theta}|\psi\rangle$ defines a global action of $U(1)$ on it. (Here multiplication by $e^{i\theta}$ is just the scalar multiplication that is part of the definition of the vector space H.)

We can summarize the relations among Q, $\bar{Q}$, and H in the following diagram:

$$\begin{array}{c} U(1) \rightarrow \bar{Q} \subseteq H \\ \downarrow^{\pi} \\ Q \end{array} .$$

Until now we have been dealing with an arbitrary quantum system not necessarily related to any classical system, and in particular not necessarily possessing any meaningful "classical limit." However, if we now do imagine that Q is the configuration space appropriate to such a limit, then we can describe the possibility of different bundles $\bar{Q}$ over the same Q as a species of "quantization ambiguity" inherent in the passage from the given classical system to one of its quantum analogues. Now it is known [15] that the distinct $U(1)$-bundle structures over Q are parametrized by the elements of $H^2(Q;Z)$, the second cohomology group of Q with coefficients in the integers. It is also known [16] that there is a (non-natural) isomorphism between $H^2(Q,Z)$ and the direct sum

$$H_2(Q;Z)^* \oplus \mathrm{Tor}\, H_1(Q;Z),$$

where the torsion subgroup is defined in general by $\mathrm{Tor}(A) := \{a \in A \mid na = 0$ for some integer $n\}$, and $H_2(Q)^* := \mathrm{Hom}(H_2(Q), Z)$ is[2] effectively the *non*-torsion part of $H_2(Q)$. Thus there are in a certain sense distinct "quantization ambiguities" associated with the groups $H_2(Q)^*$ and $\mathrm{Tor}(H_1(Q))$.

In the treatment of spin and statistics by Finkelstein and Rubinstein [13] only $H_1(Q)$ comes into play. Such a treatment suffices for the $SU(2)$ chiral model which they considered, but it cannot handle more than two flavors. But taking the cases $N_f \geqq 3$ into account would seem to be particularly desirable, because the $SU(N_f)$ chiral models for $N_f \geqq 3$ have the advantage of explaining in a certain sense why the proton *must* – and not only *can* – be a fermion. In the following sections we shall extend the proof of [13] to general kinks, including in particular the higher chiral models just mentioned.

In concluding this section let us remark that a principal $U(1)$-bundle $\bar{Q}$ embedded in the quantum hilbert space H can arise in ways other than that we have just been considering. In fact such a bundle will automatically come into play whenever one has reason to refer to some distinguished subset Q of the *pure-states* of some quantum system: an arbitrary normalized hilbert-space vector $|\psi\rangle$ is then an element of $\bar{Q}$ iff it represents one of the elements of Q. For example if Q is a set of adiabatically related stationary states of a variable Hamiltonian $H(\lambda)$, then we have the situation of [17]. In that case the $|\psi\rangle$ are genuinely normalizable and $\bar{Q}$ is not only a $U(1)$-bundle, but one with a *connection*. For still another case – one where $\bar{Q}$ inherits a Lorentzian metric rather than a connection – see Sect. 7 of [18].

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[2] Unless otherwise mentioned, all homology groups will have coefficients in the integers Z

III. Characterization of Spin-Type and Statistics in the Present Framework

The distinction between the *spinorial* angular momenta ($j=1/2, 3/2, 5/2, \ldots$) and the *tensorial* ones ($j=0, 1, 2, \ldots$) would ordinarily be expressed in terms of the distinction between double-valued and single-valued representations of the rotation group $SO(3, R)$. To re-express it in the language we will need, consider first any pure state of tensorial angular momentum and a corresponding state vector $|\psi_0\rangle$. By acting on $|\psi_0\rangle$ with an arbitrary rotation $g \in SO(3)$ we produce an association

