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Group Theoretic Approach to the Open Bosonic String Multi-Loop S-Matrix

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Abstract. The new approach to string scattering proposed by the authors is generalized to include multi-loop contributions. As an example, the planar one-loop contribution, including its integration measure, to the open bosonic string S-matrix is computed. The external state dependence for any multi-loop contribution is computed and found to be determined by one group theoretic function which is derived.

1. Introduction

In a recent paper [1], hereafter referred to as I, a new approach to string theory was given. This method relies on the observation that the remarkable simplicity of string scattering amplitudes is a consequence of duality, overlap conditions and unitarity.

The principal character in this approach is the vertex which depends on the moduli, the actual scattering amplitude being obtained by integrating over the moduli with a suitable measure. By moduli we mean the external moduli, i.e., Koba-Nielsen co-ordinates [2] and the parameters associated with loops. As we shall see these arise in connection with the duality properties of the vertex. In a Feynman graph type of approach [3], which the method discussed here is not, the latter parameters arise from the usual parametric form of the propagator used in string theory [4]. While in the sum over Riemann surface approach [5], they arise as the Teichmüller parameters [6, 7].

The duality property well known in string theory states that one can permute the legs of an amplitude and the result is the same provided that one maintains their cyclic order [8] at least for the case of the open string. In the approach advocated here, the vertex can have its dependence on the external moduli cycled by the application of appropriate conformal transformations on all its legs. There also exists, for each loop, a cyclic transformation which leaves the vertex inert and

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corresponds to going around the given loop. In fact, the moduli of the vertex are really the parameters of the cycling transformations which characterize the vertex. A general conformal transformation when applied to the vertex will change the values of the moduli on which the vertex depends. Indeed the behaviour under the cycling transformations allows one to compute the specific conformal transformation which changes the moduli in a given way. Such a change can only be achieved by a conformal transformation that contains constant terms or poles. This is analogous to the procedure of boosts in the theory of induced representations, the rôle of the little group being played by the transformations which leave the moduli alone.

A convenient representation of the vertex is in an oscillator basis. In this case, the conformal transformation is implemented by the L_n's, and one that changes the moduli is a transformation which involves L_n, $n \leqq 0$, as well as L_n, $n > 0$.

It is the knowledge of this moduli changing transformation and the enforcement of unitarity that allows for a deduction of the measure with which the vertex is integrated to yield the actual scattering amplitude. Demanding unitarity implies that if physical external states are applied to the actual scattering vertex and only one of which is spurious $(L_{-n}|\Omega > n \geqq 1)$ then the result is zero. For physical states L_n, $n > 1$ vanish and $L_0 = 1$. Consequently, using an infinitesimal moduli changing transformation one finds, after integration by parts, a relation between the derivative of the measure with respect to the moduli and a known function of the moduli. This computation of the measure was explicitly demonstrated for the case of the open bosonic trees in I and the planar one-loop tadpole in [9] hereafter called II. The reader will have realized that, as the approach is on-shell, no ghosts can be used and indeed none are required to find the correct result.

An example of the overlap condition is provided by a relation which expresses the momentum density on one leg in terms of its action on the other legs. The cycling transformations, since they rotate the legs, place severe constraints on the form of such identities and essentially determine them once their generic form is known. Corresponding to the cycling transformations for a given loop, one can find relations between, say, the momentum density on the same line but at different points on the string. As the operators which occur in the overlap identities involve both creation and annihilation operators, these identities lead to a very straightforward derivation of the external dependence, i.e., oscillator and momentum, of the vertex.

Overlap considerations outside string field theory have not been extensively used before. In string field theory, the interaction is an overlap δ function, and as a result, one finds overlap identities relating operators on different legs. In fact, these identities provide the quickest way to derive the oscillator form of the string field theory vertex. As is well known, one can map any open string light-cone diagram into a region of the upper half plane by an analytic conformal mapping. By applying this mapping to the light-cone overlap identities, one arrives at identities for vertices defined on the upper half-plane. Since the upper half-plane is only preserved by $SL(2, R)$, it is this group which plays the most important rôle. Consequently, from the light-cone point of view, one sees that there always exist string overlaps of the type required in this paper. From the old dual model approach, this is also obvious, as both building blocks of a dual diagram, namely the Caneschi-Schwimmer-Veneziano vertex and the propagator satisfy overlap identities involving $SL(2, R)$ transformations. Hence the new results which we derive in this paper can only agree with those that more standard oscillator algebra would provide, if it could be carried through. Factorization could also be proved directly within the framework of this paper, by examining the cycling transformations which arise when two lines are glued together and showing that these are indeed the cycling transformations associated to the new graph (up to gauge transformations on the external lines). Of course, it is our hope that the method advocated here will provide not only a faster calculational tool, but also a firmer conceptual base from which to understand string theory.

In Sect. 2 we give the general strategy for computing multi-loop contributions. Section 3 proves the Q^{μ} overlap condition for the open bosonic trees and accumulates some of the techniques necessary for the multi-loop case. The one-loop planar contribution to the S-matrix for the open bosonic string is derived in Sect. 4 and the external dependence of the multi-loop is given in Sect. 5.

2. General Multi-Loop Strategy [1]

In I the assumptions required to find the string S-matrix were given within the context of the open bosonic trees. It was apparent, however, that such a strategy was also applicable to loop contributions to the S-matrix, and we now give the generalizations required for loop contributions. The central object in our consideration is a vertex V which depends on parameters z_i, $i=1, \ldots, N$, corresponding to the N external strings and sets of parameters v_r, $r=1, \ldots, M$, corresponding to the M loops. The number of parameters depends on the type of strings being considered. For the open and closed bosonic strings, for example, the z_i are real and complex, respectively, and each v_r denotes three real and three complex parameters respectively for each loop.

The most useful representation of the vertex for current purposes is in the oscillator basis, namely it will be of the generic form

$$V(z_i, v_r) = \langle {}_1 0| \ldots \langle {}_N 0| \, h(\alpha_m^{\mu(r)}; z_i, v_r). \tag{2.1}$$

The on-shell scattering amplitude is obtained by integrating the above vertex over the parameters, i.e.,

$$W = \int \prod_i dz_i \prod_r dv_r \, \bar{f}(z_i, v_r) \, V(z_i, v_r), \tag{2.2}$$

where the method also determines the function $\bar{f}$.

The on-shell scattering amplitude follows from the assumptions.

A1) There exist cycling transformations associated with cycling the external legs and going around each loop. The cycling of the external legs is achieved by

$$V(z_1, \ldots, z_N, v_r) \prod_{k=1}^{N} (T_{k+1}^{(k)})^{-1} = V(z_2, \ldots, z_N, z_1, v_r). \tag{2.3}$$

The T's obey the equation

$$T_N \ldots T_2 T_1 = 1, \tag{2.4}$$

and T_j is obtained from T_{j-1} by cycling its dependence on the z's.

For each loop, the vertex satisfies a relation of the form

$$V(z_j, v_r) P_n^{(i)} = V(z_j, v_r), \qquad n=1, \ldots, M. \tag{2.5}$$

The $P_n^{(i)}$ differ, for a given n, from line to line and this is labelled by the upper index. As we shall see, the v_n are in fact parameters of the conformal transformation P_n^i.

A 2) The vertex obeys overlap identities. The one corresponding to cycling the external legs has the generic form

$$V(z_i, v_r) \left\{ \sum_{i=1}^{N} A_i(z) R^{(i)}(T_i T_{i-1} \ldots T_1(z)) \right\} = 0, \tag{2.6}$$

subject to one condition on the A_i's which is of the form

$$\sum_{i=1}^{N} \frac{A_i(z)}{f_i(z)} = 0. \tag{2.7}$$

The f_i are determined once the conformal weight of R is known.

For each loop we have an overlap identity of the form

$$V(z_i, v_r) \left\{ \sum_m B_m^{(i)} R^{(i)}((P_n^{(i)})^m (z)) \right\} = 0, \tag{2.8}$$

where the B_m are also subject to one condition

$$\sum_m \frac{B_m^{(i)}(z)}{g_m^{(i)}(z)} = 0. \tag{2.9}$$

A 3) The theory is unitary.

Following the same arguments as in I one finds that

$$f_i(z) = S_{T_i \ldots T_3}(T_2 T_1 z), f_2 = 1, f_1 = \frac{1}{S_{T_2}(T_1 z)}, \tag{2.10}$$

and

$$g_m^{(i)}(z) = S_{(P_n^i)^m}(z), \tag{2.11}$$

where

$$S_g(z) = \left[\frac{z}{g(z)} \frac{d}{dz} g(z) \right]^d. \tag{2.12}$$

Care must be taken to include only so many non-zero A's or B's that lead to convergent expressions in z for some region of z. In practice this means that only two of them are non-zero. For the conformal operator

$$Q^\mu(z) = -\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu z^{-n} + \alpha_0^\mu \ln z + \frac{\overleftarrow{\partial}}{\partial \alpha_0^\mu}, \tag{2.13}$$

which has conformal dimension zero, the above overlaps are

$$V\{Q^{(i)\mu}(z) - Q^{(j)\mu}(T_j \ldots T_{i+1}(z))\} = 0; \qquad 1 \leqq i < j \leqq N, \tag{2.14}$$

and

$$V\{Q^{(k)\mu}(z) - Q^{(k)\mu}(P_n^k(z))\} = 0 \qquad \forall n, k. \tag{2.15}$$

The corresponding identities for $P^{\mu}(z)=\sum_{n=-\infty}^{\infty}\alpha_n^{\mu}z^{-n}$ are easily found by computing the above S's for $d=1$. This result is in agreement with differentiating the above identities with respect to $z\,d/dz$. The identities for $Q^{\mu}(z)$ and $P^{\mu}(z)$, unlike for $L(z)=\sum_n L_n z^{-n}$, do not involve normal ordering and so no constants occur in the identities. However, $L(z)$ is not, due to the central change, exactly an object of weight 2, and we must add in constants whose values will be found below. As for the tree case, one can find integrated identities. These have a particularly elegant form and will be discussed later in this section.

Let us now consider the form of the cycling transformations in more detail. In fact, the loop cycling transformations are simplest to discuss since they are just arbitrary members of $SL(2,R)$ for the open bosonic string. These may be parametrized by three real parameters for each loop which can be chosen to be their two fixed points α_n^i and β_n^i and their multiplier ω_n^i.

The external state cycling transformations, T's are also elements of $SL(2,R)$ for the open bosonic string, and we will take them to be given by

$$T_j=\frac{z\to 1}{z}\frac{1}{(1-u_{j-1,j})}, \tag{2.16}$$

where $u_{j-1,j}$ is the Chan variable between lines j and $j-1$ computed by considering not only the external points z_j but also all their images under the action of the loop cycling transformations. Since the T's depend only on the cross-ratios of the z's, the resulting expression will be $SL(2,R)$ invariant.

It is instructive to rewrite Eq. (2.16) in the form

$$T_j=(V^j)^{-1}(V^{j-1}), \tag{2.17}$$

where

$$V^j=\begin{pmatrix}\infty & 0 & 1\\ z_{j-1} & z_j & z_{j+1}\end{pmatrix}. \tag{2.18}$$

One then realizes that

$$T_j\dots T_{i+1}=(V^j)^{-1}V^i, \tag{2.19}$$

and in particular if we define

$$\xi_j=T_j\dots T_2(\xi_1),$$

then (2.20)

$$\xi_j=((V^j)^{-1}V^i)(\xi_i)\qquad \forall i,j.$$

We were guided to the above cycling transformations by examining the Lovelace [10] and Olive [11] vertex for open bosonic string scattering.

