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Rarefactions and Large Time Behavior for Parabolic Equations and Monotone Schemes*

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Abstract. We consider the large time behavior of monotone semigroups associated with degenerate parabolic equations and monotone difference schemes. For an appropriate class of initial data the solution is shown to converge to rarefaction waves at a determined asymptotic rate.

1. Introduction

Our main point of interest is the large time behavior of two solution operators, one continuous, the other discrete, when acting on a certain class of initial data.

The continuous example is the solution to the class of degenerate parabolic equations of the type

$$u_t + f(u)_x = A(u)_{xx}, \tag{1.1}$$

where u is scalar, f is convex and $A'(u) \geqq 0$. When $A(u) = |u|^{\gamma} \cdot u, \gamma > 0$, we have the convective porous medium equation.

The discrete example is the class of monotone difference schemes for the scalar conservation law ((1.1) with $A \equiv 0$). We write the scheme in the following way:

$$u^{n+1}(x) = u^n(x) - \lambda \Delta_d(g(u^n(x - p_0 d), \ldots, u^n(x + q_0 d)), \tag{1.2}$$

where we chose $x \in \mathbf{R}$ rather than on a mesh

$$\lambda = \frac{\Delta x}{\Delta t}, \quad (\Delta_d u)(x) = u(x) - u(x-d), \quad p_0 \geqq 0, \quad q_0 > 0, \quad d > 0,$$

and several conditions on the numerical flux g will be specified. The parameter d is not necessarily small.

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The scalar conservation law

$$u_t + f(u)_x = 0 \tag{1.3}$$

is invariant under the transformation

$$x = \frac{\tilde{x}}{v}, \quad t = \frac{\tilde{t}}{v}, \quad v > 0, \tag{1.4}$$

and it has continuous, self-similar solutions of the form

$$r(x,t) = R\left(\frac{x}{t}\right) = \begin{cases} u_- & \frac{x}{t} < a(u_-) \\ a^{-1}\left(\frac{x}{t}\right) & a(u_-) \leqq \frac{x}{t} \leqq a(u_+), \\ u_+ & a(u_+) < \frac{x}{t} \end{cases} \tag{1.5}$$

where $a(u) = f'(u)$ and $u_- < u_+$ are the values at $\mp\infty$; these solutions are called rarefactions. With respect to the variables $\tilde{x}, \tilde{t}$ defined in (1.4) Eq. (1.1) changes to

$$u_{\tilde{t}} + f(u)_{\tilde{x}} = vA(u)_{\tilde{x}\tilde{x}},$$

and its solutions are close to solutions of (1.3) when v is small.

For the monotone schemes in (1.2) this scaling procedure, in effect, changes d to vd, and the consistency with (1.3) is merely a consequence of the consistency of g with f.

We will prove that for a fairly large class of initial data, the error between solutions in (1.1) and (1.2) and the appropriate rarefactions tends to zero in $L^p, 1 < p \leqq \infty$. More specifically $u = R + K$ and $|K|_{L^p(dx)} \leqq c(\ln t)^{(1/2)+(1/2p)} t^{-(1/2)+(1/2p)}, 1 \leqq p \leqq \infty$, and the rate of decay for K, without the $\ln t$ term, is the real rate for Burger's equation (when $f(u) = 1/2u^2$ and $A(u) = u$).

In the next section we will prove a proposition which states the result in the more general framework of monotone semigroups that satisfy a consistency condition.

In L^1, the example of Burger's equation shows that we stay at a positive distance from rarefactions.

The complementary situation, when $u_- > u_+$ and (1.1), (1.2) admit travelling waves, was treated in Ref. [3] and [1]. It was shown there that these travelling wave solutions attract in L^1 a large class of initial data.

In [4], there are results about the L^∞ behavior of the equation $u_t + f(u)_x = cu_{xx}$, $c > 0$, without a rate.

2. Monotone Semigroups

For $u_- < u_+$ we define $U \subset L^\infty(R)$ by

$$U = \left\{ u \in L^\infty, u_- \leqq u(x) \leqq u_+, \int_{x<0} \left| \sup_{z \leqq x} u(z) - u_- \right| dx < \infty, \right.$$

$$\left.\int_{x>0}\left|\inf_{z\geqq x} u(z)-u_+\right|dx<\infty\right\}.$$

As in [1] we consider a semigroup $T(t)$, $t\in R_+$ or $\mathbf{Z}_+$, defined on U, and satisfying:

(1) $u\leqq v$ a.e. $\Rightarrow T(t)u\leqq T(t)v$ a.e. (monotone),
(2) $u-v\in L^1\Rightarrow T(t)u-v\in L^1$ (preserves L^1),
(3) $u-v\in L^1\Rightarrow \int_{-\infty}^{+\infty} T(t)u-T(t)v=\int_{-\infty}^{+\infty} u-v$ (conservative),
(4) $T(t)\tau_h=\tau_h T(t)$, $\tau_h u=u(x-h)$ (translation invariant).