$$g \to R(g)\,|\psi_0\rangle\,, \tag{3.1}$$

where $R(g)$ is the rotation operator representing g in H. The set of all state-vectors $|\psi\rangle$ representing rotates of the original pure-state, or more accurately the set E_{spin} of all pairs $(g, |\psi\rangle)$ such that $g \in SO(3)$ and $|\psi\rangle = uR(g)\,|\psi_0\rangle$ for some $u = e^{i\theta} \in U(1)$, is then a principal $U(1)$ bundle over $SO(3)$; and the mapping (3.1) is a cross-section of E_{spin}. Therefore E_{spin} is a trivial bundle in this case. On the other hand, if we had begun with a spinorial state $|\psi\rangle$ then we would again have obtained a bundle $E_{\text{spin}} = \{(g, |\psi\rangle)\}$ over $SO(3)$, but it would not have been trivial. Instead (3.1) would have been double-valued; and we know that there is for spinorial $|\psi_0\rangle$ no way of redefining the phases of the $R(g)$ to resolve this double-valuedness.

Now there are in fact only two distinct $U(1)$-bundles over $SO(3)$, as follows from the diffeomorphism $SO(3) \simeq P^3 \equiv S^3/Z_2$, together with the formula $H^2(P^3, Z) = Z_2$. Hence the question whether a given pure state is tensorial or spinorial is the same as the question whether the bundle E_{spin} over $P^3 = SO(3)$ to which it gives rise is trivial or twisted ($\equiv$non-trivial).[3]

Let us now specialize this conclusion to the situation where $|\psi_0\rangle$ represents a single "configuration" eigenstate belonging to Q, as introduced in the previous section. Then (3.1) induces a map

$$f_{\text{spin}} : P^3 \to Q$$

and E_{spin} is by definition the pullback[4] of $\bar{Q}$ via this map: $E_{\text{spin}} = \bar{f}_{\text{spin}}(\bar{Q})$. More generally, if $|\psi_0\rangle$ is any linear combination of elements of $\bar{Q}$ all of which give rise to the twisted (respectively trivial) bundle over P^3, then by continuity $|\psi_0\rangle$ also gives rise to the twisted (respectively trivial) bundle.

For the general non-linear σ-model, these relationships will reduce the determination of kink spin-type (namely spinorial vs. tensorial) to the question whether the pullback of $\bar{Q}$ via a certain class of maps $f_{\text{spin}} : P^3 \to Q$ is trivial or twisted. In the same way the determination of kink statistics will be reduced to the question whether the pullback of $\bar{Q}$ via a certain map

$$f_{\text{stat}} : P^2 \to Q$$

is trivial or not, where $P^2 = S^2/Z_2$ is real projective 2-space.

[3] Notice how only the topology of $SO(3)$ figures in this formulation, but not the multiplication structure which makes it a group

[4] A definition of pullback can be found in Appendix B of [1]

To understand this latter reduction, recall [4] that the statistical type of a particle species can be deduced by referring to the bundle of position eigenvectors of a pair of such particles both of which are in the same, freely chosen but fixed, internal state. The base-space of this bundle (which need be defined only for sufficiently widely separated particles) is contractible onto P^2, with the contraction having the meaning of going to the center of mass system and placing the particles at a fixed separation. The inclusion of this P^2 into Q defines the map f_{stat}; and the bundle $\overleftarrow{f}_{\mathrm{stat}}(\bar{Q})$ (in other language, the restriction of $\bar{Q}$ to P^2) is non-trivial iff the particles are fermions.

We have thus reduced both the tensorial-spinorial dichotomy and the boson-fermion dichotomy to the triviality or not of certain bundles $\overleftarrow{f}_{\mathrm{spin}}(\bar{Q})$ and $\overleftarrow{f}_{\mathrm{stat}}(\bar{Q})$ respectively.[5] In these terms the spin-statistics correlation will be expressed by the statement that the one bundle is trivial if and only if the other one is trivial. To prove this statement we must define the maps f_{spin} and f_{stat} more specifically for the case where Q is (the above defined subset of) Map(R^3, Φ).

