Commun. Math. Phys. 111, 593–612 (1987)

Communications in Mathematical Physics

Long Range Dynamics and Broken Symmetries in Gauge Models. The Stückelberg-Kibble Model*

G. Morchio[1] and F. Strocchi[2]

[1] Dipartimento di Fisica dell'Università, I-56100 Pisa, Italy
[2] International School for Advanced Studies, International Centre for Theoretical Physics, I-34100 Trieste, Italy

Abstract. The general algebraic features associated to long range dynamics like the problem of removing the infrared cutoff, the definition of the algebraic dynamics and the occurrence of variables at infinity, the essential localization (seizing of the vacuum), the effective dynamics and its covariance group (dynamical symmetry group), the generalization of Goldstone's theorem and the non-trivial Goldstone spectrum, the mass/energy gap generation by the non-trivial classical motion of the variables at infinity are explicitly shown in the Kibble model as a prototype of gauge models exhibiting the Higgs phenomenon. The relation between mass generation in the Higgs phenomenon and the plasma energy gap is also discussed.

1. Introduction

The general properties of the dynamics of systems with long range interactions have been discussed in previous papers [1–4] with emphasis on the occurrence of variables at infinity in the time evolution of (quasi) local variables. The evidence that the above structures are indeed realized has been shown in various cases like a class of spin models [1], the BCS model [1], the Coulomb gas in uniform background [5]. From the above examples it appears that the occurrence of variables at infinity is a generic feature associated to Coulomb like interactions and therefore present in gauge field theories (in positive gauges). This has been explicitly shown for the Schwinger model, usually regarded as a prototype of gauge theories with unbroken gauge symmetry, [6] and the present paper provides a detailed analysis of the Kibble model, in four dimensions, usually regarded as the prototype of gauge theories exhibiting the Higgs phenomenon [7–9]. The model can be obtained from the abelian Higgs-Kibble model by neglecting the quantum fluctuations of the modulus of the Higgs field $\chi=|\chi|e^{i\varphi}$ and by actually freezing $|\chi|$ to 1. In this case the Higgs-Kibble Lagrangian becomes

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}^2-(\partial_\mu\varphi-eA_\mu)^2+\text{gauge fixing} \tag{1.1}$$

* Work supported in part by INFN, Sezione di Pisa

and in the Coulomb gauge the model is described by the following formal Hamiltonian [7–9]:

$$H = \tfrac{1}{2}\int [(\nabla\varphi)^2 + \pi^2] d^3x + \tfrac{1}{2}\int d^3x\, d^3y \pi(x)\pi(y) V(x-y). \tag{1.2}$$

The interaction term is not as strange as it could appear; it actually occurs also in the full QED Hamiltonian, provided one realizes the analogy between π and the charge density $\varrho(x)$ [the actual derivation of Eq. (1.2) in the Coulomb gauge clarifies this point].

The model exhibits also strong analogies with the Coulomb electron gas in uniform background, with the correspondence $\pi \to \varrho(x) - \varrho_B$, $\varrho(x)$ = electron densities, ϱ_B = background density, $\partial_i\varphi \to j_i$, j = the electric current, and therefore it can also be considered as a prototype of Coulomb systems in uniform background (see also Sect. 7 below), showing the deep link between mass generation in the Higgs effect and the plasmon energy gap.

The model is soluble and apparently simple; however, the standard treatment [7–9], without a careful handling of the infrared cutoff and its very delicate removal, misses the basic algebraic structures mentioned above and in any case leaves many questions of principle open: i) How does it happen that starting from a symmetric Hamiltonian one ends up with non-symmetric equations of motion? ii) Which is the basic mechanism by which the conclusions of the Goldstone's theorem are evaded? iii) Is there a generalized Goldstone's theorem by which spontaneous symmetry breaking entails the existence of generalized Goldstone bosons with non-trivial energy spectrum, and it actually allows one to predict their mass?

The general framework discussed in [1–4] allows a full (and rigorous) control of the model as well as an answer to the above questions. The following treatment of the model will also provide a non-trivial example in which the variables at infinity are not obviously present in the starting Hamiltonian (as in mean field models) but they arise as a result of delicate interplay between the long range $1/r$ Coulomb-like interaction and the kinetic term. The model will also shed light on the general mathematical structures associated to long range dynamics like a) the weak topology of the physically relevant states which allows the removal of the infrared cutoff, b) the symmetries of the algebraic dynamics and the occurrence of variables at infinity, c) the algebra of essential localization, d) the effective dynamics and its covariance group (dynamical symmetry group), e) the mass gap generation by symmetry breaking and its relation with the "classical motion of the variables at infinity."

2. Canonical Field Algebra and Quasi Free States

The basic algebraic setting on which the following discussion is based, is the (local) algebra $\mathscr{A}$ generated by the canonical variables φ, π smeared with test functions in $\mathscr{S}(R^3)$. We denote by $W(F)$ the Weyl operators, $F = (f_1, f_2) \in \mathscr{T} \equiv \mathscr{S}_{\text{real}}(R^3) x \mathscr{S}_{\text{real}}(R^3)$ and by $\langle \cdot , \cdot \rangle$ the usual symplectic form on $\mathscr{T}$,

$$[\Phi(F), \Phi(G)] = i\langle F, G\rangle = i[(f_1, g_2) - (f_2, g_1)] \tag{2.1}$$

(for more details see [10, 6]).

