

The Global Markov Property for Equilibrium States Which are Determined by Correlations in a Strip

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Abstract. Consider an interaction-round-face potential on the lattice \mathbb{Z}^d , which may include nearest-neighbor and next-nearest-neighbor pair interactions, as well as more general plaquette terms. Assuming some periodicity of the potential it is shown that, under the condition that equilibrium states can be distinguished by expectation values of sufficiently local observables, equilibrium states possess a global Markov property. The condition under which this Markov property is shown to hold is met in particular in case the equilibrium state is unique or determined by the magnetization. The proof is based on an application of the variational principle to states which are constructed like Markov chains.

1. Introduction

We consider systems of classical spins on \mathbb{Z}^d with, per lattice point, values in a compact metric space. The Markov property entails that upon fixing the configuration on the boundary of a volume the distribution of the spin configurations inside the boundary becomes independent of the spin configuration outside the boundary. The very definition of Gibbs states for a finite range potential states that Markov properties hold for finite volumes, see e.g. [1]. In this case one speaks of a local Markov property. In case such a property holds for an infinite volume this is called a global Markov property.

Gibbs states for finite-range potentials possess thus local Markov properties. However, the question whether or not Gibbs states possess global Markov properties is not easy to answer. Indeed, for the three-dimensional Ising model Goldstein has constructed an example of a Gibbs state that does not have the global Markov property with respect to a plane, [1]. Other examples of this type are given in [10].

Global Markov properties are an important ingredient in Nelson's scheme of Euclidean field theory because these properties are essential in constructing the Hamiltonian of the theory (see e.g. [2]). Despite their importance, global Markov properties are known to hold in relatively few cases. These cases can be divided into two main groups:

- A) The interaction is attractive or such that F.K.G. inequalities hold. The maximal Gibbs measure (+ magnetization) and the minimal Gibbs measure (− magnetization) then have global Markov properties. Such cases are discussed in [1, 3, and 4].
- B) The potential satisfies Dobrushin’s uniqueness criterion, implying that there exists a unique Gibbs state, cf. [9]. This unique Gibbs state then has the global Markov property with respect to all surfaces, [3 and 5].

One consequence of Theorem 1 below is that these results can be extended: If the interaction has the properties specified in Theorem 1 and admits a unique Gibbs state then this state has a global Markov property specified below.

This paper is inspired by methods used by Schlijper in [6]. In particular the technique of entropy inequalities used in Lemma 2 is taken from [6].

The proof of the main result (Theorem 1) of this paper uses the variational principle. Briefly outlined the argument goes as follows. Consider a state on two planes of the lattice. A construction is presented which yields a state on all of the lattice having a global Markov property and which is an extension of the state on the two planes. [In fact two versions of such a construction are presented in this paper.] This construction is then applied to the restriction μ_2 , to two planes of an equilibrium state μ on \mathbb{Z}^d yielding a (new) state $\bar{\mu}$ on all of \mathbb{Z}^d . If μ already had the global Markov property then $\mu = \bar{\mu}$, but irrespective of this it is shown that $\bar{\mu}$ always is an equilibrium state (with a global Markov property and coinciding with μ on the two planes).

The proof that $\bar{\mu}$ is an equilibrium state rests on two observations,

- 1) the energy density of μ equals the energy density of $\bar{\mu}$,
- 2) the entropy density of $\bar{\mu}$ is not smaller than the entropy density of μ , since these two observations, by the variational principle (Theorem 0), imply that $\bar{\mu}$ is also an equilibrium state. The second of the just mentioned observations is obtained making use of two lemmas (Lemma 1 and Lemma 2). Theorem 2, for the case of a two-dimensional lattice, states a further relation between the constructed state $\bar{\mu}$ and the original equilibrium state μ .

We introduce some notation. Let Ω_0 be a compact metric space. For $\Lambda \subseteq \mathbb{Z}^d$ we put $\Omega_\Lambda = (\Omega_0)^\Lambda$. We write $\Omega_{\mathbb{Z}^d} = \Omega$. We equip Ω_Λ with the product topology and denote by $C(\Omega_\Lambda)$ the space of continuous functions on Ω_Λ . Let $\alpha_{\Lambda, \Lambda'}: \Omega_{\Lambda'} \rightarrow \Omega_\Lambda$ be the restriction map for $\Lambda' \supset \Lambda$. We write $\alpha_\Lambda = \alpha_{\Lambda, \mathbb{Z}^d}$. Define $\mathfrak{F}_\Lambda = \{f: \Omega \rightarrow \mathbb{C}; \text{there exists } f_\Lambda: \Omega_\Lambda \rightarrow \mathbb{C} \text{ such that } f = f_\Lambda \cdot \alpha_\Lambda\}$. Denote by $\mathfrak{F}(\Lambda)$ the set of complex valued functions on Ω_Λ .

The map

$$\mathfrak{F}(\Lambda) \ni f_\Lambda \rightarrow f_\Lambda \alpha_\Lambda \in \mathfrak{F}_\Lambda$$

gives then a one-to-one correspondence between $\mathfrak{F}(\Lambda)$ and \mathfrak{F}_Λ . Henceforth we will identify $\mathfrak{F}(\Lambda)$ and \mathfrak{F}_Λ by this one-to-one correspondence. Under this correspondence

$$C(\Omega_\Lambda) = C_\Lambda \equiv \{f \in C(\Omega) \mid \text{there exists } f_\Lambda \in C(\Omega_\Lambda) \text{ such that } f = f_\Lambda \cdot \alpha_\Lambda\}.$$

Denote by \mathfrak{B}_Λ the smallest σ -algebra of subsets of Ω with the property that functions from C_Λ are measurable with respect to it. \mathfrak{B}_Λ is the σ -algebra generated by cylinder sets with finite base in Λ . Write $\mathfrak{B}_{\mathbb{Z}^d} = \mathfrak{B}$. Let $\mathfrak{B}(\Lambda)$ be the Borel σ -algebra

of Ω_A . Upon identifying sets with their characteristic function, $\mathfrak{B}(A)$ as a subset of $\mathfrak{F}(A)$ is identified with $\mathfrak{B}_A \subset \mathfrak{F}_A$. States on $C_A = C(\Omega_A)$ are identified with probability measures on $\mathfrak{B}_A = \mathfrak{B}(A)$, through the Riesz–Markov Theorem.

