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A Block Spin Construction of Ondelettes* Part II: The QFT Connection

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Abstract. We apply the Lemarié basis of ondelettes to the Battle–Federbush cluster expansion for the ϕ_3^4 quantum field theory. Since there is no infrared problem for this model, we also show how the large-scale ondelettes can be thrown away and replaced by unit-scale functions. Finally, we apply the block spin machine of Part I to the construction of exponentially localized ondelettes orthogonal with respect to the free, massless action of the scalar field.

1. Introduction

Ondelettes [1–5] have an important application to phase cell cluster expansions in quantum field theory [6–11]. They provide a very natural setting for the expansion of any interacting vacuum expectation that can be regarded as a small perturbation of a Gaussian expectation. The minimum-scale and finite-volume cutoffs are simultaneously removed by such an expansion because elementary stability is the only kind of stability needed. Moreover, such a decomposition of phase space has renormalization group ideas built into it.

A cluster expansion was developed in [7] which was based on expansion functions (a certain Bessel potential of certain L^2-ondelettes) with respect to which the free Euclidean boson field ϕ with mass m has a diagonal covariance. Thus the expansion decouples only the interaction when applied to the ϕ_d^{2n} and Y_d models. The specific ondelettes that were used in [7], however, suffered from a serious lack of regularity that affected the positivity of the ϕ_3^4 interaction with respect to the random variables corresponding to such expansion functions. Consequently, the diagonal-covariance expansion was applied to a hierarchical version [8] of the ϕ_3^4 model. Williamson [9] controlled the ϕ_3^4 model by using the L^2-ondelettes directly as expansion functions and decoupling the resulting non-diagonal covariance as well as the interaction. This choice solves the positivity problem for the Battle–Federbush ondelettes.

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Very recently Lemarié [4] discovered a new basis of L^2-ondelettes that happens to eliminate the error in [7] and makes the Battle–Federbush expansion directly applicable to the ϕ_3^4 field theory. We presented a block-spin construction of the Lemarié basis in [5]. In this paper we show how these functions affect the details in [7]. The point is that the subsequent changes in [7] are almost trivial, so the Battle–Federbush work [7, 8] essentially anticipates the existence of Lemarié functions.

In this paper we also introduce a basis of ondelettes orthogonal with respect to the massless Sobelev norm $\|\vec{\nabla}\varphi\|_2$ and having the same regularity and long-distance decay properties as Lemarié functions. This result is non-trivial because the $|\vec{\nabla}|^{-1}$ potential of the Lemarié functions cannot have the same long-distance decay properties. Our motivation for constructing such a basis is that it should be useful in analyzing small field modes for the group-valued σ-model. The relevance of this model to an understanding of Yang–Mills theory is discussed in [12, 13]. We state the small-field stability expected for our variables in Sect. 4.

We start with a preliminary description of how the ondelettes of Lemarié are applied to the boson field ϕ of mass m. Since there is no infrared problem to consider here, we impose a large-scale cutoff on the basis—say at the unit-scale level—and complete the basis on that scale with modified functions that have the same regularity and long-distance decay properties as the smaller scale functions, but do not necessarily have vanishing moments. This general game has already been discussed in [10], and the original basis used by Battle and Federbush is precisely of this modified type [7]. In the next section we describe this modification for Lemarié functions.

We expand the field in modes:

$$\phi(x) = \sum_k \alpha_k u_k(x), \tag{1.1}$$

where

$$u_k = (-\Delta + m^2)^{-1/2}\psi_k \tag{1.2}$$

and $\{\psi_k\}$ is the modified basis of L^2-ondelettes. Equivalently, the phase cell variables α_k are defined by

$$\alpha_k = (\phi, (-\Delta + m^2)^{1/2}\psi_k), \tag{1.3}$$

and so

$$\langle \alpha_k \alpha_l \rangle_0 = \delta_{kl}, \tag{1.4}$$

where $\langle \cdot \rangle_0$ denotes the free (Gaussian) boson expectation. If A is a finite set of modes, then the corresponding regularization ϕ_A of the field ϕ is given by

$$\phi_A(x) = \sum_{k\in A} \alpha_k u_k(x). \tag{1.5}$$

In Sect. 2 we take $\{\psi_k\}$ to be the modified Lemarié basis and verify the following inequalities.