IV. The Maps f_{stat} and f_{spin} for Kinks

Henceforth let us restrict ourselves to the case of *(ungauged)* σ-models, in which the only dynamical variable is a scalar field valued in a manifold we are calling Φ. For the subcase of primary interest, kinks will be "Skyrmions" and $\Phi = SU(n)$ will be the "target manifold" of an $SU(n)$-"chiral model." (See [19] for background on such models.)

In all such cases Φ contains a distinguished value e which ϕ assumes in its vacuum state and which, therefore *all* $\phi \in Q$ must approach at spatial infinity. In this way the configuration space Q becomes as usual a space of "pointed" continuous maps of ${}_{\infty}S^3$ into ${}_{e}\Phi$ [i.e. maps $\phi: S^3 \to \Phi$ such that $\phi(\infty) = e$]. Assuming that the third homotopy group $\pi_3(\Phi)$ is non-trivial, there will be elements ϕ of Q not homotopic to the constant map $\phi(x) \equiv e$; and a ϕ whose equivalence class is one of the generators of $\pi_3(\Phi)$ is conventionally called a kink [13]. Here we will be slightly more liberal, allowing a kink to be *any* specified configuration $\phi_1 : S^3 \to \Phi$. More precisely we will assume that $\phi_1(x) = e$ for $|x| > L$ and call "one-kink configuration centered at $x = b$" any ϕ obtained from ϕ_1 by an overall rigid motion which carries the origin to $x = b$:

$$\phi(g \cdot x + b) \equiv \phi_1(x) \tag{4.1}$$

[5] One may wonder why the notion of fermion, say, can't be defined even more simply in terms of the sign resulting from adiabatic exchange of two identical particles. The problem with such a definition is that adiabatic transport around any loop – even one which doesn't exchange the particles – will in general introduce a phase which depends as much on the details of the transport process as on the particle statistics themselves [17]. For example, such "irrelevant phases" will certainly arise if electrically charged particles are transported through a magnetic field. However, this does not mean that it is impossible to define statistical- (and similarly spin-) type in terms of adiabatic transport. On the contrary *any* determinate mechanism of transport will introduce a connection on $E_{\mathrm{stat}} = \overleftarrow{f}_{\mathrm{stat}}(\bar{Q})$ and in that sense *will* fully determine the structure of E_{stat} since any bundle with connection can be reconstructed from the holonomy elements of that connection

for some rotation $g \in SO(3)$. A *two*-kink configuration ϕ will be one whose support [by definition the closure of the complement of $\phi^{-1}(e)$] is enclosed by two disjoint spheres of radius L within each of which ϕ coincides with a 1-kink configuration. Finally, an anti-kink configuration will be one obtained from ϕ_1 by a rigid motion together with a reflection.

Notice that a kink as we have defined it need not possess any form of dynamical stability.[6] Some degree of stability is of course needed – though not necessarily at the classical level – for the kink to be interpretable as a particle; but it will not play any direct role here. A quantum kink will in any case be a superposition of many configurations, ϕ, more or less similar to ϕ_1. In this respect our use of a single configuration to represent "one-kink" is a convenient fiction, justified only because the quantum fuzziness in ϕ does not lead to any ambiguity in the spin-type or statistical-type of the associated quantum particle.

The definition of the maps f_{spin} and f_{stat} is now fairly straightforward. To define the former we need merely set, for $g \in P^3 = SO(3)$, $f_{\text{spin}}(g) = \phi_1 \circ g$, which is just the ϕ of (4.1) with $b=0$. More generally, we need not assume that the "initial" configuration $\phi = f_{\text{spin}}(1)$ is precisely ϕ_1, but only that it coincides with a translate of the latter in some ball of center b and radius $>L$. We then define $\phi = f_{\text{spin}}(g)$ for $g \in SO(3)$ by

$$\phi(x) = \phi(b + g^{-1}(x-b)) \quad \text{for} \quad |x-b| \leqq L,$$

$$\phi(x) = \phi(x) \quad \text{for} \quad |x-b| \geqq L.$$

In this way we can make sense of the spin-type of a kink without making the unrealistic assumption that it is surrounded by nothing but vacuum. The specific ϕ we will need is that which has kinks at $\pm b = (0, 0, \pm 2L) \in R^3$:

$$\phi_2(x, y, z) = \phi_1(x, y, z \mp 2L)$$

for (4.2)

$$x^2 + y^2 + (z \pm 2L)^2 > L^2.$$

Henceforth f_{spin} will always be defined relative to this specific configuration which I will call ϕ_2.