For $k \in \mathbb{Z}$ we introduce $P_k = \{x \in \mathbb{Z}^d \mid x_1 = k\}$ and

$$(\mathbb{Z}^d)_{k,\delta} = \{x \in \mathbb{Z}^d \mid \delta x_1 \geq 0\} \quad \text{for } \delta = \pm.$$

We say that a state μ on $C(\Omega)$ has the Markov property with respect to P_k , if $\mathbb{E}_k^\mu f = \mathbb{E}_{\leq k}^\mu f$ for all $f \in L^2(\mu; \mathfrak{B}_{(\mathbb{Z}^d)_{k,+}})$, where \mathbb{E}_k^μ and $\mathbb{E}_{\leq k}^\mu$ are orthogonal projections on $L^2(\mu; \mathfrak{B})$ with ranges given respectively by $L^2(\mu; \mathfrak{B}_{P_k})$ and $L^2(\mu; \mathfrak{B}_{(\mathbb{Z}^d)_{k,-}})$.

For $a \in \mathbb{Z}^d$ consider $\tau^{(a)}: \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ which maps $x \in \mathbb{Z}^d$ onto $\tau^{(a)}x \in \mathbb{Z}^d$ given by $\tau^{(a)}x = x + a$. By transpositions (with the inverse) \mathbb{Z}^d acts on Ω , and $\mathfrak{F}^{\mathbb{Z}^d}$. Note that $C(\Omega)$ is invariant under this action. Thus by one more transposition \mathbb{Z}^d acts on $E_{C(\Omega)}$, where $E_{C(\Omega)}$ is the set of states on $C(\Omega)$. All these actions of $a \in \mathbb{Z}^d$ on the various spaces are also denoted by $\tau^{(a)}$. Then, for instance, $\tau^{(a)}\mathfrak{F}_A = \mathfrak{F}_{A+a}$.

$$\text{Let } \Phi: \bigcup_{\substack{A \subset \mathbb{Z}^d \\ A \text{ finite}}} \Omega_A \rightarrow \mathbb{R},$$

with $\Phi \circ \alpha_A \in C(\Omega_A)$ for all A finite, be a potential which is nonzero only round faces of \mathbb{Z}^d , i.e., $\Phi \circ \alpha_x \equiv 0$ whenever there exist $x, y \in X$ such that $\max\{|x_i - y_i| \mid i = 1, \dots, d\} > 1$.

We will make use of the variational principle. To this end we assume that Φ is H -invariant, i.e., $\Phi \circ \tau^{(a)} = \Phi$ for all $a \in H$, where $H \subseteq \mathbb{Z}^d$ is a subgroup of finite index in \mathbb{Z}^d . Let μ_0 be a probability measure on Ω_0 . Denote by $\mu_0^{(\Lambda)}$ the product measure on $\Omega_\Lambda = (\Omega_0)^\Lambda$ constructed from μ_0 .

The pressure P_Φ is defined by

$$P_\Phi = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \ln Z_\Lambda,$$

where

$$Z_\Lambda = \int_{\Omega_\Lambda} \mu_0^{(\Lambda)}(d\xi_\Lambda) \exp[-H_\Lambda(\xi_\Lambda)]$$

with

$$H_\Lambda(\xi_\Lambda) = \sum_{X \subseteq \Lambda} \Phi \circ \alpha_{X,\Lambda}(\xi_\Lambda).$$

Lim denotes limit in the sense of van Hove and $|\Lambda|$ denotes the number of points $\Lambda \uparrow \mathbb{Z}^d$ of $\Lambda \subset \mathbb{Z}^d$.

If μ is a state on $C(\Omega)$ such that its restriction to $C(\Omega_\Lambda)$ is absolutely continuous with respect to $\mu_0^{(\Lambda)}$ we denote its Radon–Nikodym derivative by $\mu^{(\Lambda)}$. The average entropy density per lattice point, $s(\mu)$, of a G -invariant state μ where G is a subgroup of \mathbb{Z}^d with finite index in \mathbb{Z}^d , is defined by

$$s(\mu) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} S_\Lambda(\mu)$$

with $S_\Lambda(\mu) = - \int_{\Omega_\Lambda} \mu^{(\Lambda)}(d\xi_\Lambda)(\mu^{(\Lambda)}(\xi_\Lambda) \ln \mu^{(\Lambda)}(\xi_\Lambda))$ if the restriction of μ to $C(\Omega_\Lambda)$ is absolutely continuous with respect to $\mu_0^{(\Lambda)}$ and $S_\Lambda(\mu) = -\infty$ otherwise.

Let $M_0 \subset \mathbb{Z}^d$ be such that $\{\tau^{(a)}M_0\}_{a \in G}$ is a partition of \mathbb{Z}^d . Note that $|M_0| < \infty$ since G has finite index in \mathbb{Z}^d . Suppose $\{\Lambda_n\}_{n \in \mathbb{N}}$ is a van Hove sequence such that each Λ_n is a union of sets from the collection $\{\tau^{(a)}M_0\}_{a \in G}$. Then $s(\mu) = \inf_{n \in \mathbb{N}} (1/|\Lambda_n|)S_{\Lambda_n}(\mu)$. In particular $s(\mu) \leq (1/|\Lambda_n|)S_{\Lambda_n}(\mu)$ for all $n \in \mathbb{N}$. Furthermore the functional s is upper-semicontinuous on the set of G -invariant states.

The proofs of these assertions for a general subgroup $G \subseteq \mathbb{Z}^d$ with finite index are a modification of the proof in case $G = \mathbb{Z}^d$, cf. [8]. For $x \in \mathbb{Z}^d$ define $A_\Phi^{(x)} \in C(\Omega)$ by

$$A_\Phi^{(x)} = \sum_X^{(x)} \Phi \circ \alpha_X,$$

where $\sum_X^{(x)}$ means taking the sum over finite subsets of X of \mathbb{Z}^d with x as first element of X for the lexicographic order of \mathbb{Z}^d .

The variational principle is stated in the next theorem where $E_{C(\Omega)}^G \subset E_{C(\Omega)}$ is the set of G -invariant states and where \mathcal{G}_Φ is the set of Gibbs states for the potential Φ .

Theorem 0.