Theorem 1.1. *For $\varepsilon, \varepsilon' > 0$ there is a constant $c > 0$ independent of A such that*

$$\int |\vec{\nabla}\phi_A|^2 + m^2 \int \phi_A^2 + \int \phi_A^4 \geqq c \sum_{k\in A} L_k^{4-d+\varepsilon'} |\alpha_k|^{4-\varepsilon}, \tag{1.6}$$

where $L_k = 2^{-r_k}$ is the scale of ψ_k.

Theorem 1.2. *There is a constant $c > 0$ independent of k such that*

$$|u_k(x)| \leqq c e^{-\varepsilon|x - x^{(k)}|} \tag{1.7}$$

and

$$|u_k(x)| \leqq c L_k^{1-d/2}\left(1 + \frac{|x - x^{(k)}|}{L_k}\right)^{-N}, \tag{1.8}$$

where $x^{(k)}$ is the center of the cube associated with k and N can be made arbitrarily large by adjusting a construction parameter.

Remark. Theorem 1.1 appears as Theorem 6.1 in [7], and it is false for the expansion functions used there. This is the positivity problem mentioned above. We do not re-visit the stability result (Theorem 7.2 in [7]) because its proof is more abstract and does not depend on the orthonormal basis used. (For further results in that direction, see Lieb [14].) Theorem 1.2 is just a re-phrasing of Estimates 3.2 and 3.3 in [7], but the proof cannot be the same because the Lemarié functions are not sharply localized.

Note. A couple of months after circulating the preprint version of this paper, the author was informed that Ingrid Daubechies had just discovered an L^2-orthonormal basis of class C^r ondelettes with compact support! While this basis is not vital for our purposes, it is interesting that the kind of basis that we originally wanted in [7] really exists! We do not know whether this remarkable result can be extended to massless Sobelev norms.

2. Application to ϕ_3^4

Before we prove the theorems, we describe the modified Lemarié functions. For $L_k \leqq 1$ we have

$$\Phi_k(x) = L_k^{-d/2}\Phi_l(L_k^{-1}x) \tag{2.1}$$

for some mode l for which $L_l = 1$. Φ_l is given by l for which $L_l = 1$. Φ_l is given by

$$\hat{\Phi}_l(p) = \exp\left(i2\sum_{\mu=1}^{s} m_\mu p_\mu + i\sum_{\mu=s+1}^{d} m_\mu p_\mu\right)\hat{\Phi}_s(p), \tag{2.2}$$

where $l = (s, m)$ and Φ_s is given by (4.7) in [5]. We now propose to complete the basis at the $L_k = 1$ level. Recall that in [5] the first stage in the construction of the Φ_l was to solve the $\|\varphi\|_2^2$-minimization problem for block spin assignments that look like

$$\text{o o o o} + - \text{o o o o o o}$$

in the preferred coordinate x_s, like

$$\text{o o o o} + + \text{o o o o o o}$$

in the coordinates x_μ for $\mu < s$, and like

$$\text{o o o o} + \text{o o o o o o o}$$

in the coordinates x_μ for $\mu > s$. What we do now is use only the last type of block spin assignment in all coordinates at the $L_k = 1$ level. By checking the argument in

Sect. 6 of [5], one can easily convince oneself that the resulting basis is complete. Our basis $\{\psi_k\}$ is given by

$$\psi_k = \Phi_k, \quad L_k < 1, \tag{2.3}$$

$$\hat{\psi}_k(p) = \exp\left(i \sum_{\mu=1}^{d} m_\mu p_\mu \right) \hat{\Phi}(p), \quad L_k = 1, \tag{2.4}$$

where $k = m$ in the latter case and

$$\hat{\Phi}(p) = \prod_{\mu=1}^{d} \frac{(-1)^{(1/2)M} e^{(1/2)Mp_\mu} \hat{\chi}(p_\mu)^{M+1}}{\left[\sum_{n=-\infty}^{\infty} |\hat{\chi}(p_\mu + 2\pi n)|^{2M+2} \right]^{1/2}} \tag{2.5}$$

with χ as the characteristic function of $[0, 1]$ and M the even integer adjusted to obtain whatever degree of smoothness one desires for the basis functions. The only price paid for this modification is that the unit scale functions do not have vanishing moments.