By continuity, the bundle $\overleftarrow{f}_{\text{spin}}(\bar{Q})$ is constant for ϕ_1 within a fixed homotopy class in $\pi_3(\Phi)$. Its spinorial or tensorial character is therefore unambiguous within a given such class or "kink-sector," being independent within such a sector of the amplitudes $\psi(\phi) \equiv \langle \phi | \psi \rangle$ with which individual configurations ϕ occur in the actual quantum state. In particular, the spin-type of a quantum kink does indeed not depend on the details of the configuration ϕ_1, but only on the latter's homotopy class.

In the same way the statistical type (bose or fermi) of a quantum kink also depends only on the element of $\pi_3(\Phi)$ to which ϕ_1 belongs. Recalling that f_{stat} must be defined with reference to a fixed "internal state" of the kink, we may construct it as follows. To begin with, we identify P^2 with the set of pairs of vectors $\{+b, -b\} \subseteqq R^3$ such that $|b| = 2L$. Then for each such pair $\{\pm b\} \in P^2$, we define $f_{\text{stat}}(\{\pm b\})$ to be the 2-kink configuration $\phi(x)$ which coincides with $\phi_1(x-b)$ for

[6] Precisely for this reason I have used the word "kink" rather than the word "soliton"

$|x+b| \geqq L$ and with $\phi_1(x+b)$ for $|x-b| \geqq L$. The spin-statistics correlation can now be formulated as follows:

For any principal $U(1)$-bundle $\bar{Q}$ over Q, the bundles $\bar{f}_{\text{spin}}(\bar{Q})$ and $\bar{f}_{\text{stat}}(\bar{Q})$ are either both trivial or both non-trivial.

In the next section we will establish this correlation by relating the maps f_{spin} and f_{stat} to each other. Before doing so, however, it may be helpful to dwell a moment on the "chiral model" case where $\Phi = SU(n)$, with e being the identity within $SU(n)$. For all values of the "flavor number," $n=2,3,4,\ldots$ we have $\pi_0(Q) = \pi_{0+3}(\Phi) = \pi_3(SU(n)) = Z$. Hence the kink-sectors are labelled by a single integer [the "winding number" given by

$$\mathrm{tr}(1/24\pi^2)\int(\phi^{-1}\partial_a\phi)(\phi^{-1}\partial_b\phi)(\phi^{-1}\partial_c\phi)\varepsilon^{abc}d^3x]$$

which we may call "baryon number."

The possibilities for $\bar{Q}$, however, are very different for $n=2$ from what they are for $n \geqq 3$. For $n=2$ we have $\pi_1(Q) = \pi_{1+3}(\Phi) = \pi_4(SU(2)) = Z_2$, whence [16] $H_1(Q) = Z_2$, whence $\mathrm{Tor}(H_1(Q)) = Z_2$. The corresponding twofold ambiguity in $\bar{Q}$ can also be described [13] without direct reference to $U(1)$ bundles as the possibility of generalizing the quantum ψ-functional from a function with domain Q to one defined on a twofold covering space of Q. Depending on the "parity" of this ψ, the kinks (Skyrmions) it describes will be either spinorial fermions or tensorial bosons.

