Commun. Math. Phys. 112, 423–446 (1987)

Communications in
Mathematical
Physics

Multisoliton Solutions in the Scheme for Unified Description of Integrable Relativistic Massive Fields. Non-Degenerate $sl(2, \mathbb{C})$ Case

I. V. Barashenkov and B. S. Getmanov

Joint Institute for Nuclear Research, LCTA, Head Post Office P.O. Box 79, Moscow, USSR

Abstract. A scheme allowing systematic construction of integrable two-dimensional models of the Lorentz-invariant Lagrangian massive field theory is presented for the case when the associated linear problem is formulated on $sl(2, \mathbb{C})$ algebra. A natural dressing procedure is developed then for the generic system of two (either scalar or spinor) fields inherent in the scheme and an explicit N-soliton solution on zero background is calculated. Solutions of reduced systems which include both familiar and new equations are extracted from the solution of the generic system, not all of these reductions being related immediately to $sl(2, \mathbb{C})$ real forms. Finally, in the case of scalar equations we present the Miura-type transformations relating solutions with different boundary conditions.

Introduction

In the present paper[1] we derive exact multisoliton solutions within the framework of the Unified Integrable Lorentz Fields (UNILOF) description scheme. This scheme provides an Inverse Scattering formalism appropriate for construction and solution of all two-dimensional integrable relativistic massive systems (both spinor and scalar) in a unified way. (The *massless* systems have been analysed in detail by Zakharov and Mikhailov [1, 2].) A brief account of the UNILOF scheme has been given by one of the authors [4, 5]. The starting point is the Zakharov-Shabat equations for the relativistic case (1.1) in a new, *triangular* gauge (this is a key point of the scheme). Selection of this special gauge not only provides the unification but also produces non-linear equations in manifestly Lagrangian form [5].

An important degenerate case in the UNILOF scheme corresponds to the two-dimensional Toda lattices. These have been explored previously by Mikhailov, Olshanetsky, and Perelomov [6, 7], Fordy and Gibbons [25] (periodic lattices) and by Leznov and Saveliev (unclosed chains, ref. [8]). Here we study the non-degenerate case.

---

[1] Some of these results have been announced in [3]

Starting from the generic linear 2×2 matrix problem (1.1), we derive a system of two fields (or, rather, a one-parameter family of gauge-equivalent systems) which may be considered spinor. This model will be referred to as "the generic system associated with the algebra $\mathcal{G} = sl(2, \mathbb{C})$", or simply as "the $\mathcal{G}$-system." We may easily reformulate it in terms of two complex *scalar* fields. Reducing each of the two formulations of the $\mathcal{G}$-system, we obtain both known models such as the massive Thirring model and the complex sine-Gordon equation and new ones e.g., the second massive spinor model and $O(1,1)$ sine-Gordon equation.

This communication's main purpose is to supplement the regular scheme for construction of integrable systems with an adequate procedure of finding their soliton solutions. To do this, we extend Zakharov-Shabat-Mikhailov's dressing method [9, 1, 6] to the linear problems of type (2.2). Here the difficulty is that one can utilize the *canonical* normalization of the corresponding Riemann problem (very convenient and normally used) only for a certain particular representative of the aforementioned gauge-equivalent class. Of course, provided the solution for this special case is known, solutions to other $\mathcal{G}$-systems may be obtained merely through a gauge transformation. However, this strategy seems to be inefficient since the latter implies non-local substitutions for the field variables. In order to avoid these, we take a different line and do not impose any a priori normalization conditions on the dressing matrix. Although calculations become more involved, this enables us to "dress" the whole family of gauge-equivalent $\mathcal{G}$-systems simultaneously, N-soliton solutions appearing in a unified closed determinant form.

Solutions to the reduced equations are obtained by constraining parameters of solutions to the $\mathcal{G}$-system. At this stage, the difficulty is encountered in the case of the Minkowskian complex sine-Gordon equation. The problem is that unlike the other reductions, this one is not related directly to any real form of the $sl(2, \mathbb{C})$ algebra. Consequently, we have to introduce an auxiliary gauge which induces a rather complicated mapping of the dressing matrices manifold onto itself. Nevertheless, as soon as this mapping is found, the reduction conditions are straightforward.

In this paper we confine ourselves to the "dressing" of the *zero* seed solution (zero background). However, in the case of scalar fields these solutions provide an immediate information about the solitons on the *nonzero* constant background. The latter may be obtained via the Miura-type transformations taking each of the two complexified sine-Gordon equations to the same equation, but with the opposite sign of the mass term.

The paper is organised as follows. The $\mathcal{G}$-system is derived and reduced in Sect. 1 and its N-soliton solution is constructed in Sect. 2. In the subsequent sections we specialize the parameters of this solution so as to satisfy the following reduced systems: In Minkowski space – the (extended) massive Thirring model (MTM, Sect. 3); the usual complex sine/sinh-Gordon equation [referred to as $O(2)$ SGE, Sect. 6]; a new massive spinor model and a new complexified version of SGE [called $O(1,1)$ SGE], Sect. 4. In Euclidean space (Sect. 5) – extended $O(2)$ SGE and the Euclidean MTM. In Sect. 7 the Miura maps are presented, and in the last section we discuss connections between scalar and spinor systems, including the correspondence between SGE and MTM.

1. The Model

Below all the quantities are assumed to be complex unless the opposite is specified. Consider the set of linear equations:

$$i\partial_+\Psi=(\lambda^2U_2^+ + U_0^+)\Psi, \qquad i\partial_-\Psi=(\lambda^{-2}U_2^- + U_0^-)\Psi, \tag{1.1}$$

where $U_2^\pm(z_+,z_-)$, $U_0^\pm(z_+,z_-)$, and $\Psi(\lambda;z_+,z_-)$ are 2×2 matrix-valued functions of complex variables z_+ and z_-, $\partial_\pm\equiv\partial/\partial z_\pm$, and λ is a spectral parameter. The integrability conditions for (1.1) are:

$$i\partial_\pm U_2^\mp + [U_2^\mp, U_0^\pm]=0, \tag{1.2$^\pm$}$$

$$i\partial_- U_0^+ - i\partial_+ U_0^- + [U_2^+, U_2^-] + [U_0^+, U_0^-]=0. \tag{1.3}$$

Subtracting the trace multiplied by the identity matrix from each of the four matrices $U_2^\pm, U_0^\pm$ leaves (1.2)–(1.3) invariant. Hence, without loss of generality, we may consider $U_2^\pm, U_0^\pm\in sl(2,\mathbb{C})$. Next, the set (1.1) is covariant under the gauge transformation [9, 1]:

$$\Psi=g\tilde{\Psi}, \qquad U_2^\pm=g\tilde{U}_2^\pm g^{-1}, \qquad U_0^\pm=g\tilde{U}_0^\pm g^{-1}+i\partial_\pm g\cdot g^{-1}, \tag{1.4}$$

$g(\lambda;z_+,z_-)\in SL(2,\mathbb{C})$. In accordance with the central idea of the UNILOF scheme, let us fix the gauge by choosing U_2^+ upper-triangular matrix and U_2^- lower-triangular one: $(U_2^+)_{21}=(U_2^-)_{12}=0$. Then we find from (1.2$^\pm$):

$$(U_0^+)_{12}\operatorname{tr}(U_2^-\sigma_3)=0, \qquad (U_0^-)_{21}\operatorname{tr}(U_2^+\sigma_3)=0. \tag{1.5}$$

First, let us assume $(U_0^+)_{12}=(U_0^-)_{21}=0$. Then Eqs. (1.2$^\pm$) imply $\partial_\pm\operatorname{diag}U_2^\mp=0$, and we may introduce complex functions $a^\pm(z_\pm)$ such that $\operatorname{diag}U_2^\pm=\frac{1}{2}a^\pm(z_\pm)\sigma_3$. For the traceless U_2^+ respectively U_2^- the choice $\operatorname{tr}(U_2^+\sigma_3)=0$ respectively $\operatorname{tr}(U_2^-\sigma_3)=0$ in (1.5) corresponds to what we call the degenerate case: $a^+(z_+)\equiv 0$ respectively $a^-(z_-)\equiv 0$. In this paper we adopt that $a^\pm(z_\pm)\neq 0$ for all $z_\pm$.

Now let us denote matrix elements as follows:

$$\begin{aligned} U_2^+&=\begin{pmatrix} a^+/2 & q_1 \\ 0 & -a^+/2\end{pmatrix}, \qquad U_0^+=\begin{pmatrix} F^+/2 & 0 \\ q_2 & -F^+/2\end{pmatrix}, \\ U_2^-&=\begin{pmatrix} a^-/2 & 0 \\ q_4 & -a^-/2\end{pmatrix}, \qquad U_0^-=\begin{pmatrix} F^-/2 & q_3 \\ 0 & -F^-/2\end{pmatrix}. \end{aligned} \tag{1.6}$$

In this notation the compatibility conditions (1.2)–(1.3) are written as

$$\begin{aligned} &(i\partial_- - F^-)q_1 + a^+q_3=0, \qquad (i\partial_- + F^-)q_2 - a^+q_4=0, \\ &(i\partial_+ - F^+)q_3 + a^-q_1=0, \qquad (i\partial_+ + F^+)q_4 - a^-q_2=0, \end{aligned} \tag{1.7}$$

$$i\partial_- F^+ - i\partial_+ F^- + 2(q_1q_4 - q_2q_3)=0. \tag{1.8}$$

Redefining the fields: $q_{1,2}\to a^+q_{1,2}$, $q_{3,4}\to a^-q_{3,4}$, $F^\pm\to a^\pm F^\pm$ and changing the variables $z_\pm$ so that $\partial_\pm\to a^\pm(z_\pm)\partial_\pm$, we may, without loss of generality, fix $a^\pm\equiv 1$. Next, the system (1.7)–(1.8) possesses a "residual" $\mathbb{C}^*$ gauge invariance ($\mathbb{C}^*=\mathbb{C}\setminus\{0\}$):

$$q_{1,3}=e^{\Theta}\tilde{q}_{1,3}, \qquad q_{2,4}=e^{-\Theta}\tilde{q}_{2,4}, \qquad F^\pm=\tilde{F}^\pm + i\partial_\pm\Theta, \tag{1.9}$$

which amounts to the selection of $g=\exp(\frac{1}{2}\Theta\sigma_3)$ in (1.4). On the other hand, Eqs. (1.7)–(1.8) imply: $\partial_-(F^+ +\omega_+ q_1q_2)=\partial_+(F^- +\omega_- q_3q_4)$, where $\omega_\pm$ are any two constants verifying

$$\omega_+ + \omega_- = 2. \tag{1.10}$$

Hence, there exists potential π such that $F^+ +\omega_+ q_1q_2=\partial_+\pi$, $F^- +\omega_- q_3q_4=\partial_-\pi$. In view of the invariance (1.9) we may set $\pi\equiv 0$, thereby obtaining a family of gauge-equivalent systems:

$$\begin{aligned} &(i\partial_- - F^-)q_1 + q_3 = 0, \quad (i\partial_- + F^-)q_2 - q_4 = 0, \\ &(i\partial_+ - F^+)q_3 + q_1 = 0, \quad (i\partial_+ + F^+)q_4 - q_2 = 0, \end{aligned} \tag{1.11$'$}$$

$$F^+ = -\omega_+ q_1 q_2, \qquad F^- = -\omega_- q_3 q_4. \tag{1.11$''$}$$

For each pair $\omega_\pm$ obeying (1.10) the system (1.11) will be referred to as the "generic system," or merely as "the $\mathscr{G}$-system." Let us also note that (1.11′) yields a conservation law $\partial_-(q_1q_2)+\partial_+(q_3q_4)=0$, whence

$$q_1q_2=\partial_+\Lambda, \qquad q_3q_4=-\partial_-\Lambda. \tag{1.12}$$

Recovering Λ from here, we can specify the transformation (1.9) mapping the $\mathscr{G}$-system with $\omega_\pm$ into that with $\tilde{\omega}_\pm$. Namely, the corresponding Θ is:

$$\Theta = i(\tilde{\omega}_- - \omega_-)\Lambda. \tag{1.13}$$

For two distinct choices of $\omega_\pm$ the $\mathscr{G}$-system (1.11) is manifestly Lagrangian. From the field-theoretic point of view, the most interesting case appears to be that with $\omega_\pm = 1$, the corresponding Lagrangian being given by

$$\mathscr{L} = iq_2\partial_- q_1 + iq_4\partial_+ q_3 + q_1q_4 + q_2q_3 + q_1q_2q_3q_4 + (\text{compl. conj.}). \tag{1.14}$$

The second choice is $\omega_- = 0$ (or $\omega_+ = 0$). At $\omega_- = 0$, $\omega_+ = 2$ eliminating q_3 and q_4 we obtain Mikhailov's model [10], derivable from

$$\mathscr{L} = -\partial_+ q_1 \partial_- q_2 + q_1q_2 + iq_1^2 q_2 \partial_- q_2 + (\text{c.c.}). \tag{1.15}$$

Notation. In this paper we discuss field theories both in Minkowski (denoted M^2) and Euclidean (E^2) spaces. The Greek indices will be reserved to label the corresponding vector components, with the usual summation convention being adopted. In M^2 the laboratory coordinates are x^0 and x^1, and the metric signature is $(+\,-)$, i.e., $k_\mu x^\mu = k^0x^0 - k^1x^1$. Also the light cone variables will be used: $\eta=\frac{1}{2}(x^0+x^1)$, $\xi=\frac{1}{2}(x^0-x^1)$. In E^2 the laboratory coordinates are x_1 and x_2, $k_\mu x_\mu = k_1x_1+k_2x_2$, and we shall use the Laplace coordinates $z=\frac{1}{2}(x_1+ix_2)$, $z^*=\frac{1}{2}(x_1-ix_2)$ instead of η and ξ. γ-matrices are defined through the Pauli σ-matrices: $\gamma^0=\gamma_0=\sigma_1$, $\gamma^1=-\gamma_1=i\sigma_2$, $\gamma^5=\gamma^0\gamma^1$ in M^2, and $\gamma_\mu=\sigma_\mu$ in E^2. Finally, * denotes complex conjugation, ^{T} transposition, and † Hermitian conjugation.