In fact the P's and T's are not all independent. Applying P_n^i to V and using Eqs. (2.3) and (2.5) or equivalently using the Q^{μ} overlaps of Eqs. (2.14) and (2.15), we find that

$$P_n^{i+1}=T_{i+1}P_n^i(T_{i+1})^{-1}.$$

Iterating this equation gives

$$P_n^j=(T_j\dots T_{i+1})P_n^i(T_j\dots T_{i+1})^{-1}. \tag{2.21}$$

Consequently, given $P_n{}^i$, i.e.; the n^{th} loop cycling transformation on leg i, we can find it on all legs once the T's are specified. The most convenient way to express this fact is to take $P_n{}^i = (V^i)^{-1} P_n^{-1}(V^i)$, where P_n is arbitrary, and then we find

$$P_n{}^j = (V^j)^{-1} P_n{}^{-1} V^j \qquad \forall j\,. \tag{2.22}$$

The parameters required to label P_n can be taken to be the v_n. Written in this way we can see the interpretation of P_n. Let us write $P_n V^j = V^{j+N+1}$ in the sense that the points z_j are moved by the transformation P_n to their images z_{j+N+1}. As a result, the formula for $P_n{}^j$ is of the same form as Eq. (2.19) except that it is between the original points and their immediate images. The resulting transformation $P_n{}^j$ is the same even if we compute it by considering image points and points related to these by P_n, as the transformations T_j only involve cross ratios.

Let us now count the number of parameters that occur in V. There are N real z_i's and three M real parameters in the P_n's. However, the overall result is $SL(2,R)$ invariant, and so we may make three choices. An $SL(2,R)$ transformation, however, induces a similarity transformation on P_n (i.e., $P_n \to S P_n S^{-1}$), which transforms the fixed points $\alpha_n \to S\alpha_n$, $\beta_n \to S\beta_n$, but leaves the multiplier ω_n inert (see Appendix A). It is customary to choose $z_1 = 1$ and to choose the fixed points of one of the loops, say 0 and ∞. Consequently, we find that V depends on $N+3M-3$ real parameters. The case $N=0$, $M=1$ is special in the sense that we have two fixed points which we may choose and one multiplier (one parameter) which cannot be chosen. The fact that the number changes by three if we change M by one is in accord with the trivial observation that changing M means adding three propagators in the dual model graph and so in its parametric form adding three extra parameters.

The above discussion generalizes to the closed bosonic string and superstring by replacing $SL(2,R)$ by $SL(2,C)$ for the closed bosonic string and by prefacing these groups with the word graded for the superstring. For the closed string, one finds $N+3M-3$ complex parameters in V. The case $N=0$, $M=1$ is again special and one can only choose both the fixed points but not the multiplier, giving in all one complex parameter. This of course agrees with the propagator count and also the number of inequivalent complex structures on a Riemann surface.

The behaviour of the vertex under the $SL(2,R)$ transformations given above allows us also to find the effects of general conformal transformations which change the parameters z_r and v_r. This is reminiscent of the method of induced representations. Let us consider a general change in all the parameters which is implemented by a conformal transformation,

$$V(\hat{z}_i, \hat{v}_r) = V(z_i, v_r) \prod_{i=1}^{N} \mathscr{M}^i\,. \tag{2.23}$$

The vertex $V(\hat{z}, \hat{v}_r)$ must satisfy the appropriate cycling equations for $\hat{T} = T(\hat{z}, \hat{v}_r)$ and $\hat{P} = P(\hat{z}, \hat{v}_r)$ in Eqs. (2.3) and (2.5). Consequently we find that

$$T_{i+1}{}^{-1} \mathscr{M}^{i+1} \hat{T}_{i+1} = \mathscr{M}^i \qquad \forall i \tag{2.24}$$

and

$$(P_n{}^i)^{-1} \mathscr{M}^i \hat{P}_n{}^i = \mathscr{M}^i \qquad \forall i, n\,. \tag{2.25}$$

If we take $\mathcal{M}^i$ to be an infinitesimal conformal transformation, i.e.,

$$\mathcal{M}^i(z)=z+\varepsilon f_i(z), \tag{2.26}$$

then the above equations become

$$P_n{}^i(z)+\varepsilon f_i(z)\frac{dP_n{}^i}{dz}=\hat{P}_n{}^i(z)+\varepsilon f_i(P_n{}^i z) \tag{2.27}$$

and

$$T_{i+1}(z)+\varepsilon f_i(z)\frac{dT_{i+1}(z)}{dz}=\hat{T}_{i+1}(z)+\varepsilon f_{i+1}(T_{i+1}z) \tag{2.28}$$

at lowest order. It is clear from the above equation and in accord with previous arguments that given $f_k(z)$ on the k^{th} leg which must satisfy (2.27) then all the other f_i's are determined by (2.28). The remaining Eqs. (2.27) for $i=k$ are automatically satisfied. For the infinitesimal case, Eq. (2.23) becomes

$$\left(\sum_{i=1}^{N}\delta z_i\frac{\partial}{\partial z_i}+\sum_{r=1}^{M}\delta v_r\frac{\partial}{\partial v_r}\right)V(z_i,v_r)$$
$$=V(z_i,v_r)\sum_{k=1}^{N}\left\{\oint\frac{dz}{z}\varepsilon f_k(z)\frac{L^{(k)}(z)}{z}+\varepsilon C^{(k)}\right\}. \tag{2.29}$$

There also exist conformal transformations which do not change the moduli. These are given in Eqs. (2.27) and (2.28) by setting $T=\hat{T}$ and $P=\hat{P}$.

The above equations are used to compute the measure that occurs in Eq. (2.2). This comes about because the $f_i(z)$ which shifts z_i or v_r has a pole or constant on one leg and is an analytic function which vanishes at $z=0$ on all the other legs. Translated in L_n's this means on one leg that we have $a_{-2}L_{-2}+a_{-1}L_{-1}+\sum_{n=0}^{\infty}a_nL_n$, while on the other legs we have only $L_n\, n\geqq 0$. Unitarity, however, tells us that one spurious state acting on the physical vertex W does not couple on-shell if all states are physical, and hence the terms $a_{-2}L_{-2}+a_{-1}L_{-1}$ should vanish. This relates a derivative of the vertex, and by integration by parts the measure, with respect to the parameters (i.e. moduli) to a known function of the parameters. This was used to find the measure for trees in I and a tadpole one-loop diagram in II. We shall use this method in Sect. 4, for an arbitrary one-loop diagram. It is important to realize that no mention of ghosts is required.

We now obtain the external state dependence of the vertex. On a given leg, say leg j, we have loop cycling transformations $P_n{}^j\, n=1,\dots,M$. These we take to form a Schottky group (see Appendix B).

The points $z_j\; j=1,\dots,N$ for the external particles lie in the fundamental domain of the Schottky group. Let us consider the quantity

$$V\oint_{\xi_j=0}\frac{d\xi_j}{\xi_j}\phi^j(\xi_j)\,P^{\mu(j)}(\xi_j)=0, \tag{2.30}$$

where $\xi_j=0$ corresponds to the points z_j and the function ϕ^j is to be discussed.

We may deform this contour around z_j so that it goes around the edge of the fundamental region, i.e., around the $2M$ isometric circles and also around the other points z_i $i \neq j$. The validity of this contour distortion relies on the fact that $\phi^j(\xi_j)$ only has poles at $\xi_j = 0$, that is, it can be written in the form

$$\phi^j(\xi_j) = \frac{\xi_j^{-n}}{n} + \sum_{m=0}^{\infty} E_n{}^j{}_m (\xi_j)^m . \tag{2.31}$$

As for the case of the trees [1, 12, 13], one must use the P^μ overlap identities to retain an expression which is still convergent as one moves across the complex plane. However, Eq. (2.14) shows that this change of $P_\mu^{(j)}(\xi_j)$ to $P_\mu{}^{(j+1)}(T_{j+1}\xi_j)$ involves the factor $S_{T_{j+1}}(\xi_j)$ which is precisely the factor required to change the integration variable from ξ_j to ξ_{j+1}. Consequently, we find that

$$\begin{aligned} V \oint_{\xi_j=0} \frac{d\xi_j}{\xi_j} \phi^j(\xi_j) P^{\mu(j)}(\xi_j) = V \Bigg\{ & - \sum_{\substack{i=1 \\ i \neq j}}^{N} \oint_{\xi_i=0} \frac{d\xi_i}{\xi_i} \phi^j(\xi_j) P^{\mu(i)}(\xi_i) \\ & + \sum_{n=1}^{M} \left\{ \oint_{C_n} \frac{d\xi_j}{\xi_j} \phi^j(\xi_j) P^{\mu(j)}(\xi_j) + \oint_{C_n'} \frac{d\xi_j}{\xi_j} \phi^j(\xi_j) P^{\mu(j)}(P_n{}^j \xi_j) \right\} \Bigg\} . \end{aligned} \tag{2.32}$$

We now examine the conditions under which the last term vanishes. We recall that $P_n{}^j$ maps C_n to C_n' and so if we change a variable, these two terms will cancel provided

$$\phi^j(P_n{}^j \xi_j) = \phi^j(\xi_j) . \tag{2.33}$$

We therefore adopt this condition.

However, when performing the above contour integrals, we must beware of any multi-valued behaviour of ϕ^j. Let us suppose that ϕ^j is not single-valued but when going around a circle C_n, and so C_n', it changes by a constant, c. We can implement this by taking a cut between the circles C_n and C_n'. The integration contour shown in Fig. 1 is the one that actually results from the contour deformation which lies around C_n and C_n'. We may write this contribution as

$$\begin{aligned} & V \left\{ \int_A^B \frac{d\xi_j}{\xi_j} \phi^j(\xi_j) P^{\mu(j)}(\xi_j) + \int_C^D \frac{d\xi_j}{\xi_j} \phi^j(\xi_j) P^{\mu(j)}(\xi_j) \right. \\ & \left. = Vc \int_A^B \frac{d\xi_j}{\xi_j} P^{\mu(j)}(\xi_j) = cV \{ Q^{\mu(j)}(A) - Q^{\mu(j)}(B) \} \right\} \\ & = 0 . \end{aligned} \tag{2.34}$$

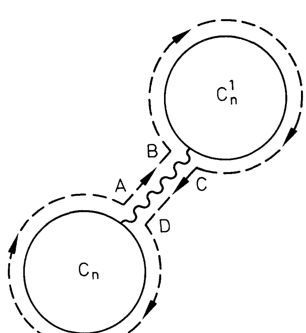

Fig. 1

However, A and B may be chosen so that they are related by the action of $P_n{}^j$ and then by Eq. (2.14) we obtain zero. We can summarize the result as:

$$V\left\{\sum_{k=1}^{N} \oint_{\xi_k=0} \frac{d\xi_k}{\xi_k} \phi^j(\xi_j)\, P^{\mu(k)}(\xi_k)\right\} = 0\,, \tag{2.35}$$

provided ϕ^j satisfies Eq. (2.33) and changes by a constant when its argument is taken around any isometric circle.

To demand that ϕ^j be single-valued would require it to be an automorphic function not only with respect to the group generated by the multi-loop cycling transformations, $P_n{}^j$, but also by a rotation about the centres of the isometric circles, and, in general, such functions with the desired pole structure do not exist.