For $n=3, 4, 5, \ldots$ on the other hand we have $\pi_1(Q) = \pi_4(SU(n)) = 0$, whence $H_1(Q)$ also vanishes. For these n, Q is its own universal covering space (all loops in Q being contractible) but spinorial quantizations are nonetheless possible because of the non-triviality of $H_2(Q)^*$ [20, 18, 21]. In fact, it is only for $n \geqq 3$ that one sees clearly the utility of defining f_{spin} and f_{stat} with their full domains P^3 and P^2. For $n=2$ one can study kink-spin and kink-statistics entirely in terms of the loops in Q defined by restricting these two maps to convenient non-trivial loops within their respective domains [13]. For $n \geqq 3$, in contrast, the corresponding restrictions are trivial because $\pi_1(Q)$ itself is trivial.

One last point worth mentioning is that for $n \geqq 3$ there is in practice less ambiguity in the choice of $\bar{Q}$ than there is for $n=2$, the reason being the general fact that elements of $H^2(Q)$ stemming from $H_2(Q)^*$ are typically determined by the equations of motion (field equations) whereas those stemming from $\mathrm{Tor}(H_1(Q))$ are not. For $n \geqq 3$ the equations of motion exert their influence via the "chiral anomaly," which, being known from other considerations, forces baryons to be spinorial via the effect of the corresponding "Wess-Zumino term" in the action [20, 18]. This is precisely analogous to the way in which the equations of motion for an electric charge moving in a coulombic magnetic field force the "orbital angular momentum" to be spinorial for odd values of eg/h [11, 18].

V. A Homotopy Between f_{stat} and $\hat{f}_{\text{spin}}$

We wish to show that $E_{\text{spin}} = \bar{f}_{\text{spin}}(\bar{Q})$ is trivial iff $E_{\text{stat}} = \bar{f}_{\text{stat}}(\bar{Q})$ is trivial; or what is the same thing, that the former is the unique non-trivial bundle over P^3 precisely when the latter is the unique non-trivial bundle over P^2.

To that end let us note that a principal $U(1)$-bundle over P^3 is non-trivial if and only if its restriction to an equatorial $P^2 \subsetneqq P^3$ is non-trivial. This follows directly from the circumstance that these non-trivial bundles are respectively the quotient of $S^3 \times U(1)$ by $\Pi_3 \times (-1)$ and of $S^2 \times U(1)$ by $\Pi_2 \times (-1)$, where Π_m is the antipodal inversion of the m-sphere S^m $(m=2, 3)$, and where $U(1)$ has been identified with the unit complex circle. Defining, then the inclusion i: $P^2 \to P^3$ of $P^2 = S^2/\Pi_2$ as the equator of $P^3 = S^3/\Pi_3$, and setting $\hat{f}_{\text{spin}} = f_{\text{spin}} \circ i$, we have that $\overleftarrow{f}_{\text{spin}}(\bar{Q})$ is trivial iff $\overleftarrow{\hat{f}}_{\text{spin}}(\bar{Q}) = \overleftarrow{(f_{\text{spin}} \circ i)}(\bar{Q}) = \overleftarrow{i} \circ \overleftarrow{f}_{\text{spin}}(\bar{Q})$ is trivial. The spin-statistics correlation is thus reduced to the assertion that $\overleftarrow{\hat{f}}_{\text{spin}}(\bar{Q})$ and $\overleftarrow{f}_{\text{stat}}(\bar{Q})$ are either both trivial or both non-trivial. By introducing $\hat{f}_{\text{spin}}$ in place of f_{spin} we have referred both spin-type and statistical-type to maps f with the same domain, P^2. If these two maps can be shown to be homotopic then, by continuity, the corresponding pulled-back bundles over P^2 will have been proved equivalent, and the spin-statistics correlation established for arbitrary kinks.

To construct a homotopy from f_{stat} to $\hat{f}_{\text{spin}}$ we can proceed in two steps. First we fiber P^2 into individual loops λ and then – in a manner varying continuously with λ – deform $f_{\text{stat}} | \lambda$, the restriction of f_{stat} to λ, into the corresponding restriction $\hat{f}_{\text{spin}} | \lambda$. Now let us recall that we have represented P^2 as a quotient of the unit sphere in R^3 by Π_2. As such it can be conveniently parametrized by the usual spherical coordinates θ, ϕ except that now both θ *and* ϕ run from 0 to π. The loops $\lambda(\phi)$ of fixed ϕ then cover P^2 without overlap, except at their common base-point (namely the equivalence class uniting the north pole, $\theta=0$ with the south pole, $\theta=\pi$); and except for the fact that $\lambda(\pi)$ is really just $\lambda(0)$ parameterized oppositely by θ.