If we want the $\mathscr{G}$-system to represent a model of relativistic (or Euclidean) field theory the transformation properties of $q_1,\ldots,q_4$ should be specified. There are two possibilities related to scalar and spinor fields.

1.1. *Conventional and Extended MTM in M^2 Space.* In M^2 let us set $z_+ = \eta$, $z_- = \xi$ and denote $q_1 = u_1$, $q_2 = u_2^*$, $q_3 = v_1$, $q_4 = v_2^*$. Then the Lagrangian (1.14) is:

$$\mathscr{L}_1 = iu_2^* u_{1\xi} + iv_2^* v_{1\eta} + u_2^* v_1 + v_2^* u_1 + u_1 u_2^* v_1 v_2^* + (\text{c.c.}). \tag{1.16}$$

If $\psi_1 = (u_1, v_1)^T$ and $\psi_2 = (u_2, v_2)^T$ belong to the two-dimensional vector space that forms the spinor representation of the Lorentz group, the $\mathscr{G}$-system (1.16) becomes a model of two spinor fields:

$$L_1 = i\bar{\boldsymbol{\psi}}(\gamma^\mu \otimes \sigma_1)\partial_\mu \boldsymbol{\psi} + \bar{\boldsymbol{\psi}}(\mathbb{1} \otimes \sigma_1)\boldsymbol{\psi} + \tfrac{1}{8}\{[\bar{\boldsymbol{\psi}}(\gamma^\mu \otimes \sigma_1)\boldsymbol{\psi}]^2 - [\bar{\boldsymbol{\psi}}(\gamma^\mu \otimes \sigma_2)\boldsymbol{\psi}]^2\}, \tag{1.17}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$, $\bar{\boldsymbol{\psi}} = (\bar{\psi}_1, \bar{\psi}_2)$, $\bar{\psi}_i = \psi_i^\dagger \gamma_0$, $\bar{\boldsymbol{\psi}}\boldsymbol{\psi} = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$. Identification[2] $\psi_1 = \psi_2 \equiv \psi$ reduces (1.17) to the massive Thirring model (MTM) [11]):

$$L_2 = i\bar{\psi}\gamma^\mu \partial_\mu \psi + \bar{\psi}\psi + \tfrac{1}{4}(\bar{\psi}\gamma_\mu\psi)^2, \tag{1.18}$$

with $\psi = (u, v)^T$. In terms of u and v, Eq. (1.18) is rewritten as

$$\mathscr{L}_2 = iu_\xi u^* + iv_\eta v^* + uv^* + u^* v + |uv|^2. \tag{1.19}$$

MTM may be extended to the (generically) non-Lagrangian model [12],

$$iu_\xi + v + \omega_- |v|^2 u = 0, \qquad iv_\eta + u + \omega_+ |u|^2 v = 0, \tag{1.20}$$

which emerges from the system (1.11) under the reduction

$$q_1 = q_2^* \equiv u, \qquad q_3 = q_4^* \equiv v, \qquad \omega_\pm \in \mathbb{R}. \tag{1.21}$$

MTM corresponds to $\omega_\pm = 1$. Specialization (1.21) preserves, of course, the gauge equivalence between (1.11) and (1.14). As a result, the extended MTM (1.20) is transformable into the conventional one (1.19) through the change of variables (1.9), (1.12), (1.13). At $\omega_- = 0$ the system (1.20) is the reduced form of Eq. (1.15) derivable from $\mathscr{L}_3 = -u_\eta u_\xi^* + |u|^2 + iu^2 u^* u_\xi^*$.

Remark 1.1. Under the definition of η, ξ through x^μ given above, the choice $z_+ = \eta$, $z_- = \xi$ leads to "infraluminic" (i.e., travelling at velocities $v: |v| \leqq 1$) solitons of MTM. If we set $z_- = -\xi$, we would obtain tachyon solutions of the model (1.18) with $i\bar{\psi}\gamma^\mu \partial_\mu \psi$ replaced by $-i\bar{\psi}\gamma^\mu\gamma^5 \partial_\mu \psi$. As both types of solutions are connected through the trivial substitution $\xi \to -\xi$, we confine ourselves to the former case.

1.2. *The Second Massive Spinor Model in M^2.* Let $z_+ = i\eta$, $z_- = -i\xi$, and let $\omega_\pm$, $q_1, \ldots, q_4 \in \mathbb{R}$. Defining a covariant spinor $\psi = (u, v)^T$, where

$$q_1 = u + u^*, \qquad q_2 = i(u - u^*), \qquad q_3 = i(v - v^*), \qquad q_4 = v + v^*, \tag{1.22}$$

we reduce the $\mathscr{G}$-system (1.11) to another spinor model in Minkowski space:

$$iu_\xi + v + \omega_-(v^2 - v^{*2})u^* = 0, \qquad iv_\eta + u + \omega_+(u^2 - u^{*2})v^* = 0. \tag{1.23}$$

By means of the substitution (1.9), (1.12), (1.13), Eq. (1.23) may be transformed into $(\omega_\pm = 1)$-form, derivable from the Lagrangian

$$\mathscr{L}_4 = iu_\xi u^* + iv_\eta v^* + uv^* + u^* v - \tfrac{1}{2}(u^2 - u^{*2})(v^2 - v^{*2}), \tag{1.24'}$$

[2] The sign of the nonlinearity being nonessential [11], the identifications $\psi_2 = \pm\psi_1$ are equivalent

or into ($\omega_-=0$)-form, defined by $\mathscr{L}_5=iu_\eta u_\xi-iu^2+(u+u^*)^2(uu^*)_\xi+$(c.c.). Integrability of the model (1.24′) has been first suggested by V. E. Kovtun, who has found it to possess a higher conserved current (private communication). In the covariant notation Eq. (1.24′) reads ($\tilde{\psi}\equiv\psi^T\gamma_0$):

$$L_4=i\bar{\psi}\gamma_\mu\partial^\mu\psi+\bar{\psi}\psi+\tfrac{1}{8}(\tilde{\psi}\gamma_\mu\psi-\bar{\psi}\gamma_\mu\psi^*)^2\,. \tag{1.24″}$$

1.3. *Euclidean Thirring Model.* In the Euclidean domain we set $z_+=z$, $z_-=\varepsilon z^*$, $\varepsilon=\pm1$. In contrast to M^2 space, here we cannot confine ourselves to a certain particular choice of ε, say $\varepsilon=1$ (cf. Remark 1.1). Solutions of the system (1.11) with $\varepsilon=-1$ and $\varepsilon=1$ are unrelated and will be treated independently. Let us denote $q_1=u_1$, $q_2=\varepsilon v_2^*$, $q_3=v_1$, $q_4=u_2^*$ and require that the columns $\psi_1=(u_1,v_1)^T$ and $\psi_2=(u_2,v_2)^T$ transform as $O(2)$ spinors. Then Eq. (1.16) represents a system of two Euclidean spinor fields:

$$L_6=i\boldsymbol{\psi}^\dagger(\gamma_\mu\otimes\sigma_1)\partial_\mu\boldsymbol{\psi}+\boldsymbol{\psi}^\dagger(\mathscr{E}^2\otimes\sigma_1)\boldsymbol{\psi}+\tfrac{1}{8}\varepsilon\{[\boldsymbol{\psi}^\dagger(\gamma_\mu\otimes\sigma_1)\boldsymbol{\psi}]^2-[\boldsymbol{\psi}^\dagger(\gamma_\mu\otimes\sigma_2)\boldsymbol{\psi}]^2\}\,. \tag{1.25}$$

Here $\boldsymbol{\psi}=(\psi_1,\psi_2)^T$, $\boldsymbol{\psi}^\dagger=(\psi_1^\dagger,\psi_2^\dagger)$, $\mathscr{E}=\operatorname{diag}\{1,\varepsilon^{1/2}\}$. Imposing the condition $\tau\psi_2=\psi_1\equiv\psi$ ($\tau=\pm1$) reduces the system (1.25) to the Euclidean MTM:

$$L_7=i\psi^\dagger\gamma_\mu\partial_\mu\psi+\psi^\dagger\mathscr{E}^2\psi+\tfrac{1}{4}\tau\varepsilon(\psi^\dagger\gamma_\mu\psi)^2\,. \tag{1.26}$$

On the other hand, if we start from the $\mathscr{G}$-system (1.11) and require that

$$\tau q_4^*=q_1\equiv u\,,\qquad \varepsilon\tau q_2^*=q_3\equiv v\,,\qquad \tau=\pm1;\qquad \omega_+^*=\omega_-\equiv\omega\,, \tag{1.27}$$

we shall arrive at the non-Lagrangian model

$$iu_{z^*}+\varepsilon v+\tau\varepsilon\omega v|u|^2=0,\qquad iv_z+u+\tau\varepsilon\omega^*u|v|^2=0\,, \tag{1.28}$$

containing MTM (1.26) as a special case of $\omega=1$ [$\psi=(u,v)^T$].

1.4. *The Second Spinor Reduction in E^2.* If we set $q_1=u-v^*$, $q_2=-(u+v^*)$, $q_3=v-u^*$, $q_4=u^*+v$, $z_+=z$, $z_-=-z^*$ in Eq. (1.14), we shall obtain another spinor model in the Euclidean domain. In the covariant notation $\psi=(u,v)^T$, $\tilde{\psi}\equiv\psi^T\gamma_1$, $\bar{\psi}\equiv\psi^\dagger\gamma_1$, $\gamma_5=-i\gamma_1\gamma_2$, it looks like:

$$L_8=i\psi^\dagger\gamma_\mu\partial_\mu\psi+\psi^\dagger\gamma_5\psi-\tfrac{1}{8}(\tilde{\psi}\gamma_\mu\psi-\bar{\psi}\gamma_\mu\psi^*)^2\,.$$

1.5. *$O(2)$ sine/sinh-Gordon Equations.* Let us define new fields $\varphi_1=e^{-iv}q_1$ and $\varphi_2=e^{-iv}q_4^*$ ($v=\text{const}\in\mathbb{R}$) which are required to be scalars both in M^2 and E^2 cases. q_2 and q_3 may be expressed through φ_1,φ_2^* by means of the first and fourth equations in (1.11′):

$$q_2=ie^{-iv}(1+\omega_+\varphi_1\varphi_2^*)^{-1}\partial_+\varphi_2^*,\qquad q_3=-ie^{iv}(1+\omega_-\varphi_1\varphi_2^*)^{-1}\partial_-\varphi_1\,. \tag{1.29}$$

Inserting these expressions into the remaining two equations, we obtain a system of two complex scalar fields, i.e., a scalar formulation of the $\mathscr{G}$-system (1.11):

$$\begin{gathered}\partial_+\partial_-\varphi_1+\varphi_1\mathscr{D}_-+\delta\cdot\varphi_1\partial_-\varphi_1\partial_+\varphi_2^*(\mathscr{D}_+\mathscr{D}_-)^{-1}-\omega_-\varphi_2^*\partial_-\varphi_1\partial_+\varphi_1\mathscr{D}_-^{-1}=0\,,\\ \partial_+\partial_-\varphi_2+\varphi_2\mathscr{D}_+^*-\delta^*\cdot\varphi_2\partial_-^*\varphi_1^*\partial_+^*\varphi_2(\mathscr{D}_+^*\mathscr{D}_-^*)^{-1}-\omega_+^*\varphi_1^*\partial_-\varphi_2\partial_+\varphi_2(\mathscr{D}_+^*)^{-1}=0\,,\end{gathered} \tag{1.30}$$

where $\delta=\omega_+-\omega_-$, $\mathscr{D}_\pm=1+\omega_\pm\varphi_1\varphi_2^*$. At $\omega_\pm=1$ it is derivable from

$$\mathscr{L}_9=\partial_-\varphi_1\,\partial_+\varphi_2^*(1+\varphi_1\varphi_2^*)^{-1}-\varphi_1\varphi_2^*+(\text{c.c.})\,. \tag{1.31}$$

First, let us consider the model (1.31) in M^2 space: $z_+=\eta$, $z_-=\xi$. Imposing the restriction $\tau\varphi_2=\varphi_1\equiv\varphi$, $\tau=\pm1$ reduces (1.31) to

$$\mathscr{L}_{10}=\varphi_\xi\varphi_\eta^*(1+\tau|\varphi|^2)^{-1}-|\varphi|^2+(\text{c.c.})\,. \tag{1.32}$$

Equation (1.32) defines the complex sine- and sinh-Gordon equations [13–15, 26] for $\tau=-1$ and $\tau=1$, respectively. In this paper they are referred to as $O(2)$ SGE in order to be distinguished from $O(1,1)$ SGE (Subsect. 1.6).

Remark 1.2. As in Subsect. 1.1, we restrict ourselves to the choice $z_-=\xi$, which leads to the subluminal solitons of $O(2)$ SGE. Substitution $\xi\to-\xi$ changes the mass term sign in (1.32) and these are converted into tachyons (cf. Remark 1.1).

Now let us pass to the Euclidean domain and put $z_+=z$, $z_-=\varepsilon z^*$, $\varepsilon=\pm1$. Imposing the conditions $\tau\varphi_2=\varphi^1\equiv\varphi$, $\omega_+^*=\omega_-\equiv\omega$ in Eq. (1.30), we obtain the (non-Lagrangian) extended $O(2)$ SGE which is lacking in M^2:

$$\varphi_{zz^*}+\varepsilon\varphi\mathscr{D}-\tau\omega\varphi^*\varphi_z\varphi_{z^*}\mathscr{D}^{-1}+\tau(\omega^*-\omega)\varphi\varphi_z^*\varphi_{z^*}|\mathscr{D}|^{-2}=0\,, \tag{1.33}$$

$\mathscr{D}=1+\tau\omega|\varphi|^2$. At $\omega=1$ Eq. (1.33) may be derived from the Lagrangian

$$\mathscr{L}_{10}=\varphi_z^*\varphi_{z^*}(1+\tau|\varphi|^2)^{-1}-\varepsilon|\varphi|^2\,. \tag{1.34}$$

Remark 1.3. Due to the coincidence of the reduction conditions ($q_1=\tau q_4^*$, $q_3=\tau\varepsilon q_2^*$), the Euclidean MTM (1.26) is completely equivalent to $O(2)$ SGE (1.34), the same also being true for their extended versions (1.28) and (1.33). Thus, solutions for the two systems will be constructed simultaneously.