Such a ϕ^j will be found in Sect. 4, for one-loop and in Sect. 5 for multi-loops. Recalling Eq. (2.20), we find that Eq. (2.35) becomes

$$\begin{aligned} V\Bigg\{&\frac{a_{-n}^{\mu(j)}}{\sqrt{n}} + \sum_{m=1}^{\infty} E_n{}^{jm} a_m^{\mu j}\sqrt{m} + \sum_{\substack{k=1\\k\neq j}}^{N}\Bigg\{\sum_{p=1}^{\infty}\frac{1}{\sqrt{n}}(\Gamma(V^j)^{-1}V^k)_{np}\,a_p^{\mu(k)} \\ &+ \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} E_{nm}\sqrt{m}[(V^j)^{-1}V^k]_{mp}\,a_p^{\mu(k)} \\ &+ \frac{1}{\sqrt{n}}[\Gamma(V_j)^{-1}V^k(0)]^n a_0^{\mu(k)} + \sum_{m=1}^{\infty} E_n{}^j{}_m((V^j)^{-1}V^k(0))^m a_0^{\mu(k)}\Bigg\}\Bigg\} \\ &= 0\,. \end{aligned} \tag{2.36}$$

The vertex is then found to be

$$\begin{aligned} V = \langle_1 0| \dots \langle_N 0| \exp - &\Bigg\{\sum_{i,j=1}^{N} [\tfrac{1}{2}(a^i|\bar{E}^i(V^i)^{-1}V^j|a^j) \\ &+ (a^i|\bar{E}^i|(V^i)^{-1}V^j(0))a_0^j + \tfrac{1}{2}a_0^i\mathscr{M}_{00}^{ij}a_0^j\Bigg\} \\ &\cdot V^{\text{tree}}\,, \end{aligned} \tag{2.37}$$

where

$$\bar{E}_n{}^j{}_m = \sqrt{n}\,E_n{}^j{}_m\sqrt{m}\,.$$

By V^{tree}, we mean the expression which occurs for the corresponding tree graph, once one replaces the tree external line cycling transformation by those appropriate for the loop. This factorization comes about since the third and fifth terms of Eq. (2.36) are of the same type as occurs for the tree graph which is discussed in Sect. 3. We have yet to determine $\mathscr{M}_{00}^{ij}$ which requires the Q^μ overlap. This is done in Sect. 4 for one loop and Sect. 5 for the multi-loop case, where the reader will also find more details of the above derivation for planar graphs.

Equation (2.37) is a very general result, holding for the open bosonic string scatterings, for arbitrary number and type of physical external states. It will hold for all planar and non-planar graphs and one may expect its generic form to generalize to all string theories.

3. The Q^μ Overlap for Tree Graphs

In I the scattering vertex for on-shell bosonic trees [10, 11] was derived from the overlap conditions. It will be instructive in what follows for loops to give an explicit proof of the Q^μ overlap. The structure of such an overlap implies that the vertex is of the form of an exponential, namely in the notation of Appendix A it is of the form

$$V=\langle_1 0|\dots\langle_N 0|\exp-\Big\{\sum_{1\leq i<j\leq N}[(a^i|\mathscr{M}^{ij}|a^j) + (a^i|M^{ij})a_0^j + a_0^i(M^{ji}|a^j) + R^{ij}a_0^i a_0^j]\Big\}. \tag{3.1}$$

We found [1] that the integrated P^μ identity immediately gave that

$$\mathscr{M}^{ij}=\Gamma(V^i)^{-1}V^j \tag{3.2}$$

and

$$|M^{ij})=|\Gamma(V^i)^{-1}V^j(0)). \tag{3.3}$$

We note that

$$\tilde{\mathscr{M}}^{ij}=\Gamma(\mathscr{M}^{ij})^{-1}\Gamma=\mathscr{M}^{ji}. \tag{3.4}$$

The Q^μ overlap reads

$$V\{Q^{\mu(i)}(z)-Q^{\mu(j)}((V^j)^{-1}V^i(z))\}=0, \qquad \forall i,j. \tag{3.5}$$

This equation is most easily analyzed by first considering the terms which do not contain the zero modes and originate from the action of the non-zero modes in Q^μ,

$$V\Big\{-\sum_{\substack{1\leq k\leq N\\ k\neq i}}(a^k|\Gamma(V^k)^{-1}V^i|z) + \sum_{\substack{1\leq k\leq N\\ k\neq j}}(a^k|\Gamma(V^k)^{-1}V^j|(V^j)^{-1}V^i(z)) - (a^i|\Gamma z)+(a^j|\Gamma(V^j)^{-1}V^i z)\Big\}. \tag{3.6}$$

Using Eq. (A.4) in the second term we find that these terms give

$$V\{(a^j|\Gamma(V^j)^{-1}V^i(0))-(a^i|\Gamma(V^i)^{-1}V^j(0))\}. \tag{3.7}$$

The above terms are accounted for by the derivative with respect to momentum (a_0^μ) in Q^μ acting on the $a_0^\mu a_n^\mu$ terms in the vertex.

All the remaining terms from Eq. (3.5) are

$$V\Big\{-\sum_{\substack{1\leq k\leq N\\ k\neq i}}(z|\Gamma(V^i)^{-1}V^k(0))a_0^k + \sum_{\substack{1\leq k\leq N\\ k\neq j}}((V^j)^{-1}V^i(z)|\Gamma(V^j)^{-1}V^k(0))a_0^k + a_0^i\ln z - a_0^j\ln[(V^j)^{-1}V^i(z)] + \sum_{1\leq k\leq N}(-R^{ik}+R^{jk})a_0^k\Big\}, \tag{3.8}$$

where we can take $R^{kk}=0$. Using Appendix A, one can carry out the following manipulation:

$$\begin{aligned}&\sum_{\substack{1\leq k\leq N\\ k\neq i\,\mathrm{or}\,j}} a_0^k(\Gamma(V^j)^{-1}V^k(0)|(V^j)^{-1}V^i(z))\\ &=\sum_{\substack{1\leq k\leq N\\ k\neq i\,\mathrm{or}\,j}} a_0^k\{(z|\Gamma(V^i)^{-1}V^k(0))-(z|\Gamma(V^i)^{-1}V^j\Gamma(0))\\ &\quad+(\Gamma(V^j)^{-1}V^k(0)|(V^j)^{-1}V^i(0))\}\,. \end{aligned}\tag{3.9}$$

Consequently, the Q^μ overlap now results in the expression,

$$\begin{aligned}V\Big\{&-(z|\Gamma(V^i)^{-1}V^j(0))a_0^j+a_0^i\ln z-a_0^j\ln[(V^j)^{-1}V^i(z)]\\ &+((V^j)^{-1}V^i(z)|\Gamma(V^j)^{-1}V^i(0))a_0^i+(a_0^i+a_0^j)(z|\Gamma(V^i)^{-1}V^j\Gamma(0))\\ &-\sum_{1\leq k\leq N}(R^{ik}-R^{jk})a_0^k\\ &+\sum_{\substack{1\leq k\leq N\\ k\neq i\,\mathrm{or}\,j}}(\Gamma(V^j)^{-1}V^k(0)|(V^j)^{-1}V^i(0))a_0^k\Big\}\,,\end{aligned}\tag{3.10}$$

after using momentum conservation. To evaluate this expression one can write

$$\Gamma(V^i)^{-1}V^j(z)=\frac{a^{ij}z+b^{ij}}{c^{ij}z+d^{ij}}\,.\tag{3.11}$$

One immediately finds that the z dependent terms drop and taking

$$\begin{aligned}R^{ij}&=\tfrac{1}{2}\ln\frac{d}{dz}[\Gamma(V^i)^{-1}V^j(z)]|_{z=0}\\ &=-\ln\left[\frac{d^{ij}}{\sqrt{a^{ij}d^{ij}-b^{ij}c^{ij}}}\right],\end{aligned}\tag{3.12}$$

the expression reduces to zero. The $a_0^\mu a_0^\mu$ piece agrees with that given in [4]. When performing the above calculation, one encounters R^{ji} for $j>i$ which is defined by being set equal to R^{ij}. For future use, we now summarize the result which is the Olive-Lovelace vertex [10, 11],

$$\begin{aligned}V=\langle_1 0|\dots\langle_N 0|\exp-\Big\{&\sum_{1\leq i<j\leq N}\Big\{(ai|\Gamma(V^i)^{-1}V^j|a^j)\\ &+(a^i|\Gamma(V^i)^{-1}V^j(0))a_0^j+a_0^i(\Gamma(V^j)^{-1}V^i(0)|a^j)\\ &+\tfrac{1}{2}\ln\left[\frac{d}{dz}[\Gamma(V^i)^{-1}V^j(z)]\right]\Big|_{z=0}a_0^i a_0^j\Big\}\Big\}\,.\end{aligned}\tag{3.13}$$

The (V^i)'s are given in Eq. (2.18) and are discussed more fully in I for the tree case.

Clearly, the integrated Q^μ overlap identity will determine the $a_0^i a_0^j$ piece. This equation, however, contains $\ln z$'s which are integrated and one must take into account the contributions of their cuts. A preliminary investigation indicates that, taking into account momentum conservation, this method would work.

4. One-Loop Planar Graphs

4.1. Cycling Transformations

We begin by computing the cycling transformation from the charges and their images as shown in Fig. 2. The points 1 to N correspond to the external lines and so $z_i = x_{i-1} \dots x_1 \equiv \bar{x}_{i-1}$. The remaining image charges are found by applying the $SL(2, R)$ transformation which may be taken to be simply multiplication by ω, that is $P(z) = \omega z$ in the notation of Eq. (2.22). As explained in Sect. 2, the cycling transformation is given by

$$T_j = \frac{z-1}{z(1-u_{j-1,j})} \tag{4.1}$$

where $u_{j-1,j}$ is the Chan variable between the $j-1^{\text{th}}$ and j^{th} legs, namely

$$u_{j-1,j} = \frac{(z_{j+1}-z_{j-2})}{(z_{j+1}-z_{j-1})} \frac{(z_j - z_{j-1})}{(z_j - z_{j-2})}. \tag{4.2}$$

A straightforward computation shows that

$$T_j = (V^j)^{-1} (V^{j-1}), \tag{4.3}$$

where V^j is given in Eq. (2.18). One finds that

$$\begin{aligned} T_j \dots T_{i+1}(z) &= (V^j)^{-1} V^i(z) \\ &= \frac{[z(1-\bar{x}_{j-1,i-1}) + a_i(x_{i-1} - \bar{x}_{j-1,i-1})]}{[z(1-\bar{x}_{j-2,i-1}) + a_i(x_{i-1} - \bar{x}_{j-2,i-1})]} \frac{(x_j x_{j-1} - 1)}{(x_{j-1})(x_j{}^{-1})}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \bar{x}_{j,i} &= x_j \dots x_i, \qquad \bar{x}_j = x_j \dots x_1, \\ a_i &= \frac{(1-x_i x_{i-1})}{(x_i - 1)} \frac{1}{x_{i-1}}. \end{aligned} \tag{4.5}$$

The case $i=1$ is given in the above formula by setting

$$\begin{aligned} x_0 &= x_n; \qquad \bar{x}_{j-1}, 0 = \bar{x}_{j-1} x_n, \\ a_1 &= \frac{(1-x_1 x_n)}{(x_1 - 1)} \frac{1}{x_n}. \end{aligned} \tag{4.6}$$

The cycling transformation corresponding to the loop as seen from the k^{th} line is given by

$$P_n{}^k = (V^{k+N+1})^{-1} V^k = (V^k)^{-1} P^{-1} V^k. \tag{4.7}$$

One finds that

$$P_n{}^k(z) = \frac{[z(1-\omega x_{k-1}) + a_k x_{k-1}(1-\omega)]}{[z(1-\omega) + a_k(x_{k-1} - \omega)]} (-a_k). \tag{4.8}$$

Again, the $k=1$ case is found by making the substitutions in Eq. (4.6).