Now our definitions are actually not complete until we have specified the inclusion-map i of P^2 into $SO(3) = P^3$. This we do by taking $i(\theta, \phi)$ to be the rotation of angle 2θ about the axis $e(\phi)$ whose spherical coordinates are $(\pi/2, \phi + \pi/2)$, or equivalently whose Cartesian components are $(-\sin\phi, \cos\phi, 0)$. With this, the restrictions $f_{\text{stat}} | \lambda(\phi)$ and $\hat{f}_{\text{spin}} | \lambda(\phi)$ become definite maps of S^1 into Q, i.e. definite loops in Q. We may call the former loop the "exchange with axis $e(\phi)$" and the latter loop the "2π-rotation with axis $e(\phi)$."

With these definitions let us suppose that there exists for each $\phi \in [0, \pi]$ a based[7] homotopy deforming $f_{\text{stat}} | \lambda(\phi)$ into $\hat{f}_{\text{spin}} | \lambda(\phi)$ and let us suppose further that this homotopy depends continuously on the angle ϕ. Then by assembling all these homotopies into one, we can construct the desired deformation of the complete map f_{stat} into the complete map $\hat{f}_{\text{spin}}$. Thus let $f_s(\phi)$ for $s \in [0, 1]$ be the homotopy of $f_{\text{spin}} | \lambda(\phi)$ into $\hat{f}_{\text{stat}} | \lambda(\phi)$. By definition we have $f_0(\phi) = f_{\text{stat}} | \lambda(\phi)$, $f_1(\phi) = \hat{f}_{\text{spin}} | \lambda(\phi)$, and, for all s and ϕ, $f_s(\phi)|_{\theta=0} = f_s(\phi)|_{\theta=\pi} = f_0(\phi)|_{\theta=0} = \phi_2$. Now for each s and ϕ, $f_s(\phi)$ is a loop in Q parameterized by the angle θ. Making this dependence on θ explicit yields a continuous sequence of functions

$$F_s(\theta, \phi) \equiv [f_s(\phi)](\theta)$$

[7] That is, a homotopy throughout which the loops' base-point, $[f_{\text{stat}} | \lambda(\phi)]|_{\theta=0}$ remains unaltered. Notice in this connection that $\hat{f}_{\text{spin}} | \lambda(\phi)$ has the same base-point as $f_{\text{stat}} | \lambda(\phi)$ by virtue of the definitions of $f_{\text{spin}}(1) = \phi_2$ and of the inclusion i

from P^2 to Q such that $F_0 = f_{\text{stat}}$ and $F_1 = \hat{f}_{\text{spin}}$. This sequence of mappings F_s is the homotopy we wanted. Of course we have supposed, in constructing the F_s, that we had available the individual homotopies deforming the exchange with axis $e(\phi)$ to the 2π-rotation with the same axis. Fortunately such homotopies are indeed available, having been constructed in [13]. To see that they depend continuously on $e(\phi)$ we need only observe that they depend continuously on what here has been called ϕ_1 and invoke rotational symmetry.