A Lemma of Crandall and Tartar [9] shows that, given (2) and (3), the property (1) is equivalent to

(5) $\quad |T(t)u-T(t)v|_{L^1}\leqq|u-v|_{L^1}$, if $u-v\in L^1$ (L^1 – contractive).

With this we form $T^\alpha_\theta=\delta_{1/h}T(\alpha)\delta_h$, where $0<\alpha\leqq 1$, $\delta_h u=u(hx)$ and note that T^α_h is also an L^1-contraction. If $t\in\mathbf{Z}_+$, then α is by definition equal to 1.

The next condition makes $T(t)$ consistent with a self-similar solution. Suppose there exists $\rho(x)\in U$ which is Lipschitz continuous, $\rho'\geqq 0$, $|\rho'|_{L^\infty}<\infty$, and such that

(6) $$|T^\alpha_h\rho-\delta_{1/(1+\alpha h)}\rho|_{L^1}\leqq Ch^2.$$

Then,

Proposition. $|T(t)u-\delta_{1/t}\rho|_{L^p}\leqq C(\ln t)^{(1/2)+(1/2p)}t^{-(1/2)+(1/2p)}$, $t\geqq 1$, $1\leqq p\leqq\infty$, $u\in U$.

Remark. The constants C are not the same and they don't depend on h or t etc. Before proving the proposition, a few remarks about (6): We note that an equation which is invariant under (1.4) has a solution operator $T(t)$ which satisfies $\delta_{1/h}T(\alpha)\delta_h=T(\alpha h)$, and therefore the left-hand side of (6) is identically zero if $T(t)\rho=\delta_{1/(1+h)}\rho$. For Eq. (1.1), (6) represents the following local condition:

Let v satisfy

$$\begin{aligned} &v_t+f(v)_x=hA(v)_{xx},\\ &v(0,x)=R(x)=r(x,1)\ (\text{see } 1.5). \end{aligned}\tag{2.1}$$

Then

$$|v(\alpha h,x)-r(1+\alpha h,x)|_{L^1}\leqq Ch^2.$$

Here $\rho(x)=R(x)$.

For (1.2), to be consistent, we take

$$\rho(x)=R\left(\frac{x}{\lambda d}\right).\tag{2.2}$$

Condition (6) now amounts to the requirement that the local truncation error for consistent monotone schemes is of $O(h^2)$ in L^1, where h is the mesh-size. Since rarefactions are Lipschitz continuous with bounded derivatives, we will be able to prove this in Sect. 4.

Proof of Proposition. First, let $p=1$ and define:

$$u^t(x)=(T(t)u)(tx)=\delta_t T(t)u. \tag{2.3}$$

We then have the identity:

$$|u^t-\rho|_{L^1}=\frac{1}{t}|T(t)u-\delta_{1/t}\rho|,$$

and what we need to show is:

$$|u^t-\rho|_{L^1}\leqq C\frac{\ln t}{t}. \tag{2.4}$$

It suffices to consider $t=n\in Z^+$, since, with $t=n+\alpha$, for some $0<\alpha<1$,

$$\begin{aligned}|T(n+\alpha)u-\delta_{1/(n+\alpha)}\rho|_{L^1}&\leqq|T(n)u-\delta_{1/n}\rho|_{L^1}+|T(\alpha)\delta_{1/n}\rho-\delta_{1/(n+\alpha)}\rho|_{L^1}\\&=|T(n)u-\delta_{1/n}\rho|_{L^1}+n|\delta_n T(\alpha)\delta_{1/n}\rho-\delta_{n/(n+\alpha)}\rho|_{L^1}\\&\leqq|T(n)u-\delta_{1/n}\rho|_{L^1}+\frac{C}{n}.\end{aligned}$$