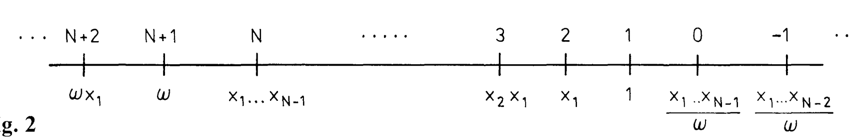

Fig. 2

Although it would be perfectly correct to take the transformations above as the cycling transformations, it is more convenient to carry out a gauge transformation on them as discussed in II. We carry out on the k^{th} leg the transformation $(\alpha_k)^{L_0 - L_1 - 1}$, and this results in changing

$$V^k \to V^k A_k^{-1}, \tag{4.9}$$

where

$$A_k(z) = \frac{\alpha_k z}{1-(1-\alpha_k)z}. \tag{4.10}$$

The effect of the transformation is most easily found by using the 2×2 matrix multiplication rule for these transformations. There are two gauge transformations that yield a loop cycling transformation without a z in its denominator and of the two we choose

$$\alpha_k = \frac{1+a_k}{a_k}. \tag{4.11}$$

The resulting gauge transformed cycling transformations, using the same symbols as before, are

$$P^k(z) = \omega z + \frac{(1-\omega)}{(1-x_k)} \tag{4.12}$$

and

$$T_j \dots T_{i+1}(z) = \frac{(x_{i-1} - \bar{x}_{j-1,i-1})}{(x_j-1)(\bar{x}_{j-1,i-1})} + z\frac{(x_i-1)}{(x_j-1)}\frac{1}{\bar{x}_{j-1,i}} \tag{4.13}$$

or

$$T_j = A_j(V^j)^{-1}(V^{j-1})(A_{j-1})^{-1} = \frac{(z-1)}{x_{j-1}}\frac{(1-x_{j-1})}{1-x_j}. \tag{4.14}$$

The overlap identities for Q^μ are given by substitution in Eqs. (2.14) and (2.15).

It is instructive to consider the product of the T's. Here there is a choice depending on how one defines T_1, one can choose either

$$T_1 = A_1(V^1)^{-1} V^N A_N^{-1} \quad \text{or} \quad T_1 = A_1(V^1)^{-1} V^0 A_N^{-1}. \tag{4.15}$$

In the first case, the product of the T's is obviously one while in the latter case

$$T_N \dots T_1 = A_N(V^N)^{-1} V^0 (A_N)^{-1} = P^N. \tag{4.16}$$

Since P^N leaves the vertex inert it is irrelevant which choice one takes and we adopt the former possibility. In what follows, we will often write $V^j A_j^{-1}$ as V^j.

4.2. The Measure

We now implement the strategy set out in Sect. 2. Let us first consider a general variation of the x_i's, and so ω which is given by $\omega = \bar{x}_N \equiv x_N \dots x_1$. Equation (2.27) which guarantees the correct loop cycling properties for the new vertex reads

$$\varepsilon\omega f_k(z) = z\,\delta\omega - \frac{1}{1-x_k}\delta\omega + \frac{(1-\omega)}{(1-x_k)^2}\delta x_k + \varepsilon f_k\left(z\omega + \frac{1-\omega}{1-x_k}\right). \tag{4.17}$$

If we vary only x_N by

$$\delta x_N = \frac{\varepsilon x_N}{\omega},\ \delta\omega = \varepsilon \quad \text{and} \quad \delta x_k = 0, \qquad 1 \leqq k < N, \tag{4.18}$$

we find that

$$\omega f_k(z) = \left(z - \frac{1}{1-x_k}\right) + f_k\left(z\omega + \frac{1-\omega}{1-x_k}\right), \qquad k = 1, 2, \ldots, N-1, \tag{4.19}$$

$$\omega f_N(z) = \left(z - \frac{1}{1-x_N}\right) + \frac{1-\omega}{(1-x_N)^2}\frac{x_N}{\omega} + f_N\left(z\omega + \frac{1-\omega}{1-x_N}\right). \tag{4.20}$$

Equation (2.28) which ensures the correct cycling of the external legs for a general change $x_k \to \hat{x}_k$ reads in this case,

$$\varepsilon f_{k+1}\left[\frac{(z-1)(1-x_k)}{x_k(1-x_{k+1})}\right] + (z-1)\left[\frac{(1-\hat{x}_k)}{\hat{x}_k(1-\hat{x}_{k+1})} - \frac{(1-x_k)}{x_k(1-x_{k+1})}\right] = \frac{\varepsilon f_k(z)(1-x_k)}{x_k(1-x_{k+1})}. \tag{4.21}$$

Making the particular changes of Eq. (4.18) this equation becomes

$$f_{k+1}\left[\frac{(z-1)(1-x_k)}{x_k(1-x_{k+1})}\right] = f_k(z)\frac{(1-x_k)}{x_k(1-x_{k+1})}, \qquad k = 1, \ldots, N-1, \tag{4.22}$$

$$f_N\left[\frac{(z-1)(1-x_{N-1})}{x_{N-1}(1-x_N)}\right] + \frac{(z-1)(1-x_{N-1})}{(1-x_N)^2 x_{N-1}}\frac{x_N}{\omega} = \frac{f_{N-1}(z)(1-x_{N-1})}{(1-x_N)x_{N-1}} \tag{4.23}$$

and

$$f_1\left[\frac{(z-1)(1-x_N)}{x_N(1-x_1)}\right] - \frac{(z-1)}{x_N(1-x_1)}\frac{1}{\omega} = \frac{f_N(z)(1-x_N)}{x_N(1-x_1)}. \tag{4.24}$$

The last equation can be derived from the one above it, due to the fact that the product of the T's is 1.

The conformal transformations which leave the vertex inert are found by setting $\delta\omega$ and δx_k to zero in Eq. (4.17) and $x_k = \hat{x}_k$ in Eq. (4.21). One solution is given by

$$f_k(z) = e^{t_k} \equiv \left(z - \frac{1}{1-x_k}\right), \tag{4.25}$$

which implies the equation

$$V\left(\sum_{k=1}^{N}\left[L_0^{(k)}-\frac{L_{-1}^{(k)}}{(1-x_k)}\right]\right)=0. \tag{4.26}$$

To find the conformal transformation $\mathscr{M}$ which induces the change in the moduli of Eq. (4.18), we must solve the above equations for f_k. Let us define

$$f_k=e^{t_k}j_k(t_k), \qquad k=1,\dots,N-1, \tag{4.27}$$

$$f_N=e^{t_N}j_N(t_N)-\frac{x_N}{(1-x_N)^2}\frac{1}{\omega}, \tag{4.28}$$

where

$$t_k=\ln\left(z-\frac{1}{1-x_k}\right). \tag{4.29}$$

Substituting into Eqs. (4.19) and (4.20), we find that

$$\frac{1}{\omega}+j_k(t+\ln\omega)=j_k(t), \qquad k=1,\dots,N, \tag{4.30}$$

and demanding j to be a single valued function of z implies that

$$j_k(t+2\pi i)=j_k(t). \tag{4.31}$$

As a result, one such solution is

$$j_k=\frac{1}{\omega}\sum_i c_i^{(k)}\bar{\zeta}(t-s_i^{(k)})+d^k, \tag{4.32}$$

where

$$\bar{\zeta}(t)=\zeta(t)-\frac{\eta}{\pi i}t, \sum_i c_i^k=1, \eta=\zeta(i\pi), \tag{4.33}$$

$s_i^{(k)}$ and $d^{(k)}$ are arbitrary and ζ is the Weierstrass ζ-function [14] with periods $2i\pi$ and $\ln\omega$.

We now substitute Eqs. (4.27) and (4.28) into Eqs. (4.22) and (4.23). One finds that

$$\begin{gathered} j_{k+1}\left(t+\ln\frac{(1-x_k)}{x_k(1-x_{k+1})}\right)=j_k(t); \qquad k=1,\dots,N-1, \\ j_N\left(t+\ln\frac{(1-x_{N-1})}{(x_{N-1})(1-x_N)}\right)+\frac{x_N}{\omega(1-x_N)}=j_{N-1}(t). \end{gathered} \tag{4.34}$$

Consequently, given $j_1(t)$ we know all the other j_i's. Let us take

$$j_1(t)=\frac{1}{\omega}\bar{\zeta}(t-\ln(-(1-x_1))), \tag{4.35}$$

then

$$j_k(t)=\frac{1}{\omega}\bar{\zeta}\left(t-\ln\left[-\frac{(1-x_k)}{\bar{x}_{k-1}}\right]\right),\qquad k=1,\dots,N-1\,,\tag{4.36}$$

$$j_N(t)=\frac{1}{\omega}\bar{\zeta}\left(t-\ln\left[-\frac{(1-x_N)}{x_N}\right]\right)-\frac{x_N}{\omega(1-x_N)}\,.\tag{4.37}$$

Substituting these results in Eq. (2.29), we find

$$\begin{aligned}\frac{x_N}{\omega}\frac{\partial V}{\partial x_N}=V\Bigg\{&\Bigg[\frac{L^{(1)}_{-2}}{(1-x_1)^2\omega}-\frac{3}{2\omega}\frac{L^{(1)}_{-1}}{(1-x_1)}+\frac{L^{(1)}_0}{2\omega}+\frac{L^{(1)}_0}{\omega}\left\{2\omega\frac{\partial}{\partial\omega}\ln f\right\}\\&+c^{(1)}\Bigg]+\sum_{k=2}^{N}\frac{1}{\omega}\Bigg[-\frac{L^{(k)}_{-1}}{(1-x_k)}\bar{\zeta}(\ln\bar{x}_{k-1})+L^{(k)}_0\bar{\zeta}(\ln\bar{x}_{k-1})\\&+\bar{\zeta}'(\ln\bar{x}_{k-1})L^{(k)}_0\Bigg]+\left[\frac{-x_N}{\omega(1-x_N)}L^{(N)}_0\right]+(\text{terms involving } L_n;n\geqq 1)\Bigg\}\,.\end{aligned}\tag{4.38}$$

Here we have used the fact that

$$\frac{\eta}{\pi i}\equiv\bar{\zeta}(i\pi)=-\left(2\omega\frac{\partial}{\partial\omega}\ln f(\omega)+\frac{1}{12}\right),\tag{4.39}$$

where

$$f(\omega)=\prod_{n=1}^{\infty}(1-\omega^n)\,.\tag{4.40}$$

The constant can be found by putting the vacuum state at zero momentum on the right-hand side:

$$0=V\left\{\frac{a^{\mu(1)}_{-1}a^{\mu(1)}_{-1}}{2(1-x_1)^2\omega}+c^{(1)}\right\}\{|0\rangle_1\,|0\rangle_2\dots|0\rangle_N\}\,.\tag{4.41}$$

This is just the coefficient of the $a^{\mu(1)}_1 a^{\mu(1)}_1$ term in the exponential which is found in the next section. The result is

$$c^{(1)}=D\frac{\partial}{\partial\omega}\ln f(\omega)+\frac{D}{2\omega}\ln\omega\,.\tag{4.42}$$

We observe that any solution for f must involve a non-analytic function. This is to be expected as we are changing the moduli.