1. For all $\mu \in E_{C(\Omega)}^G, s(\mu) - \frac{1}{|M_0|} \sum_{x \in M_0} \mu(A_\Phi^{(x)}) \leq P_\Phi$.
2. If $\mu \in E_{C(\Omega)}^G$, then

$$\mu \in \mathcal{G}_\Phi \Leftrightarrow s(\mu) - \frac{1}{|M_0|} \sum_{x \in M_0} \mu(A_\Phi^{(x)}) = P_\Phi. \quad \square$$

2. The Main Result

Introduce $P_k(N) = \{x \in \mathbb{Z}^d | x_1 = k; -N \leq x_i < N \text{ for } i = 2, \dots, d\}$, where $k \in \mathbb{Z}$ and $N \in \mathbb{N} \cup \{\infty\}$. Let $\mu_2^{(N)}$ be a state on $C_{P_0(N) \cup P_1(N)}$ and let $\mathbb{E}_1^{(N)}$ and $\mathbb{E}_1^{(N)}$ be the orthogonal projections in $L^2(\mu_2^{(N)}; \mathfrak{B}_{P_0(N) \cup P_1(N)})$ with ranges given respectively by $L^2(\mu_2^{(N)}; \mathfrak{B}_{P_0(N)})$ and $L^2(\mu_2^{(N)}; \mathfrak{B}_{P_1(N)})$. Put $\bar{\Lambda}^{(N)} = \{x \in \mathbb{Z}^d | -N \leq x_i < N \text{ for } i = 2, \dots, d\}$ and put $\tau_1 = \tau^{((1,0,\dots,0))}$ for $(1, 0, \dots, 0) \in \mathbb{Z}^d$.

Next we introduce two states $\bar{\mu}_{\text{alt}}^{(N)}$ and $\bar{\mu}_{\text{inv}}^{(N)}$ which are extensions of $\mu_2^{(N)}$ to $C_{\bar{\Lambda}^{(N)}}$. The state $\bar{\mu}_{\text{alt}}^{(N)}$ will be constructed such that it is reflectionally invariant. In fact $\bar{\mu}_{\text{alt}}^{(N)}$ is the unique state on $C_{\bar{\Lambda}^{(N)}}$ that is a reflectionally invariant extension of $\mu_2^{(N)}$ which has the Markov property with respect to $P_0(N)$ and $P_1(N)$. Formally, let $\bar{\mu}_{\text{alt}}^{(N)}$ be the state on $C_{\bar{\Lambda}^{(N)}}$ defined by

$$\begin{aligned} \bar{\mu}_{\text{alt}}^{(N)} \left(\prod_{k=-L}^L f_l \right) &= \int d\mu_2^{(N)} \{ [\mathbb{E}_1^{(N)}(\tilde{f}_{-1} \mathbb{E}_1^{(N)}(\tilde{f}_{-2} \cdots \mathbb{E}_{(-1)^{-l_1}}(\tilde{f}_{-L}) \cdots))] \\ &\quad \cdot f_0 f_1 [\mathbb{E}_1^{(N)}(\tilde{f}_2 \mathbb{E}_1^{(N)}(\tilde{f}_3 \cdots \mathbb{E}_{(-1)^{l_1}}(\tilde{f}_L) \cdots))] \} \end{aligned}$$

for $L \in \mathbb{N}$ and $f_l \in C_{P_l(N)}$; $l = -L, \dots, L$.

We have introduced $\tilde{f}_l = \tau_1^{-l+\delta-1,(-1)^l} f_l$ and also $-1 \uparrow = \downarrow$ and $+1 \uparrow = \uparrow$. In case $\mu_2^{(N)}$ satisfies $\mu_2^{(N)}(\tau_1 f) = \mu_2^{(N)}(f)$ for all $f \in C_{P_0(N)}$ we also introduce the state $\bar{\mu}_{\text{inv}}^{(N)}$ on $C_{\bar{\Lambda}^{(N)}}$. The state $\bar{\mu}_{\text{inv}}^{(N)}$ is the unique state on $C_{\bar{\Lambda}^{(N)}}$ that is a τ_1 -invariant extension of $\mu_2^{(N)}$ to $C_{\bar{\Lambda}^{(N)}}$ which has the Markov property with respect to $P_0(N)$ and is defined by

$$\bar{\mu}_{\text{inv}}^{(N)} \left(\prod_{l=-L}^L f_l \right) = \int d\mu_2^{(N)} \{ [\tau_1^{-1} \mathbb{E}_{\uparrow}^{(N)}(f'_{-1} \tau_1^{-1} \mathbb{E}_{\uparrow}^{(N)}(f'_{-2} \cdots \tau_1^{-1} \mathbb{E}_{\uparrow}^{(N)}(f'_{-L}) \cdots))] \cdot f_0 f_1 [\tau_1 \mathbb{E}_{\downarrow}^{(N)}(f'_2 \tau_1 \mathbb{E}_{\downarrow}^{(N)}(f'_3 \cdots \tau_1 \mathbb{E}_{\downarrow}^{(N)}(f'_L) \cdots))] \}$$

for $L \in \mathbb{N}$ and $f_l \in C_{P_l(N)}$; $l = -L, \dots, L$. We have introduced

$$f_l = \begin{cases} \tau_1^{-l} f_l & \text{when } l \leq 0 \\ \tau_1^{-l+1} f_l & \text{when } l > 0. \end{cases}$$

The method given above for constructing $\bar{\mu}_{\text{alt}}^{(N)}$ and $\bar{\mu}_{\text{inv}}^{(N)}$ relates to a method discussed in [2], see pages 122–125 and references given there. Let θ_k be the reflection of the lattice in the plane P_k . Thus θ_k maps $x \in \mathbb{Z}^d$ onto $\theta_k x \in \mathbb{Z}^d$, where $\theta_k x$ is given by $(\theta_k x)_1 = 2k - x_1$ and $(\theta_k x)_i = x_i$ for $i = 2, \dots, d$. A reflection θ_k acts in a natural way on $\Omega_{\bar{\Lambda}^{(N)}}$, $C_{\bar{\Lambda}^{(N)}}$ and on the space of states on $C_{\bar{\Lambda}^{(N)}}$. These actions are denoted by θ_k too. Note that θ_k is also well-defined for $k \in \mathbb{Z} + \frac{1}{2}$. The state $\bar{\mu}_{\text{alt}}^{(N)}$ is θ_k -invariant for all $k \in \mathbb{Z}$, while the state $\bar{\mu}_{\text{inv}}^{(N)}$ is τ_1 -invariant. Note that if $\mu_2^{(N)}$ is $\theta_{1/2}$ -invariant then $\bar{\mu}_{\text{alt}}^{(N)} = \bar{\mu}_{\text{inv}}^{(N)}$. If $\tau^{(a)} \mu^{(\infty)} = \mu_2^{(\infty)}$ for some $a \in \mathbb{Z}^d$ with $a_1 = 0$, then also $\tau^{(a)} \bar{\mu}_{\text{alt}}^{(\infty)} = \bar{\mu}_{\text{alt}}^{(\infty)}$ and $\tau^{(a)} \bar{\mu}_{\text{inv}}^{(\infty)} = \bar{\mu}_{\text{inv}}^{(\infty)}$. By construction the states $\bar{\mu}_{\text{alt}}^{(\infty)}$ and $\bar{\mu}_{\text{inv}}^{(\infty)}$ have the Markov property with respect to P_k for all $k \in \mathbb{Z}$. [For instance considering $\bar{\mu}_{\text{alt}}^{(\infty)}$ one readily verifies that $\mathbb{E}_{\bar{\mu}_{\text{alt}}^{(\infty)}}^{(f_0 f_1 \cdots f_n)} = f_0 \mathbb{E}_{\uparrow}^{(\infty)}(\tilde{f}_1 (\mathbb{E}_{\uparrow}^{(\infty)}(\tilde{f}_2 \cdots \mathbb{E}_{(-1)^n}(\tilde{f}_n) \cdots)))$, where $f_k \in C_{P_k}$ and $n \in \mathbb{N}$ and where the same definition of \tilde{f}_l has been used as above for $l = 1, \dots, n$.] For $n \in \mathbb{N}$ and $\bar{\mu}$ a state on $C_{\bar{\Lambda}^{(n)}}$ we denote by $\bar{\mu}^{\otimes}$ the product state on $C(\Omega)$ defined by