We will consider quasi free states on $\mathscr{A}$ of the form

$$\omega_M(W(F)) = \exp\{-\tfrac{1}{4}[F, F]_M\}, \tag{2.2}$$

where

$$[F, G]_M = \sum_{i,j} \int d^3k \bar{\tilde{f}}_i(k) M_{ij}(k) \tilde{g}_j(k), \tag{2.3}$$

with M of the form

$$M(k) = \begin{pmatrix} (1+\alpha^2)^{1/2} r^{-1} & \alpha \\ \alpha & (1+\alpha^2)^{1/2} r \end{pmatrix}, \tag{2.4}$$

$\alpha = \alpha(k)$ real and $r(k) > 0$ (see [6, Proposition 2.1 and Eq. (2.18)]).

A simple extension of the above family of states is given by

$$\omega_{M,\lambda}(W(F)) = \omega_M(W(F)) e^{i\lambda \tilde{f}_1(0)}, \quad \lambda \in \mathbb{R}, \tag{2.5}$$

which define representations with a non-vanishing expectation value of the field φ, $\langle \varphi \rangle = \lambda$.

3. Infrared Cutoff Dynamics and its Symmetries

To give a precise meaning to the formal Hamiltonian (1.2) one has to introduce an infrared regularization and then discuss the removal of the infrared cutoff. We choose the following regularization:

$$V(x) \to V_L(x) \equiv \frac{e^2}{|\mathbf{x}|} f(|\mathbf{x}|/L), \tag{3.1}$$

where $f(x) \in \mathscr{D}(R^1)$, $f(x) = 1$ for $|x| < 1$, $f(x) = 0$ for $|x| > 1 + a$. The results do not depend on the specific choice of the infrared regularization.

The cutoff Hamiltonian H_L defines an infrared cutoff dynamics α_L^t on the local canonical algebra $\mathscr{A}$. In fact, H_L gives rise to the following equations of motion:

$$\dot{\varphi}_t(f) = \pi_t(f) + \pi_t(V_L * f),$$
$$\dot{\pi}_t(f) = \varphi_t(\Delta f),$$

i.e.

$$\Phi_t(F) = \Phi(F_L^t), \quad F \in \mathscr{T}, \tag{3.2}$$

where

$$\tilde{F}_L^t(k) = \tilde{A}_L^t(k) \tilde{F}(k), \tag{3.3}$$

$$\tilde{A}_L^t(k) = \begin{pmatrix} \cos\omega_L t & -(k^2/\omega_L)\sin\omega_L t \\ (\omega_L/k^2)\sin\omega_L t & \cos\omega_L t \end{pmatrix}, \tag{3.4}$$

$$\omega_L = \omega_L(k) \equiv [k^2(1 + \tilde{V}_L(k))]^{1/2}, \tag{3.5}$$

(in the following the subscript L will often be omitted for simplicity of notation). Furthermore, since the matrix elements of $A_L^t(k)$ are C^∞-functions of $\omega_L^2(k)$, which is a C^∞-function of k by Eq. (3.5) (thanks to the infrared cutoff!), and they are bounded by polynomials in k, it follows that $F^t_L \in \mathscr{T}$. Finally, it is easy to check that the linear transformation $F \to F^t$ preserves the symplectic form and therefore Eq. (3.2) defines an automorphism α^t_L of $\mathscr{A}$:

$$\alpha^t_L(W(F)) \equiv W(A^t_L F) \tag{3.6}$$

(infrared cutoff dynamics).

An obvious symmetry of the infrared cutoff dynamics is that corresponding to *space-translations*

$$\alpha_{\mathbf{a}} W(F) = W(F_{\mathbf{a}}), \qquad F_{\mathbf{a}}(\mathbf{x}) \equiv F(\mathbf{x} - \mathbf{a}). \tag{3.7}$$

It is easy to see that the following transformation

$$\beta^\lambda : W(F) \to W(F) \exp[i\lambda \textstyle\int f_1(x) d^3x], \qquad \lambda \in \mathbb{R}, \tag{3.8}$$

$F = (f_1, f_2) \in \mathscr{T}$, (corresponding to $\varphi \to \varphi + \lambda$, $\pi \to \pi$) preserves the algebraic relations and therefore it defines an automorphism of $\mathscr{A}$ *(rigid gauge transformations)*. Furthermore, since

$$\tilde{f}^t_1(0) = \tilde{f}_1(0), \tag{3.9}$$

one has

$$\alpha^t_L \beta^\lambda W(F) = e^{i\lambda \tilde{f}_1(0)} \alpha^t_L W(F) = \beta^\lambda \alpha^t_L W(F), \tag{3.10}$$

i.e. β^λ defines a symmetry of the infrared cutoff dynamics.

A less obvious symmetry of α^t_L is that corresponding to *linear gauge transformations (of the second kind)*

$$\beta^\lambda_i W(F) = W(F) \exp[i\lambda \textstyle\int f_1(x) x_i d^3x], \tag{3.11}$$

corresponding to

$$\varphi \to \varphi + \lambda x_i, \qquad \pi \to \pi. \tag{3.12}$$

As a matter of fact, one easily checks that the transformation (3.11) preserves the algebraic relations (in k-space it amounts to the multiplication of $\tilde{F}(k)$ by $\exp[\lambda(\partial_i \tilde{f}_1)(0)]$) and therefore it defines an automorphism of $\mathscr{A}$; furthermore

$$\beta^\lambda_i \alpha^t_L = \alpha^t_L \beta^\lambda_i, \tag{3.13}$$

as a consequence of $\partial_i \tilde{f}^t_1(0) = \partial_i \tilde{f}_1(0)$.