We now compute the conformal transformation that changes the x's but leaves $\omega=\bar{x}_n$ inert, that is we consider

$$\frac{\delta x_k}{x_k}=-\frac{\delta x_{k+1}}{x_{k+1}}=\varepsilon\,.\tag{4.43}$$

Substituting in Eq. (4.17) we find that

$$\omega f_i(z)=\frac{1-\omega}{(1-x_k)^2}x_k\,\delta_{k,i}+f_i\left(z\omega+\frac{(1-\omega)}{(1-x_k)}\right)-\frac{(1-\omega)}{(1-x_{k+1})^2}x_{k+1}\,\delta_{k+1,i}\,.\tag{4.44}$$

Substituting in Eq. (4.21), we find the relation between f_{i+1} and f_i. It is simple to show that a solution of this constraint and Eq. (4.44) is given by

$$\begin{aligned} f_i &= 0; \quad i=1,\dots,k-1,k+2,\dots,N, \\ f_k &= e^{t_k}\left(\frac{-x_k}{1-x_k}\right) - \frac{x_k}{(1-x_k)^2}, \\ f_{k+1} &= e^{t_{k+1}}\left(\frac{1}{1-x_{k+1}}\right) + \frac{x_{k+1}}{(1-x_{k+1})^2}. \end{aligned} \tag{4.45}$$

We have neglected the homogeneous solution as this just corresponds to a conformal transformation that does not change the moduli. Consequently, we find that

$$x_k\frac{\partial V}{\partial x_k} - x_{k+1}\frac{\partial V}{\partial x_{k+1}} = V\left\{-L_0^{(k)}\frac{x_k}{1-x_k} + \frac{L_0^{(k+1)}}{(1-x_{k+1})} - \frac{L_{-1}^{(k+1)}}{(1-x_{k+1})}\right\}. \tag{4.46}$$

We can now deduce the measure which occurs in the physical vertex

$$W = \int \prod_{i=1}^{N} dx_i\, \bar{f}(x_i)\, V(x_i). \tag{4.47}$$

We have chosen to treat x_n rather than ω as a fundamental variable. To put the vertex on-shell we apply

$$\tilde{P} = \prod_{k=1}^{N} \tilde{P}_k, \quad \text{where} \quad \tilde{P}_k = \oint_{z=0} \frac{dz}{z} z^{(L_0^{(k)}-1)}. \tag{4.48}$$

Using Eq. (4.46) we find that

$$\begin{aligned} 0 = W\tilde{P}L_{-1}^{(k+1)} = \int \prod_{i=1}^{N} dx_i (1-x_{k+1})\bar{f}\left\{x_k\frac{\partial V}{\partial x_k} - x_{k+1}\frac{\partial V}{\partial x_{k+1}}\right. \\ \left. + (1-x_{k+1})\frac{x_k}{1-x_k}\right\}; \quad k=1,\dots,N-1. \end{aligned} \tag{4.49}$$

Integrating by parts one finds that

$$\left(x_k\frac{\partial}{\partial x_k} - x_{k+1}\frac{\partial}{\partial x_{k+1}}\right)\ln\bar{f} = \frac{x_k}{1-x_k} - \frac{x_{k+1}}{1-x_{k+1}}. \tag{4.50}$$

Using Eq. (4.38), on the other hand, we find that

$$\begin{aligned} 0 = W\tilde{P}L_{-2}^{(1)} = \int \prod_{i=1}^{N} dx_i \omega(1-x_1)^2 \bar{f}\left\{\frac{x_n}{\omega}\frac{\partial V}{\partial x_n} + \frac{3}{2\omega}\frac{L_{-1}^{(1)}}{(1-x_1)}\right. \\ + \frac{1}{2\omega} - \frac{1}{\omega}\sum_{k=2}^{N}\left\{(\bar{\zeta}(\ln\bar{x}_{k-1}) + \bar{\zeta}'(\ln\bar{x}_{k-1}) - \frac{L_{-1}^{(k)}}{1-x_k}\bar{\zeta}(\ln\bar{x}_{k-1})\right\} \\ \left. - \left((D-2)\frac{\partial}{\partial\omega}\ln f + \frac{D}{2\omega}\frac{\partial}{\partial\omega}\ln\omega + \frac{x_N}{\omega(1-x_N)}\right)\right\}. \end{aligned} \tag{4.51}$$

To evaluate this expression, we consider

$$
\begin{aligned}
&-\int \prod_{i=1}^{N} dx_i \bar{f} V \frac{L_{-1}^{(k)}}{(1-x_k)} \bar{\zeta}(\ln \bar{x}_{k-1})\,\omega(1-x_1)^2 \\
&= \int \prod_{i=1}^{N} dx_i \bar{f}(1-x_1)^2 \left\{ \left(x_{k-1}\frac{\partial}{\partial x_{k-1}} - x_k \frac{\partial}{\partial x_k}\right) V + V\frac{x_{k-1}}{1-x_{k-1}} \right. \\
&\left. -2V\delta_{k,2}\frac{x_1}{1-x_1} - \frac{V}{1-x_k} \right\} \bar{\zeta}(\ln(\bar{x}_{k-1})) \\
&= -\int \prod_{i=1}^{N} dx_i \bar{f}(1-x_1)^2 V\{\bar{\zeta}'(\ln \bar{x}_{k-1}) + \bar{\zeta}(\ln \bar{x}_{k-1})\}\,.
\end{aligned}
\tag{4.52}
$$

As a result the terms summed from $k=2$ to N vanish. The term with $L_{-1}^{(1)}$ is processed by first using Eq. (4.26) and then using the argument in Eq. (4.52) but with $\bar{\zeta}$ replaced by 1. The net result is to replace $L_{-1}/1-x_1$ by (-1). Putting all this together and integrating by parts one finds that

$$
\frac{x_n}{\omega}\frac{\partial}{\partial x_n}\ln(\bar{f}\omega) = -\frac{\partial}{\partial \omega}\ln[\omega f^{D-2}(\omega)(\ln\omega)^{D/2}]\,. \tag{4.53}
$$

It is straightforward to show that the unique solution to the first-order differential equations (4.50) and (4.53) is

$$
\bar{f} = [\omega^2 f^{D-2}(\omega)(\ln\omega)^{D/2}]^{-1} \prod_{k=1}^{N} \frac{1}{(1-x_k)}\,. \tag{4.54}
$$

This concludes the demonstration of the measure. We note that taking $N=1$ we gain agreement with the result found in II.

4.3. The External Dependence

We now apply the discussion of Sect. 2 to find the dependence of the vertex on the oscillators and momenta of the external lines. A suitable representation of the fundamental region of P^j is the region between the circles in Fig. 3. The reader is encouraged to carry out the derivation of Eq. (2.25) for this one-loop case. The function ϕ^j with the required properties is given by

$$
\phi^j(z) = -\frac{(1-x_j)^n}{n!}\left[\left(-\frac{\partial}{\partial y}\right)^n \ln\psi\left[\frac{1-(1-x_j)z}{y}\right]\right]\Bigg|_{y=1}, \tag{4.55}
$$

where [15]

$$
\psi(x,\omega) = -2\pi i \exp\left[\frac{(\ln x)^2}{2\ln\omega}\right]\frac{\theta_1(v,\tau)}{\theta_1'(0,\tau)},
$$

and

$$
v = \frac{\ln x}{2\pi i}, \qquad \tau = \frac{\ln\omega}{2\pi i}\,. \tag{4.56}
$$

Under $x \to \omega x$ one can show, using standard properties [14] of θ functions, that

$$
\psi(\omega x, \omega) = -\psi(x,\omega)\,. \tag{4.57}
$$

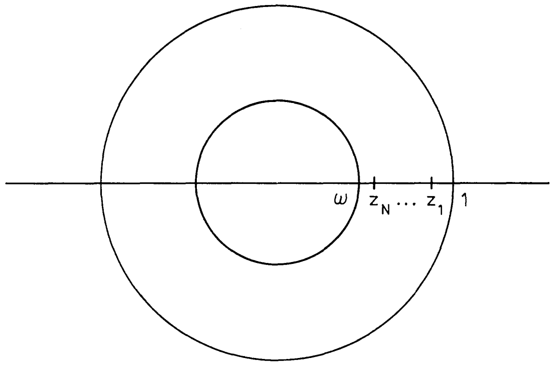

Fig. 3

Although under $v \to v+1$, θ_1 is inert, the exponential prefactor changes and consequently we find that

$$\begin{aligned}\phi^j(P^j z) &= \phi^j(z)\,,\\ \phi^j(t_j+2\pi i) &= \phi^j(t_j) - \frac{(1-x_j)^n}{\ln\omega}\,\frac{2\pi i}{n}\,,\end{aligned} \tag{4.58}$$

where

$$t_j = \ln\left(z - \frac{1}{1-x_j}\right).$$

These are the required properties. This function also has the desired pole structure as it can be written as

$$\begin{aligned}\phi^j(z) &= \frac{z^{-n}}{n} + \left(-\frac{\partial}{\partial y}\right)^n \left[\frac{(1-x_j)^n}{n!}\ln\frac{(y-(1-(1-x_j))z)}{\psi\left(\dfrac{1-(1-x_j)z}{y}\right)}\right]_{y=1}\\ &= \frac{z^{-n}}{n} + \sum_{m=0}^{\infty} E_{n\ m}^{\ j} z^m\,.\end{aligned} \tag{4.59}$$

The second term is analytic due to the fact that $\theta_1(v,\tau)$ has only one zero, at $v=0$, in its fundamental domain. We observe that

$$E_{n\ m}^{\ j} = \frac{(1-x_j)^n}{n!}\,\frac{(1-x_j)^m}{m!}\left(-\frac{\partial}{\partial x}\right)^m\left(-\frac{\partial}{\partial y}\right)^n \ln\left[\frac{y-x}{\psi\left(\dfrac{x}{y}\right)}\right]\Bigg|_{\substack{x=1\\ y=1}}\,. \tag{4.60}$$

It is instructive to express $E_{n\ m}^{\ j}$ in terms of a matrix which contains the group operations $P_n^{\ j}$. One finds that

$$\bar{E}_{nm}^{\ \ j} \equiv \sqrt{n}\,E_{n\ m}^{\ j}\sqrt{m} = [\Gamma(\mathscr{P}^j-1)]_{nm} + \frac{1}{\ln\omega}\,\frac{(1-x_j)^n}{\sqrt{n}}\,\frac{(1-x_j)^m}{\sqrt{m}}\,, \tag{4.61}$$

where

$$\mathscr{P}^j = \sum_{n=-\infty}^{\infty} (P^j)^n\,. \tag{4.62}$$

This is established along the same lines as in [16] where one uses the result

$$\psi(x,\omega)=\frac{(1-x)}{\sqrt{x}}\exp\left[\frac{(\ln x)^2}{2\ln\omega}\right]\prod_{n=1}^{\infty}\frac{(1-\omega^n x)(1-\omega^n/x)}{(1-\omega^n)^2}, \tag{4.63}$$

and evaluates

$$\left(\frac{1-x}{1-x_j}\middle|\bar{E}^j\middle|\frac{1-y}{1-x_j}\right)=\ln\left[\frac{y-x}{\psi\left(\frac{x}{y}\right)}\right]+\ln\frac{\psi(x)}{1-x}+\ln\frac{\psi(y)}{(1-y)}=\ln\Omega(x,y). \tag{4.64}$$

Using the right-hand side of Eq. (4.61), one notices that

$$E_{n\ m}^{\ j}=\frac{(1-x_j)^n}{n!}\frac{(1-x_j)^m}{m!}\left(-\frac{\partial}{\partial x}\right)^n\left(-\frac{\partial}{\partial y}\right)^m\ln\Omega(x,y)|_{x=y=1} \tag{4.65}$$

is in agreement with Eq. (4.60). Clearly we may shift

$$\ln\Omega(x,y)\to\ln\Omega(x,y)+f(x)+f(y), \tag{4.66}$$

and $E_{n\ m}^{\ j}$ is unaffected.