Under the restriction $\varphi=\varphi^*$, Eqs. (1.32), (1.34) define the real SGE,

$$\mathscr{L}_{11}=\varphi_\xi\varphi_\eta/(1+\tau\varphi^2)-\varphi^2\,, \tag{1.35'}$$

$$\mathscr{L}_{11}=\varphi_z\varphi_{z^*}/(1+\tau\varphi^2)-\varepsilon\varphi^2 \tag{1.35''}$$

in M^2 and E^2, respectively. At $\tau=1$, setting $\varphi=\sinh f$ yields $\mathscr{L}_{11}=\partial_+f\,\partial_-f-\sinh^2 f$. At $\tau=-1$ there are two cases: at $|\varphi|\leqq1$, we put $\varphi=\sin f$ and obtain $\mathscr{L}_{11}=\partial_+f\,\partial_-f-\sin^2 f$, while at $|\varphi|\geqq1$ substitution $\varphi=\pm\cosh f$ leads to $\mathscr{L}_{11}=\partial_+f\,\partial_-f+\sinh^2 f$.

1.6. *Sine-Gordon Equation with* O(1,1) *Isotopic Symmetry.* Let us substitute $z_+\to iz_+$, $z_-\to-i\varepsilon z_-$, $\varepsilon=\pm1$, and require that $\omega_\pm=1$, $q_1,\ldots,q_4\in\mathbb{R}$. In both the M^2 and E^2 case we introduce scalar fields $\varphi^\pm$ and $\varphi_{1,2}$ such that $\varphi^-=q_1$, $\varphi^+=q_4$, $\varphi^\pm=\varphi_1\pm\varphi_2$. Eliminating q_2,q_3 from (1.11) as in Subsect. 1.5 produces a new system of two real scalar fields derivable from

$$\mathscr{L}_{12}=\partial_+\varphi^+\,\partial_-\varphi^-(1+\varphi^+\varphi^-)^{-1}-\varepsilon\varphi^+\varphi^-=\partial_+\boldsymbol{\varphi}\cdot\partial_-\boldsymbol{\varphi}(1+\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^{-1}-\varepsilon\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}\,. \tag{1.36}$$

Here $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)$ belongs to isotopic space with $O(1,1)$ invariant scalar product: $\boldsymbol{\varphi}\cdot\boldsymbol{\phi}\equiv\varphi_1\phi_1-\varphi_2\phi_2$, whence the name: $O(1,1)$ SGE. Similarly to $O(2)$ SGE, it admits a complex formulation [see Eq. (4.7)]. Further, in M^2 the system (1.36) can be extended to a non-Lagrangian equation (here we set $z_+ = \eta$, $z_- = \xi$ and fix $\varepsilon = 1$):

$$\varphi^{\pm}_{\eta\xi}\mathscr{D}_{\pm}^{-1} + \varphi^{\pm} - \omega_{\pm}\varphi^{\pm}_{\eta}\varphi^{\pm}_{\xi}\varphi^{\mp}\mathscr{D}_{\pm}^{-2} \pm \delta \cdot \varphi^{-}_{\xi}\varphi^{+}_{\eta}\varphi^{\pm}\mathscr{D}_{\pm}^{-2}\mathscr{D}_{\mp}^{-1} = 0\,, \qquad (1.37)$$

with $\omega_{\pm} \in \mathbb{R}$, $\delta = \omega_- - \omega_+$, $\mathscr{D}_{\pm} = 1 + \omega_{\pm}\varphi^+\varphi^-$. The reduction restrictions coinciding, Eq. (1.37) is equivalent to Eq. (1.23) while the Minkowskian $O(1,1)$ SGE (1.36) is equivalent to the second spinor model (1.24). Imposing of the constraint $\varphi^+ = \tau\varphi^- \equiv \varphi$ $(\tau = \pm 1)$ on the Lagrangian (1.36) provides a deeper reduction to the real SGE (1.35).

2. N-Soliton Solutions for the Generic (Nonreduced) System

The gauge transformation generated by the matrix

$$g_s = \mathrm{diag}\{\lambda^{1/2}, \lambda^{-1/2}\} \qquad (2.1)$$

converts the linear problem (1.1), (1.6) to the following form[3]:

$$i\partial_{\pm}\Psi = (\lambda^{\pm 2}A_2 + \lambda^{\pm 1}A_1^{\pm} + A_0^{\pm})\Psi \equiv A^{\pm}\Psi\,, \qquad (2.2)$$

where $\Psi \in GL(2,\mathbb{C})$, $A_2 = \frac{1}{2}\sigma_3$, $A_0^{\pm} = \frac{1}{2}F^{\pm}\sigma_3$, and

$$A_1^+ = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, \quad A_1^- = \begin{pmatrix} 0 & q_3 \\ q_4 & 0 \end{pmatrix}. \qquad (2.3)$$

The compatibility conditions (1.11) being invariant under the transformation (2.1), we can use this stratified gauge instead of (1.1), (1.6). The motivation is that in constructing solutions it provides us with an effective way to take into account the special form of the linear problem matrices. Indeed, the linear problem (2.2) with diagonal A_2, $A_0^{\pm}$ and off-diagonal $A_1^{\pm}$ [so that $\sigma_3 A^{\pm}(\lambda)\sigma_3 = A^{\pm}(-\lambda)$] results from the $\mathbb{Z}_2$-reduction [6] of the general quadratic bundle (in which all the matrices $A^{\pm}_{0,1,2}$ are generic). Hence, the manifold $\{\Psi(\lambda)\}$ of fundamental solutions $\Psi(\lambda)$ to Eq. (2.2) is invariant under the involutory transformation $\Psi(\lambda) \to \sigma_3\Psi(-\lambda)\sigma_3$ [i.e., $\sigma_3\Psi(-\lambda, z_{\pm})\sigma_3 = \Psi(\lambda, z_{\pm})H(\lambda)$ for some constant $H(\lambda) \in GL(2,\mathbb{C})$]. In this paper we construct solutions q_i vanishing at infinity so that $\Psi(\lambda)$ can be chosen to obey simply

$$\sigma_3\Psi(-\lambda)\sigma_3 = \Psi(\lambda)\,. \qquad (2.4)$$

To check this, take $q_i(z_{\pm}) \equiv 0$ as the "bare" solution of Eq. (1.11) and choose the related $\Psi(\lambda)$ in the form $\Psi_0(\lambda) = \exp\{-i(\lambda^2 z_+ + \lambda^{-2}z_-)A_2\}$, evidently verifying (2.4). If the dressed fields satisfy $q_i(z_{\pm}) \to 0$ at infinity we can select Ψ such that $\Psi(\lambda, z_{\pm}) \to c\Psi_0(\lambda, z_{\pm})$, $c \in \mathbb{C}$. Then $\Psi(\lambda)$ verifies (2.4) asymptotically and, therefore, identically.

The Dressing Procedure. In construction of soliton solutions we utilize the idea of Zakharov-Shabat-Mikhailov's "dressing method" which is equivalent to solving a rational Riemann problem [9, 1, 6]. Define the $GL(2,\mathbb{C})$-valued function $\chi(\lambda, z_{\pm})$

---

[3] This transition has been advised by A. V. Mikhailov. Another way of obtaining the stratified gauge (2.2) is delineated in [5]

("dressing matrix"), meromorphic in λ, with meromorphic inverse, regular at $\lambda=\infty$, through the formula

$$\chi(\lambda)=\Psi(\lambda)\,\Psi_0^{-1}(\lambda). \tag{2.5}$$

Equation (2.4) implies $\sigma_3\chi(-\lambda)\sigma_3=\chi(\lambda)$, $\sigma_3\chi^{-1}(-\lambda)\sigma_3=\chi^{-1}(\lambda)$, whence

$$\chi(\lambda)=R\left(\mathbb{1}+\sum_{i=1}^{N}\frac{P^i}{\lambda-\nu_i}-\sum_{i=1}^{N}\frac{\sigma_3P^i\sigma_3}{\lambda+\nu_i}\right), \tag{2.6'}$$

$$\chi^{-1}(\lambda)=\left(\mathbb{1}+\sum_{i=1}^{N}\frac{Q^i}{\lambda-\mu_i}-\sum_{i=1}^{N}\frac{\sigma_3Q^i\sigma_3}{\lambda+\mu_i}\right)R^{-1}, \tag{2.6''}$$

where R, P^i, Q^i are 2×2 matrices. By the Liouville formula it may be inferred from (2.2) that $\partial_\pm \det\Psi=0$, and we can select Ψ in such a way, that $R=\chi(\infty)\in SL(2,\mathbb{C})$. Moreover, as $\sigma_3\chi(\infty)\sigma_3=\chi(\infty)$, R belongs to the diagonal subgroup $\mathbb{C}^*\subset SL(2,\mathbb{C}): R=\operatorname{diag}\{r,r^{-1}\}$. Next, it is straightforward to verify that when $\omega_-=0$, r is constant [see Eqs. (2.28), (2.29) below] and we may normalize $\chi(\lambda)$ *canonically*: $\chi(\infty)=\mathbb{1}$. Generally speaking, it is sufficient to determine solutions only for this particular case. Solutions to other $\mathscr{G}$-systems could then be obtained through the gauge transformation (1.9), (1.12), (1.13). However, a serious drawback to these latter solutions would be the presence of the nonlocal multiplier e^{Θ}. Therefore we prefer not to fix the gauge (and, consequently, the normalization) and construct solutions for the whole family of $\mathscr{G}$-systems simultaneously. In other words, we supplement the solution for the case $\omega_-=0$ (which is of limited importance itself) with a *closed* expression for $r(z_\pm)$ (or, equivalently, for e^{Θ}).

We shall be concerned with the generic situation of $\nu_i\neq\pm\mu_k$. Requiring that residues of the left-hand side of the identity $\chi\chi^{-1}=\mathbb{1}$ vanish gives

$$P^i\chi^{-1}(\nu_i)=\chi(\mu_i)Q^i=0,\qquad i=1,\dots,N. \tag{2.7}$$

Without loss of generality, choose the degenerate P^i, Q^j matrices as

$$P^i_{AB}=x^i_At^i_B,\qquad Q^i_{AB}=s^i_Ay^i_B,\qquad A,B=1,2. \tag{2.8}$$

Here $\mathbf{x}^i$, $\mathbf{y}^i$, $\mathbf{s}^i$, $\mathbf{t}^i\in\mathbb{C}^2$, $i=1,\dots,N$. The components of these vectors may be rearranged to form the vectors $|x_A\rangle, |y_A\rangle, |s_A\rangle, |t_A\rangle\in\mathbb{C}^N$, $A=1,2$. For instance, $\langle x_A|\equiv(x^1_A,\dots,x^N_A)$ while $\mathbf{x}^i\equiv(x^i_1,x^i_2)$. Here and below the small Latin indices run over $1,\dots,N$, whereas the capital Latins take only two values, 1 and 2. Also note that $\langle u_A|v_B\rangle\equiv u^1_Av^1_B+\dots+u^N_Av^N_B$. Insertion of (2.8) into (2.7) yields

$$2a_{1,2}|x_{1,2}\rangle=|s_{1,2}\rangle,\qquad 2\langle y_{1,2}|a_{2,1}=-\langle t_{1,2}|, \tag{2.9}$$

where $N\times N$ matrices a_1 and a_2 are defined through

$$a_1^{ij}=(\nu_j^2-\mu_i^2)^{-1}(\nu_js^i_1t^j_1+\mu_is^i_2t^j_2),\qquad a_2^{ij}=(\nu_j^2-\mu_i^2)^{-1}(\mu_is^i_1t^j_1+\nu_js^i_2t^j_2). \tag{2.10}$$

These matrices obey the obvious identities

$$|s_1\rangle\langle t_1|=a_1\langle\nu|-|\mu\rangle a_2,\qquad |s_2\rangle\langle t_2|=a_2\langle\nu|-|\mu\rangle a_1, \tag{2.11}$$

where $\langle\nu|\equiv(\nu_1,\dots,\nu_N)$, $\langle\mu|\equiv(\mu_1,\dots,\mu_N)$. Equation (2.9) implies:

$$|x_{1,2}\rangle=\tfrac{1}{2}a_{1,2}^{-1}|s_{1,2}\rangle,\qquad \langle y_{1,2}|=-\tfrac{1}{2}\langle t_{1,2}|a_{2,1}^{-1}. \tag{2.12}$$

Coordinate Dependence of $\mathbf{s}^i$ *and* $\mathbf{t}^i$. Using (2.5), the linear problem (2.2) may be rewritten in terms of χ:

$$i\partial_\pm\chi\cdot\chi^{-1}+\lambda^{\pm 2}[\chi,A_2]\chi^{-1}=\lambda^{\pm 1}A_1^\mp+A_0^\pm\,. \tag{2.13$^\pm$}$$

Inserting Eqs. (2.6) and (2.8) into (2.13) and requiring that residues of the left-hand side at $\lambda=v_i,\mu_i$ vanish, we obtain in view of (2.7):

$$(i\partial_\pm+v_i^{\pm 2}A_2)\mathbf{t}^i=0,\qquad (i\partial_\pm-\mu_i^{\pm 2}A_2)\mathbf{s}^i=0\,, \tag{2.14}$$

whence the dependence of $\mathbf{t}^i$, $\mathbf{s}^i$ on $z_\pm$ is found to be ($\mathbf{m}^i,\mathbf{n}^i=\mathrm{const}\in\mathbb{C}^2$):

$$\mathbf{t}^i=\exp\{i(v_i^2z_++v_i^{-2}z_-)A_2\}\mathbf{m}^i,\qquad \mathbf{s}^i=\exp\{-i(\mu_i^2z_++\mu_i^{-2}z_-)A_2\}\mathbf{n}^i\,. \tag{2.15}$$

Recovering of the "Potentials" q_i, $F^\pm$. As soon as the constraints (2.14) are imposed on χ and χ^{-1}, the expression $f_-(\lambda)=i\partial_-\chi\cdot\chi^{-1}+\lambda^{-2}[\chi A_2]\chi^{-1}$ on the left-hand side of (2.13$^-$) defines a rational function of λ with a single pole at $\lambda=0$. Below, expanding $f_-(\lambda)$ in the Laurent series in the vicinity of $\lambda=0$, we shall determine A_1^- and A_0^- as the coefficients at λ^{-1} and λ^0, respectively. On the other hand, expanding $f_-(\lambda)$ at $\lambda=\infty$, we shall arrive at (formally) different expressions for A_1^- and A_0^-. Finally, comparison of the two results produces a priori valid identities which will then be efficiently utilized. Equation (2.13$^+$) will be treated in the same way.