For the following, it is also useful to have the commutation relations between β^λ_i and the space translations

$$\beta^\lambda_i \alpha_{\mathbf{a}} = \beta^{\lambda a_i} \alpha_{\mathbf{a}} \beta^\lambda_i. \tag{3.14}$$

4. Physically Relevant States and the Removal of the Infrared Cutoff. Algebraic Dynamics

According to the general approach discussed in [1–3], in order to remove the infrared cutoff and to define in the limit $L \to \infty$ an algebraic dynamics, one has to make reference to a family $\mathbb{F}$ of "physically relevant states" with the following properties:

i) $\mathbb{F}$ is closed under linear combinations.

ii) $\mathbb{F}$ is norm closed and separating, i.e. $\phi(A)=0$, $\forall\phi\in\mathbb{F}$ implies $A=0$.

iii) $\mathbb{F}$ is "stable under local operations" in the sense that if $\phi\in\mathbb{F}$ also $\phi_{AB}(\cdot) =\phi(A\cdot B)$, with $A,B\in\mathscr{A}$, belongs to $\mathbb{F}$.

Thus, the positive part $\mathbb{F}^+$ of $\mathbb{F}$ is a full folium as in [11]. One then discusses the (ultra) strong convergence of α_F^t, as $L\to\infty$, with respect to the topology induced on $\mathscr{A}$ by $\mathbb{F}$.

As a family $\mathbb{F}$ of physically relevant states we choose that obtained from the (pure) quasi-free states of Sect. 2 by application of the symmetrics β^λ and β_i^λ. More explicitly we consider the states associated to the representations of $\mathscr{A}$ defined by the following states:

$$\begin{aligned}\omega_{M,\lambda,\boldsymbol{\lambda}}(W(F))&=\omega_M(\beta^\lambda\beta_1^{\lambda_1}\beta_2^{\lambda_2}\beta_3^{\lambda_3}W(F))\\&=\omega_M(W(F))\exp\Big[i\lambda\tilde{f}_1(0)+\sum_j\lambda_j(\partial_j\tilde{f})(0)\Big],\end{aligned}\tag{4.1}$$

with $\lambda\in\mathbb{R}$, $\boldsymbol{\lambda}\in\mathbb{R}^3$,

$$\omega_M(W(F))=\exp(-\tfrac{1}{4}[F,F]_M)\tag{4.2}$$

and M satisfying conditions (2.4). In the following, to simplify the notation, the states $\omega_{M,\lambda,\boldsymbol{\lambda}}$ will be denoted by ω_Λ, $\Lambda=\{\lambda,\lambda_i;\ i=1,2,3\}$, with the index M understood; similarly β^Λ will denote a generic element $\beta^\lambda\beta_1^{\lambda_1}\beta_2^{\lambda_2}\beta_3^{\lambda_3}$.

Furthermore we restrict the space of states by the requirement that $\alpha(k)$, $r(k)$ of Sect. 2 are regular functions for $k\neq0$, $\alpha(k)$, $r(k)$, $r(k)^{-1}$ are bounded by polynomials at infinity and

$$M_{11}(k),\qquad k^{-4}M_{22}(k)\in L^1_{\mathrm{loc}}.\tag{4.3}$$

(We will denote by $\mathbb{M}$ the set of M satisfying the above conditions.) By definition, $\mathbb{F}$ is therefore stable under $(\beta^\lambda)^*$ and $(\beta_i^\lambda)^*$. Since by Eqs. (3.3), (3.4) $F_L^t\in\mathscr{T}$ whenever F does, the family $\mathbb{F}$ is also stable under $(\alpha_L^t)^*$, (in general, however, there will be only one Fock vacuum stable under $(\alpha_L^t)^*$, depending on L, so that the corresponding representations will be disjoint for different L's).

We can now remove the infrared cutoff.

Proposition 4.1. *Let* $\mathbb{F}$ *be a family of states defined above. Then*

a) $\alpha_L^t(A)$, *A in the norm closure of* $\mathscr{A}$, *is (ultra) strongly convergent as* $L\to\infty$ *with respect to the topology defined by* $\mathbb{F}$.

b) *The (weak) convergence of* α_L^t *defines a one parameter group of mappings of* $\mathbb{F}$ *into itself*

$$(\alpha^t)^*=w^*\text{-}\lim_{L\to\infty}(\alpha_L^t)^*,\qquad t\in\mathbb{R}.$$

c) $(\alpha^t)^*$ *uniquely determines a one-parameter group of automorphisms* α^t, $t\in\mathbb{R}$, *of* $\mathscr{M}=\bar{\mathscr{A}}$, *where the bar denotes the weak closure in the topology induced by* $\mathbb{F}$ *on* $\mathscr{A}$, ***(algebraic dynamics)***.

The proof requires some technicality, which sheds light on the problem of removing the infrared cutoff also in more realistic models (as in the Coulomb gas with uniform background [5]); for the proof we refer the reader to Appendix A.