Equipped with the required ϕ's we may evaluate the oscillator form of the vertex as was done in Sect. 2 to arrive at Eq. (2.37). In this derivation, use was made of the equation

$$\bar{E}^j(V^j)^{-1}(V^i)=\Gamma[\bar{E}^j(V^j)^{-1}(V^i)]^{-1}\Gamma=\bar{E}^i(V^i)^{-1}V^j \tag{4.67}$$

which is straightforwardly proved by taking the form of $\bar{E}$ given in Eq. (4.61) and realizing that

$$\mathscr{P}^j(V^j)^{-1}V^k=(V^j)^{-1}V^k\mathscr{P}^k \tag{4.68}$$

as a consequence of Eq. (2.21). The expression for V^{tree} is given by Eq. (3.13) up to substitution of the V's of Eq. (4.14).

We now use the overlap

$$V\{Q^i(z)-Q^j((V^j)^{-1}V^i(z))\}=0,\qquad \forall i,j, \tag{4.69}$$

to find the $a_0^\mu a_0^\mu$ piece. If we consider the V^{tree} to contain the usual $a_0^\mu a_0^\mu$ piece, then the above equation becomes

$$\begin{aligned}V\Bigg\{&\sum_{k=1}^{N}a_0^k\{-((V^i)^{-1}V^k(0)|\tilde{E}^j|z)\\&+((V^j)^{-1}V^k(0)|\bar{E}^j|(V^j)^{-1}V^i(z))\\&-\bar{\mathscr{M}}_{00}^{ik}+\bar{\mathscr{M}}_{00}^{jk}\}\Bigg\}=0,\end{aligned} \tag{4.70}$$

where $\bar{\mathscr{M}}_{00}^{kk}$ is taken to be zero. We process the second term in a similar way to that given in Eq. (3.9), and the above equation becomes

$$\begin{aligned}V\Bigg\{&\sum_{k=1}^{N}a_0^k\{-((V^i)^{-1}V^k(0)|\tilde{E}^j|z)\\&+((V^j)^{-1}V^k(0)|\tilde{E}^j|(V^j)^{-1}V^i(z))-\bar{\mathscr{M}}_{00}^{ik}+\bar{\mathscr{M}}_{00}^{jk}\}\Bigg\}=0.\end{aligned}$$

To evaluate this expression we use the form of E^j of Eq. (4.60) and recognize that we have a Taylor expansion in two variables apart from the zeroth terms which must be added and subtracted by hand. Using in addition momentum conservation we find that

$$\begin{aligned}0=V\sum_{\substack{k=1\\k\neq i\,\mathrm{or}\,j}}^{N} a_0^k\Bigg\{&-\bar{\mathscr{M}}^{ik}_{00}+\bar{\mathscr{M}}^{jk}_{00}+\ln\psi(\bar{x}_{j-1,k})\\&-\ln\psi(\bar{x}_{i-1,k})-\ln\left(1-\frac{1}{\bar{x}_{j-1,k}}\right)+\ln\left(1-\frac{1}{\bar{x}_{i-1,k}}\right)\Bigg\}\\&+(a_0^i-a_0^j)\left(\ln\psi(\bar{x}_{j-1,i})-\ln\left(1-\frac{1}{\bar{x}_{j-1,i}}\right)\right)-\bar{\mathscr{M}}_{00}{}^{ij}a_0^j\\&+\bar{\mathscr{M}}^{ji}_{00}a_0^i+(a_0^i+a_0^j)\ln\bar{x}_{j-1,i},\end{aligned}\tag{4.71}$$

and so we conclude that

$$\bar{\mathscr{M}}^{ij}_{00}=-\ln\psi(\bar{x}_{j-1,i})+\tfrac{1}{2}\ln\bar{x}_{j-1,i}+\ln\left(1-\frac{1}{\bar{x}_{j-1,i}}\right).\tag{4.72}$$

In the above, we have used the equations

$$\begin{aligned}(V^j)^{-1}V^i(z)&=\frac{1}{1-x_j}\left(1-\frac{(1-z(1-x_i))}{\bar{x}_{j-1,i}}\right)\qquad\text{for}\quad j>i,\\(V^j)^{-1}V^i(z)&=\frac{1}{1-x_j}(1-\bar{x}_{i-1,j}(1-(1-x_i)z))\quad\text{for}\quad j<i.\end{aligned}\tag{4.73}$$

The contribution of this type coming from V^{tree} is

$$\exp\sum_{1\leqq i<j\leqq N}a_0^ia_0^j\ln\frac{(1-\bar{x}_{j-1,i})}{\sqrt{(1-x_i)(1-x_j)(\bar{x}_{j-1,i})}}.\tag{4.74}$$

Combining these results and collecting the zero mode pieces we find the planar one-loop vertex is given by

$$\begin{aligned}V=\langle_1 0|\dots\langle_N 0|\exp-\Bigg\{&\sum_{ijj=1}^{N}\{\tfrac{1}{2}(a^i|\bar{E}^i(V^i)^{-1}V^j|a^j)\\&+(a^i|\bar{E}^i|(V^i)^{-1}V^j(0))a_0^j\}+\sum_{1\leqq i<j\leqq N}\{(a^i|\Gamma(V^i)^{-1}V^j|a^j)\\&+(a^i|\Gamma(V^i)^{-1}V^j(0))a_0^j+a_0^i(\Gamma(V^j)^{-1}V^i(0)|a^j)\}\Bigg\}\\&\cdot\prod_{i=1}^{N}(1-x_i)^{\frac{p_i^2}{2}}\prod_{1\leqq i<j\leqq N}[\psi(\bar{x}_{j-1,i})]^{p_i\cdot p_j},\end{aligned}\tag{4.75}$$

where $\bar{E}^i$ is given in Eq. (4.61) and $(V^i)^{-1}V^j$ is in Eq. (4.72). This agrees with the previously announced result of II. The actual planar amplitude is given by

$$\int\prod_{i=1}^{N}\bar{f}V,\tag{4.76}$$

where $\bar{f}$ is given in Eq. (4.54). For the previously known tachyonic case, we recover the correct answer. The external momentum dependence [17] and the measure is in accord with the results of [18, 19]. Taking $\prod_{i=1}^{N} \xi^{i\mu}\alpha_{-1}^{\mu(i)}|\;\rangle$, with $p^{\mu i}\xi_{\mu}{}^{i}=0$, as our external state, we find the scattering of N photons which is discussed in [20].

In Appendix C, some non-planar one-loop graphs are calculated.

5. Further Multi-Loop Considerations

Let us continue the derivation of the external state dependence of an arbitrary loop graph given in Sect. 2. We must first find a function ϕ^j such that

$$\phi^j(P_n{}^j z)=\phi^j(z); \qquad \phi^j(A_n^j z)=\phi^j(z)+c\,, \tag{5.1}$$

where c is a constant.

In Appendix B, we encountered the function $\phi(z,z')$ which is inert under the A_n cycles (A_n is a rotation by 2π about the centre of C_n) but transforms according to Eq. (B.6) under $P_n{}^j$, which are to be identified with R_n. One can imagine taking the derivatives of $\phi(z,z')$ with respect to z' at $z'=0$. This function would be inert under $P_n{}^j$ were it not for the $v_n(z')$ term. We have, however, at our disposal the object $v_n(z)$ which transforms according to Eqs. (B.5) and (B.3), and thus we can construct the object

$$\begin{aligned}\ln\bar{\chi}_m{}^j(z,z')+\ln(z'-z)&\equiv\ln(z'-z)-\ln\phi^j(z',z)\\ &\quad-\tfrac{1}{2}\sum_{r,s=1}^{m}(v_r^j(z')-v_r^j(z))\,(\tau^{-1})_r^s(v_s^j(z')-v_s^j(z))\\ &\equiv\ln\chi^j(z,z')\,.\end{aligned} \tag{5.2}$$

We note that this object transforms as

$$\ln\bar{\chi}^j(P_n{}^j z,z')-\ln\bar{\chi}^j(z,z')=\ln(c_n^j z+d_n^j)\,, \tag{5.3}$$

$$\ln\bar{\chi}^j(A_n{}^j z,z')-\ln\bar{\chi}^j(z,z')=2\pi^2(\tau^{-1})_{nn}-2\pi i(v_s^j(z)-v_s^j(z'))\,(\tau^{-1})_s{}^n\,, \tag{5.4}$$

and as a result if we define

$$\phi^j(z)\equiv\frac{1}{n!}\left(\frac{\partial}{\partial z'}\right)^n\ln\bar{\chi}^j(z,z')|_{z'=0}\,, \tag{5.5}$$

we have a function that transforms according to Eq. (5.1).

It will be clear that the final answer is independent of the value of the constant c under an A_n cycle. Up to this ambiguity, ϕ^j is unique as the difference of two of them is A_N and B_N periodic and analytic in the fundamental domain.

This function also has the desired pole structure for

$$\phi^j(z)=\frac{z^{-n}}{n}+\frac{1}{n!}\left(\frac{\partial}{\partial z'}\right)^n\ln\chi^j(z,z')|_{z'=0} \tag{5.6}$$

and $\chi^i(z,z')$ is an analytic function of z and z'. This follows from the fact that $T_\gamma{}^j(z)$ lies within the circle corresponding to the leftmost factor occurring in $T_\gamma{}^j$; as a result, a factor like $(T_\gamma{}^j(z)-z')$ can never vanish if z and z' belong to the fundamental region. As a result, $\ln\{(z-z')/\phi^j(z',z)\}$ is the analytic function of z and z'. The same argument applies to $v_r(z)$ since α_r and β_r lie in C_r and C'_r and $\sum^r$

explicitly excludes a T_γ^{j-1} with a leftmost factor that is T_r^j. As a result

$$E_{n\ m}^{\ j} = \frac{1}{n!}\frac{1}{m!}\left(\frac{\partial}{\partial z}\right)^n\left(\frac{\partial}{\partial z'}\right)^m \ln\chi^j(z,z')|_{\substack{z=0\\ z'=0}}. \tag{5.7}$$

We observe that the replacement

$$\ln\chi^j(z,z')\to\ln\chi^j(z,z')+f(z')+f(z) \tag{5.8}$$

does not affect $E_{n\ m}^{\ j}$ $n,m\neq 0$. It does change E_{no} and E_{00}, but from momentum conservation this drops out of Eq. (2.30); the index zero, here, meaning no derivative at all.