Next, dropping the L^1 subscript, and by (2.3),

$$\begin{aligned}|u^{n+1}-\rho|&=|\delta_{n+1}T(n+1)u-\rho|\\&=\frac{n}{n+1}|\delta_n T(1)\delta_{1/n}\delta_n T(n)u-\delta_{n/(n+1)}\rho|\\&\leqq\frac{n}{n+1}|T^1_{1/n}u^n-T^1_{1/n}\rho|+\frac{n}{n+1}|T^1_{1/n}\rho-\delta_{n/(n+1)}\rho|\\&\leqq\frac{n}{n+1}|u^n-\rho|+C\frac{n}{n+1}\cdot\frac{1}{n^2}\text{ by (6).}\end{aligned}$$

Assume that $|u^j-\rho|\leqq C(\ln j)/j$ for $2\leqq j\leqq n$ with C independent of u_0, where $|u_0-\rho|_{L^1}\leqq M$. This is true for $n=2$,

$$\begin{aligned}|u^{n+1}-\rho|&\leqq C\frac{\ln n}{n+1}+C\frac{1}{n(n+1)}\\&=C\left(\frac{\ln(n+1)}{n+1}+\frac{1}{n+1}\left(\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)\right)\right)\leqq C\frac{\ln(n+1)}{n+1},\end{aligned}$$

and the induction step is complete.

To prove the case $p=\infty$, we first observe that we can restrict our attention to $u\in U, u$ increasing, since for any $u\in U$, our definition of U allows for two functions $\varphi_l,\varphi_u\in U$, increasing, such that $\varphi_l\leqq u\leqq\varphi_u$. The monotonicity of $T(t)$ then yields

$$|T(t)u-r|_{L^\infty}\leqq|T(t)\varphi_l-r|_{L^\infty}+|T(t)u-T(t)\varphi_l|_{L^\infty}$$

$$\leqq |T(t)\varphi_l - r|_{L^\infty} + |T(t)\varphi_u - T(t)\varphi_l|_{L^\infty}, \quad \text{by Condition (1)}$$
$$\leqq 2|T(t)\varphi_l - r|_{L^\infty} + |T(t)\varphi_u - r|_{L^\infty}. \tag{2.5}$$

To continue the proof for $p=\infty$, we fix x_1 and let $l=\rho(x_1)-u^t(x_1)$ and without loss of generality let $l \geqq 0$. We also let $M=|\rho'|_{L^\infty}$ and $x_0 = x_1 - l/M$. Then,

$$\rho(x) \geqq u^t(x_1) + M(x-x_0), \quad x_0 \leqq x \leqq x_1,$$

since they are equal at $x=x_1$ and the derivative of the function on the right side of the equality is always bigger.

Since $u^t(x)$ is increasing ($T(t)$ preserves monotonicity),

$$u^t(x_1) + M(x-x_0) \geqq u^t(x_1) \geqq u^t(x), \quad x_0 \leqq x \leqq x_1,$$

and therefore,

$$|u^t - \rho|_{L^1} \geqq \int_{x_0}^{x_1} (\rho - u^t)dx \geqq \int_{x_0}^{x_1} M(x-x_0)dx = \frac{1}{2}\frac{l^2}{M}.$$

And now, since the L^∞ norm is invariant under δ_t,

$$|T(t)u - \delta_{1/t}\rho|_{L^\infty} = |u^t - \rho|_{L^\infty} \leqq \sqrt{2M|u^t-\rho|_{L^\infty}} \leqq C\left(\frac{\ln t}{t}\right)^{1/2}.$$

Finally, in L^p

$$|T(t)u - \delta_{1/t}\rho|_{L^\infty} \leqq |T(t)u - \delta_{1/t}\rho|_{L^\infty}^{1-1/p} |T(t)u - \delta_{1/t}\rho|_{L^1}^{1/p}$$
$$\leqq C(\ln t)^{(1/2p)+(1/2)} t^{-(1/2)+(1/2p)}.$$

3. Quasilinear Parabolic Equations

We consider (1.1) when $A(u)$ is smooth in (u_-, u_+), and it is differentiable with A' nonnegative and Lipschitz continuous in $[u_-, u_+]$. We have thus included the porous medium equation when $u_- = 0$. The results in Ref. [2], and the extensions in Ref. [1] show that there exists a unique solution operator satisfying (1)–(5) of Sect. 2.

Volpert and Hudjaev regularize the equation by adding artificial viscosity and they obtain estimates independent of the viscosity parameter. In the statement of their theorem they need more smoothness on $A(u)$. Osher and Ralston overcome this difficulty by modifying the initial data.