5. Algebraic Dynamics. Variables at Infinity

We can now answer some of the questions raised in Sect. 1. Due to the long range Coulomb type interaction the introduction of an infrared regularization is necessary and the removal of the infrared cutoff requires special care (see the previous section). The resulting algebraic dynamics exhibits very peculiar features (as expected on the basis of the general ideas discussed in [1–3]). The first interesting property is that α_L^t does not converge in norm and the algebraic dynamics $\alpha^t = w\text{-}\lim \alpha_L^t$ does not leave the quasi local (canonical) algebra $\mathscr{A}$ stable. The situation is more delicate than in the Schwinger model [6], where stability under the algebraic dynamics was obtained by considering the algebra generated by the quasi local canonical algebra $\mathscr{A}$ and an algebra at infinity $\mathscr{A}_\infty$. Here, the infrared interaction has stronger effects and the time evolution of local variables does not only involve variables at infinity but also a substantial delocalization: as we will see more explicitly below α^t leaves stable the algebra $\mathscr{G}(\mathscr{A}_\ell \cup \mathscr{A}_\infty)$ generated by an *algebra* $\mathscr{A}_\ell \subset \mathscr{M}$, *with trivial center* and by an algebra at infinity $\mathscr{A}_\infty$.

To discuss the above features more explicitly, we start by considering the following enlargement of the Weyl algebra $\mathscr{A}$. We extend the Weyl system by considering Weyl operators $W(F)$ defined for

$$F = (f_1, f_2), \quad f_1 \in \mathscr{S}(R^3), \quad \mathbf{k}^2 \tilde{f}_2(k) \in \mathscr{S}(R^3), \tag{5.1}$$

briefly we will say $F \in \mathscr{T}_{\text{ext}}$.

One can easily see that the antisymmetric form $\langle \cdot, \cdot \rangle$ is still defined on $\mathscr{T}_{\text{ext}}$, and therefore the CCR's [Eq. (2.1)] extend to the enlarged Weyl system. We denote by $\mathscr{A}_\ell$ *the algebra generated by the extended Weyl system* $W(F)$, $F \in \mathscr{T}_{\text{ext}}$ [Eq. (5.1)]. Clearly with respect to the quasilocal canonical algebra $\mathscr{A}$, generated by $W(F)$, $F \in \mathscr{T} = \mathscr{S} \times \mathscr{S}$ the enlarged algebra $\mathscr{A}_\ell$ has a long range Coulomb type delocalization. One can easily show that $\mathscr{A}_\ell$ can be identified with a subalgebra of $\mathscr{M}$; $W(F)$, $F = (f_1, f_2) \in \mathscr{T}_{\text{ext}}$ is identified with the strong limit of $W(F_n)$, $f_{1n} = f_1$, $k^2 \tilde{f}_{2n}$ converging in $\mathscr{S}$ to $k^2 \tilde{f}_2$. For the following it is convenient to introduce the following *variables at infinity:*

i) the variable at infinity φ_∞: we start by considering $W(sF_R)$, $s \in \mathbb{R}$, with $F_R = (f_{1R}, 0)$, f_{1R} satisfying the following conditions

$$\begin{gathered} \tilde{f}_{1R}(0) = 1, \quad (\partial_j \tilde{f}_{1R})(0) = 0, \quad f_{1R} \geqq 0, \\ \|F_R\|_M \to 0 \quad \text{as} \quad R \to \infty, \quad \forall M \in \mathbb{M}. \end{gathered} \tag{5.2}$$

Then, by using an explicit calculation as in the Appendix Eq. (A.7), we get

$$\lim_{R \to \infty} \omega_{M,\Lambda}(W_1 W(sF_R) W_2) = e^{i\lambda s} \omega_{M,\Lambda}(W_1 W_2). \tag{5.3}$$

This implies weak convergence of $W(sF_R)$ by a density argument and uniform boundedness of $W(sF_R)$.

Actually, as a consequence of $\|F_R\|_M \to 0$ and $(\partial_j \tilde{f}_{1R})(0)=0$, $W(sF_R)$ converges strongly (in each $\mathscr{H}^{M,\Lambda}$), and we may define

$$U_\infty(s) \equiv s\text{-}\lim_{R\to\infty} W(sF_R) \tag{5.4}$$

(as an element of $\mathscr{M}$). $U(s)$ is a strongly continuous one parameter group [see Eq. (5.3)] and we may write

$$U_\infty(s) \equiv e^{is\varphi_\infty} \tag{5.5}$$

with

$$\omega_{M,\lambda,\lambda}(U(s)) = e^{is\lambda}. \tag{5.6}$$

ii) Similarly, by choosing $W(sF_R^j)$, $s \in \mathbb{R}$, with $F_R^j = (f_{1R}^j, 0)$, $f_{1R}^j \in \mathscr{S}(R^3)$, $j=1,2,3$, satisfying[1]

$$\tilde{f}_{1R}^j(0) = 0, \qquad (\partial_i \tilde{f}_{1R}^j)(0) = i\delta_{ij}, \tag{5.7}$$

$$\lim_{R\to\infty} \|F_R^j\|_M = 0, \qquad \forall M \in \mathbb{M},$$

we get

$$\lim_{R\to\infty} \omega_{M,\Lambda}(W_1 W(sF_R^j) W_2) = e^{is\lambda_j} \omega_{M,\Lambda}(W_1 W_2).$$