Using Eq. (A.12), we find that

$$\begin{aligned} -\left(z|\Gamma\sum_\gamma{}^{jr} T_\gamma^j\left\{|\beta_r^j\right) - |\alpha_r^j)\right\} &= \sum_\gamma{}^r\left\{\ln\left(\frac{z-T_\gamma^j\beta_r^j}{z-T_\gamma^j\alpha_r^j}\right)\right. \\ &\left. -\ln\frac{T_\gamma^j\beta_r^j}{T_\gamma\alpha_r^j}\right\} = v_r^{\ j}(z)-v_r^{\ j}(0). \end{aligned} \tag{5.9}$$

Similarly, one finds that

$$\begin{aligned}\sum_{\substack{\gamma\\ \gamma\neq I}}(z'|\Gamma T_\gamma^{\ j}|z) &= \tfrac{1}{2}\sum_{\substack{\gamma\\ \gamma\neq I}}\{(z'|\Gamma T_\gamma^{\ j}|z)+(z\leftrightarrow z')\}\\ &= +\ln\chi^j(z,z')+\ln(z-z')+g(z)+g(z'),\end{aligned} \tag{5.10}$$

where g is a function of z or z' alone.

The period matrix can also be expressed in terms of the fixed points:

$$\tau_r^{\ s} = +\sum_\gamma{}^{r,s}\left\{\left[\left(\frac{1}{\beta_r^j}\right| - \left(\frac{1}{\alpha_r^j}\right|\right] T_\gamma^{\ j}[|\beta_s^j) - |\alpha_s^j)] + \delta_{r,s}\ln\omega_s\right\}. \tag{5.11}$$

It is straightforward to show that if we define the infinite dimensional matrix

$$\begin{aligned}\bar{E}_{n\ m}^{\ j} &= [\Gamma(\mathscr{P}^j-1)]_{nm} + \sum_{r,s}\left\{\sum_\gamma{}^r \Gamma T_\gamma^{\ j}(|\beta_r^j)-|\alpha_r^j))\right\}_n\\ &\quad\cdot(\tau^{-1})_r^{\ s}\sum_\delta{}^s\{[(\beta_s^j|-(\alpha_s^j|]\,\Gamma(T_\delta^j)^{-1}\},\end{aligned} \tag{5.12}$$

where

$$\mathscr{P}^j = \sum_\gamma T_\gamma^{\ j}, \tag{5.13}$$

then

$$(z|\bar{E}^j|z') = \ln\chi(z,z') + \{\text{a function of } z \text{ or } z' \text{ alone}\}, \tag{5.14}$$

which establishes that

$$\bar{E}_{n\ m}^{\ j} = \sqrt{n}\,E_{n\ m}^{\ j}\sqrt{m}. \tag{5.15}$$

The loop-cycling transformations for different lines are related by Eq. (2.21), and hence

$$\begin{gathered} V^k\mathscr{P}_n^{\ k}(V^k)^{-1} = V^j\mathscr{P}_n^{\ j}(V^j)^{-1},\\ \alpha_n^j = (V^j)^{-1}(V^k)\alpha_n^k;\qquad \beta_n^j = (V^j)^{-1}V^k\beta_n^k,\\ V^jT_\gamma^j(V^j)^{-1} = V^kT_\gamma^k(V^k)^{-1}. \end{gathered} \tag{5.16}$$

The period matrix, being a cross-ratio, is independent of j as the lack of such an index anticipated.

It is easily shown from Eq. (5.12) that

$$\widetilde{(\bar{E}^j(V^j)^{-1}V^k)}=\bar{E}^k(V^k)^{-1}V^j. \tag{5.17}$$

This completes all the additional steps required in order to establish Eq. (2.35).

To find the $a_0^i a_0^j$ piece we consider the Q^μ overlap:

$$0=V\{Q^{\mu(i)}(z)-Q^{\mu(j)}((V^j)^{-1}V^i(z))\}. \tag{5.18}$$

This proceeds as for the planar one-loop case. We consider V to contain V^{tree} of Eq. (3.13) with the appropriate V^i's and this takes care of the $a_0^\mu \ln$ terms in Q^μ. We are left with

$$\begin{aligned}0=V\Bigg\{&\sum_{k=1}^{N} a_0^k\{-((V^i)^{-1}V^k(0)|\bar{E}^i|z)\\&+((V^j)^{-1}V^k(0)|\bar{E}^j|(V^j)^{-1}V^i(z))-\mathscr{M}_{00}^{ik}+\mathscr{M}_{00}^{jk}\}\Bigg\}.\end{aligned} \tag{5.19}$$

Using similar manipulations to those in Eq. (3.9) we find that

$$0=V\sum_{k=1}^{N} a_0^k\{((V^j)^{-1}V^k(0)|\bar{E}^j|(V^j)^{-1}V^i(0))-\mathscr{M}_{00}^{ik}+\mathscr{M}_{00}^{jk}\}. \tag{5.20}$$

Equation (5.7) allows us to rewrite the parts of this equation which do not contain v_r's as

$$\begin{aligned}0=\sum_{k=1}^{N} a_0^k\Bigg\{&\ln\left[\frac{(V^j)^{-1}V^k(0)-(V^j)^{-1}V^i(0)}{\phi^j((V^j)^{-1}V^k(0),(V^j)^{-1}V^i(0))}\right]\\&-\ln\left[\frac{(V^j)^{-1}V^k(0)}{\phi^j((V^j)^{-1}V^k(0),0)}\right]-\ln\frac{(V^j)^{-1}V^i(0)}{\phi^j((V^j)^{-1}V^i(0),0)}\\&-\mathscr{M}_{00}^{ik}+\mathscr{M}_{00}^{jk}\Bigg\}.\end{aligned} \tag{5.21}$$

We note, however, that the first term in brackets can be written as

$$\begin{aligned}&-\tfrac{1}{2}\sum_{\substack{\gamma\\ \gamma\neq I}}\ln\frac{(T_\gamma^j(V^j)^{-1}V^i(0)-(V^j)^{-1}V^k(0))}{(T_\gamma^j(V^j)^{-1}V^i(0)-(V^j)^{-1}V^i(0))}\\&\cdot\frac{(T_\gamma^j(V^j)^{-1}V^k(0)-(V^j)^{-1}V^i(0))}{(T_\gamma^j(V^j)^{-1}V^k(0)-(V^j)^{-1}V^k(0))}=\ln\left[\frac{(V^i)^{-1}V^k(0)}{\phi^i((V^i)^{-1}V^k(0),0)}\right],\end{aligned} \tag{5.22}$$

since the terms in the first line are a cross-ratio and T_γ^j obeys Eq. (5.16). Applying a similar argument to the v_r piece we find that

$$\mathscr{M}_{00}^{ij}=\ln\chi^i((V^i)^{-1}V^j(0),0). \tag{5.23}$$

We summarize the result. The M-loop string scattering vertex for N external arbitrary excited states is given by

$$V = \langle_1 0| \dots \langle_N 0| \exp - \left[\sum_{ijj=1}^{N} \{\tfrac{1}{2}(a^i|\bar{E}^i(V^i)^{-1} V^j|a^j) + a_0^i((V^j)^{-1} V^i(0)|\bar{E}^j|a^j)\} + \sum_{1 \leqq i < j \leqq N} a_0^i a_0^j \ln \chi^i((V^i)^{-1} V^j(0), 0) \right] \cdot V^{\text{tree}}, \tag{5.24}$$

where

$$E_{n\ m}^{\ j} = \frac{1}{n!} \frac{1}{m!} \left(\frac{\partial}{\partial z}\right)^n \left(\frac{\partial}{\partial z'}\right)^m \ln \chi^j(z, z')|_{z'=z=0} \, . \tag{5.25}$$

χ^j is given in Eq. (5.2). We observe that the entire external state dependence is controlled by one function χ. The $a_0^i a_0^i$ piece is in generic agreement with [6, 21] and the two-loop case treated in [22] for the closed string.

To find the result for a particular diagram one must deduce the cycling transformations according to the discussion of Sect. 2. For the planar case we may use the $SL(2, R)$ invariance to fix the point $z_1 = 1$ and choose the two fixed points of the one-loop cycling transformation to be 0 and ∞. Hence the T_j are as for the one-loop case which are given in Eq. (4.14), the $P_1{}^j$ are as in Eq. (4.12), and the P_n, $n \geqq 1$ are arbitrary. The external state dependence for a tadpole diagram agrees after use of Eq. (5.12) with [23]. It only remains to compute the measure for the multi-loop case. This will be done elsewhere; however, we can already see its major features. One of the most important terms comes from the constant C^K of Eq. (2.29), which is proportional to E_{11}^K.

6. Conclusions

We have derived, following the new approach given in I, the one-loop planar contribution including the integration measure for the scattering of N arbitrarily excited string states. We have also found the external state dependence for any contribution in terms of one group theoretic function.

The method of I can also be applied to superstrings, and the resulting computation beginning with the tree diagrams will be reported elsewhere [24].

Although the method does not require ghost fields even for the computation of the measure, all the techniques used here could be extended to include them as indeed has already been carried out for some vertices [13, 25]. In this case the measure should emerge as a result of demanding that Q rather than L_n vanish on external legs when all states are physical.

The ease with which one can compute each perturbative effect leads one to hope that one can sum perturbation theory. Another approach to this would be to identify the associated "cycling transformations" for the summed result and deduce it directly. In gauge covariant string field theory [26], as in any Lagrangian field theory, there is a concrete procedure to compute non-perturbative effects. It would therefore be of interest to be able to translate back and forth between these methods and hopefully learn how non-perturbative methods look in the new approach.

At first sight, the need to add several contributions, i.e., planar non-planar, etc., to find the S-matrix for the open bosonic string may seem an undesirable feature. However, these various contributions have vertices which are analytic continuations of each other and so can be written as one term only. With a better understanding of the cycling transformations, this would probably occur naturally. It still remains to be investigated what are the most general classes of allowed cycling transformations. The loop cycling transformations are arbitrary. The external cycling transformations must themselves cycle into each other, their product multiply to one, lead to an $SL(2,R)$ result, i.e., be functions of cross-ratios, and lead to factorizable amplitudes. Further, we can alter them by an external gauge transformation involving L_0-1 and L_1. It would seem likely that the ones we in fact use can always be obtained from an arbitrary $SL(2,R)$ transformation when subject to the above constraints.

The method studied in this series of papers would seem to provide a quick calculational tool. It also has the advantage that it carries what would seem to be the minimum amount of baggage required to satisfactorily define string theory. To illustrate this point we note that in both gauge covariant string theory and the Polyakov approach, one is forced, almost at the outset, at the classical level to set $D=26$. However, one knows that in fact there is no unitarity problem with tree graphs, and the problem first occurs at the one-loop level where one encounters cuts and not poles, when one factorizes if $D=26$. This leads one to hope that the new method can be generalized to define new string theories such as Liouville string theories. One could also consider changing some of the group theory, such as changing $SL(2,R)$ by other subgroups of the conformal group, in particular $SL(2,Z)$ may be of interest for number theory considerations. It is interesting to observe that the method does not involve any use of a space-time metric. Clearly, it is not involved in the cycling transformations or in demanding unitarity; and although, the overlap identities involve μ indices they are not contracted. Of course, when one adopts the oscillator basis representation of the vertex in Minkowski space, the metric appears, but this is only a representation. In fact, the appearance of the Minkowski metric can be traced to its occurrence in the commutation relation of two oscillators.

The above speculation relates to a further point. Perhaps one of the most remarkable and little understood results is the vertex operator construction [27] of Lie groups. This feature is easy to see from the point of view of oscillator vertices, but is far from apparent from the point of view of overlap δ functions. This suggests that string theory may be best formulated in an algebraic manner. At least to these authors, one of the more attractive features of string theory is that it is beginning to look, especially within the approach studied here, that one can find a theory of physics in which a space-time manifold is not one of its prerequisites, but is, hopefully, a macroscopic effect.