We let $\tilde{u}$ satisfy the following equation which incorporates both regularizations:

$$\tilde{u}_t + f(\tilde{u})_x = (A(\tilde{u}) + \nu\tilde{u})_{xx},$$
$$\tilde{u}(0,x) \in U_\varepsilon = \{\varphi \in U : u_- + \varepsilon \leqq \varphi \leqq u_+ - \varepsilon\}. \tag{3.1}$$

Standard results on parabolic equations yield smooth classical solutions to (3.1) [5].

Our claim is that it suffices to verify property (6), i.e. Eq. (2.1) for smooth solutions $\tilde{u}$ with a constant independent of ε and ν. To show this we use the estimates in Ref. [2] together with the arguments in Ref. [1] which yield

$$\int_{-\infty}^{+\infty} |u(x,t) - \tilde{u}(x,t)| w_\lambda(x) dx \leqq e^{K_\lambda t} \int |u(x,0) - \tilde{u}(x,0)| w_\lambda(x) dx,$$

where

$$w_\lambda(x) = \exp(-\lambda(1+x^2)^{1/2})$$

and

$$K_\lambda = \lambda \sup_{[u_-,u_+]} (|f'(u)| + (1+\lambda)A'(u)).$$

Now take $\tilde{u}(0,x) = R_\varepsilon(x) = r_\varepsilon(1,x)$, the rarefaction which connects $u_- + \varepsilon$ to $u_+ - \varepsilon$. Then

$$\begin{aligned} |u(\alpha h, x) - r(1+\alpha h, x)|_{L^1(w_\lambda dx)} &\leqq |u(\alpha h, x) - \tilde{u}(\alpha h, x)|_{L^1(w_\lambda dx)} \\ &\quad + |\tilde{u}(\alpha h, x) - r_\varepsilon(1+\alpha h, x)|_{L^1(w_\lambda dx)} \\ &\quad + |r_\varepsilon(1+\alpha h, x) - r(1+\alpha h, x)|_{L^1(w_\lambda dx)} \\ &\leqq e^{K_\lambda \alpha h} |r(1,x) - r_\varepsilon(1,x)|_{L^1(w_\lambda dx)} + Ch^2 \\ &\quad + |r_\varepsilon(1+\alpha h, x) - r(1+\alpha h, x)|_{L^1(w_\lambda dx)}. \end{aligned}$$

Now let $\varepsilon \to 0$ and then $\lambda \to 0$.

It remains, therefore, to show Eq. (2.1) when $A(u)$ is smooth and $A'(u) \geqq a_0 > 0$ in $[u_-, u_+]$ and that the constant C doesn't depend on a_0. This last part will become evident from the proof.

We let $\phi = v - r$. Then ϕ is a Lipschitz continuous function which satisfies

$$\begin{aligned} \phi_t + (f(r+\varphi) - (f(r))_x &= hA(v)_{xx}, \\ \phi(0,x) = 0, &\quad \text{see (2.1)).} \end{aligned} \tag{3.2}$$

We multiply (3.2) by a regularized sign function of ϕ[1], which is the derivative of a regularized absolute value function denoted by L_ε and defined as follows:

$$L_\varepsilon(z) = \begin{cases} |z| - \dfrac{\varepsilon}{2}, & |z| > \varepsilon \\ \dfrac{1}{2\varepsilon} z^2, & |z| < \varepsilon \end{cases}.$$

Then the regularized sign function is given by

$$L'_\varepsilon(z) = \begin{cases} \operatorname{sgn} z, & |z| > \varepsilon \\ \dfrac{1}{\varepsilon} z, & |z| < \varepsilon \end{cases},$$

[1] We thank the referee for suggesting the use of a regularized sign function. This replaces the original less elegant argument

and, from (3.2) we obtain:

$$\int_0^h\int_{-\infty}^{+\infty} L_\varepsilon(\phi)_t+\int_0^h\int_{-\infty}^{+\infty} L'_\varepsilon(\phi)[f(r+\phi)-f(r)]_x=h\int_0^h\int_{-\infty}^{+\infty} L'_\varepsilon(\phi)A(\phi+r)_{xx}. \quad (3.3)$$

The first term on the left, in (3.3), is $\int_{-\infty}^{+\infty} L_\varepsilon(\phi)(h)$ which tends to $|\phi|_{L^1(dx)}(h)$ as $\varepsilon\to 0$ by Lebesgue's Dominated Convergence Theorem.