We turn our attention to the proof of Theorem 1.1. The main part of the argument is exactly as in [7] and reduces to

Lemma 2.1. *For $D = \sqrt{-\Delta + m^2}$ we have*

$$|(D^j \psi_k)(x)| \leqq c L_k^{-j-d/2} \left(1 + \frac{|x - x^{(k)}|}{L_k} \right)^{-\bar{N}} \tag{2.6}$$

if $j < M$.

The parallel statement in [7] is Estimate 3.5, which is false for the ondelettes used there; the functions are discontinuous, so fractional derivatives cannot lie in L^∞. For the ondelettes used here we have as much regularity as we want; we need only adjust M (which also determines the size of $\bar{N}$).

Proof of Lemma 2.1. The estimate is obvious for $L_k = 1$. Indeed we have exponential decay, as can be seen from the real analyticity of $D(p)^j \hat{\Phi}(p)$. Now consider the case $L_k < 1$. By (2.1–2.3) the desired estimate reduces to establishing the bound

$$|(D_k^j \Phi_s)(x)| \leqq c(1 + |x|)^{-\bar{N}}, \tag{2.7}$$

where

$$D_k = \sqrt{-\Delta + L_k^2 m^2}, \tag{2.8}$$

so the key is to look at the behavior of

$$\frac{\partial^{|\alpha|}}{\partial p^\alpha} (D_k(p)^j \hat{\Phi}_s(p)) \tag{2.9}$$

for multi-indices α. The point is that for large p the quantity is integrable for $j < M$ and for small p it is integrable for $|\alpha| < M + 1 + j + d$. The latter observation is a consequence of the property

$$\hat{\Phi}_s(p) = O(p_s^{M+1}), \tag{2.10}$$

which is easily inferred from (4.7) of [5]. (We are also appealing to the fact that

$$\left|\frac{\partial^{|\beta|}}{\partial p^{\beta}} D_k(p)^j\right| \leqq c|p|^{j-\beta} \tag{2.11}$$

uniformly in k.) □

Finally we consider Theorem 1.2 and see that there is no more work to be done. The above argument with $j=-1$ establishes the bounds (1.7) & (1.8).

We close this section with a remark on the importance of exponential localization for the functions ψ_k. Actually it is not the scale-commensurate exponential decay of the ψ_k that we care about in this context. The unit-scale exponential decay of the functions u_k—i.e., Estimate (1.7)—is what we really want, and even that property is needed only to establish exponential decay of vacuum correlations. One can easily check [7] to see that only the scale-commensurate power law decay (1.8) is needed for convergence of the phase cell cluster expansion.

In [7] one never sees the issue of exponential localization because the ψ_k functions introduced there are sharply localized. Estimate (1.7) is even more trivial in that case. Indeed, it was not even mentioned, because in [7] one is exclusively interested in convergence of the multi-scale cluster expansion. If one wishes to verify exponential decay of correlations, it is necessary to estimate quantities of the form

$$\left\langle \prod_{i=1}^{m} \phi(f_i) \prod_{j=1}^{n} \phi(g_j) \right\rangle = \sum_{\substack{k_1,\dots,k_m \\ l_1,\dots,l_n}} \prod_i (u_{k_i}, f_i) \prod_j (u_{l_j}, g_j) \cdot \langle \prod_i \alpha_{k_i} \prod_j \alpha_{l_j} \rangle, \tag{2.12}$$

where $f_1,\dots,f_m,\ g_1,\dots,g_n \in C_0^\infty$ and (following Glimm and Jaffe [15]) we may assume m, n odd, so that

$$\langle \prod_i \phi(f_i) \rangle = \langle \prod_j \phi(g_j) \rangle = 0. \tag{2.13}$$

The name of the game is to obtain an exponential decay bound:

$$|\langle \prod_i \phi(f_i) \prod_j \phi(g_j) \rangle| \leqq c e^{-\varepsilon \operatorname{dist}(A,B)}, \tag{2.14}$$

where $A = \bigcup_i \operatorname{supp} f_i$, $B = \bigcup_j \operatorname{supp} g_j$. Now by (1.7) we have

$$|(u_k, f_k)|^{1/2} \leqq c e^{-\varepsilon \operatorname{dist}(A, x^{(k)})}, \tag{2.15}$$

$$|(u_l, g_j)|^{1/2} \leqq c e^{-\varepsilon \operatorname{dist}(x^{(l)}, B)}. \tag{2.16}$$