$$\bar{\mu}^{\otimes} = \prod_{(n_2, \dots, n_d) \in \mathbb{Z}^{d-1}}^{\otimes} \tau^{((0, 2n_2, \dots, 2n_d))} \bar{\mu}.$$

Consider the situation where $\mu_2^{(\infty)}$ is a state on $C_{P_0 \cup P_1}$ and $\mu_2^{(n)}$ is, for $n \in \mathbb{N}$, defined as the state on $C_{P_0(n) \cup P_1(n)}$ obtained by restricting $\mu_2^{(\infty)}$, i.e.,

$$\mu_2^{(n)} = \mu_2^{(\infty)} \upharpoonright_{C_{P_0(n) \cup P_1(n)}}.$$

Like above define the states $\{\bar{\mu}_{\text{alt}}^{(n)}\}_{n \in \mathbb{N}}$, $\bar{\mu}_{\text{alt}}^{(\infty)}$ and, assuming $\tau_1(\mu_2^{(\infty)} \upharpoonright_{C_{P_0}}) = \mu_2^{(\infty)} \upharpoonright_{C_{P_1}}$, also the states $\{\bar{\mu}_{\text{inv}}^{(n)}\}_{n \in \mathbb{N}}$ and $\bar{\mu}_{\text{inv}}^{(\infty)}$.

Lemma 1. *Let $n_0 \in \mathbb{N}$, $n_0 \geq 1$. Let H be the subgroup of \mathbb{Z}^d generated by $\{a^{(i)}\}_{i=2, \dots, d}$, where $a_j^{(i)} = n_0 \delta_{i,j}$. Assume $\mu_2^{(\infty)}$ is H -invariant.*

1) *Define for $k \in \mathbb{N}$, $k \geq 1$ the state $\rho_{\text{alt}}^{(k)}$ on $C(\Omega)$ by*

$$\rho_{\text{alt}}^{(k)} = \frac{1}{(2k)^{d-1}} \sum_{a \in H \cap P_0(kn_0)} \tau^{(a)} \bar{\mu}_{\text{alt}}^{(kn_0)\otimes}.$$

Then

$$\text{weak}^* \lim_{k \rightarrow \infty} \rho_{\text{alt}}^{(k)} = \bar{\mu}_{\text{alt}}^{(\infty)}.$$

2) Assume $\tau_1(\mu_2^{(\infty)} \upharpoonright_{C_{P_0}}) = \mu_2^{(\infty)} \upharpoonright_{C_{P_1}}$. Define for $k \in \mathbb{N}$, $k \geq 1$ the state $\rho_{\text{inv}}^{(k)}$ on $C(\Omega)$ by

$$\rho_{\text{inv}}^{(k)} = \frac{1}{(2k)^{d-1}} \sum_{a \in H \cap P_0(kn_0)} \tau^{(a)} \bar{\mu}_{\text{inv}}^{(kn_0) \otimes}.$$

Then

$$\text{weak}^* \lim_{k \rightarrow \infty} \rho_{\text{inv}}^{(k)} = \bar{\mu}_{\text{inv}}^{(\infty)}.$$

Proof. The proof of 2) parallels the proof of 1). We therefore only present the proof of 1).

1) Take $f = f_0 f_1, \dots, f_L$ with $f_l \in C_{P_l(N)}$ for $l = 0, \dots, L$ and some $N < \infty$. It is sufficient to prove

$$\lim_{k \rightarrow \infty} \rho_{\text{alt}}^{(k)}(f) = \bar{\mu}_{\text{alt}}^{(\infty)}(f).$$

Choose $\varepsilon: 0 < \varepsilon \in \mathbb{R}$.

Since the monotone sequences of projections $\{\mathbb{E}_{\pm \uparrow}^{(n)}\}_{n \in \mathbb{N}}$ converge strongly to $\mathbb{E}_{\pm \uparrow}^{(\infty)}$ there exists $\tilde{m} \in \mathbb{N}$ such that for $m \geq \tilde{m}$,

$$\|(\mathbb{E}_{(-1) \uparrow}^{(m)} - \mathbb{E}_{(-1) \uparrow}^{(\infty)})(\tilde{f}_l \mathbb{E}_{(-1) \uparrow}^{(\infty)l+1}(\tilde{f}_{l+1} \cdots \tilde{f}_{L-1} \mathbb{E}_{(-1) \uparrow}^{(\infty)L}(\tilde{f}_L) \cdots))\|_{L^2} \leq \varepsilon$$

for all $l = 2, \dots, L$. Thus for $a \in H$ and $m \geq \tilde{m}$,

$$\|(\tau^{(a)} \circ \mathbb{E}_{(-1) \uparrow}^{(m)} \circ \tau^{(-a)} - \mathbb{E}_{(-1) \uparrow}^{(\infty)})(\tau^{(a)} \tilde{f}_l \mathbb{E}_{(-1) \uparrow}^{(\infty)l+1}(\cdots(\tau^{(a)} \tilde{f}_{L-1}) \mathbb{E}_{(-1) \uparrow}^{(\infty)L}(\tau^{(a)} \tilde{f}_L) \cdots))\|_{L^2} \leq \varepsilon,$$

where we used $\mathbb{E}_{(-1) \uparrow}^{(\infty)} \circ \tau^{(a)} = \tau^{(a)} \circ \mathbb{E}_{(-1) \uparrow}^{(\infty)}$ and the fact that H acts isometrically on the L^2 space.