Unfortunately [13] does not exhibit the exchange→rotation homotopy very explicitly, although it does in effect give (in its Appendix C) a step by step analytic representation of a homotopy related to the former as follows. Both $f_{\text{stat}} \mid \lambda(\phi)$ and $\hat{f}_{\text{spin}} \mid \lambda(\phi)$ are loops in the 2-kink sector of Q; for both of them the initial and final configurations are ϕ_2, which has kinks at $(0,0,2L)$ and $(0,0-2L)$. Such a loop is symbolized in Fig. 1, in which the coordinates x and y (say) have been suppressed and the loop parameter θ (let us call it "time") runs upwards. (Thus the initial and final values of θ are to be identified.) In this figure the directed lines represent the initial and final 2-kink configuration ϕ_2, while the "blob" represents the non-trivial portion of the homotopy. A corresponding "vacuum sector" loop (or "process" if we think of θ as a time) is symbolized in Fig. 2. In it the 2 kinks are no longer eternal, as in Fig. 1, but instead are paired with anti-kinks, from which they separate at an early "time" and with which they recombine after the homotopy is finished. Now the homotopy given (in effect) in Appendix C of [13] connects with each other not $f_{\text{stat}} \mid \lambda(\phi)$ and $\hat{f}_{\text{spin}} \mid \lambda(\phi)$, but the vacuum-sector processes which correspond to them in the way that Fig. 2 corresponds to Fig. 1.

To convert such a vacuum-sector homotopy to one in the 2-kink sector we may proceed as indicated by the scheme

$$1 \to 3 \to 4 \simeq 2 \to 2' \simeq 4' \to 3' \to 1'$$

(where the numbers refer to the diagrams). Let us imagine for example that the blob in Fig. 1 is an exchange with axis $e(\phi)$. By an initial homotopy (which can be induced by a deformation (isotopy) of the x-y-z-θ manifold) we deform Fig. 1 into Fig. 3. Then, by a homotopy which affects only the region within the broken rectangle, we deform Fig. 3 into Fig. 4. But Fig. 4 contains within it the process of Fig. 2, which by assumption, we can deform into the vacuum sector correspondent of the process which rotates *one* of the kinks through angle 2π about axis $e(\phi)$,

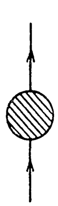

Fig. 1

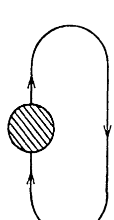

Fig. 2

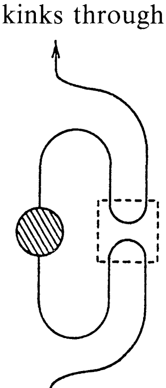

Fig. 3

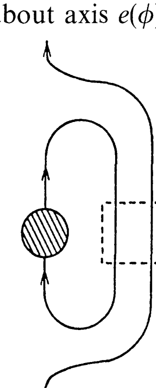

Fig. 4

leaving the other kink alone. Carrying out this deformation on the vacuum "subprocess" in Fig. 4 we arrive at a process which obviously can be deformed into $f_{\rm spin} \,|\, \lambda(\phi)$ by reversing the steps which led from Fig. 1 to Fig. 4. It is clear that the homotopy we finally obtain will vary continuously with ϕ_2 insofar as this is true of the one "given to us" (the one operating on Fig. 2).

Perhaps the one step in the above sequence whose implementation is not so evident is the homotopy 3→4, which in effect only pertains to the region or "subprocess" within the box in Fig. 3. Taking the origin of coordinates to be the center of this box we may analytically define a homotopy of the required effect by the formulas

$$\begin{aligned} \phi(x,y,z,\theta,s) &\equiv \phi_2(x,y,Z(z,\theta/\pi,s)), \\ Z(z,t,s) &\equiv |z| + 6L[8t(1-t)(1-s)-1], \end{aligned} \tag{5.1}$$

where both $t := \theta/\pi$ and the homotopy parameter s run from 0 to 1. [ϕ_2 has been defined in Eq. (4.2).]

To see what this does, let us first examine its "end result" by setting $s=1$. For this s, $Z=|z|-6L$ and

$$\phi(x,y,z,\pi t,1) \equiv \phi_2(x,y,|z|-6L), \tag{5.2}$$

which is actually independent of $\theta=\pi t$. For $z>0$ this t-independent configuration coincides with ϕ_2, or to be precise, a translate of ϕ_2 through $(0,0,6L)$. For $z<0$ it coincides with the parity reverse of ϕ_2, a configuration of two anti-kinks corresponding to downward-directed line in the diagrams. Taken together the $z \gtrless 0$ portions of (5.2) thus agree with the portion of Fig. 4 within the box.