First of all, let us note an elementary relation

$$\det(a+|u_1\rangle\langle u_2|)=\det a+\langle u_2|\mathscr{A}|u_1\rangle\,. \tag{2.16}$$

Here a is any non-degenerate $N\times N$ matrix, $\mathscr{A}$ stands for the augmented matrix ($\mathscr{A}=\det a\cdot a^{-1}$) and $|u_{1,2}\rangle\in\mathbb{C}^N$. Using (2.16) one easily proves that[4]

$$\mathbb{1}-2\Sigma Q_\|^i\mu_i^{-1}=\Pi(v_j\mu_j^{-1})(\Delta_1\Delta_2^{-1})^{\sigma_3},\qquad \mathbb{1}-2\Sigma P_\|^i v_i^{-1}=\Pi(\mu_j v_j^{-1})(\Delta_2\Delta_1^{-1})^{\sigma_3}, \tag{2.17}$$

where $\Delta_{1,2}\equiv\det a_{1,2}$. Now, expanding (2.13$^\pm$) at $\lambda=\infty$ produces

$$A_1^+=-2R\sigma_3\Sigma P_\perp^i R^{-1}\,, \tag{2.18}$$

$$A_0^+=i\partial_+R\cdot R^{-1}-4\sigma_3\Sigma P_\perp^i\Sigma Q_\perp^j\,, \tag{2.19}$$

$$A_1^-=2i\{\partial_-R\cdot\Sigma(Q^i+P^i)_\perp+R\,\partial_-\Sigma P_\perp^i\}\,R^{-1}\,, \tag{2.20}$$

$$A_0^-=i\partial_-R\cdot R^{-1}\,. \tag{2.21}$$

On the other hand, expanding (2.13$^\pm$) at $\lambda=0$ yields

$$\begin{aligned}A_1^+=&-2i\{\Pi\mu_j v_j^{-1}\,\partial_+[R(\Delta_2\Delta_1^{-1})^{\sigma_3}]\,\Sigma(Q_\perp^i\mu_i^{-2})R^{-1}\\&+\Pi v_j\mu_j^{-1}\,\partial_+[\Sigma P_\perp^i v_i^{-2}R]\,(\Delta_1\Delta_2^{-1})^{\sigma_3}R^{-1}\}\,,\end{aligned} \tag{2.22}$$

$$A_0^+=i\partial_+R\cdot R^{-1}+i(\Delta_1\Delta_2^{-1})^{\sigma_3}\,\partial_+(\Delta_2\Delta_1^{-1})^{\sigma_3}\,, \tag{2.23}$$

$$A_1^-=2\Pi(v_j\mu_j^{-1})R\sigma_3\Sigma(P_\perp^i v_i^{-2})(\Delta_1\Delta_2^{-1})^{\sigma_3}R^{-1}\,, \tag{2.24}$$

$$A_0^-=i\partial_-[R(\Delta_2\Delta_1^{-1})^{\sigma_3}]\,(\Delta_1\Delta_2^{-1})^{\sigma_3}R^{-1}-4R\sigma_3\Sigma(P_\perp^i v_i^{-2})\,\Sigma(Q_\perp^j\mu_j^{-2})R^{-1}, \tag{2.25}$$

[4] From now on we use the notations: $\Sigma f_i\equiv f_1+\ldots+f_N$, $\Pi f_j\equiv f_1\cdot\ldots\cdot f_N$. The subscript $\|$ respectively $\perp$ indicates the diagonal respectively off-diagonal part of a matrix

where the identities (2.17) have been utilized. At this stage we can write the N-soliton solution to the system (1.7)–(1.8) depending on an arbitrary functional parameter $r(z_\pm)$. Using the notation $\langle t_A v^{-2}| \equiv (t_A^1 v_1^{-2}, \dots, t_A^N v_N^{-2})$, $A=1,2$ we find from (2.18), (2.19), (2.21), and (2.24):

$$\begin{aligned} q_4 &= -\Pi(v_j\mu_j^{-1})r^{-2}\Delta_1\Delta_2^{-1}\langle t_1 v^{-2}|a_2^{-1}|s_2\rangle\,, \\ q_3 &= \Pi(v_j\mu_j^{-1})r^2\Delta_2\Delta_1^{-1}\langle t_2 v^{-2}|a_1^{-1}|s_1\rangle\,, \qquad (2.26) \\ q_2 &= r^{-2}\langle t_1|a_2^{-1}|s_2\rangle, \qquad q_1 = -r^2\langle t_2|a_1^{-1}|s_1\rangle\,, \end{aligned}$$

$$F^+ = 2ir^{-1}\partial_+ r + 2\langle t_2|a_1^{-1}|s_1\rangle\langle t_1|a_2^{-1}|s_2\rangle, \qquad F^- = 2ir^{-1}\partial_- r\,. \qquad (2.27)$$

Calculation of the Function $r(z_\pm)$. In order to determine solutions of the $\mathscr{G}$-system (1.11) we have to specify the function $r(z_\pm)$ by the requirement that Eq. (1.11″) hold. Substituting Eqs. (2.26) into (1.11″) and comparing to (2.27) produces

$$2ir^{-1}\partial_+ r = -\omega_-\langle t_1|a_2^{-1}|s_2\rangle\langle t_2|a_1^{-1}|s_1\rangle\,. \qquad (2.28)$$

$$2ir^{-1}\partial_- r = \omega_-\Pi(v_j\mu_j^{-1})^2\langle t_1 v^{-2}|a_2^{-1}|s_2\rangle\langle t_2 v^{-2}|a_1^{-1}|s_1\rangle\,. \qquad (2.29)$$

To recover $r(z_\pm)$ from here, we shall need certain auxiliary identities.

Lemma 2.1. *Let the matrices* a_1, a_2 *be defined by Eq.* (2.10) *and* $\mathscr{A}_1, \mathscr{A}_2$ *stand for the augmented matrices. Then the following relations hold for any* n, ℓ:

$$\langle t_2 v^\ell|\mathscr{A}_2|s_1\mu^{-1}\rangle = \Pi(v_j\mu_j^{-1})\langle t_2 v^{\ell-1}|\mathscr{A}_1|s_1\rangle\,, \qquad (2.30)$$

$$\langle t_2 v^{-1}|\mathscr{A}_2|s_1\mu^n\rangle = \Pi(\mu_j v_j^{-1})\langle t_2|\mathscr{A}_1|s_1\mu^{n-1}\rangle\,. \qquad (2.31)$$

Proof. Consider an auxiliary expression

(a) $$\mathscr{S} = \Delta_2 + \langle t_2 v^\ell|\mathscr{A}_2|s_1\mu^{-1}\rangle$$

and transform it by means of the identity (2.16):

$$= \det(a_2 + |s_1\mu^{-1}\rangle\langle t_2 v^\ell|) = \Pi v_j\mu_j^{-1}\det(|\mu\rangle a_2\langle v^{-1}| + |s_1\rangle\langle t_2 v^{\ell-1}|)\,.$$

Applying the first relation in (2.11) yields then

$$\mathscr{S} = \Pi v_j\mu_j^{-1}\det(a_1 - |s_1\rangle\langle t_1 v^{-1}| + |s_1\rangle\langle t_2 v^{\ell-1}|)\,,$$

while the identity (2.16) implies:

$$\begin{aligned} \mathscr{S} &= \Pi v_j\mu_j^{-1}\{\Delta_1 + \langle t_2 v^{\ell-1} - t_1 v^{-1}|\mathscr{A}_1|s_1\rangle\} \\ &= \Pi v_j\mu_j^{-1}\{\det(a_1 - |s_1\rangle\langle t_1 v^{-1}|) + \langle t_2 v^{\ell-1}|\mathscr{A}_1|s_1\rangle\}\,. \end{aligned}$$

Finally, in view of Eq. (2.11) we have

(b) $$\mathscr{S} = \Pi v_j\mu_j^{-1}\{\Pi\mu_j v_j^{-1}\Delta_2 + \langle t_2 v^{\ell-1}|\mathscr{A}_1|s_1\rangle\}\,.$$

Comparing (a) to (b) we establish (2.30). Equation (2.31) is proved by analogy.

Corollary. *The following identities hold:*

$$\Pi v_j\mu_j^{-1}\langle t_2 v^{-2}|a_1^{-1}|s_1\rangle = \Pi\mu_j v_j^{-1}\langle t_2|a_1^{-1}|s_1\mu^{-2}\rangle\,, \qquad (2.32)$$

$$\Pi\mu_j v_j^{-1}\langle t_2 v|a_2^{-1}|s_1\mu^{-1}\rangle = \Pi v_j\mu_j^{-1}\langle t_2 v^{-1}|a_2^{-1}|s_1\mu\rangle\,, \qquad (2.33)$$

$$\Pi v_j\mu_j^{-1}\langle t_1 v^{-2}|a_2^{-1}|s_2\rangle = \Pi\mu_j v_j^{-1}\langle t_1|a_2^{-1}|s_2\mu^{-2}\rangle\,. \qquad (2.34)$$

Proof. Putting $\ell=-1$ in Eq. (2.30) and comparing to Eq. (2.31) with $n=-1$ produces the relation (2.32). Similarly, the identity (2.33) is proved by combining Eq. (2.30) with $\ell=1$ and Eq. (2.31) with $n=1$. Next, let us note that new identities may be generated from (2.30)–(2.33) merely by the permutation of indices $1 \rightleftarrows 2$. For example, Eq. (2.34) is the permuted Eq. (2.32).

Lemma 2.2.
$$i\partial_+(\Delta_1\Delta_2^{-1})\Delta_2\Delta_1^{-1} = -\langle t_2|a_1^{-1}|s_1\rangle\langle t_1|a_2^{-1}|s_2\rangle, \tag{2.35}$$
$$i\partial_-(\Delta_1\Delta_2^{-1})\Delta_2\Delta_1^{-1} = \langle t_2\nu^{-2}|a_1^{-1}|s_1\rangle\langle t_1|a_2^{-1}|s_2\mu^{-2}\rangle. \tag{2.36}$$

Proof. Equations (2.23) and (2.25) provide alternative expressions for $F^\pm$:
$$F^+ = 2ir^{-1}\partial_+ r + 2i\Delta_1\Delta_2^{-1}\partial_+(\Delta_2\Delta_1^{-1}),$$
$$F^- = 2i(r\Delta_2\Delta_1^{-1})^{-1}\partial_-(r\Delta_2\Delta_1^{-1}) + 2\langle t_2\nu^{-2}|a_1^{-1}|s_1\rangle\langle t_1|a_2^{-1}|s_2\mu^{-2}\rangle.$$
Comparing these to (2.27), we are led to Eqs. (2.35), (2.36).

Now, applying the identity (2.34) in Eq. (2.36) and comparing (2.35), (2.36) to (2.28), (2.29) results in the following

Proposition 2.3. *Solutions* (2.26), (2.27) *satisfy the identities* (1.11″) *if and only if up to an arbitrary multiplicative constant*
$$r(z_+, z_-) = (\Delta_1\Delta_2^{-1})^{\omega_-/2}. \tag{2.37}$$

Soliton Solutions in Explicit Form. In order to have a determinant formulation of solutions, let us note the elementary identity [3]
$$\langle u_1|a^{-1}|u_2\rangle = \frac{1}{\det a}\left|\begin{array}{c|c} 0 & \langle u_1| \\ \hline |u_2\rangle & a \end{array}\right|. \tag{2.38}$$

On the right-hand side of (2.38) there is a determinant of $(N+1)\times(N+1)$ matrix composed of $N\times N$ matrix a, N-column $|u_2\rangle$ and N-row $\langle u_1|$. Now, substituting Eq. (2.37) into (2.26) yields the N-soliton solution to the $\mathcal{G}$-system (1.11). Symmetrizing the found expressions by means of the identities (2.32)–(2.34) and employing the representation (2.38), we arrive at the main result of this section[5]:

Theorem 2.4. *The general N-soliton solution of the $\mathcal{G}$-system* (1.11), *propagating on zero background, is given by*
$$q_1 = -\left(\frac{\Delta_1}{\Delta_2}\right)^{\omega_-}\langle t_2|a_1^{-1}|s_1\rangle = -\frac{\Delta_1^{\omega_- - 1}}{\Delta_2^{\omega_-}}\left|\begin{array}{c|c} 0 & \langle t_2| \\ \hline |s_1\rangle & a_1 \end{array}\right|, \tag{2.39}$$
$$q_2 = \left(\frac{\Delta_2}{\Delta_1}\right)^{\omega_-}\langle t_1|a_2^{-1}|s_2\rangle = \frac{\Delta_2^{\omega_- - 1}}{\Delta_1^{\omega_-}}\left|\begin{array}{c|c} 0 & \langle t_1| \\ \hline |s_2\rangle & a_2 \end{array}\right|, \tag{2.40}$$
$$q_3 = \left(\frac{\Delta_2}{\Delta_1}\right)^{\omega_+}\left\langle\frac{t_2}{\nu}\middle|a_2^{-1}\middle|\frac{s_1}{\mu}\right\rangle = \frac{\Delta_2^{\omega_+ - 1}}{\Delta_1^{\omega_+}}\left|\begin{array}{c|c} 0 & \langle t_2\nu^{-1}| \\ \hline |s_1\mu^{-1}\rangle & a_2 \end{array}\right|, \tag{2.41}$$
$$q_4 = -\left(\frac{\Delta_1}{\Delta_2}\right)^{\omega_+}\left\langle\frac{t_1}{\nu}\middle|a_1^{-1}\middle|\frac{s_2}{\mu}\right\rangle = -\frac{\Delta_1^{\omega_+ - 1}}{\Delta_2^{\omega_+}}\left|\begin{array}{c|c} 0 & \langle t_1\nu^{-1}| \\ \hline |s_2\mu^{-1}\rangle & a_1 \end{array}\right|. \tag{2.42}$$

[5] Although in the remainder of this paper we do not write determinants explicitly [as in (2.39)–(2.42)], in view of the identity (2.38) all solutions should be understood as determinant ratios

Remark 2.1. If we are interested in solutions to the second order system (1.30), then only q_1 and q_4 are needed. In this case the following formulas turn out to be more efficient[5]:

$$\begin{aligned} q_1 &= -\Pi(\mu_j \nu_j^{-1})(\Delta_1 \Delta_2^{-1})^{\omega_- -1} \langle t_2 \nu | a_2^{-1} | s_1 \mu^{-1} \rangle \\ &= -\Pi(\nu_j \mu_j^{-1})(\Delta_1 \Delta_2^{-1})^{\omega_- -1} \langle t_2 \nu^{-1} | a_2^{-1} | s_1 \mu \rangle , \\ q_4 &= -\Pi(\nu_j \mu_j^{-1})(\Delta_1 \Delta_2^{-1})^{\omega_+ -1} \langle t_1 \nu^{-2} | a_2^{-1} | s_2 \rangle \\ &= -\Pi(\mu_j \nu_j^{-1})(\Delta_1 \Delta_2^{-1})^{\omega_+ -1} \langle t_1 | a_2^{-1} | s_2 \mu^{-2} \rangle . \end{aligned} \tag{2.43}$$

These are obtained by using Eq. (2.30) for $\ell = \pm 1$, Eqs. (2.32) and (2.34).