As above, one can actually show that $W(sF_R^i)$ converges strongly in each $\mathscr{H}^{M,\Lambda}$, and one may define

$$T_\infty^i(s) = s\text{-}\lim_{R\to\infty} W(sF_R^i), \tag{5.8}$$

as an element of $\mathscr{M}$. $T_\infty^i(s)$ defines a strongly continuous one parameter group and its generator defines a variable at infinity $(\partial_i\varphi)_\infty$, with $\omega_{M,\Lambda}(e^{is(\partial_i\varphi)_\infty}) = e^{is\lambda_i}$; the algebra generated (through norm closures) by $U_\infty(s)$ and $T_\infty^i(s)$ will be denoted by $\mathscr{A}_\infty$. We then have

Proposition 5.1. *The algebraic dynamics α^t maps the algebra $\mathscr{G}(\mathscr{A}_\ell \cup \mathscr{A}_\infty)$, generated by $\mathscr{A}_\ell$ and by the algebra at infinity $\mathscr{A}_\infty$ (through linear combinations and norm closures), into itself.*

In particular, $\forall F \in \mathscr{T}_{\mathrm{ext}}$,

$$\alpha^t W(F) = W(F^t) \exp i[(\tilde{f}_1(0) - \tilde{f}_1^t(0))\varphi_\infty] \times \prod_{j=1}^{3} \exp i[(\textstyle\int d^3x\, x_j(f_1 - f_1^t)(\partial_j\varphi)_\infty], \tag{5.9}$$

where φ_∞ and $(\partial_i\varphi)_\infty$ are the variables at infinity corresponding to the fields φ and $\partial_i\varphi$ and are invariant under α^t.

[1] The above conditions (5.2) (5.7) are fulfilled e.g. by taking $f_{1R}(x) = R^{-3} f(|x|/R)$, $f(0)=1$, $f \geq 0$, and $f_{1R}^j = -R^{-3}\partial_j f(|x|/R)$

Remark. The above proposition provides a rigorous control on the solution of the field equations: Eq. (5.9) corresponds to

$$\alpha^t(\varphi(x)) = \widetilde{\cos\omega t} * \varphi + [\widetilde{(\omega/k^2)\sin\omega t}] * \pi + (1-\cos\omega_0 t)\varphi_\infty + (1-\cos\omega_0 t)x_j(\partial_j\varphi)_\infty\,, \tag{5.10}$$

and it clarifies the occurrence of variables at infinity in the time evolution of the local variables. Clearly, in representations characterized by a translationally (or rotationally) invariant ground state, the variable at infinity $(\partial_i\varphi)_\infty$ vanishes and one is left only with the variable at infinity φ_∞.

For the proof of Proposition 5.1, see Appendix B.

6. Variables at Infinity and Symmetries

To investigate the symmetries of the algebraic dynamics it is convenient to discuss how the symmetries β^λ, $\lambda\in\mathbb{R}$, β^{λ_i}, $\lambda_i\in\mathbb{R}$, and $\alpha_{\mathbf{x}}$, $\mathbf{x}\in\mathbb{R}^3$, defined on the quasilocal algebra $\mathscr{A}$, have a unique weakly continuous extension to (the weak closure) $\mathscr{M}=\bar{\mathscr{A}}$ (see [1–3] for the general strategy).

i) *Rigid Gauge Transformations* β^λ. Since by definition $(\beta^\lambda)^*$ maps $\mathbb{F}$ into $\mathbb{F}$, β^λ is $\mathbb{F}$-weakly continuous, and therefore

$$\beta^\lambda U_\infty(s) = w\text{-}\lim_{R\to\infty} \beta^\lambda W(sF_R) = e^{i\lambda s}U_\infty(s)\,, \tag{6.1}$$

i.e.

$$\beta^\lambda e^{is\varphi_\infty} = e^{is(\varphi_\infty+\lambda)}\,. \tag{6.2}$$

Furthermore, by condition (5.7), $\tilde{f}^i_{1R}(0)=0$, we have

$$\beta^\lambda T^i_\infty(s) = w\text{-}\lim_{R\to\infty} \beta^\lambda W(sF^i_R) = T^i_\infty(s)\,. \tag{6.3}$$

ii) *Linear Gauge Transformations* $\beta^{\boldsymbol{\lambda}}$. Again, $(\beta^{\boldsymbol{\lambda}})^*$ leaves $\mathbb{F}$ stable so that $\beta^{\boldsymbol{\lambda}}$ is $\mathbb{F}$-weakly continuous and by Eq. (5.4),

$$\begin{aligned}\beta^{\lambda_i} U_\infty(s) &= w\text{-}\lim_{R\to\infty} \beta^{\lambda_i} W(sF_R)\\ &= \lim_{R\to\infty} W(sF_R)\exp\sum_i s\lambda_i(\partial_i\tilde{f}_{1R})(0) = U_\infty(s)\,,\end{aligned} \tag{6.4}$$

i.e.

$$\beta^{\lambda_i}(e^{is\varphi_\infty}) = e^{is\varphi_\infty}\,. \tag{6.5}$$

Similarly,

$$\begin{aligned}\beta^{\lambda_i} T^j_\infty(s) &= w\text{-}\lim_R \beta^{\lambda_i} W(sF^j_R)\\ &= w\text{-}\lim_R W(sF^j_R)e^{i\lambda_i\delta_{ij}s} = T^j_\infty(s)e^{is\lambda_i\delta_{ij}}\,,\end{aligned} \tag{6.6}$$

i.e.