Both of these last properties are consequences of the H -invariance of $\mu_2^{(\infty)}$. If $\tau^{(a)} P_0(m) \subseteq P_0(n)$ then,

$$\tau^{(a)} \circ \mathbb{E}_{(-1) \uparrow}^{(m)} \circ \tau^{(-a)} \leq \mathbb{E}_{(-1) \uparrow}^{(n)}.$$

Hence, for $a \in X_H(\tilde{m}; n)$ where $X_H(\tilde{m}; n) = \{a \in H \mid \tau^{(a)} P_0(\tilde{m}) \subseteq P_0(n)\}$,

$$\begin{aligned} & \|(\tau^{(a)} f_0)(\tau^{(a)} f_1) \mathbb{E}_{\uparrow}^{(n)}((\tau^{(a)} \tilde{f}_2) \mathbb{E}_{\downarrow}^{(n)}((\tau^{(a)} f_3) \cdots (\tau^{(a)} \tilde{f}_{l-1}) (\mathbb{E}_{(-1) \uparrow}^{(n)} - \mathbb{E}_{(-1) \uparrow}^{(\infty)})) \\ & \cdot ((\tau^{(a)} \tilde{f}_l) \mathbb{E}_{(-1) \uparrow}^{(\infty)l+1}(\cdots(\tau^{(a)} \tilde{f}_{L-1}) \mathbb{E}_{(-1) \uparrow}^{(\infty)L}(\tau^{(a)} \tilde{f}_L) \cdots)) \cdots))\|_{L^2} \leq \varepsilon \prod_{k=0}^{l-1} \|f_k\|_{\infty} \end{aligned}$$

for $l = 2, \dots, d$. For $a \in X_H(\tilde{m}; n)$ one thus readily deduces,

$$\begin{aligned} & \|(\tau^{(a)} f_0)(\tau^{(a)} f_1) \mathbb{E}_{\uparrow}^{(n)}((\tau^{(a)} \tilde{f}_2) \mathbb{E}_{\downarrow}^{(n)}(\cdots \mathbb{E}_{(-1) \uparrow}^{(n)L}(\tau^{(a)} \tilde{f}_L) \cdots)) \\ & - (\tau^{(a)} f_0)(\tau^{(a)} f_1) \mathbb{E}_{\uparrow}^{(\infty)}((\tau^{(a)} \tilde{f}_2) \mathbb{E}_{\downarrow}^{(\infty)}(\cdots \mathbb{E}_{(-1) \uparrow}^{(\infty)L}(\tau^{(a)} \tilde{f}_L) \cdots))\|_{L^2} \\ & \leq \|f_0\|_{\infty} \|f_1\|_{\infty} \left[\varepsilon + \varepsilon \sum_{l=3}^L \prod_{k=2}^{l-1} \|f_k\|_{\infty} \right] \\ & \equiv \varepsilon C_{f_0, \dots, f_L}. \end{aligned}$$

From the definitions of $\bar{\mu}_{\text{alt}}^{(n) \otimes}$ and $\bar{\mu}_{\text{alt}}^{(\infty)}$ it follows therefore

$$|\bar{\mu}_{\text{alt}}^{(n) \otimes}(\tau^{(a)} f) - \bar{\mu}_{\text{alt}}^{(\infty)}(\tau^{(a)} f)| \leq \varepsilon C_{f_0, \dots, f_L}$$

whenever $a \in X_H(\tilde{m}; n)$ and $n \geq N$. Consider the estimate

$$\begin{aligned} |\rho_{\text{alt}}^{(k)}(f) - \bar{\mu}_{\text{alt}}^{(\infty)}(f)| &\leq \frac{1}{(2k)^{d-1}} \sum_{-a \in H \cap P_0(kn_0) \cap X_H(\tilde{m}; kn_0)^c} |(\tau^{(a)} \bar{\mu}_{\text{alt}}^{(n)\otimes})(f) - \bar{\mu}_{\text{alt}}^{(\infty)}(f)| \\ &\quad + \frac{1}{(2k)^{d-1}} \sum_{-a \in X_H(\tilde{m}; kn_0)} |(\tau^{(a)} \bar{\mu}_{\text{alt}}^{(n)\otimes})(f) - \bar{\mu}_{\text{alt}}^{(\infty)}(f)| \\ &\leq \frac{1}{(2k)^{d-1}} \sum_{-a \in H \cap P_0(kn_0) \cap X_H(\tilde{m}; kn_0)^c} 2 \|f\|_{\infty} \\ &\quad + \frac{1}{(2k)^{d-1}} \sum_{-a \in X_H(\tilde{m}; kn_0)} \varepsilon C_{f_0, \dots, f_L}. \end{aligned}$$

The first sum in the last upper bound is a sum over a number of terms bounded by a constant times $(2k)^{d-2}$.

Hence,

$$\limsup_{k \rightarrow \infty} |\rho_{\text{alt}}^{(k)}(f) - \bar{\mu}_{\text{alt}}^{(\infty)}(f)| \leq \varepsilon C_{f_0, \dots, f_L},$$

and the lemma follows. \square

For $m, L, K \in \mathbb{N}$ define $A(m; L, K) \subset \mathbb{Z}^d$ as

$$A(m; L, K) = \{x \in \mathbb{Z}^d \mid -L \leq x_1 \leq K+1 \text{ and } -m \leq x_i < m \text{ for } i=2, \dots, d\}.$$

Let M_{+1} be the Radon–Nikodym derivative of the restriction of $\bar{\mu}_{\text{alt}}^{(n)}$ (= the restriction of μ) to $C_{A^{(n)}}$ with respect to $\mu_0^{(A^{(n)})}$. Define $M_{-1} = \theta_{1/2} M_{+1}$. Furthermore define

$$\underline{M}_{+1} = \int_{\Omega_{P_1(n)}} d\mu_0^{(P_1(n))} M_{+1},$$

and

$$\underline{M}_{-1} = \int_{\Omega_{P_1(n)}} d\mu_0^{(P_1(n))} M_{-1}.$$

Then the Radon–Nikodym derivative, $\bar{\mu}^{(m)(A(m;L,K))}$ of the restriction of $\bar{\mu}_{\text{alt}}^{(n)}$ to $C_{A(m;L,K)}$ with respect to $\mu_0^{(A(m;L,K))}$ is given by

$$\bar{\mu}_{\text{alt}}^{(n)(A(m;L,K))} = \frac{\tau_1^{-L} \underline{M}_{(-1)^{-L}} \dots \tau_1^{-1} M_{-1}}{\tau_1^{-L+1} \underline{M}_{(-1)^{-L+1}} \dots \underline{M}_{+1}} \cdot M_{+1} \cdot \frac{\tau_1 M_{-1}}{\tau_1 \underline{M}_{-1}} \dots \frac{\tau_1^K M_{(-1)^K}}{\tau_1^K \underline{M}_{(-1)^K}}.$$