Remark 2.2. Solutions in the form of determinants ratio are usually supposed to be hardly verifiable. In order to simplify the verification, we shall provide simple closed expressions for the derivatives of (2.39)–(2.42), which are involved in the equations of the $\mathscr{G}$-system. Consider first an alternative representation for the solutions:

$$\begin{aligned} q_1 &= -i(\Delta_2 \Delta_1^{-1})^{\omega_+} \partial_+ \langle t_2 \nu^{-1} | a_2^{-1} | s_1 \mu^{-1} \rangle , \\ q_2 &= -i(\Delta_1 \Delta_2^{-1})^{\omega_+} \partial_+ \langle t_1 \nu^{-1} | a_1^{-1} | s_2 \mu^{-1} \rangle , \\ q_3 = i(\Delta_1 \Delta_2^{-1})^{\omega_-} \partial_- \langle t_2 | a_1^{-1} | s_1 \rangle , &\qquad q_4 = i(\Delta_2 \Delta_1^{-1})^{\omega_-} \partial_- \langle t_1 | a_2^{-1} | s_2 \rangle , \end{aligned} \tag{2.44}$$

which follow from (2.20), (2.22), and (2.37). Comparing then (2.44) to (2.39)–(2.42) produces the necessary derivatives:

$$\begin{aligned} i\partial_- \langle t_2 | a_1^{-1} | s_1 \rangle &= (\Delta_2 \Delta_1^{-1})^2 \langle t_2 \nu^{-1} | a_2^{-1} | s_1 \mu^{-1} \rangle , \\ i\partial_- \langle t_1 | a_2^{-1} | s_2 \rangle &= -(\Delta_1 \Delta_2^{-1})^2 \langle t_1 \nu^{-1} | a_1^{-1} | s_2 \mu^{-1} \rangle , \\ i\partial_+ \langle t_2 \nu^{-1} | a_2^{-1} | s_1 \mu^{-1} \rangle &= (\Delta_1 \Delta_2^{-1})^2 \langle t_2 | a_1^{-1} | s_1 \rangle , \\ i\partial_+ \langle t_1 \nu^{-1} | a_1^{-1} | s_2 \mu^{-1} \rangle &= -(\Delta_2 \Delta_1^{-1})^2 \langle t_1 | a_2^{-1} | s_2 \rangle . \end{aligned} \tag{2.45}$$

In view of Eqs. (2.35), (2.36), and (2.45) the verification is straightforward.

Remark 2.3. Redefinition $\mathbf{x}^i \to p_i \mathbf{x}^i$, $\mathbf{t}^i \to p_i^{-1} \mathbf{t}^i$, $\mathbf{s}^i \to q_i \mathbf{s}^i$, $\mathbf{y}^i \to q_i^{-1} \mathbf{y}^i$, $p_i, q_i \in \mathbb{C}$ leaves P^i, Q^i and therefore the solutions, unchanged. Below this invariance will be used to normalize $\mathbf{s}^i$ and $\mathbf{t}^i$ in a suitable way.

3. Extended Massive Thirring Model in Minkowski Space

In M^2 we put $z_+ = \eta$, $z_- = \xi$. The reduction to extended and conventional MTM is defined by the restrictions (1.21), which amount to the requirement that $iA_1^{\pm}$ and $iA_0^{\pm}$ lie in the real form $su(2)$ of $sl(2, \mathbb{C})$ algebra: $(A_1^{\pm})^\dagger = A_1^{\pm}$, $(A_0^{\pm})^\dagger = A_0^{\pm}$. Since in this case $(\Psi^{-1}(\lambda^*))^\dagger$ also satisfies Eqs. (2.2), an additional involution $\Psi(\lambda) \to (\Psi^{-1}(\lambda^*))^\dagger$ is defined on the manifold $\{\Psi(\lambda)\}$. In other words, a coordinate-independent matrix $H(\lambda)$ exists such that $\Psi(\lambda) = (\Psi^{-1}(\lambda^*))^\dagger H(\lambda)$. For χ (2.5) this implies

$$\chi(\lambda^*; \eta, \xi)^\dagger \chi(\lambda; \eta, \xi) = \Psi_0(\lambda; \eta, \xi) H(\lambda) \Psi_0^{-1}(\lambda; \eta, \xi) . \tag{3.1}$$

For general $H(\lambda)$ the right-hand side of Eq. (3.1) possesses essential singularities at $\lambda=0$ and $\lambda=\infty$ while the left-hand side is rational in λ. These singularities are removed if and only if $H(\lambda)$ is diagonal. Furthermore, in the generic case $H(\lambda)$ may be easily shown to be actually λ-independent:

Lemma 3.1. *Assume $m_1^i, m_2^i \neq 0$, $i=1,\dots,N$. Then $H(\lambda)$ is a constant matrix.*

Proof. Equation (3.1) implies that $H(\lambda)=\chi(\lambda^*)^\dagger\,\chi(\lambda)$ is a rational function with simple poles at $\lambda=\pm\nu_i$, $\pm\nu_i^*$, regular at $\lambda=\infty$. Consider, e.g., the residue at $\lambda=\nu_i$: $\operatorname{res}\{H(\lambda),\nu_i\}=\mathbf{p}^i\otimes\mathbf{t}^i$, where $\mathbf{p}^i=\chi(\nu_i^*)^\dagger R\mathbf{x}^i$, and $\mathbf{t}^i$ is given by (2.15). The residue is (η,ξ)-independent only if for any constant $\mathbf{c}\in\mathbb{C}^2$ the vector $\mathbf{c}'=\mathbf{c}\cdot\operatorname{res}\{H(\lambda),\nu_i\}$ is constant as well. However, provided $\mathbf{c}'\neq 0$ and in view of the assumption, the expression $c_1'/c_2'=(m_1^i/m_2^i)\exp\{i(\nu_i^2\eta+\nu_i^{-2}\xi)\}$ does depend on the coordinates. Therefore, $\mathbf{c}'=0$ for any $\mathbf{c}\in\mathbb{C}^2$ and the residue vanishes. Q.E.D.

For a diagonal constant matrix H Eq. (3.1) implies $\chi(\lambda)=(\chi^{-1}(\lambda^*))^\dagger H$. Equating poles and the corresponding residues in the left-hand side of this equation to those in the right-hand side produces, without loss of generality

$$H=R^\dagger R; \tag{3.2}$$

$$\nu_i=\mu_i^*, \qquad i=1,\dots,N; \tag{3.3}$$

$$HP^i=Q^{i\dagger}H, \qquad i=1,\dots,N. \tag{3.4}$$

From Eq. (3.4) it ensues that $\mathbf{t}^i=\ell_i\mathbf{s}^{i*}H$, $H\mathbf{x}^i=\ell_i^{-1}\mathbf{y}^{i*}$, $\ell_i\in\mathbb{C}$. By Remark 2.3 we may set $\ell_i=1$, $i=1,\dots,N$. Substituting then $\mathbf{s}^{i*}H$ for $\mathbf{t}^i$ and μ_i^* for ν_i in Eq. (2.10), we note that $a_1^\dagger=-a_2$ and $\Delta_1^*=(-1)^N\Delta_2$. By means of Eqs. (2.37) and (3.2) H is evaluated to be the unit matrix, and finally we find:

$$m_1^i=n_1^{i*}, \qquad m_2^i=n_2^{i*}, \qquad i=1,\dots,N. \tag{3.5}$$

Thus, we are able to formulate the following

Proposition 3.2. *The general N-soliton solution to the (extended) Massive Thirring Model* (1.20) *is extracted from the solution* (2.39)–(2.42) *of the $\mathcal{G}$-system by imposing the constraints* (3.3), (3.5).

Now let us exhibit the N-soliton solution of MTM in covariant form. Under the proper Lorentz transformations we have:

$$x^\mu\to O^{\mu\nu}x^\nu,\ O^{11}=O^{22}=\cosh\phi,\ O^{12}=O^{21}=\sinh\phi. \tag{3.6}$$

In spinor representation the rotation (3.6) is given by the matrix $S=\exp(-\tfrac{1}{2}\phi\sigma_3)$, while the reflection $x^1\to -x^1$ is represented by $S=\sigma_1$. To specify the transformation properties of solutions, let us adopt that the column $\Psi_i=(\mu_i,\mu_i^{-1})^T$ transforms as a covariant spinor. That is, if $\mu_i=e^{\beta_i}$, then we have $\beta_i\to\beta_i-\tfrac{1}{2}\phi$ under $SO(1,1)$ rotations (3.6), and $\beta_i\to-\beta_i$ under the reflection $x^1\to -x^1$. Next, it appears useful to introduce a unit complex space-like vector $k_i^\mu=-\tfrac{1}{2}i\tilde{\Psi}_i\gamma^\mu\Psi_i$ $\in M^2(\tilde{\Psi}_i=\Psi_i^T\gamma_0)$ so that $k_i^0=-i\cosh 2\beta_i$, $k_i^1=i\sinh 2\beta_i$ and a scalar $\zeta_i^0:\exp(\zeta_i^0)$ $=n_1^i\mu_i^{-1/2}$. Lastly, by Remark 2.3 we may, without loss of generality, impose the restriction $n_1^i n_2^i=\mu_i$. Then N-soliton solution of (extended) MTM is[5]:

$$\begin{aligned} u=q_1&=[(-1)^N\Delta_1/\Delta_1^*]^{\omega_-}\,\langle\exp(\tfrac{1}{2}\beta^*-\zeta^*)|a_1^{-1}|\exp(\zeta+\tfrac{1}{2}\beta)\rangle\,,\\ v=q_3&=[(-1)^N\Delta_1^*/\Delta_1]^{\omega_+}\,\langle\exp(-\tfrac{1}{2}\beta^*-\zeta^*)|(a_1^\dagger)^{-1}|\exp(\zeta-\tfrac{1}{2}\beta)\rangle\,, \end{aligned} \tag{3.7}$$

where $\zeta_i \equiv \frac{1}{2} k_i^\mu x_\mu + \zeta_i^0$, and a_1 matrix acquires the form

$$a_1^{ij} = \cosh(\zeta_i + \zeta_j^* - \tfrac{1}{2}\beta_i + \tfrac{1}{2}\beta_j^*)/\sinh(\beta_j^* - \beta_i).$$

Under $SO(1,1)$ rotation (3.6) $a_1 \to a_1$ and $\psi = (u, v)^T$ transforms like a covariant spinor: $\psi \to e^{-\phi\sigma_3/2}\psi$. Under the reflection $x^1 \to -x^1$ we have $a_1 \to a_1^\dagger$ and (for $\omega_\pm = 1$) $\psi \to \sigma_1 \psi$ [6].

In conclusion let us remark that N-soliton solution of the conventional ($\omega_\pm = 1$) MTM was obtained first in [16], in a different (non-determinant) form.

4. $O(1,1)$-Sine-Gordon Equation and the Second Massive Spinor Model in M^2

Let us consider Minkowski space and set $z_+ = i\eta$, $z_- = -i\xi$ (cf. Sect. 3). The systems (1.23) and (1.37) emerge under the condition of reality of $\omega_\pm, q_1, \ldots, q_4$. In this case $A_0^\pm$ and $A_1^\pm$ lie in $sl(2, \mathbb{R})$ which is equivalent to existence of the following involution on the manifold $\{\chi(\lambda)\}$:

$$\chi^*(\lambda^*) = \chi(\lambda) H, \quad (\chi^{-1}(\lambda^*))^* = H^{-1}\chi^{-1}(\lambda), \tag{4.1}$$

H being diagonal and constant in analogy with Sect. 3. Equations (4.1) imply:

$$H = \operatorname{diag}\{h, h^*\} = R^* R^{-1}; \tag{4.2}$$

$$\begin{aligned} &v_i^* = \iota_i v_{(i)}, \quad m_1^{i*} = h m_1^{(i)}, \quad m_2^{i*} = \iota_i h^* m_2^{(i)}, \\ &\mu_i^* = \gamma_i \mu_{[i]}, \quad n_1^{i*} = h^* n_1^{[i]}, \quad n_2^{i*} = \gamma_i h n_2^{[i]}, \end{aligned} \tag{4.3}$$

$i = 1, \ldots, N$; $\iota_i, \gamma_i = \pm 1$. Here we have introduced two independent permutations of N numbers: $\{1, \ldots, N\} \to \{(1), \ldots, (N)\}$ and $\{1, \ldots, N\} \to \{[1], \ldots, [N]\}$ ((i) and $[i]$ denote the corresponding images of i) such that $((i)) = [[i]] = i \ \forall i \in \{1, \ldots, N\}$. By means of Eqs. (4.3), (2.37), and (4.2) h is calculated to be

$$h = (\Pi \iota_j \gamma_j)^{\omega_-/2} \tag{4.4}$$

(from now on the value of h is fixed). So we have

Proposition 4.1. *The general N-soliton solutions to the (extended) second massive spinor model* (1.23) *and (extended)* $O(1,1)$ *SGE* (1.37) *are extracted from the solution* (2.39)–(2.43) *of the $\mathscr{G}$-system by imposing the restrictions* (4.3)–(4.4).