$$\beta^{\lambda_i}(\exp is(\partial_j\varphi)_\infty) = \exp is[(\partial_j\varphi)_\infty + \delta_{ij}\lambda_j]\,. \tag{6.7}$$

iii) *(Space Translations)*, $\alpha_{\mathbf{x}}$. We start by showing that $\mathbb{F}$ is stable under $(\alpha_{\mathbf{x}})^*$, $\mathbf{x}\in\mathbb{R}^3$; in fact, $\mathbb{F}$ is generated from the translationally invariant states ω_M through applications of $(\beta^{\lambda})^*$ and $(\beta^{\lambda_i})^*$ and

$$\begin{aligned}(\alpha_{\mathbf{x}})^*(\beta^{\lambda})^*\prod_j(\beta^{\lambda_j})^*\omega_M&=(\beta^{\lambda})^*\left(\beta^{\sum_i\lambda_i x_i}\right)^*\prod_j(\beta^{\lambda_j})^*(\alpha_{\mathbf{x}})^*\omega_M\\&=\left(\beta^{\lambda+\sum_i\lambda_i x_i}\right)^*\prod_j(\beta^{\lambda_j})^*\omega_M\,.\end{aligned}\tag{6.8}$$

The stability of $\mathbb{F}$ implies that $\alpha_{\mathbf{x}}$ is $\mathbb{F}$-weakly continuous and therefore

$$\alpha_{\mathbf{x}}U_\infty(s)=w\text{-}\lim_R\alpha_{\mathbf{x}}W(sF_R)=U_\infty(s)\prod_j T^j(sx_j)\,,\tag{6.9}$$

i.e.

$$\alpha_{\mathbf{a}}(\exp is\varphi_\infty)=\exp is\left(\varphi_\infty+\sum_i(\partial_i\varphi)_\infty a_i\right).\tag{6.10}$$

Similarly, one has

$$\alpha_{\mathbf{a}}T^i_\infty(s)=w\text{-}\lim_R\alpha_{\mathbf{a}}W(sF^i_R)=T^i_R(s)\,,\tag{6.11}$$

i.e.

$$\alpha_{\mathbf{a}}(\exp is(\partial_k\varphi)_\infty)=\exp is(\partial_k\varphi)_\infty\,.\tag{6.12}$$

Using these formulas one can check that the so extended symmetries from $\mathscr{A}$ to $\mathscr{G}(\mathscr{A}_\ell\cup\mathscr{A}_\infty)$ still obey the original group laws, e.g. Eqs. (3.14).

We can also check that the above symmetries β^{λ}, β^{λ_i}, $\alpha_{\mathbf{a}}$ are symmetries of the algebraic dynamics α^t. In fact, we have

$$\begin{aligned}\beta^{\lambda}\alpha^t W(F)&=\beta^{\lambda}(W(F^t)\exp[i\varphi_\infty(\tilde{f}_1(0)-\tilde{f}^t_1(0))]\\&\quad\times\exp\Big\{\sum_j(\partial_j\varphi)_\infty[(\partial_j\tilde{f}_1)(0)-(\partial_j\tilde{f}^t_1)(0)]\Big\}\\&=W(F^t)e^{i\lambda\tilde{f}^t_1(0)}\exp[i(\varphi_\infty+\lambda)(\tilde{f}_1(0)-\tilde{f}^t_1(0))]\\&\quad\times\exp\Big\{\sum_j(\partial_j\varphi)_\infty[(\partial_j\tilde{f}_1)(0)-(\partial_j\tilde{f}^t_1)(0)]\Big\}\\&=\alpha^t W(F)e^{i\lambda\tilde{f}_1(0)}=\alpha^t\beta^{\lambda}W(F)\,.\end{aligned}$$

Similarly, one proves

$$\beta^{\lambda_i}\alpha^t=\alpha^t\beta^{\lambda_i}\,,\qquad\alpha_{\mathbf{a}}\alpha^t=\alpha^t\alpha_{\mathbf{a}}\,.$$

It should be noted that the variables at infinity are crucial for the above symmetries of the dynamics.

7. Effective Dynamics and its Covariance Group

In each factorial representation Π of $\mathscr{A}$, defined by states of $\mathbb{F}$, the variables at infinity get frozen to their expectation values in Π and therefore $\forall A\in\mathscr{A}_\ell$ the dynamics α^t reduces to an *effective dynamics* α^t_Π; symbolically

$$\alpha^t(A) = F_t(\mathscr{A}_\ell, \mathscr{A}_\infty),$$
$$\Pi(\alpha^t(A)) = \Pi(\alpha^t_\Pi(A)) = \Pi(F_t(\mathscr{A}_\ell, \langle \mathscr{A}_\infty \rangle_\Pi).$$

From Eq. (5.9) with φ_∞ and $(\partial_j\varphi)_\infty$ replaced by their expectation values in Π, it follows easily that α^t_Π *leaves* $\mathscr{A}_\ell$ *stable* and *in fact it defines an automorphism* of $\mathscr{A}_\ell$. More explicitly for the field variables, one has, $\forall F \in \mathscr{T}_{\text{ext}}$,

$$\begin{aligned} \alpha^t\Phi(F) &= \Phi(F^t) + \varphi_\infty[\tilde{f}_1(0)(1-\cos\omega_0 t) \\ &\quad + \mathbf{k}^2\tilde{f}_2(0)\sin\omega_0 t/\omega_0] - i\sum_j(\partial_j\varphi)_\infty[\partial_j\tilde{f}_1(0)(1-\cos\omega_0 t) \\ &\quad + (\partial_j(\mathbf{k}^2\tilde{f}_2))(0)\sin\omega_0/\omega_0], \\ \alpha^t_\Pi\Phi(F) &= \Phi(F^t) + (\tilde{f}_1(0) - \tilde{f}^t_1(0)]\Pi(\varphi_\infty) \\ &\quad - i\sum_j[(\partial_j\tilde{f}_1)(0) - (\partial_j\tilde{f}^t_1)(0)]\Pi((\partial_j\varphi)_\infty). \end{aligned} \tag{7.1}$$