The second term on the left, in (3.3), after integrating by parts, is equal to:

$$-\int_0^h\int_{-\infty}^{+\infty} L''_\varepsilon(\phi)\phi_x[f(r+\phi)-f(r)]=-\int_0^h\int_{-\infty}^{+\infty}\frac{1}{\varepsilon}\chi_{|\phi|<\varepsilon}[f(\phi+r)-f(r)]\phi_x.$$

The integrand above tends to zero pointwise and is dominated by $\sup_{u\in[u_-,u_+]} f'(u)\cdot|\phi_x|$. Therefore, by the Dominated Convergence Theorem the integral tends to zero as $\varepsilon\to 0$.

Finally, we consider the term on the right in (3.3). After integrating by parts and differentiating, we obtain:

$$-h\int_0^h\int_{-\infty}^{+\infty} L''_\varepsilon(\phi)\phi_x A'(r+\phi)(\phi_x+r_x)\leqq -h\int_0^h\int_{-\infty}^{+\infty} L''_\varepsilon(\phi)\phi_x A'(r+\phi)r_x,$$

since $L''_\varepsilon(\phi)\phi_x^2 A'(r+\phi)\geqq 0$. (We note that this is the only place where we used $A'\geqq 0$ and that we didn't need A' strictly positive.)

We now have

$$-h\int_0^h\int_{-\infty}^{+\infty} L'_\varepsilon(\phi)_x A(r)_x-h\int_0^h\int_{-\infty}^{+\infty} L''_\varepsilon(\phi)(A'(r+\phi)-A'(r))\phi_x r_x.$$

The second term tends to zero as $\varepsilon\to 0$ by virtue of the same Dominated Convergence Theorem, and the first term is estimated by

$$h\int_0^h |A(r)_x|_{BV(dx)}(\tau)d\tau\leqq h^2\sup_{0\leqq\tau\leqq h}|A(r)_x|_{BV(dx)},$$

where the $BV(dx)$ norm is defined by

$$|g|_{BV(dx)}=\sup_{h>0}\frac{1}{h}\int_{-\infty}^{+\infty}|g(x+h)-g(x)|dx.$$

In conclusion, after letting $\varepsilon\to 0$, (3.3) yields $|\phi|_{L^1(dx)}(h)\leqq Ch^2$, which is the desired estimate.

We close this section with the example mentioned in the introduction which uses Burger's equation:

$$u_t+(\tfrac{1}{2}u^2)_x=u_{xx},$$

$$u(0,x)=H(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases}.$$

Here one can solve explicitly, using the Cole–Hopf transformation, and obtain

$$u(t,x)=\frac{\int_0^{\infty} e^{[(x/t)-1]y/2}e^{-y^2/4t}\,dy}{\int_{-\infty}^{+\infty} e^{yx/2t-y_+/2}e^{-y^2/4t}\,dy},$$

where

$$y_+=\begin{cases} y & y\geqq 0\\ 0 & y<0\end{cases}.$$

Here the rarefaction is given by

$$r(x,t)=\begin{cases} 0 & \frac{x}{t}\leqq 0\\ \frac{x}{t} & 0\leqq \frac{x}{t}\leqq 1.\\ 1 & \frac{x}{t}\geqq 1\end{cases}$$

To obtain the asymptotic expansion of $u(x,t)$ for $x\leqq 0$ we let $s=x/t$ and integrate by parts in the numerator to obtain

$$\int_0^{\infty} e^{(s-1)y/2}e^{-y^2/4t}\,dy=\frac{2}{s-1}\left\{-1+O\left(\frac{1}{t}\right)\right\},$$

where $O(1/t)$ is uniform in $s\leqq 0$.