N-Soliton Solution of $O(1,1)$ *SGE in Covariant Form.* Below we limit ourselves to the case $\omega_\pm = 1$. Let us define two Lorentz scalars, $\exp(\zeta_i^0) = n_1^i$ and $\exp(z_i^0) = (m_2^i)^{-1}$, and two unit vectors, k_i^μ and q_i^μ ($i = 1, \ldots, N$) such that $k_i^0 = \frac{1}{2}(\mu_i^2 - \mu_i^{-2})$, $k_i^1 = -\frac{1}{2}(\mu_i^2 + \mu_i^{-2})$, $q_i^0 = \frac{1}{2}(v_i^{-2} - v_i^2)$, $q_i^1 = \frac{1}{2}(v_i^2 + v_i^{-2})$. Due to the Remark 2.3, we may impose the constraints $n_1^i n_2^i = \upsilon_i \mu_i$ and $m_1^i m_2^i = \delta_i v_i$, where $\upsilon_i = \{\pm 1 \text{ for } i = [i]; 1 \text{ otherwise}\}$, while $\delta_i = \{\pm 1 \text{ for } i = (i); 1 \text{ otherwise}\}$. Using then Eqs. (2.43) for $q_1 = \varphi^-$ and $q_4 = \varphi^+$, we obtain [5]:

$$\varphi^+ = \Pi(\mu_j v_j^{-1}) \langle e^z \delta | b_2^{-1} | \upsilon e^{-\zeta} \rangle, \quad \varphi^- = \Pi(\mu_j v_j^{-1}) \langle e^{-z} | b_2^{-1} | e^{\zeta} \rangle. \tag{4.5}$$

---

[6] Below we, for brevity, restrict ourselves to the proper Lorentz/orthogonal transformations

Here $b_2^{ij}=(\delta_j e^{\zeta_i+z_j}+v_i e^{-\zeta_i-z_j})/(v_j^2\mu_i^{-2}-1)$ and $\zeta_i=\frac{1}{2}k_i^\mu x_\mu+\zeta_i^0$, $z_i=\frac{1}{2}q_i^\mu x_\mu+z_i^0$. Since $v_j^2\mu_i^{-2}=(q_j^\mu+\varepsilon^{\mu\nu}q_{j\nu})k_{i\mu}$, solution (4.5) is indeed invariant under the Lorentz transformations (3.6)[6].

One-Soliton Solution. For $N=1$, introducing $e^{\tilde\zeta}\equiv\delta_1^{-1/2}v_1^{-1/2}\exp(-z_1^0-\zeta_1^0)$, $e^{\tilde z}\equiv i\delta_1^{1/2}v_1^{-1/2}\exp(z_1^0-\zeta_1^0)$, $e^\alpha\equiv i\mu_1 v_1$, $e^\beta\equiv i\mu_1 v_1^{-1}$ we rewrite (4.5) as

$$\varphi^\pm=\pm\cosh\beta\frac{\exp\{\pm[\sinh\beta(\cosh\alpha\cdot x^0+\sinh\alpha\cdot x^1)+\tilde z]\}}{\cosh\{\cosh\beta(\sinh\alpha\cdot x^0+\cosh\alpha\cdot x^1)+\tilde\zeta\}}. \tag{4.6}$$

Here, in view of Eqs. (4.3)–(4.4), $e^{\alpha*}=-\tau\tilde\tau e^\alpha$, $e^{\beta*}=-\tau\tilde\tau e^\beta$, $e^{\tilde\zeta*}=\tilde\tau e^{\tilde\zeta}$, $e^{\tilde z*}=-\tau e^{\tilde z}$ (we have denoted $\gamma_1 v_1\delta_1\equiv\tau$, $\iota_1 v_1\delta_1\equiv\tilde\tau$). At $\tilde\tau=1$ we have $\operatorname{Im}\tilde\zeta=0$, the denominator of (4.6) vanishes nowhere and the soliton is regular in the finite part of (x^0,x^1) plane. At $\tilde\tau=-1$, conversely, $\varphi^\pm$ is singular there. In the generic case of $\sinh\beta\neq 0$, $\varphi^\pm$ is, in addition, unbounded as $|x^1|$ or $|x^0|\to\infty$. To make sure that solution (4.6) indeed represents a localised object, it is advantageous to pass from $\varphi^\pm$ to new variables.

Namely, provided $\tau=1$, Eq. (4.6) implies $\varphi^+\varphi^-\geqq 0$, and we can introduce complex field $\varphi=\varrho e^{i\vartheta}$ with $\varrho\equiv(\varphi^+\varphi^-)^{1/2}$ and $\vartheta\equiv\operatorname{arc\,tanh}[(\varphi^+-\varphi^-)/(\varphi^++\varphi^-)]$. If $\tau=-1$, the soliton (4.6) obeys $\varphi^+\varphi^-\leqq 0$, and we define $\varrho\equiv(-\varphi^+\varphi^-)^{1/2}$, $\vartheta\equiv\operatorname{arc\,tanh}[(\varphi^++\varphi^-)/(\varphi^+-\varphi^-)]$. Transforming to φ, Lagrangian (1.36) becomes

$$\mathscr{L}_{12}=\varphi_\eta\varphi_\xi\varphi^*\varphi^{-1}(1+\tau|\varphi|^2)^{-1}-|\varphi|^2+(\text{c.c.}),\qquad \tau=\pm 1, \tag{4.7$'$}$$

or, in the covariant notation $[J_\mu=i(\varphi^*\,\partial_\mu\varphi-\varphi\,\partial_\mu\varphi^*)]$:

$$L_{12}=\frac{|\partial_\mu\varphi|^2}{1+\tau|\varphi|^2}-|\varphi|^2-\frac{1}{2}\frac{J_\mu^2}{|\varphi|^2(1+\tau|\varphi|^2)}. \tag{4.7$''$}$$

In terms of the new variable solution (4.6) decays rapidly as $|x^1|\to\infty$ (or as $|x^0|\to\infty$). This justifies its being referred to as a soliton. At $\tau\tilde\tau=-1$ the soliton is infraluminic, while at $\tau\tilde\tau=1$ it is a tachyon.

The Real SGE. Among the solutions (4.5) there are ones remaining finite as $|x^0|,|x^1|\to\infty$. This important subclass satisfies the constraint $\tau\varphi^+=\varphi^-\equiv\varphi$ $(\tau=\pm 1)$, with φ verifying the real SGE (1.35′).

Proposition 4.2. *The general N-soliton solution to the real SGE* (1.35′) *is extracted from the solution* (4.5) *of* $O(1,1)$ *SGE by imposing the restrictions*

$$v_i=i\mu_i\Rightarrow k_i^\mu=q_i^\mu,\quad \exp(\zeta_i^0)=\tau^{1/2}i^{N+1}\exp(z_i^0),\quad v_i=\delta_i. \tag{4.8}$$

Proof. Under the reduction (4.3)–(4.4) the identity (2.33) acquires the form

$$\langle e^{-z}v^{-2}|b_2^{-1}|\mu^2e^\zeta\rangle=\Pi(\mu_j v_j^{-1})^2\langle e^{-z}|b_2^{-1}|e^\zeta\rangle. \tag{4.9}$$

On the other hand, Eqs. (4.8) imply $b_2^{ij}=-\mu_i^2\delta_i b_2^{ji}v_j^{-2}\delta_j$, whence $\varphi^+=\tau i^N\langle e^{-z}v^{-2}|b_2^{-1}|\mu^2e^\zeta\rangle$. Making use of (4.9), we obtain $\varphi^+=\tau(-i)^N\langle e^{-z}|b_2^{-1}|e^\zeta\rangle=\tau\varphi^-$. Q.E.D.

Combining Eq. (4.8) with (4.3)–(4.4), we can cast the N-soliton solution of the real SGE (1.35′) into the following ultimate form[5]:

$$\varphi=\langle e^{-z}|b_2^{-1}|e^z\rangle, \tag{4.10}$$

where $b_2^{ij}=\{e^{z_i+z_j}-\tau e^{-z_i-z_j}\}/(\mu_j^2\mu_i^{-2}+1)$ and $\exp\{(z_i^0)^*\}=\iota_i\exp\{z_{(i)}^0\}$.

5. Massive Thirring Model and $O(2)$ sine/sinh-Gordon Equations in Euclidean Space

In E^2 we set $z_+=z$, $z_-=\varepsilon z^*$, $\varepsilon=\pm 1$. Reduction to the (extended) $O(2)$ SGE (1.33) and, simultaneously, to the extended MTM (1.28) is defined by the requirements (1.27) which amount to the following constraints:

$$(A_1^-)^\dagger=\tau\mathscr{E}^2 A_1^+, \qquad (A_0^-)^\dagger=\varepsilon A_0^+, \tag{5.1}$$

with $\mathscr{E}=\operatorname{diag}\{1,\sqrt{\varepsilon}\}$. Unlike the cases discussed above, each of the conditions (5.1) relate *two different* matrices. Consequently, this reduction is not associated with any real form of the $sl(2,\mathbb{C})$ algebra; nevertheless, its solutions are extracted in the same way. From (5.1) it ensues that a diagonal matrix H exists such that

$$\chi(\lambda)=\mathscr{E}^{-1}[\chi^{-1}(\tau\sqrt{\varepsilon}/\lambda^*)]^\dagger H, \tag{5.2}$$

in the generic case $(m_1^i, m_2^i \neq 0,\ i=1,\dots,N)$ H being constant. Also it may be inferred from (5.2) that

$$H^\dagger=\mathscr{E}^2 H. \tag{5.3}$$

Now, comparing the left-hand side of (5.2) to the right-hand side, we have

$$\nu_i=\tau\varepsilon\sqrt{\varepsilon}(\mu_i^*)^{-1}, \qquad i=1,\dots,N; \tag{5.4}$$

$$RP^i=-\tau\varepsilon\sqrt{\varepsilon}(\mu_i^*)^{-2}\mathscr{E}^{-1}(R^\dagger)^{-1}Q^{i\dagger}; \tag{5.5}$$

$$H=\Pi(\mu_j\nu_j^{-1})^*(\Delta_2^*/\Delta_1^*)^{\sigma_3}\mathscr{E}RR^\dagger, \tag{5.6}$$

where we have used (2.17). Expressing $\mathbf{t}^i$ from (5.5): $\mathbf{t}^i=k_i H\mathbf{s}^{i*}$, $k_i\in\mathbb{C}$ and inserting into the matrices (2.10) gives, with the help of (5.3):

$$e^{i\delta}=\pm(\sqrt{\varepsilon})^N, \tag{5.7}$$

where $\delta\equiv\arg(\Delta_1\Delta_2^{-1})$. Combining Eqs. (5.6) and (5.7), we obtain $H=\pm\tau^N\Pi|\mu_j|^2\mathscr{E}^{2N+1}\exp(-\delta\Omega\sigma_3)$ with $\Omega\equiv\operatorname{Im}\omega$. Finally, picking $k_i=\pm\tau^N\Pi|\mu_j|^{-2}$, $i=1,\dots,N$ yields

$$m_1^i=e^{-\delta\Omega}n_1^{i*}, \qquad m_2^i=\varepsilon^N\sqrt{\varepsilon}\,e^{\delta\Omega}n_2^{i*}, \tag{5.8}$$

the values of $\sqrt{\varepsilon}$ and δ being fixed. Thus, we arrive at

Proposition 5.1. *The general N-soliton solution to the (extended) Euclidean MTM* (1.28) *and* $O(2)$ *SGE* (1.33) *is extracted from the solution* (2.39)–(2.43) *to the* $\mathscr{G}$*-system by imposing the restrictions* (5.4), (5.7), (5.8).

Let us cast the Euclidean $O(2)$ SGE N-soliton solution into a covariant form [6]. Define Euclidean unit vector k_μ^i through the relations $k_1^i=-\frac{1}{2}i(\mu_i^2+\varepsilon\mu_i^{-2})$, $k_2^i=\frac{1}{2}(\mu_i^2-\varepsilon\mu_i^{-2})$, $i=1,\dots,N$ and a scalar ζ_i^0 by $\exp(\zeta_i^0)\equiv\exp(-\frac{1}{2}\delta\Omega)n_1^i$. According to Remark 2.3 we may impose $n_1^i n_2^i=\mu_i$. Then solution to $O(2)$ SGE is obtained from the first formula in (2.43) [5]:

$$\varphi=q_1=\Pi|\mu_j|^2(\pm\det b_1/\det b_2)^{i\Omega}\langle e^{-\zeta^*}|b_2^{-1}|e^{\zeta}\rangle. \tag{5.9}$$

Here $\zeta_i = \frac{1}{2} k^i_\mu x_\mu + \zeta^0_i$, and matrices $b_1 = \tau\sqrt{\varepsilon}|\mu\rangle a_1$ and $b_2 = |\mu\rangle a_2$ are given by

$$b_1^{ij} = \{(\mu_i\mu_j^*)^{-1} \exp(\zeta_i + \zeta_j^*) + \tau\varepsilon^{N+1}\mu_i\mu_j^* \exp(-\zeta_i - \zeta_j^*)\}/[\varepsilon(\mu_i\mu_j^*)^{-2} - 1],$$
$$b_2^{ij} = \{\exp(\zeta_i + \zeta_j^*) + \tau\varepsilon^N \exp(-\zeta_i - \zeta_j^*)\}/[\varepsilon(\mu_i\mu_j^*)^{-2} - 1].$$

The quantity $(\mu_i\mu_j^*)^{-2} = (k^i_\mu + i\varepsilon_{\mu\nu}k^i_\nu)k^{j*}_\mu$ being invariant with respect to $SO(2)$-rotations of E^2 space, b_1, b_2 and, eventually, φ are $SO(2)$ scalars.

The one-soliton solution at $\Omega = 0$ looks like

$$\varphi = (\varepsilon|\mu|^{-2} - |\mu|^2)\exp(\zeta - \zeta^*)\{\exp(\zeta + \zeta^*) + \tau\varepsilon\exp(-\zeta - \zeta^*)\}^{-1}.$$

Hence, in contrast to the M^2 case (cf. Sect. 6), the Euclidean $O(2)$ sinh-Gordon equation [Eq. (1.34) at $\tau = 1$] possesses both singular ($\varepsilon = -1$) and regular ($\varepsilon = 1$) solitons.