On the other hand,

$$\begin{aligned} \sum_k |(u_k, f_i)|^{1/2} &= \sum_k |(u_k, f_i)|^{1/4} |((-\Delta+m^2)^{-M} u_k, (-\Delta+m^2)^M f_i)|^{1/4} \\ &\leqq \sum_{\vec{n}} e^{-\varepsilon|\vec{n}|} \sum_{k: x^{(k)} \in \operatorname{supp} f_i + \vec{n}} \|(-\Delta+m^2)^{-M} u_k\|_\infty^{1/4} \|(-\Delta+m^2)^M f_i\|_1^{1/4} \\ &\leqq c \sum_{k: x^{(k)} \in \operatorname{supp} f_i} \|(-\Delta+m^2)^{-M} u_k\|_\infty^{1/4}. \end{aligned} \tag{2.17}$$

By the obvious generalization of (1.8), (2.17) implies

$$\sum_k |(u_k, f_i)|^{1/2} \leqq c \sum_{k: x^{(k)} \in \operatorname{supp} f_i} L_k^{d+\varepsilon} \leqq c |\operatorname{supp} f_i| \tag{2.18}$$

if M is chosen large enough. Taking these elementary estimates together, we see that (2.14) would follow from

$$|\langle \prod_i \alpha_{k_i} \prod_j \alpha_{l_j} \rangle| \leqq c e^{-\varepsilon \min |x^{(k_i)} - x^{(l_j)}|} \tag{2.19}$$

This estimate can be proven by a re-examination of the phase cell cluster expansion, where the convergence proof is modified by judicious use of (1.7) and the observation

$$\langle \prod_i \alpha_{k_i} \rangle = \langle \prod_j \alpha_{l_j} \rangle = 0. \tag{2.20}$$

The details are case-dependent, but straightforward.

3. Massless Sobelev Ondelettes

In this section we produce a basis of ondelettes (with no large-scale cutoff) that share all of the properties of the Lemarié functions but are orthogonal with respect to the norm $\|\vec{\nabla}\varphi\|_2$. To put it another way, we construct expansion functions for the scalar field ϕ with respect to which the free massless action $\int(\vec{\nabla}\phi)^2$ is diagonal. Since the $|\vec{\nabla}|^{-1}$ potential of the Lemarié functions are not exponentially localized, we must return to our block spin machine to obtain such a basis.

The first stage in our block spin construction is to minimize $\|\vec{\nabla}\varphi\|_2^2$ with respect to the constraints

$$\prod_{\mu=1}^{d} \left(\int dp_\mu\right)\left(\prod_{\mu=1}^{d} p_\mu^{-1}\right)^M \hat{\varphi}(p)\overline{\hat{\chi}_\Gamma(p)} = \sigma_s(\Gamma), \tag{3.1}$$

where—as in [5]—we consider without loss the s^{th} sub-level of the unit-scale level and the 0^{th} translate of the standard block spin assignment at that level. Our notation here is exactly the same as in [5]; so are the constraints, for that matter. Only the quadratic form we seek to minimize is different.

By checking the corresponding derivation in [5], it is easy to see that the solution φ_s is given by

$$\hat{\varphi}_s(p) = g_s(p)^{-1} p^{-2} \prod_{\mu=1}^{s-1} [\hat{\chi}(2p_\mu)^{M+1}(1-e^{-i2p_\mu})^M] \hat{\chi}(p_s)^{M+1}(1-e^{-ip_s})^{M+1}$$
$$\prod_{\mu=s+1}^{d} [\hat{\chi}(p_\mu)^{M+1}(1-e^{-ip_\mu})^M], \tag{3.2}$$

where χ is the characteristic function of $[0,1]$ and

$$g_s(p) = \sum_n \frac{\prod_{\mu=1}^{s-1} |\hat{\chi}(2p_\mu + 2\pi n_\mu)|^{2M+2} \prod_{\mu=s}^{d} |\hat{\chi}(p_\mu + 2\pi n_\mu)|^{2M+2}}{(p_1+\pi n_1)^2 + \cdots + (p_{s-1}+\pi n_{s-1})^2 + (p_s + 2\pi n_s)^2 + \cdots + (p_d + 2\pi n_d)^2}. \tag{3.3}$$

Remark. The constraints (3.1) are not given directly by bounded linear functionals, but (3.2) is based on the bounded linear constraints implied. The "well-posed form" of the problem reads almost the same as (2.8) in [5], except there is an additional factor of $1 - e^{ip_s}$ in the integrand, and so $P_M^-(-m_s) - P_M^-(1-m_s)$ is replaced by

$P^-_{M+1}(-m_s) - P^-_{M+1}(1-m_s)$. Thus the linear functionals we use here are bounded with respect to $\|\partial_s\varphi\|_2 \leqq \|\vec{\nabla}\varphi\|_2$. However, this does not affect the power of $1-e^{-ip_s}$ appearing in the expression for $\hat{\varphi}_s(p)$ above.