The meaning of (5.1) for $s=0$, i.e. the "initial process" of the homotopy may be analyzed similarly. For $s=0$ we find

$$Z = |z| - 6L[8(t-1/2)^2 - 1],$$

which yields a $\phi(x,y,z;\,\pi t,0)$ describing a process like that in the box in Fig. 3 (though one not as smooth as that figure might suggest). In particular it coincides with the $s=1$ process for $t=0,1$, while for $t=1/2$, we have

$$\phi(x,y,z;\,\pi/2,0) = \phi_2(x,y,|z|+6L) = e,$$

which is pure vacuum.

Finally we should check that our homotopy becomes trivial on the boundary of the box. For the boundaries at $t=0,1$ we have $Z(z,t=0,1,s)=|z|-6L$, which is manifestly independent of s. For the "vertical" boundaries we merely note that, for any t, $Z \geqq |z|-6L$, whence we will always have $\phi=e$ on these boundaries, assuming the box is large enough in the x, y, and z directions to enclose the cube defined by

$$|x|, |y|, |z| \leqq 9L.$$

VI. Conceivable Extensions and Generalizations

The Spin-Statistics correlation we have obtained is perfectly general with respect to the manifold Φ in which ϕ takes its values. In particular it applies to $\Phi = SU(n)$ for all values of n: $n \geqq 3$ as well as $n=2$. However, in deriving it we have limited ourselves rather strongly by assuming that each $\phi \in Q$ is represented by a single *ray* in hilbert space and not by a subspace of dimension two or more. Such multi-dimensional representations figure prominently in canonical quantum gravity [22, 12] and also are an essential feature of any parastatistical quantization of a system containing three or more indistinguishable point particles [12].

Whether "quantum multiplicity" of this sort can exist for kinks is, to my knowledge, an open question, although the simplest such possibilities – those based on promoting ψ to a function on a covering space of Q – are absent here, because $\pi_1(Q)=\pi_4(\Phi)$ is always abelian, whence without irreducible representations of dimension 2 or greater. If there *are* non-trivial $U(n)$ bundles $\tilde{Q}$ over Q, then it would be interesting to study spin-statistics correlations in that more general case, and specifically to work out the implications there of our general homotopy between f_{stat} and $\hat{f}_{\text{spin}}$.

Another sort of generalization of our present results would be to a class of theories larger than just the nonlinear-sigma-models. In fact the question arises to what extent the topological methods pioneered in [13] furnish the basis for a unified derivation of the spin-statistics correlation in all the cases where it is known to hold.

One case in which this does appear to happen is that of the Saha Dyon. In fact by embedding the multiparticle configuration space of [11, 1] in a larger one which includes the anti-particles of the dyons' constituents, one can provide for pair creation and annihilation of the particles involved. The deformation of the analog of f_{stat} into the analog of $\hat{f}_{\text{spin}}$ then proceeds in strict analogy with what we have done above.

The case which seems least susceptible to being treated in this fashion is that of point particles created by the operator-valued distributions of "axiomatic field theory." The ingredients such as energy-positivity, Lorentz invariance, and analyticity which enter the spin-statistics proofs in that context [8] seem very different from the considerations of simple continuity which enter the topological proofs. (Indeed we never even mentioned Lorentz invariance in the derivation given above.) Nevertheless the two approaches might be harmonized if it turned out that the role of Lorentz invariance etc. was really only to guarantee the existence of the intermediate pure-states traversed in the course of deforming f_{stat} into $\hat{f}_{\text{spin}}$. Evidence for such an interpretation comes from the observation (often attributed to Feynman) that the existence of anti-particles (which we have seen to be crucial ingredients of these intermediate states) can be traced directly to the combination of energy positivity with locality and Lorentz invariance.

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Communicated by L. Alvarez-Gaumé

Received January 29, 1987