It is now obvious that, due to the freezing of the variables at infinity, the effective dynamics α^t_Π is no longer symmetric under β^λ and $\beta^{\boldsymbol{\lambda}}$. To investigate the covariance group of α^t_Π it is convenient to introduce the following automorphisms of $\mathscr{A}_\ell$.

Symmetry γ^λ, $\lambda \in \mathbb{R}$: $\forall F \in \mathscr{T}_{\text{ext}}$ we define

$$\gamma^\lambda W(F) = W(F)\exp i\lambda(\mathbf{k}^2\tilde{f}_2)(0). \tag{7.2}$$

Clearly γ^λ is trivial on $\mathscr{A}$, but not on $\mathscr{A}_\ell$.

Symmetry $\gamma^{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \in \mathbb{R}^3$: $\forall F \in \mathscr{T}_{\text{ext}}$, we define

$$\begin{aligned} \gamma^{\boldsymbol{\lambda}} W(F) &= W(F)\exp\sum_i \lambda_i(\partial_i(\mathbf{k}^2\tilde{f}_2))(0) \\ &= W(F)\exp[-\mathrm{i}\textstyle\int \Delta f_2(x)\boldsymbol{\lambda}\cdot\mathbf{x}dx]. \end{aligned} \tag{7.3}$$

Again, $\gamma^{\boldsymbol{\lambda}}$ is trivial on $\mathscr{A}$ but not on $\mathscr{A}_\ell$. The possibility of introducing the above symmetries is therefore strictly related to the necessity of extending the quasi-local algebra $\mathscr{A}$ in order to get stability under time evolution; in a certain sense the symmetries γ^λ, $\gamma^{\boldsymbol{\lambda}}$ have a dynamical origin.

It is very simple to check that the symmetries β^λ, $\beta^{\boldsymbol{\lambda}}$, γ^μ, $\gamma^{\boldsymbol{\mu}}$, all commute, i.e. they form an abelian group. Furthermore

$$\gamma^\lambda\alpha_{\mathbf{a}}W(F) = \gamma^\lambda W(F_{\mathbf{a}}) = \alpha_{\mathbf{a}}\gamma^\lambda W(F), \tag{7.4}$$

i.e. β^λ and γ^λ commute with the space translations, whereas $\gamma^{\boldsymbol{\lambda}}$ has the same commutation as $\beta^{\boldsymbol{\lambda}}$ [Eq. 3.14)]:

$$\begin{aligned} \gamma^{\boldsymbol{\lambda}}\alpha_{\mathbf{a}}W(F) &= \gamma^{\boldsymbol{\lambda}}W(F_{\mathbf{a}}) = \alpha_{\mathbf{a}}\gamma^{\boldsymbol{\lambda}}\exp i\boldsymbol{\lambda}\cdot\mathbf{a}(\mathbf{k}^2\tilde{f}_2)(0) \\ &= \alpha_{\mathbf{a}}\gamma^{\boldsymbol{\lambda}}\gamma^{\boldsymbol{\lambda}\cdot\mathbf{a}}W(F). \end{aligned} \tag{7.5}$$

With the help of the above symmetries we may now characterize the *covariance group G of the effective dynamics*, namely the group generated by β^λ, $\beta^{\boldsymbol{\lambda}}$, $\alpha_{\mathbf{a}}$, and α^t_Π. For simplicity we consider the effective dynamics corresponding to the representation Π with $\Pi(\varphi_\infty)=0=\Pi((\partial_j\varphi)_\infty)$, so that $\alpha^t_\Pi W(F)=W(F^t)$. From Eq. (7.1), we then get

$$\begin{aligned}\beta^\lambda\alpha^t_\Pi W(F)&=\beta^\lambda W(F^t)=W(F^t)\exp i\lambda\tilde{J}^t_1(0)\\&=W(F^t)\exp i\lambda[\tilde{J}_1(0)\cos\omega_0 t-(\mathbf{k}^2\tilde{J}_2)(0)\sin\omega_0 t/\omega_0]\\&=\alpha^t_\Pi\beta^{\lambda\cos\omega_0 t}\gamma^{-\lambda\sin\omega_0 t/\omega_0}W(F)\,,\end{aligned}\tag{7.6}$$

and

$$\beta^{\boldsymbol{\lambda}}\alpha^t_\Pi W(F)=W(F^t)\exp\boldsymbol{\lambda}(\boldsymbol{\partial}\tilde{J}^t_1)(0)=\alpha^t_\Pi\beta^{\boldsymbol{\lambda}\cos\omega_0 t}\gamma^{-\boldsymbol{\lambda}\sin\omega_0 t/\omega_0}W(F)\,.\tag{7.7}$$