Actually, what we have done here can be regarded as a continuum version of what Gawedzki and Kupiainen [16] originally did to the free part of the lattice $(\vec{\nabla}\phi)^4$ model. Of course, the constraints are modified here to obtain solutions with the desired degree of smoothness.

The second stage of our construction is to carry out the translation-invariant orthogonalization on each of the orthogonal levels obtained from this constrained minimization. Again, without loss we consider only the s^{th} sub-level of the unit-scale level—i.e., we consider the subspace generated by the $(2m_1,\ldots,2m_s,m_{s+1},\ldots,m_d)$-translates of φ_s. The overlap matrix for our Sobelev inner product is

$$a_{m-m'} = \prod_{\mu=1}^{d} \left(\int dp_\mu\right) p^2 \hat{\varphi}_{s,m}(p)\overline{\hat{\varphi}_{s,m'}(p)}, \tag{3.4}$$

where

$$\hat{\varphi}_{s,m}(p) = \exp\left(i2\sum_{\mu=1}^{s} m_\mu p_\mu + i\sum_{\mu=s+1}^{d} m_\mu p_\mu \right)\hat{\varphi}_s(p). \tag{3.5}$$

Taking the inverse square root of the overlap matrix, we obtain a basis for the subspace that is orthogonal with respect to the Sobolev inner product. The functions are

$$\hat{\Phi}_{s,m}(p) = h_s(p)^{-1/2}\hat{\varphi}_{s,m}(p), \tag{3.6}$$

where

$$h_s(p) = \sum_l \left[\sum_{\mu=1}^{s} (p_\mu + \pi l_\mu)^2 + \sum_{\mu=s+1}^{d} (p_\mu + 2\pi l_\mu)^2 \right] \cdot |\hat{\varphi}_s(p_1+\pi l_1,\ldots,p_s+\pi l_s,p_{s+1}+2\pi l_{s+1},\ldots,p_d+2\pi l_d)|^2. \tag{3.7}$$

Since M is chosen to be an even integer, it follows that

$$\hat{\Phi}_s(p) = (-1)^{(1/2)(d-1)M}\eta_s(p)^{-1/2} g_s(p)^{-1} p^{-2} \prod_{\mu=1}^{s-1} \left(e^{iMp_\mu}\hat{\chi}(2p_\mu)^{M+1}\right) \cdot (1-e^{-ip_s})^{M+1}\hat{\chi}(p_s)^{M+1} \prod_{\mu=s+1}^{d} \left(e^{i(1/2)Mp_\mu}\hat{\chi}(p_\mu)^{M+1}\right), \tag{3.8}$$

$$\eta_s(p) = \sum_l \left[\sum_{\mu=1}^{s} (p_\mu+\pi l_\mu)^2 + \sum_{\mu=s+1}^{d} (p_\mu+2\pi l_\mu)^2 \right]^{-1} \cdot g_s(p_1+\pi l_1,\ldots,p_s+\pi l_s,p_{s+1}+2\pi l_{s+1},\ldots,p_d+2\pi l_d)^{-2} \cdot \left(\prod_{\mu=1}^{s-1} |\hat{\chi}(2p_\mu+2\pi l_\mu)|^{2M+2} \right) (|\hat{\chi}(p_s+\pi l_s)||1-e^{i(p_s+\pi l_s)}|)^{2M+2} \cdot \left(\prod_{\mu=s+1}^{d} |\hat{\chi}(p_\mu+2\pi l_\mu)|^{2M+2} \right). \tag{3.9}$$

Having constructed our basis, we have to convince ourselves that it has all of the desired properties. Most of the arguments for Lemarié functions carry over with trivial changes. Indeed, only the proof of exponential localization is more subtle, and we devote the remainder of the section to this issue.