Reduction to the real Euclidean SGE. At $\omega = 1$ the real solutions in (5.9) satisfy the usual SGE (1.35″). In order to isolate the real φ's, let us first recall the identity (2.33). Under the conditions (5.4), (5.8) it reads

$$\langle \mu^{*2} e^{-\zeta^*} | b_2^{-1} | e^{\zeta} \mu^2 \rangle = \varepsilon^{N+1} \langle e^{-\zeta^*} | b_2^{-1} | e^{\zeta} \rangle. \tag{5.10}$$

Next, consider a permutation $\{1, \ldots, N\} \to \{(1), \ldots, (N)\}$ such that $((i)) = i \ \forall i \in \{1, \ldots, N\}$. Imposing the reduction conditions:

$$(\mu_i^*)^2 = -\varepsilon\mu_{(i)}^{-2} \Rightarrow k^{i*}_\alpha = k^{(i)}_\alpha, \qquad \exp(\zeta_i^{0*}) = \kappa_i i^{N+1} \exp(\zeta^0_{(i)}) \tag{5.11}$$

with $\kappa_i = \pm 1$, we simplify the expression (5.9):

$$\varphi = \langle e^{-\zeta^*} | b_2^{-1} | e^{\zeta} \rangle. \tag{5.12}$$

Also we observe that $b_2^{ij*} = -\varepsilon\kappa_{(i)}\mu_{(i)}^{-2} b_2^{(i)(j)} (\mu^*_{(j)})^{-2} \kappa_{(j)}$. Using this relation and Eq. (5.10) one easily verifies that $\varphi = \varphi^*$. Thus, we have

Proposition 5.2. *The general N-soliton solution to the real SGE* (1.35″) *in E^2-space is given by Eq.* (5.12) *subject to the constraints* (5.11).

Let us say that a pair (μ_i, ζ^0_i) corresponds to a "soliton" provided $(i) = i$. If, conversely, $(i) \neq i$, then the set $(\mu_i, \zeta^0_i, \mu_{(i)}, \zeta^0_{(i)})$ parametrizes a "bion." Asymptotically, as $|z|^2 \to \infty$, solution (5.12) splits into a set of "solitons" and "bions." In the case of $\varepsilon = 1$ Eq. (5.11) implies that the "soliton" component is absent and (5.12) is a nonlinear superposition solely of "bions," regular at both τ. At $\varepsilon = -1$ both types of constituents contribute in (5.12), "solitons" and "bions" being singular at $\tau = 1$ and regular at $\tau = -1$.

In conclusion let us note that in the case $\tau = \varepsilon = -1$ the N-soliton solution to the real SGE (1.35″) is known in Hirota's form (see [22] and refs. therein).

6. *O*(2) sine-Gordon Equation in Minkowski Space

In M^2 space ($z_+ = \eta, z_- = \xi$) the reduction to $O(2)$ sine/sinh-Gordon equation (1.32) is defined by imposing the conditions ($v = \text{const} \in \mathbb{R}$):

$$q_1 = e^{iv}\varphi, \qquad q_4 = e^{-iv}\tau\varphi^*, \qquad \omega_\pm = 1. \tag{6.1}$$

Then q_2 and q_3 are automatically constrained by (1.29). Unlike the E^2 case, the above restrictions do not result in any straightforward algebraic constraints on $A_0^{\pm}$ and $A_1^{\pm}$ matrices. In this situation the simplest way to extract the specialised solutions consists in analyzing the explicit expressions. From Eq. (2.43) we have

$$q_1 = -\Pi(\mu_j v_j^{-1})\langle t_2 v | a_2^{-1} | s_1 \mu^{-1}\rangle,$$
$$\tau q_4^* = -\tau \Pi(v_j \mu_j^{-1})^* \langle s_2^* | (a_2^\dagger)^{-1} | (t_1 v^{-2})^*\rangle.$$

Identifying then

$$\mu_i = v_i^*, \qquad m_1^i = \mu_i^* n_1^{i*}, \qquad m_2^i = -\tau v_i^{-1} n_2^{i*} \tag{6.2}$$

yields $\langle t_2 v| = -\tau\langle s_2^*|$, $|s_1\mu^{-1}\rangle = |(t_1 v^{-2})^*\rangle$, $a_2^\dagger = -a_2$, and, finally, $q_1 = \tau q_4^*$.

Regular Method of Finding the Reduction Conditions. To prove that relations (6.2) extract the most general N-soliton solution to $O(2)$ SGE, we shall exhibit the involution defined on the manifold $\{\Psi(\lambda)\}$ and responsible for the discussed reduction. The restrictions this involution induces turn out to be given just by (6.2).

Let us start from the triangular gauge (1.6), the conditions (6.1), (1.29) being imposed. The gauge transformation (1.4), generated by the matrix g_τ:

$$g_\tau = \begin{pmatrix} w & i\tau^{1/2} e^{iv}\varphi w^{-1} \\ 0 & -i\tau^{1/2} w^{-1} \end{pmatrix}, \qquad w = \left(\frac{\varphi\mathscr{D}}{\varphi^*}\right)^{1/4}, \qquad \mathscr{D} = 1 + \tau|\varphi|^2 \tag{6.3}$$

($\tau^{1/2}$ fixed), converts $U_2^{\pm}, U_0^{\pm}$ matrices into the following ones:

$$\begin{aligned} \tilde{U}_2^+ &= \tfrac{1}{2}\sigma_3, \qquad \tilde{U}_0^- = (4i\mathscr{D})^{-1}[\varphi_\xi \varphi^{-1} - (\text{c.c.})]\sigma_3, \\ \tilde{U}_0^+ &= (4i\mathscr{D})^{-1}[(\varphi_\eta\varphi^{-1} + 2\tau\varphi_\eta\varphi^*) - (\text{c.c.})]\sigma_3 \\ &\quad + \tau^{1/2}\begin{pmatrix} 0 & e^{iv}\varphi_\eta w^{-2} \\ -e^{-iv}\varphi_\eta^*(w^{-2})^* & 0 \end{pmatrix}, \\ \tilde{U}_2^- &= i\tau^{1/2}\begin{pmatrix} 0 & e^{iv}\varphi(w^2)^* \\ e^{-iv}\varphi^* w^2 & 0 \end{pmatrix} + (\tfrac{1}{2} + \tau|\varphi|^2)\sigma_3. \end{aligned} \tag{6.4}$$

At $\tau = 1$, respectively $\tau = -1$, these matrices times i lie in $su(1,1)$, respectively $su(2)$ real form of $sl(2,\mathbb{C})$ algebra: $(\tilde{U}_2^{\pm})^\dagger = \mathscr{T}\tilde{U}_2^{\pm}\mathscr{T}$, $(\tilde{U}_0^{\pm})^\dagger = \mathscr{T}\tilde{U}_0^{\pm}\mathscr{T}$, with $\mathscr{T} = \text{diag}\{1, -\tau\}$. Consequently, there exists a matrix $H(\lambda)$ such that

$$\tilde{\Psi}(\lambda; \eta, \xi) = \mathscr{T}(\tilde{\Psi}^{-1}(\lambda^*; \eta, \xi))^\dagger H(\lambda). \tag{6.5}$$

Making now the composite gauge transformation with $g = g_\tau^{-1} g_s(\lambda)$, we return to the stratified gauge (2.2)–(2.3). The relation (6.5) induces then the involution

$$\Psi(\lambda; \eta, \xi) \to G^{-1}(\lambda)(\Psi^{-1}(\lambda^*; \eta, \xi))^\dagger = \Psi(\lambda; \eta, \xi) H^{-1}(\lambda) \tag{6.6}$$

on the manifold $\{\Psi(\lambda)\}$. Here

$$G(\lambda) \equiv g_s^\dagger(\lambda^*)(g_\tau \mathscr{T} g_\tau^\dagger)^{-1} g_s(\lambda) = \mathscr{D}^{-1/2}\begin{pmatrix} \lambda & e^{iv}\varphi \\ e^{-iv}\varphi^* & -\tau\lambda^{-1} \end{pmatrix}. \tag{6.7}$$

In analogy with the consideration in Sect. 3, we can demonstrate that $H(\lambda)$ is a diagonal matrix, Eq. (6.6) being reduced to

$$G(\lambda)\chi(\lambda; \eta, \xi) = [\chi^{-1}(\lambda^*; \eta, \xi)]^\dagger H(\lambda). \tag{6.8}$$

In the generic case of $m_1^i, m_2^i \neq 0$, $i=1,\dots,N$ a rational function

$$H(\lambda)=\chi^{\dagger}(\lambda^*)\,G(\lambda)\,\chi(\lambda) \tag{6.9}$$

possesses no poles at $\lambda=\pm\nu_i$, $\pm\nu_i^*$ (the proof repeats that of Lemma 3.1). Thus, the Laurent expansion of $H(\lambda)$ at $\lambda=0$ contains only a finite number of terms and can be easily evaluated from (6.9):

$$H(\lambda)=\mathscr{D}^{-1/2}|\varDelta_1\varDelta_2^{-1}|\,\mathrm{diag}\{\lambda,\, -\tau\Pi|\mu_j\nu_j^{-1}|^2\lambda^{-1}\}. \tag{6.10}$$

Finally, inserting Eqs. (2.6) and (6.10) into (6.8), and equating the corresponding poles and residues yields the reduction conditions (6.2). So we have

Proposition 6.1. *The general N-soliton solution of $O(2)$ SGE* (1.32) *is extracted from the solution* (2.39) *of the $\mathscr{G}$-system by imposing the constraints* (6.2).

In order to provide the covariant form of solutions [6], let us define unit complex space-like vectors $k_i^\mu : k_i^0=-\frac{1}{2}i(\mu_i^2+\mu_i^{-2})$, $k_i^1=\frac{1}{2}i(\mu_i^2-\mu_i^{-2})$ and scalars $\zeta_i^0 : \exp(\zeta_i^0)=n_1^i$. In view of Remark 2.3 we may impose $n_1^i n_2^i=\mu_i$. The N-soliton solution (2.43) to $O(2)$ SGE is then rewritten as [5]:

$$\varphi=\langle e^{-\zeta^*}|b_2^{-1}|e^{\zeta}\rangle, \tag{6.11}$$

where $\zeta_i=\frac{1}{2}k_i^\mu x_\mu+\zeta_i^0$ and $b_2=|\mu\rangle a_2\langle(\mu^{-1})^*|$ matrix is given by

$$b_2^{ij}=\{\exp(\zeta_i+\zeta_j^*)-\tau\exp(-\zeta_i-\zeta_j^*)\}/[(\mu_i^{-1}\mu_j^*)^2-1].$$

Since $(\mu_i^{-1}\mu_j^*)^2=(k_j^\mu+\varepsilon^{\mu\nu}k_{j\nu})k_{i\mu}$, solution (6.11) indeed represents a scalar. Finally, it should be noted that at $\tau=-1$ the 2-soliton solution is known in Hirota's form [15].

Sometimes, it is worth having a closed expression for the modulus of N-soliton solution. The modulus of the solution (6.11) reads:

$$|\varphi|^2=\tau(|\det b_1/\det b_2|^2-1), \tag{6.12}$$

where $b_1=|\mu\rangle a_1\langle(\mu^{-1})^*|$ matrix is defined by

$$b_1^{ij}=\{\mu_i^{-1}\mu_j^*\exp(\zeta_i+\zeta_j^*)-\tau(\mu_i^{-1}\mu_j^*)^{-1}\exp(-\zeta_i-\zeta_j^*)\}/[(\mu_i^{-1}\mu_j^*)^2-1].$$

To obtain (6.12) we observe that $\det G(\lambda)=-\tau$, $\det H(\lambda)=-\tau\mathscr{D}^{-1}|\varDelta_1\varDelta_2^{-1}|^2$ and $\det\chi(\infty)=1$. Comparing then the determinants of the right-hand side and left-hand side of (6.8) at $\lambda=\infty$ produces (6.12).

The Real SGE. Now let us extract solutions of the real SGE (1.35′). At this stage it is useful to fix $\upsilon=\frac{1}{4}\pi$. Then condition $\varphi=\varphi^*$ is equivalent to the equalities $(A_1^{\pm})^*=\mp iA_1^{\pm}$, $(A_0^{\pm})^*=-A_0^{\pm}$, which induce an additional involution on the manifold $\{\Psi(\lambda)\}$: $\Psi(\lambda)\to\Psi^*(i\lambda^*)\in\{\Psi(\lambda)\}$. Following the standard procedure we arrive at

Proposition 6.2. *The general N-soliton solution to the real SGE* (1.35′) *is extracted from the solution* (6.11) *of $O(2)$ SGE by imposing the restrictions*

$$\mu_i^*=i\mu_{(i)} \Rightarrow k_i^{\alpha *}=k_{(i)}^{\alpha}, \qquad \exp(\zeta_i^{0*})=\kappa_i\exp(\zeta_{(i)}^0), \tag{6.13}$$

where $\kappa_i=\pm1$ and the parentheses denote any permutation of indices such that $((i))=i$ $\forall i\in\{1,\dots,N\}$.

The quantities labelled by i satisfying $(i)=i$ correspond to single solitons, whereas at $(i)\neq i$ pairs $\{i,(i)\}$ label bions (breathers).

The N-soliton solution to the real sine-Gordon equation is, of course, well-known (see, e.g. [17–20]). The sinh-Gordon case has been treated in [21].

7. Connection Between Solutions with the Vanishing and Non-Vanishing Boundary Conditions

Let us consider the complexified sine-Gordon equations in M^2 space, i.e., (1.32) and (4.7). As we have already mentioned in Remark 1.2, the sign of the corresponding mass terms may be changed merely by substituting $\xi\to-\xi$. This substitution takes subluminal solitons to tachyons and vice versa, boundary conditions remaining the same. Below we shall exhibit a less trivial invertible transformation that also changes the mass term sign but, unlike the above substitution, relates solutions with the vanishing asymptotics $|\varphi|\to 0$ to solutions with the boundary conditions $|\tilde\varphi|\to 1$ as $|x^1|$ (or $|x^0|)\to\infty$. In particular, the previously constructed *solitons* (subluminal, decaying at infinity) are converted into subluminal *kinks*.