(Note that $M_{+1}, M_{-1}, \underline{M}_{+1}$, and \underline{M}_{-1} can be assumed to be continuous functions in case $\mu_2^{(\infty)}$ is the restriction of a Gibbs state of $\Lambda_2^{(\infty)}$.) Then

$$S_{\Lambda(m;L,L)}(\bar{\mu}_{\text{alt}}^{(m)}) = (2L+1)S_{P_0(m)} \cup P_1(m)(\mu_2^{(\infty)}) - LS_{P_0(m)}(\mu_2^{(\infty)}) - LS_{P_1(m)}(\mu_2^{(\infty)}). \quad (*)$$

Reasoning similarly one also obtains

$$S_{\Lambda(m;L,L)}(\bar{\mu}_{\text{inv}}^{(m)}) = (2L+1)S_{P_0(m)} \cup P_1(m)(\mu_2^{(\infty)}) - 2LS_{P_0(m)}(\mu_2^{(\infty)}).$$

Lemma 2

a) Consider the situation in 1) of Lemma 1. Then

$$s(\bar{\mu}_{\text{alt}}^{(\infty)}) = \lim_{n \rightarrow \infty} \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_0(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_1(n)}(\mu_2^{(\infty)})].$$

b) Consider the situation in 2) of Lemma 1. Then

$$s(\bar{\mu}_{\text{inv}}^{(\infty)}) = \lim_{n \rightarrow \infty} \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - S_{P_0(n)}(\mu_2^{(\infty)})].$$

Proof. The proof of b) is analogous to the proof of a).

We present the proof of a).

a). By Lemma 1 one obtains, using upper-semicontinuity of the entropy,

$$s(\bar{\mu}_{\text{alt}}^{(\infty)}) \geq \limsup_{k \rightarrow \infty} s(\rho_{\text{alt}}^{(k)}) = \limsup_{k \rightarrow \infty} s(\bar{\mu}_{\text{alt}}^{(kn_0)}).$$

From (*) for $n \in \mathbb{N}$, $n \geq 1$,

$$s(\bar{\mu}_{\text{alt}}^{(n)}) = \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_0(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_1(n)}(\mu_2^{(\infty)})].$$

Therefore

$$\begin{aligned} s(\bar{\mu}_{\text{alt}}^{(\infty)}) &\geq \limsup_{k \rightarrow \infty} \frac{1}{(2kn_0)^{d-1}} [S_{P_0(kn_0) \cup P_1(kn_0)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_0(kn_0)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_1(kn_0)}(\mu_2^{(\infty)})] \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_0(n)}(\mu_2^{(\infty)}) - \frac{1}{2}S_{P_1(n)}(\mu_2^{(\infty)})]. \end{aligned}$$

On the other hand, by repeated use of the strong subadditivity property of the entropy,

$$\begin{aligned} S_{\Lambda(l_0, l, l)}(\bar{\mu}_{\text{alt}}^{(\infty)}) &\leq S_{\Lambda(l_0, l, l-1)}(\bar{\mu}_{\text{alt}}^{(\infty)}) + S_{P_l(l_0) \cup P_{l-1}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - S_{P_l(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) \\ &\leq S_{\Lambda(l_0, l, l-2)}(\bar{\mu}_{\text{alt}}^{(\infty)}) + S_{P_{l-1}(l_0) \cup P_{l-2}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) \\ &\quad - S_{P_{l-1}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) + S_{P_l(l_0) \cup P_{l-1}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - S_{P_l(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) \\ &\leq \dots \dots \dots \\ &\leq \dots \dots \dots \\ &\quad \vdots \\ &\leq (2l + 1)S_{P_0(l_0) \cup P_1(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - lS_{P_0(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - lS_{P_1(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}). \end{aligned}$$

We used that by θ_k -invariance of $\bar{\mu}_{\text{alt}}^{(\infty)}$ for all $k \in \mathbb{Z}$,

$$S_{P_k(l_0) \cup P_{k+1}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) = S_{P_0(l_0) \cup P_1(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)})$$

and

$$S_{P_{2k}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) = S_{P_0(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)})$$

and

$$S_{P_{2k+1}(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) = S_{P_1(l_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}).$$

Therefore,

$$\begin{aligned} s(\bar{\mu}_{\text{alt}}^{(\infty)}) &= \lim_{l \rightarrow \infty} \left[\frac{1}{2(l+1)(2ln_0)^{d-1}} S_{\Lambda(n_0, l)}(\bar{\mu}_{\text{alt}}^{(\infty)}) \right] \\ &\leq \lim_{l \rightarrow \infty} \frac{1}{(2ln_0)^{d-1}} [S_{P_0(n_0) \cup P_1(n_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - \frac{1}{2} S_{P_0(n_0)}(\bar{\mu}_{\text{alt}}^{(\infty)}) - \frac{1}{2} S_{P_1(n_0)}(\bar{\mu}_{\text{alt}}^{(\infty)})] \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - \frac{1}{2} S_{P_0(n)}(\mu_2^{(\infty)}) - \frac{1}{2} S_{P_1(n)}(\mu_2^{(\infty)})]. \quad \square \end{aligned}$$

Consider now an interaction Φ like in the introductory section, i.e., Φ is nonzero only round faces of \mathbb{Z}^d and invariant under a group G with finite index in \mathbb{Z}^d .

Theorem 1. *Let μ be a G -invariant Gibbs state for the interaction Φ , where $|\mathbb{Z}^d/G| < \infty$. Put*

$$\mu_2^{(\infty)} = \mu \upharpoonright_{C_{P_0 \cup P_1}}.$$

- 1) *Assume that Φ and μ are θ_k -invariant for all $k \in \mathbb{Z}$. Construct the state $\bar{\mu}_{\text{alt}}^{(\infty)}$ from $\mu_2^{(\infty)}$. Then $\bar{\mu}_{\text{alt}}^{(\infty)}$ is a Gibbs state for Φ .*
- 2) *Assume that Φ and μ are invariant for translation over $(k, 0, \dots, 0) \in \mathbb{Z}^d$ for all $k \in \mathbb{Z}$. Construct $\bar{\mu}_{\text{inv}}^{(\infty)}$ from $\mu_2^{(\infty)}$. Then $\bar{\mu}_{\text{inv}}^{(\infty)}$ is a Gibbs state for Φ .*

Proof. Since G has finite index in \mathbb{Z}^d there exists $n_0 \geq 1$, $n_0 \in \mathbb{N}$ such that $\{a^{(i)}\}_{i=2, \dots, d} \subset G$, where $a_j^{(i)} = n_0 \delta_{i,j}$. Let H be the subgroup of G generated by $\{a^{(i)}\}_{i=2, \dots, d}$.