The exponential fall-off of Φ_s follows from the real analyticity of $\hat{\Phi}_s(p)$. More precisely, we need:

Theorem 3.1. *$\hat{\Phi}_s(p)$ extends to an analytic function bounded in each z_μ uniformly in $(p)_\mu$ by an integrable function of $p_\mu = \operatorname{Re} z_\mu$ on the strip $|\operatorname{Im} z_\mu| < \delta$ for some $\delta > 0$, where $(p)_\mu = (p_1, \ldots, p_{\mu-1}, p_{\mu+1}, \ldots, p_d)$.*

Proof. It is straightforward to see that the proof reduces to proving analyticity and uniform boundedness of $g_s(p)^{-1}p^{-2}$ and $\eta_s(p)^{-1/2}$ in each variable on such a strip (see the elementary estimates at the beginning of the proof of Theorem 5.1 in [5]). Now it is clear from (3.3) that $g_s(p)$ is a strictly positive, periodic function that is continuous except for second-order poles at $(\pi n_1, \ldots, \pi n_{s-1}, 2\pi n_s, \ldots, 2\pi n_d)$. On the other hand, $g_s(p)$ extends to a meromorphic function of each variable. Thus the zeros in each variable (with the other variables held fixed) lie off the real axis. Consider a variable z_μ and then appeal to:

Lemma 3.2. *Let $J((p)_\mu, z_\mu)$ be the meromorphic function defined by $J((p)_\mu, p_\mu) = g_s(p)$. There is some strip $|\operatorname{Im} z_\mu| < \delta$ with $\delta > 0$ independent of $(p)_\mu$ on which $|J((p)_\mu, z_\mu)|$ is bounded below by some small positive constant.*

Proof. Since $J((p)_\mu, z_\mu)$ is real-periodic in *all* of the variables and continuous everywhere except at the second-order poles, the $(p)_\mu$-dependent z_μ-zeros are not only off the real axis, but bounded away from it uniformly in $(p)_\mu$. To put it another way, the $(d-1)$-dimensional manifolds of zeros in $\mathbb{R}^{d-1} \times \mathbb{C}$ are separated from $\mathbb{R}^d$ by hyperplanes $\mathbb{R}^{d-1} \times (\mathbb{R} \pm i\delta)$. Appealing to the real-periodicity again, we conclude that $J((p)_\mu, z_\mu)$ is bounded away from zero between these hyperplanes. □

Proof of Theorem 3.1. (continued) Since

$$|J((p)_\mu, z_\mu)| \geqq c \left| \sum_{l \neq \mu} p_l^2 + z_\mu^2 \right|^{-1} \tag{3.10}$$

for small values of $\operatorname{Im} z_\mu$ and of such a sum, we may extend the lemma to

$$\left(\sum_{l \neq \mu} p_l^2 + z_\mu^2 \right) J((p)_\mu, z_\mu), \tag{3.11}$$

and so the uniform boundedness and analyticity of the reciprocal holds on the strip $|\operatorname{Im} z_\mu| < \delta$ because the second-order poles of (3.11) become zeros. We now have the desired conclusion for $g_s(p)^{-1}p^{-2}$.

Now in the expression (3.9) for $\eta_s(p)$ it is important to bear in mind that $p^{-2}g_s(p)^{-2}$ has zeros only at the points $(\pi n_1, \ldots, \pi n_{s+1}, 2\pi n_s, \ldots, 2\pi n_d)$, while (3.9) is a sum over $(\pi l_1, \ldots, \pi l_s, 2\pi l_{s+1}, \ldots, 2\pi l_d)$-translations. Hence $\eta_s(p)$ is strictly positive as well as periodic (as can be seen by inspecting the remainder of the expression). Let $K((p)_\mu, z_\mu)$ be defined as the analytic continuation of $K((p)_\mu, p_\mu) \equiv \eta_s(p)$. Since this function is analytic on the strip $|\operatorname{Im} z_\mu| < \delta$, it follows from its real-periodicity that $\operatorname{Re} K((p)_\mu, z_\mu)$ is bounded below by some positive constant on some strip $|\operatorname{Im} z_\mu| < \delta'$ with both constants independent of $(p)_\mu$. Thus $K((p)_\mu, z_\mu)^{-1/2}$ is analytic and bounded on this strip uniformly in $(p)_\mu$. □