In the denominator the dominant term is:

$$\int_{-\infty}^{0}\exp\left(\frac{1}{2}\frac{x}{t}y-\frac{y^2}{4t}\right)dy=2\sqrt{t}\,e^{(x/2\sqrt{t})^2}\int_{-\infty}^{-x/2\sqrt{t}} e^{-y^2}\,dy.$$

Therefore, one obtains

$$u(t,x)=\frac{1}{\sqrt{t}\,e^{(x/2\sqrt{t})^2}\int_{-\infty}^{-x/2\sqrt{t}} e^{-y^2}\,dy}+K,$$

where $|K|_{L^p(\mathbf{R}^-)}\leqq ct^{-1+1/2p}$, $R^-=(-\infty,0)$, $1\leqq p\leqq\infty$. One can therefore verify that

$$|u-r|_{L^p(\mathbf{R})}(t)\geqq|u-r|_{L^p(\mathbf{R}^-)}(t)\geqq c_0t^{(1-p)/2p},\quad 1\leqq p\leqq\infty,\quad C_0>0.$$

4. Monotone Difference Schemes

We consider (1.2) and impose the following conditions on the numerical flux $g=g(u_{-p_0},\dots,u_{+q_0})$, a function of p_0+q_0+1 variables:

(a) $g(u,\dots,u)=f(u)$.

(b) g is Lipschitz continuous everywhere and $\partial g/\partial u_i$ are Lipschitz continuous in the domain $u_{-p_0}\leqq u_{-p_0+1}\leqq\cdots\leqq u_{q_0}$.

(c) The function $u_0 - \lambda(g(u_{-p_0}, \dots, u_{q_0}) - g(u_{-(p_0+1)}, \dots, u_{q_0-1}))$ is nondecreasing in each of its arguments $u_{-p_0-1}, \dots, u_{q_0}$.

For example, the Engquist-Osher (upwind) and Lax-Friedricks (dissipative) schemes, all satisfy these conditions which imply properties (1) through (5) of Sect. 2 for the solution operator. Unfortunately, Godunov's upwind scheme does not satisfy the second part of (6)[2].

For simplicity of notation take $\lambda d = 1$ so in (2.2) $\rho(x) = R(x)$. Then, condition (6) is equivalent to

$$|r(1+h,x) - r(1,x) - \Delta_d g(r(1,x-p_0 h), \dots, r(1,x+q_0 h))|_{L^1} \leqq Ch^2,$$

with $r(t,x) = R(x/t)$ from (1.5).

First, one easily verifies that

$$|r(1+h,x) - r(1,x) - hr_t(1,x)|_{L^1} \leqq Ch^2,$$

since the expression inside the L^1 norm is compactly supported, always bounded by Ch and bounded by Ch^2 in the smooth regions which are outside some neighborhoods of $a(u_-), a(u_+)$ of measure less than Ch.

Next, by the mean value theorem

$$\begin{aligned} &\Delta_d g(R(x-p_0 h), \dots, R(x+q_0 h)) \\ &= h \sum_{i=-p_0}^{q_0} \int_0^1 \frac{\partial g}{\partial u_i}(\dots \theta R(x-ih) + (1-\theta)R(x-(i+1)h) \dots) d\theta R'(x-(i+\eta_i)h), \end{aligned}$$

for some $0 \leqq \eta_i \leqq 1$,

$$\begin{aligned} &= h \sum_i \frac{\partial g}{\partial u_i}(R(x), \dots, R(x)) \cdot R'(x) \\ &\quad + h \sum_i \int_0^1 \frac{\partial g}{\partial u_i}(\dots \theta R(x-ih) + (1-\theta)R(x-(i+1)h) \dots) \\ &\quad - \frac{\partial g}{\partial u_i}(R(x), \dots, R(x)) d\theta R'(x-(i+\eta_i)h) \\ &\quad + h \sum \frac{\partial g}{\partial u_i}(R(x), \dots, R(x))(R'(x-(i+\eta_i)h) - R'(x)) = hf(R)_x + K_0. \end{aligned}$$

Because of our assumptions on g and since $|R'|_{BV} \leqq C$, we get $|K_0|_{L^1} \leqq Ch^2$ and the result follows.

In closing, we wish to mention that the result of the Proposition yields the following L^p rate of convergence to rarefactions for monotone schemes:

$$|u^h - R|_{L^p} \leqq Ch^{(1/2)+(1/2p)} (\ln 1/h)^{(1/2)+(1/2p)}, \quad 1 \leqq p \leqq \infty,$$

where h is the mesh size (which is related to the number of iterations in time), and $u^h(0,x) = \varphi(x/h)$, $\varphi \in U$.

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[2] We thank the referee for pointing this out

The well-known results on convergence of monotone schemes [6–8] hold for general $L^1 \cap BV \cap L^\infty$ initial data. Our rate of convergence, $h\ln(1/h)$ in L^1, is an improvement over the previous rate, $h^{1/2}$, given in Refs. [6, 8]. This is because, for our special case, it was possible to adopt a more direct type of proof of convergence.

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