- 1) By the same reasoning as in the last part of the proof of Lemma 2,

$$s(\mu) \leq \lim_{n \rightarrow \infty} \frac{1}{(2n)^{d-1}} [S_{P_0(n) \cup P_1(n)}(\mu_2^{(\infty)}) - \frac{1}{2} S_{P_0(n)}(\mu_2^{(\infty)}) - \frac{1}{2} S_{P_1(n)}(\mu_2^{(\infty)})].$$

Thus $s(\mu) \leq s(\mu_{\text{alt}}^{(\infty)})$. Furthermore, for pairs (x, X) with $x \in P_0 \cup P_1$ and $X \subset \mathbb{Z}^d$ such that $x \in X$ is the first element of X (in the lexicographic order of \mathbb{Z}^d) and $\Phi \circ \alpha_x \neq 0$, we define $\hat{X}(x, X) \subset \mathbb{Z}^d$ by

- a) if $X \subset P_0 \cup P_1$, then $\hat{X}(x, X) = X$,
- b) if $X \not\subset P_0 \cup P_1$, then $\hat{X}(x, X) = \theta_1 X$.

Thus always $\hat{X}(x, X) \subset P_0 \cup P_1$. Then, since the states μ and $\bar{\mu}_{\text{alt}}^{(\infty)}$ as well as the potential Φ are invariant under the reflection θ_1 ,

$$\mu(A_{\Phi}^{(x)}) = \mu \left(\sum_X^{(x)} \Phi \circ \alpha_{\hat{X}(x, X)} \right)$$

and

$$\bar{\mu}_{\text{alt}}^{(\infty)}(A_{\Phi}^{(x)}) = \bar{\mu}_{\text{alt}}^{(\infty)} \left(\sum_X^{(x)} \Phi \circ \alpha_{\hat{X}(x, X)} \right)$$

for $x \in P_0 \cup P_1$. But

$$\mu(\Phi \circ \alpha_{\hat{X}(x, X)}) = \bar{\mu}_{\text{alt}}^{(\infty)}(\Phi \circ \alpha_{\hat{X}(x, X)})$$

for $x \in P_0 \cup P_1$. Therefore, if $x \in P_0 \cup P_1$,

$$\mu(A_\Phi^{(x)}) = \bar{\mu}_{\text{alt}}^{(\infty)}(A_\Phi^{(x)}).$$

Next note that μ and $\bar{\mu}_{\text{alt}}^{(\infty)}$ as well as Φ are invariant under the group generated by H and $(2, 0, \dots, 0) \in \mathbb{Z}^d$.

An application of Theorem 0 for this last group (with $M_0 = P_0(n_0) \cup P_1(n_0)$) gives, from the inequality

$$s(\bar{\mu}_{\text{alt}}^{(\infty)}) - \frac{1}{(2n_0)^{d-1}} \sum_{x \in P_0(n_0) \cup P_1(n_0)} \bar{\mu}_{\text{alt}}^{(\infty)}(A_\Phi^{(x)}) \geq s(\mu) - \frac{1}{(2n_0)^{d-1}} \sum_{x \in P_0(n_0) \cup P_1(n_0)} \mu(A_\Phi^{(x)}),$$

that $s(\bar{\mu}_{\text{alt}}^{(\infty)}) = s(\mu)$ and that $\bar{\mu}_{\text{alt}}^{(\infty)}$ is a Gibbs state for Φ .

2) The proof of 2) is analogous to the proof of 1). \square

Corollary 1.

1) Consider the situation described in Theorem 1.1). Let $G_0^{(2)}$ be the group generated by $G \cap P_0$ and $(2, 0, \dots, 0) \in \mathbb{Z}^d$. Assume Φ is θ_k -invariant for all $k \in \mathbb{Z}$. Consider the condition C_1 :

$$C_1 \left\{ \begin{array}{l} \text{If the Gibbs states } \mu_1 \text{ and } \mu_2 \text{ for the potential } \Phi \text{ are both } G_0^{(2)}\text{-and} \\ \theta_k\text{-invariant for all } k \in \mathbb{Z} \text{ and } \mu_1 \upharpoonright_{C_{P_0 \cup P_1}} = \mu_2 \upharpoonright_{C_{P_0 \cup P_1}}, \text{ then } \mu_1 = \mu_2. \end{array} \right.$$

If C_1 holds then every $G_0^{(2)}$ - and θ_k -invariant (for all $k \in \mathbb{Z}$) Gibbs state for the potential Φ has the Markov property with respect to P_k for all $k \in \mathbb{Z}$.

2) Consider the situation described in Theorem 1.2). Assume $(1, 0, \dots, 0) \in G$. Consider the condition C_2 :

$$C_2 \left\{ \begin{array}{l} \text{If the Gibbs states } \mu_1 \text{ and } \mu_2 \text{ for the potential } \Phi \text{ are both } G\text{-invariant and} \\ \mu_1 \upharpoonright_{C_{P_0 \cup P_1}} = \mu_2 \upharpoonright_{C_{P_0 \cup P_1}}, \text{ then } \mu_1 = \mu_2. \end{array} \right.$$

If C_2 holds then every G -invariant Gibbs state has the Markov property with respect to P_k for all $k \in \mathbb{Z}$. \square

Note that Corollary 1 implies the following. If the states determining equilibrium are recognizable by observations on a plaquette (and have some invariances), Theorem 1 implies that Markov properties hold with respect to P_k for all $k \in \mathbb{Z}$. A particular case of such situations occurs for models with a phase diagram described by the expectation value of the spin on a lattice site (magnetization).

Note. In a forthcoming paper we show that for models with nearest-neighbor interactions (and with some periodicity properties) leading to an invertible transfer matrix (for instance the q -state Potts model for nontrivial coupling) the conditions stated in Corollary 1 are met.

In two dimensions a further relation between the states μ and $\bar{\mu}_{\text{alt}}^{(\infty)}$ (or μ and $\bar{\mu}_{\text{inv}}^{(\infty)}$) can be given which shows that these states do not become disjoint over finite distances in the 1-direction. We will make use of the concept of relative entropy of states. When μ and ν are probability measures on Ω the relative entropy $S(\mu|\nu)$ is defined by

$$S(\mu|\nu) = - \int \frac{d\mu}{d\nu} \ln \left(\frac{d\mu}{d\nu} \right) d\nu$$

if $\mu \ll \nu$ and the Radon–Nikodym derivative of μ with respect to ν , $(d\mu/d\nu)$, is such that $(d\mu/d\nu)\ln(d\mu/d\nu)$ is ν -integrable. Otherwise $S(\mu|\nu) = -\infty$ by definition. If $\{v_n\}_{n \in \mathbb{N}}$ is a sequence of states converging to the weak* topology to a state ν and μ is a state such that $S(\mu|v_n) \geq -k$ for all $n \in \mathbb{N}$, where $k \geq 0$ is some finite constant, then $\mu \ll \nu$. Application of this result together with some more details can be found in [7].