Appendix A

Here we summarize for the convenience of the reader some of the group theory conventions described in the early literature [4] and heavily used in this paper.

Consider a transformation of the form:

$$T(z)=\frac{az+b}{cz+d}. \tag{A.1}$$

If the parameters are real it belongs to $SL(2, R)$ and if complex to $SL(2, C)$. We can define the infinite-dimensional matrix associated with this transformation by:

$$\frac{[T(z)]^n}{\sqrt{n}}=\sum_{m=1}^{\infty} T_{nm}\frac{z^m}{\sqrt{m}}+\frac{[T(0)]^n}{\sqrt{n}}. \tag{A.2}$$

If we define the infinite component column vector

$$|z)=\left(\frac{z}{1},\frac{z^2}{\sqrt{2}},\ldots,\frac{z^n}{\sqrt{n}},\ldots\right), \tag{A.3}$$

then Eq. (A.2) can be written:

$$|Tz)=T|z)+|T(0)), \tag{A.4}$$

where the obvious matrix multiplication is assumed. The transformation Γ is defined by

$$\Gamma(z)=\frac{1}{z}. \tag{A.5}$$

It will be more convenient when discussing vertices to use oscillators $a_n^\mu, a_n^{\mu+}$, $n\geqq 1$ which commute to 1 rather than n. That is

$$[a_n^\mu, a_m^{\nu+}]=\eta^{\mu\nu}\delta_{n,m}; \qquad n,m\geqq 1. \tag{A.6}$$

In this notation

$$Q^\mu(z)=\sum_{n=1}^{\infty}\left(-\frac{a_n^\mu z^{-n}}{\sqrt{n}}+\frac{a_n^{\mu+}z^n}{\sqrt{n}}\right)+a_0^\mu \ln z+\frac{\overleftarrow{\partial}}{\partial a_0^\mu}. \tag{A.7}$$

The shorthand

$$(a^i|T|a^j) \tag{A.8}$$

is taken to mean

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n^i T_{nm} a_m^j. \tag{A.9}$$

It can be shown that

$$(a^i|T|a^j)=(a^j|\tilde{T}|a^i), \tag{A.10}$$

where $\tilde{T}$ is the infinite matrix

$$\tilde{T}=\Gamma T^{-1}\Gamma. \tag{A.11}$$

A useful identity is given by

$$(z_1|z_2)=-\ln(1-z_1z_2). \tag{A.12}$$

As is well known the same $SU(1,1)$ transformation on all the points z_1, z_2, z_3, z_4 leaves invariant the quantity called the cross ratio,

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)}{(z_1 - z_4)} \frac{(z_2 - z_4)}{(z_2 - z_3)}. \tag{A.13}$$

A transformation of the form of Eq. (A.1) can have two, one, or only in the case of the identity, an infinite number of fixed points. In the case of two fixed points, it can be written in the form [27]

$$\frac{T(z) - \alpha}{T(z) - \beta} = \omega \frac{(z - \alpha)}{(z - \beta)}. \tag{A.14}$$

The fixed points α and β and the multiplier ω can be used to label the transformation. As an example of the use of the invariance of the cross ratio we show that $T' = STS^{-1}$ has the same multiplier as T. Clearly the fixed points of T' are $S(\alpha)$ and $S(\beta)$. Then we see that

$$\frac{(T'(z) - s(\alpha))}{(T'(z) - s(\beta))} \frac{(z - s(\beta))}{(z - s(\alpha))} = \omega' = \frac{(T(s^{-1}(z)) - \alpha)}{(T(s^{-1}(z)) - \beta)} \frac{(s^{-1}(z) - \beta)}{(s^{-1}(z) - \alpha)} = \omega. \tag{A.15}$$

The isometric circle associated with $T(z)$ of Eq. (A.1) is given by

$$|cz + d| = ad - bc,$$

and this is the unique place where infinitesimal lengths are preserved by the action of $T(z)$.

Appendix B

Consider $2M$ circles C_n and C'_n which are external to each other and let R_n be the $SL(2, C)$ transformation that takes C_n to C'_n in such a way that the region exterior to C_n is taken into the interior of C'_n (see Fig. 4). The group generated by the R_n is called a Schottky group, G_s [28, 29]. In fact the circles C_n and C'_n are nothing but the isometric circles of R_n and R_n^{-1} respectively. It is straightforward to show that since the T_n's have multipliers ω such that $|\omega| \neq 1$, the two fixed points of R_n are such that one lies in each of the circles C_n and C'_n. The fundamental region for such a group is the region exterior to all the $2M$ circles. For the case of closed strings the $2M$ circles lie anywhere in the complex plane, but for the open string the R_n's belong to $SL(2, R)$, so the circles lie with their diameters on the real axis. A further discussion of such groups can be found in [21, 28, and 29]. The relation of Schottky groups to Riemann surfaces is to simply identify the circles C_n and C'_n. This point is discussed in [6]. We now introduce some functions which are characterized by R_n and which will be used to construct the multi-loop function ϕ^j.

The first Abelian differentials are given by

$$v_n(z) = \sum_{\gamma}{}^{n} \ln \frac{(z - T_\gamma \beta_n)}{(z - T_\gamma \alpha_n)}, \tag{B.1}$$

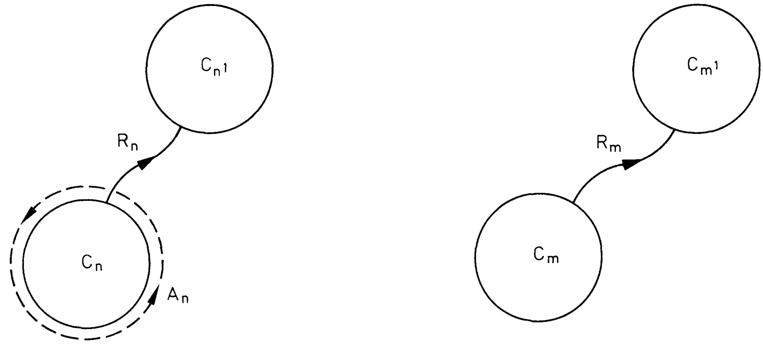

Fig. 4

where $\sum_{\gamma}^{n}$ means sum over all element of G_s except those that have a factor R_n or R_n^{-1} to the left. Another function of interest is given by

$$\ln\phi(z,z')=\ln(z-z')+\tfrac{1}{2}\sum_{\substack{\gamma\\ \gamma\neq I}}\ln\left[\frac{(T_\gamma(z)-z')}{(T_\gamma(z)-z)}\frac{(T_\gamma(z')-z)}{(T_\gamma(z')-z')}\right], \tag{B.2}$$

where $\phi(z,z')$ is essentially the prime form [30].

It is straightforward to show (see for example [21]) that

$$v_n(R_m z)-v_n(z)=\tau_{nm}, \tag{B.3}$$

where

$$\tau_{nm}=\sum_{\gamma}^{n,m}\ln\frac{(\beta_n-T_\gamma\beta_m)(\alpha_n-T_\gamma\alpha_m)}{(\alpha_n-T_\gamma\beta_m)(\beta_n-T_\gamma\alpha_m)}+\delta_{n,m}\ln\omega_n, \tag{B.4}$$

and $\sum_{\gamma}^{n,m}$ means a sum over all elements of G_s excluding those that have a R_m or R_m^{-1} on the right and a R_n and R_n^{-1} factor on the left.

As τ_{nm} is a cross-ratio it is a symmetric matrix and is called the period matrix. We also note that

$$v_n(A_m z)-v_n(z)=2i\pi\delta_{n,m}, \tag{B.5}$$

where A_m corresponds to a rotation by 2π about the centre of the isometric circle C_m.

Finally, one can show that [29, 21]

$$\phi(R_n z,z')-\phi(z,z')=-v_n(z)+v_n(z')-\frac{\tau_{nn}}{2}+\frac{i\pi}{2}-\ln(c_n z+d_n), \tag{B.6}$$

and

$$\phi(A_n z,z')-\phi(z,z')=0. \tag{B.7}$$

We have adopted a pedestrian approach to the above group theoretic functions, as it allows a simple derivation of the identities we require. The reader who has a taste for a more sophisticated language should have no difficulty in translating the above discussion.

Appendix C

In this appendix we consider the cycling transformations and measures of two non-planar one loop graphs, beginning with the non-orientable tadpole graph. We choose $z_1 = 1$ and P to have fixed points 0 and ∞; the multiplier is $-\omega$, $0 \leqq \omega \leqq 1$. The z's are shown in Fig. 5a.

One finds

$$P(z) = (V^2)^{-1}\, V^1(z) = \frac{z(1-\omega+\omega^2)-(1+\omega^2)}{-\omega + \dfrac{z\omega^2}{1+\omega^2}}. \tag{C.1}$$

Carrying out a gauge transformation with $\alpha = (1+\omega^2)^{-1}$, we obtain

$$P^{-1}(z) \to P^{-1}(z) = \frac{1}{1-\omega} - \omega z. \tag{C.2}$$

Following the discussion in Sect. 2 we find that the measure is

$$\bar{f} = \frac{1}{\omega^2}\,\frac{1}{(1-\omega^2)}\,\frac{1}{[f(-\omega)]^{D-2}}\,\frac{1}{(\ln\omega)^{D/2}}. \tag{C.3}$$

The external state dependence is given by (5.24) and is in accord with refs. [16 and 23].

Consider now the non-planar, but orientable self-energy. With $z_1 = 0$ and P with fixed points at 0 and ∞ and multiplier ω, $0 \leqq \omega \leqq 1$, the z's are found in Fig. 5b.

One finds that

$$P^{(1)}(z) = (V^1)^{-1}\,\omega^{-1}\, V^1(z) = \frac{-(1+\omega)^2}{z\omega-(1+\omega)^2}, \tag{C.4}$$

and

$$T_2(z) = (V^2)^{-1}\, V^1(z) = \frac{[z(1+x\omega)-(1+x)(1+\omega)]\,(1+\omega)}{[z(1+x\omega^2)-(1+\omega)(1+x\omega)]}. \tag{C.5}$$

Carrying out the gauge transformation $\alpha = (1+\omega)^{-1}$ on both legs one and two, one finds that the above cycling transformations become

$$P^{(1)}(z) = P^2(z) = 1 + \omega z, \tag{C.6}$$

$$T_2(z) = \frac{1+\bar{x}-(1-\omega)z\bar{x}}{(1-\omega)\,\bar{x}(1-(1-\omega)z)}, \tag{C.7}$$

where

$$\bar{x} = x^{-1}. \tag{C.8}$$

The measure comes out to be

$$\bar{f} = [\omega^2 f^{D-2}(\omega)\,[\ln\omega]^{D/2}\,(1-\omega)]^{-1}. \tag{C.9}$$

The external state dependence is given by (5.24) and it agrees with [16, 23].

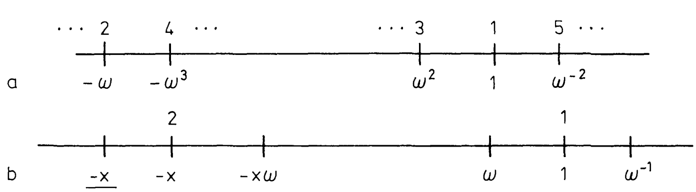

Fig. 5

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