We conclude this section with the remark that if we modify this basis in the same manner that we modified the Lemarié basis in Sect. 2, the unit-scale functions will not have exponential decay. (Indeed, we cannot expect them to, for this is a complete set of expansion functions with respect to which a *massless* action is diagonal, and a large-scale cutoff will rule out arbitrarily slow exponential fall-off.) If we complete the basis at the unit-scale level, our block spin assignments are the same as in Sect. 2, and the unit-scale functions Φ_m are given by

$$\hat{\Phi}_m(p) = \exp\left(i \sum_{\mu=1}^{d} m_\mu p_\mu \right) \hat{\Phi}(p), \tag{3.12}$$

$$\hat{\Phi}(p) = (-1)^{(1/2)Md} \eta(p)^{-1/2} g(p)^{-1} p^{-2} \prod_{\mu=1}^{d} (e^{i(1/2)Mp_\mu} \hat{\chi}(p_\mu)^{M+1}), \tag{3.13}$$

$$g(p) = \sum_n (p + 2\pi n)^{-2} \prod_{\mu=1}^{d} |\hat{\chi}(p_\mu + 2\pi n_\mu)|^{2M+2}, \tag{3.14}$$

$$\eta(p) = g(p)^{-2} \sum_l (p + 2\pi l)^{-2} \prod_{\mu=1}^{d} |\hat{\chi}(p_\mu + 2\pi l_\mu)|^{2M+2} = g(p)^{-1}. \tag{3.15}$$

Thus

$$\hat{\Phi}(p) = (-1)^{(1/2)Md} g(p)^{-1/2} p^{-2} \prod_{\mu=1}^{d} (e^{i(1/2)Mp_\mu} \hat{\chi}(p_\mu)^{M+1}), \tag{3.16}$$

which cannot be real analytic because $p^4 g(p)$ vanishes to second order at $p = 0$. The point is that the basis is tailored for an infrared problem, so this modification is undesirable.

4. Group-Valued σ-Model

We would like to say a little more about how we intend to use the massless Sobelev ondelettes. In a word, we want to develop a phase cell cluster expansion for the two-dimensional group-valued σ-model that has the same spirit as the Federbush approach [17] to Yang–Mills fields. Controlling the continuum limit of the nonlinear σ-model is a difficult problem. Recently Kupiainen and Gawedzki [18] solved it for a hierarchical kinetic term in the action. Their approach was based on iteration of the block spin transformation.

We propose to solve the problem for the real kinetic term, but for the *group-valued* case [12, 13]. We use the lattice-continuum duality of Federbush, where "small field configuration" means "configuration close to the identity." The quadratic part of the action has the form [17]

$$\lambda_0 S_0 + \sum_{r=1}^{\infty} \lambda_r (S_r - S_{r-1}) + \sum_{r=1}^{\infty} (\lambda_r - \lambda_{r-1}) S_{r-1}, \tag{4.1}$$

where λ_r is the running coupling constant for scale 2^{-r} and S_r is the lattice action (with $b = \langle l, l + 2^{-r} \vec{e}_i \rangle$)

$$S_r = \sum_{b \in \mathscr{B}_r} 2^{-rd} \operatorname{tr} (\vec{\nabla} \phi)_b^2, \tag{4.2}$$

$$(\vec{\nabla} \phi)_b = 2^{-r} (\phi^r_{l + 2^{-r} \vec{e}_i} - \phi^r_l), \tag{4.3}$$

$$\phi_l^r = 2^{-rd} \int_{\Delta_l^r} \phi(x) dx. \tag{4.4}$$

l runs over sites on the 2^{-r}-scale lattice, while b runs over the bonds. Δ_l^r is the 2^{-r}-scale cube centered at l, and the trace implies a representation of the Lie algebra elements. Equation (4.4) induces the usual block spin transformation for the scalar case [16], and in this context it applies only to the small field region. On the other hand, our phase cell decomposition is

$$\phi(x) = \sum_k \alpha_k u_k. \tag{4.5}$$

We prove in a subsequent paper a stability result analogous to what has been established in the Yang–Mills case [19]. One expects

$$\lambda_0 S_0 + \sum_{r=1}^{\infty} \lambda_r (S_r - S_{r-1}) \geqq c \sum_k \lambda_{r(k)} \alpha_k^2, \tag{4.6}$$

where $2^{-r(k)}$ is the scale of the mode k, and this infinitesimal result extends to the small field region.

The technical advantage of our ondelettes will become apparent farther down the road, when the cluster expansion is actually developed. The combinatoric problems will be simpler because the continuum covariance is never decoupled. It is already diagonal with respect to our small field excitations.

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