Theorem 2. *Let the lattice be two dimensional, i.e., \mathbb{Z}^2 . We use notation introduced in the proof of Lemma 2.*

1) *Consider the situation described in Theorem 1.1). Then*

$$\mu \uparrow_{C_{A(\infty;L,L)}} \ll \bar{\mu}_{\text{alt}}^{(\infty)} \uparrow_{C_{A(\infty;L,L)}}$$

for all $L \in \mathbb{N}$.

2) *Consider the situation described in Theorem 1.2). Then*

$$\mu \uparrow_{C_{A(\infty;L,L)}} \ll \bar{\mu}_{\text{inv}}^{(\infty)} \uparrow_{C_{A(\infty;L,L)}}$$

for all $L \in \mathbb{N}$.

Proof. We only present the proof of 1) since the proof 2) is analogous. 1) introduce $n_0: 1 \leq n_0 \in \mathbb{N}$ like in the proof of Theorem 1. Let $\mathbb{E}_k: L^2(\mu; \mathfrak{B}_{A(\infty;L,L)}) \rightarrow L^2(\mu; \mathfrak{B}_{A(kn_0;L,L)})$ be the orthogonal projection for $k \in \mathbb{N}$.

Note that the restrictions of two Gibbs states to a finite volume are mutually absolutely continuous. Thus the following definition of the state $\tilde{\mu}_k$ on $C_{A(\infty;L,L)}$ is justified:

$$\tilde{\mu}_k(f) = \int d\bar{\mu}_{\text{alt}}^{(kn_0)} \mathbb{E}_k(f)$$

for $f \in C_{A(\infty;L,L)}$. The Radon–Nikodym derivative h_k of $\mu \uparrow_{C_{A(\infty;L,L)}}$ with respect to $\tilde{\mu}_k$ is given by

$$h_k = \frac{\mu^{(A(kn_0;L,L))}}{\bar{\mu}_{\text{alt}}^{(kn_0)^{(A(kn_0;L,L))}}.$$

Therefore

$$\begin{aligned} S(\mu \uparrow_{C_{A(\infty;L,L)}} | \tilde{\mu}_k) &= - \int d\tilde{\mu}_k h_k \ln h_k = - \int d\mu \ln h_k \\ &= - \int d\mu [\ln(\mu^{(A(kn_0;L,L))}) - \ln(\bar{\mu}_{\text{alt}}^{(kn_0)^{(A(kn_0;L,L))})}]. \end{aligned}$$

From the product-type structure of the density of $\bar{\mu}_{\text{alt}}^{(kn_0)} \uparrow_{\mathfrak{B}_{A(kn_0;L,L)}}$ with respect to $\mu_0^{(A(kn_0;L,L))}$, like already used in the proof of Lemma 2 for calculating entropies, and the fact that μ coincides with $\bar{\mu}_{\text{alt}}^{(kn_0)}$ on $C_{P_{l(kn_0)} \cup P_{l+1(kn_0)}}$, one obtains

$$\int d\mu \ln(\bar{\mu}_{\text{alt}}^{(kn_0)^{(A(kn_0;L,L))}}) = \int d\mu_{\text{alt}}^{(kn_0)} \ln(\bar{\mu}_{\text{alt}}^{(kn_0)^{(A(kn_0;L,L))}}).$$

Hence,

$$\begin{aligned} S(\mu |_{C_{A(\infty;L,L)}} | \tilde{\mu}_k) &= S_{A(kn_0;L,L)}(\mu) - S_{A(kn_0;L,L)}(\bar{\mu}_{\text{alt}}^{(kn_0)}) \\ &= 4kn_0 \sum_{l=0}^{L-2} [b_l^{(kn_0)}(\mu) - b_l^{(kn_0)}(\bar{\mu}_{\text{alt}}^{(kn_0)})], \end{aligned}$$

where

$$b_l^{(kn_0)}(\cdot) = \frac{1}{4kn_0} [S_{\Lambda(kn_0, L, L-2l)}(\cdot) - S_{\Lambda(kn_0; L, L-2(l+1))}(\cdot)].$$

On the other hand

$$s(\bar{\mu}_{\text{alt}}^{(kn_0)\otimes}) = \frac{1}{2kn_0} [S_{P_0(kn_0) \cup P_1(kn_0)}(\mu) - \frac{1}{2}S_{P_0(kn_0)}(\mu) - \frac{1}{2}S_{P_1(kn_0)}(\mu)] = b_l^{(kn_0)}(\bar{\mu}_{\text{alt}}^{(kn_0)})$$

for $l = 0, \dots, L - 2$.

The variational principle implies

$$b_l^{(kn_0)}(\bar{\mu}_{\text{alt}}^{(kn_0)}) - \frac{1}{4kn_0} \sum_{x \in P_0(kn_0) \cup P_1(kn_0)} \bar{\mu}_{\text{alt}}^{(kn_0)\otimes}(A_\Phi^{(x)}) \leq s(\mu) - \frac{1}{4kn_0} \sum_{x \in P_0(kn_0) \cup P_1(kn_0)} \mu(A_\Phi^{(x)}).$$

Since the number of $x \in P_0(kn_0) \cup P_1(kn_0)$, such that there exists $X \subset \mathbb{Z}^d$ with Φ . $\alpha_X \neq 0$, X has x as first element in the lexicographic order of \mathbb{Z}^d , and $X \notin \Lambda^{(kn_0)}$, is uniformly bounded

$$s(\mu) - b_l^{(kn_0)}(\bar{\mu}_{\text{alt}}^{(kn_0)}) \geq \frac{-K}{4kn_0},$$

where $K \geq 0$ is a constant independent of k . Thus

$$S(\mu \upharpoonright_{C_{\Lambda(\infty; L, L)}} | \tilde{\mu}_k) \geq 4kn_0 \sum_{l=0}^{L-2} [b_l^{(kn_0)}(\mu) - s(\mu)] - K.$$

Moreover

$$b_l^{(kn_0)}(\mu) \geq s(\mu).$$

One obtains

$$S(\mu \upharpoonright_{C_{\Lambda(\infty; L, L)}} | \tilde{\mu}_k) \geq -K.$$

Theorem 2 1) therefore follows. \square

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