It appears useful to rewrite $O(2)$ sine-Gordon equation,

$$\varphi_{\eta\xi}+\varphi_\eta\varphi_\xi\varphi^*(1-|\varphi|^2)^{-1}+\varphi(1-|\varphi|^2)=0\,, \tag{7.1}$$

derivable from the Lagrangian (1.32) with $\tau=-1$, as

$$\varrho_{\eta\xi}+\varrho(\varrho_\eta\varrho_\xi-\vartheta_\eta\vartheta_\xi)(1-\varrho^2)^{-1}+\varepsilon\varrho(1-\varrho^2)=0\,, \tag{7.2'}$$

$$[\vartheta_\eta\varrho^2(1-\varrho^2)^{-1}]_\xi+[\vartheta_\xi\varrho^2(1-\varrho^2)^{-1}]_\eta=0\,. \tag{7.2''}$$

Here $\varepsilon=1$, $\varphi=\varrho e^{i\vartheta}$, $\vartheta\in\mathbb{R}$, $\varrho>0$. Consider solutions satisfying $\varrho\leqq 1$. In view of (7.2″) we may introduce new variables by

$$\tilde\varrho=(1-\varrho^2)^{1/2}\,,\qquad \tilde\vartheta_\eta=-\vartheta_\eta\varrho^2(1-\varrho^2)^{-1}\,,\qquad \tilde\vartheta_\xi=\vartheta_\xi\varrho^2(1-\varrho^2)^{-1} \tag{7.3}$$

(these relations define $\tilde\vartheta$ up to an additive constant). By simple substitution one verifies then that $\tilde\varrho$ and $\tilde\vartheta$ obey (7.2) with $\varepsilon=-1$. Thus we have

Proposition 7.1. *Assume that* $\varphi=\varrho e^{i\vartheta}$ *is a solution of Eq.* (7.1) *such that* $\varrho\leqq 1$. *Then* $\tilde\varphi=\tilde\varrho e^{i\tilde\vartheta}$ *with* $\tilde\varrho$ *and* $\tilde\vartheta$ *defined through* (7.3) *solves the equation*

$$\tilde\varphi_{\eta\xi}+\tilde\varphi_\eta\tilde\varphi_\xi\tilde\varphi^*(1-|\tilde\varphi|^2)^{-1}-\tilde\varphi(1-|\tilde\varphi|^2)=0\,. \tag{7.4}$$

Remark 7.1. According to Eq. (6.12) with $\tau=-1$, N-soliton solution (6.11) of Eq. (7.1) [propagating on zero background, i.e., $|\varphi(x^\mu)|\to 0$ as $|x^1|\to\infty$] verifies $\varrho\leqq 1$. Applying the transformation (7.3) one generates a solution to Eq. (7.4) consisting of N *kinks* ($|\tilde\varphi(x^\mu)|\to 1$). The formula for its modulus is straightforward from (6.12).

In the case of $\tau=1$ Eq. (1.32) defines $O(2)$ **sinh**-Gordon equation:

$$\varphi_{\eta\xi}-\varphi_\eta\varphi_\xi\varphi^*(1+|\varphi|^2)^{-1}+\varphi(1+|\varphi|^2)=0\,. \tag{7.5}$$

Let us introduce a new field $\tilde\varphi=\tilde\varrho\exp(i\tilde\vartheta)$ by the relations

$$\tilde\varrho=(1+\varrho^2)^{1/2}\,,\qquad \tilde\vartheta_\eta=-\vartheta_\eta o^2(1+\varrho^2)^{-1}\,,\qquad \tilde\vartheta_\xi=\vartheta_\xi(1+\varrho^2)^{-1}\,. \tag{7.6}$$

The following statement is then directly verified.

Proposition 7.2. *Assume that* $\varphi = \varrho e^{i\vartheta}$ *is a solution of Eq.* (7.5). *Then* $\tilde{\varphi} = \tilde{\varrho}\exp(i\tilde{\vartheta})$ *with* $\tilde{\varrho}$ *and* $\tilde{\vartheta}$ *given by* (7.6) *solves* $O(2)$ ***sine**-Gordon equation* (7.4).

Similar assertions may be proved for $O(1,1)$ SG model (1.36) as well. The corresponding equations of motion read (we put $z_+ = \eta$, $z_- = \xi$):

$$\varphi^{\pm}_{\eta\xi} + \varepsilon\varphi^{\pm}(1 + \varphi^+\varphi^-) - \varphi^{\pm}_{\eta}\varphi^{\pm}_{\xi}\varphi^{\mp}(1 + \varphi^+\varphi^-)^{-1} = 0\,. \tag{7.7}$$

In terms of the product and quotient variables, $\mathscr{P} \equiv \varphi^+\varphi^-$ and $\mathscr{Q} \equiv \varphi^+/\varphi^-$, Eqs. (7.7) are conveniently rewritten as

$$\mathscr{P}_{\eta\xi} - \tfrac{1}{2}[\mathscr{P}_\eta\mathscr{P}_\xi(1 + 2\mathscr{P}) - \mathscr{Q}_\eta\mathscr{Q}_\xi\mathscr{P}^2\mathscr{Q}^{-2}]\mathscr{P}^{-1}(1 + \mathscr{P})^{-1} + 2\varepsilon\mathscr{P}(1 + \mathscr{P}) = 0\,, \tag{7.8}$$

$$[\mathscr{Q}_\eta\mathscr{Q}^{-1}\mathscr{P}(1 + \mathscr{P})^{-1}]_\xi + [\mathscr{Q}_\xi\mathscr{Q}^{-1}\mathscr{P}(1 + \mathscr{P})^{-1}]_\eta = 0\,. \tag{7.9}$$

Equation (7.9) permits us to define the new fields $\tilde{\mathscr{P}}$ and $\tilde{\mathscr{Q}}$ through

$$\tilde{\mathscr{P}} = -(1 + \mathscr{P}),\ \tilde{\mathscr{Q}}_\eta\tilde{\mathscr{Q}}^{-1} = \mathscr{Q}_\eta\mathscr{Q}^{-1}\mathscr{P}(1 + \mathscr{P})^{-1},\ \tilde{\mathscr{Q}}_\xi\tilde{\mathscr{Q}}^{-1} = -\mathscr{Q}_\xi\mathscr{Q}^{-1}\mathscr{P}(1 + \mathscr{P})^{-1}\,. \tag{7.10}$$

Inserting then Eqs. (7.10) into Eq. (7.8), we are led to

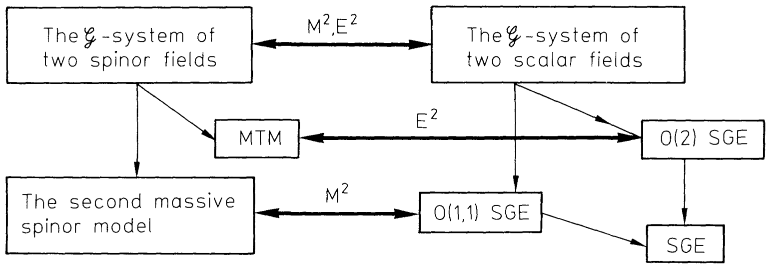
Proposition 7.3. *Assume that* $\varphi^{\pm}$ *is a solution of the system* (7.7) *with* $\varepsilon = 1$. *Then* $\tilde{\varphi}^{\pm}$ *with* $\tilde{\mathscr{P}} = \tilde{\varphi}^+\tilde{\varphi}^-$ *and* $\tilde{\mathscr{Q}} = \tilde{\varphi}^+/\tilde{\varphi}^-$ *defined by* (7.10) *obeys the same system* (7.7), *but this time with* $\varepsilon = -1$.

8. Concluding Remarks: Relationship Between the Models Discussed

One of the advantages of the UNILOF scheme is that it provides a deeper understanding of the relations between scalar and spinor integrable systems. Consider, for instance, Minkowski space. It is well known that on the quantum level the (real) sine-Gordon equation is equivalent to the massive Thirring model [23]. On the classical level the equivalence disappears [24] – at least because MTM involves twice as many field variables (taking in account the order of equations). However, one can suppose that MTM is related to some two-field generalisation of SGE. The UNILOF scheme allows us to exclude at least $O(2)$ and $O(1,1)$ SGE from the list of possible candidates: MTM and these two equations arise under the *distinct* reductions of the same more general system.

The situation changes drastically in E^2 space. According to Remark 1.3, the Euclidean MTM (1.26) is in one-to-one correspondence with $O(2)$ SGE (1.34) [expressing v from the first equation in (1.28) and inserting into the second one yields exactly (1.33)]. Instead, in Minkowski space there is a relation between other systems. Namely, in Subsect. 1.6 the second massive spinor model (1.24) has been shown to be equivalent to $O(1,1)$ SGE (1.36), (4.7). Since the real SGE is a reduction of the latter, the above observation provides the spinor model to which SGE corresponds classically (in the sense that solutions of SGE at the same time satisfy the equations of this spinor model). Lastly, both in E^2 and M^2 spaces the generic system (1.14) may be interpreted either as a model (1.17), (1.25) of two spinor fields or as an equivalent system (1.31) of two complex scalar fields.

We close this section by including, for the reader's convenience, a diagram illustrating the relationship between the systems involved in the non-degenerate $sl(2, \mathbb{C})$ case of the UNILOF scheme:



Acknowledgements. We are grateful to Prof. V. E. Zakharov, Dr. A. V. Mikhailov, and A. B. Yanovski for useful conversations and to Prof. A. B. Borisov and Dr. A. R. Its for their helpful comments about this paper. One of the authors (I. B.) would like to thank Prof. V. G. Makhankov for support of the investigation.

References

1. Zakharov, V.E., Mikhailov, A.V.: Relativistically invariant two-dimensional field theory models, integrable by the inverse problem method. JETP. **74**, 1953–1973 (1978)
2. Zakharov, V.E., Mikhailov, A.V.: On the integrability of classical spinor models in two-dimensional space-time. Commun. Math. Phys. **74**, 21–40 (1980)
3. Barashenkov, I.V., Getmanov, B.S.: Multisoliton solutions in the UNILOF scheme. Report on the III International Symposium on Selected Problems of Statistical Mechanics, Dubna, August 1984, JINR preprint D17-84-407, pp. 37–41, Dubna (1984)
4. Getmanov, B.S.: The scheme for unified description of integrable relativistic massive fields (UNILOF scheme). ibid. pp. 212–216
5. Getmanov, B.S.: UNILOF scheme for the general case of semisimple Lie algebra. ibid. pp. 217–221
6. Mikhailov, A.V.: The reduction problem and the inverse scattering method. Physica **3** D, 73–117 (1981)
7. Mikhailov, A.V., Olshanetsky, M.A., Perelomov, A.M.: Two-dimensional generalized Toda lattice. Commun. Math. Phys. **79**, 473–488 (1981)
8. Leznov, A.N., Saveliev, M.V.: Exact solutions for cylindrically-symmetric configurations of gauge fields. II. Phys. Elem. Part. Atom. Nucl. **12**, 125–161 (1981)
9. Zakharov, V.E., Shabat, A.B.: Integration of nonlinear equations of mathematical physics through the inverse scattering method. II. Funct. Anal. Appl. **13**, 13–22 (1979)
10. Gerdjikov, V.S., Ivanov, M.I., Kulish, P.P.: Quadratic bundle and nonlinear evolution equations. Theor. Math. Phys. **44**, 342–357 (1980)
11. Kuznetsov, E.A., Mikhailov, A.V.: On complete integrability of two-dimensional classical Thirring model. Theor. Math. Phys. **30**, 303–314 (1977)
12. David, D.: On an extension of the classical Thirring model. J. Math. Phys. **25**, 3424–3432 (1984)
13. Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through quadratic constraints. Commun. Math. Phys. **46**, 207–221 (1976)
14. Lund, F., Regge, T.: Unified approach to strings and vortices with soliton solutions. Phys. Rev. D **14**, 1524–1535 (1976)
15. Getmanov, B.S.: A new Lorentz-invariant system with exact multi-soliton solutions. JETP. Lett. **25**, 132–136 (1977)

16. David, D., Harnad, J., Shnider, S.: Multi-soliton solutions to the Thirring model through the reduction method. Lett. Math. Phys. **8**, 27–37 (1984)
17. Hirota, R.: Exact solution of the sine-Gordon equation for multiple collisions of solitons. J. Phys. Soc. Jpn. **33**, 1459–1463 (1972)
18. Caudrey, P.J., Eilbeck, J.C., Gibbon, J.D., Bullough, R.K.: Multiple soliton and bisoliton bound state solutions of the sine-Gordon equation. J. Phys. A: Math. Nucl. Gen. **6**, L112–L115 (1973)
19. Zakharov, V.E., Takhtadzhyan, L.A., Faddeev, L.D.: Complete description of solutions to the sine-Gordon equation. Dokl. Akad. Nauk. SSSR, **219**, 1334–1337 (1974)
20. Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: Method for solving the sine-Gordon equation. Phys. Rev. Lett. **30**, 1262–1264 (1973)
21. Pogrebkov, A.K.: Singular solitons: An example of a sinh-Gordon equation. Lett. Math. Phys. **5**, 277–285 (1981)
22. Borisov, A.B., Tankeyev, A.P., Shagalov, A.G., Bezmaternih, G.V.: Multi-vortex-like solutions of the sine-Gordon equation. Phys. Lett. **111** A, 15–18 (1985)
23. Coleman, S.: Quantum sine-Gordon equation as the massive Thirring model. Phys. Rev. D **11**, 2088–2097 (1975)
24. Kaup, D.J., Newell, A.C.: On the Coleman correspondence and the solution of the massive Thirring model. Lett. Nuovo Cim. **20**, 325–331 (1977)
25. Fordy, A.P., Gibbons, J.: Integrable nonlinear Klein-Gordon equationa and Toda Lattices. Commun. Math. Phys. **77**, 21–30 (1980)
26. Getmanov, B.S.: Integrable model of nonlinear complex scalar field with nontrivial asymptotics of soliton solutions. Theor. Math. Phys. **38**, 186–194 (1979)

Communicated by Ya. G. Sinai

Received June 10, 1986; in revised form March 12, 1987