Similarly,

$$\gamma^\lambda\alpha^t_\Pi=\alpha^t_\Pi\beta^{\lambda\omega_0\sin\omega_0 t}\gamma^{\lambda\cos\omega_0 t}\,,\tag{7.8}$$

$$\gamma^{\boldsymbol{\lambda}}\alpha^t_\Pi=\alpha^t_\Pi\gamma^{\boldsymbol{\lambda}\cos\omega_0 t}\beta^{\boldsymbol{\lambda}\omega_0\sin\omega_0 t}\,.\tag{7.9}$$

The above commutation relations (3.14), (7.4), (7.9) characterize the covariance group G. In infinitesimal form, by denoting by $\mathbf{P}$, H, G, $\mathbf{G}$, Γ, $\boldsymbol{\Gamma}$ the generator of α_a, α^t_Π, β^λ, $\beta^{\boldsymbol{\lambda}}$, γ^λ, $\gamma^{\boldsymbol{\lambda}}$, we have

$$\begin{aligned}&[G,H]=-\Gamma\,,\qquad[\Gamma,H]=\omega_0^2G\,,\\&[\mathbf{G},H]=-\boldsymbol{\Gamma}\,,\qquad[\boldsymbol{\Gamma},H]=\omega_0^2G\,,\\&[G_i,P_j]=\delta_{ij}G\,,\qquad[\Gamma_j,P_i]=\delta_{ij}\Gamma\,,\end{aligned}\tag{7.10}$$

and all the other commutators vanish. The above structure shows some resemblance with that of the jellium model [4] with the canonical field π playing the role of the net charge density $\varrho(x)-\varrho_B$, $-\partial_i\varphi$ playing the role of $\varrho_B^{-1}j_i$ ($\mathbf{j}$ the electron current) and ω_0 being the plasma frequency. For more details see [5].

Remark. We consider now the general case in which the effective dynamics is defined with reference to a representation Π' defined by a state ω' of the form $\omega_{M,\Lambda}$ (see Sect. 4), by definition of the family $\mathbb{F}$. Then, $\alpha^t_{\Pi'}=\beta^{-\lambda}\beta^{-\boldsymbol{\lambda}}\alpha^t_\Pi\beta^\lambda\beta^{\boldsymbol{\lambda}}$ (see [2]) and clearly the covariance group of $\alpha^t_{\Pi'}$ is isomorphic to that of α^t_Π.

The enlargement of the original symmetry group generated by β^λ, $\beta^{\boldsymbol{\lambda}}$, and $\alpha_{\mathbf{a}}$, to the covariance group G naturally leads to an enlargement of the family $\mathbb{F}$ of states. We therefore introduce a *new family of states* $\mathbb{S}$ on $\mathscr{A}_\ell$, obtained from the pure states ω_M (Sect. 2) by application of the covariance group G:

$$\omega_{M,\lambda,\boldsymbol{\lambda},\nu,\boldsymbol{\nu}}(A)=\omega_M(\beta^\lambda\beta^{\boldsymbol{\lambda}}\gamma^\nu\gamma^{\boldsymbol{\nu}}(A))\equiv\omega_M(\beta^\Lambda\gamma^N(A))\,,\qquad\forall A\in\mathscr{A}_\ell\,.\tag{7.11}$$

The new states $\mathbb{S}$ coincide with the states $\mathbb{F}$ when restricted to the quasi-local subalgebra $\mathscr{A}$; they define a different topology[2] on $\mathscr{A}_\ell$ with respect to the $\mathbb{F}$-topology; in particular the $\mathbb{F}$-states are not stable under α^t_Π, which is therefore not $\mathbb{F}$-weakly continuous.

The larger family of states $\mathbb{S}$ gives rise to a *larger set of variables at infinity*. The construction of the variables at infinity $U_\infty(s)$, $T^k_\infty(s)$ done in Sect. 5 as $\mathbb{F}$-weak

[2] In this topology $\mathscr{A}$ is not dense in $\mathscr{A}_\ell$

limits also works in the $\mathbb{S}$ topology namely $W(\alpha F_R)$, $F_R=(f_{1R},0)$, satisfying condition (5.2) and $W(sF_R^j)$, $F_R^j=(f_{1R}^j,0)$, satisfying condition (5.7) converge strongly on the states of the family $\mathbb{S}$, by essentially the same argument.

We have in addition two other variables at infinity corresponding to the labels v and $\mathbf{v}$ of the states $\mathbb{S}$, Eq. (7.11). They can be obtained as $\mathbb{S}$ weak limits of elements of $\mathscr{A}_\ell$ in the following way.

The variable at infinity $V_\infty(s)$, $s\in\mathbb{R}$, with

$$\omega_{M,\lambda,\boldsymbol{\lambda},v,\mathbf{v}}(V_\infty(s))=e^{ivs}, \tag{7.12}$$

can be obtained by considering

$$\begin{aligned} &W(sG_R), \qquad \tilde{G}_R=(0,\tilde{f}_R/\mathbf{k}^2),\\ &f_R(x)\equiv R^{-3}f(|x|/R), \qquad \int f(x)dx=1, \end{aligned} \tag{7.13}$$

since, as a consequence of property (4.3), $\|G_R\|_M\to 0$ as $R\to\infty$ and therefore, on the states (7.11), $W(sG_R)$ converges weakly (and actually strongly) to e^{ivs}. Thus, we may put

$$V_\infty(s)\equiv s\text{-}\lim_{R\to\infty} W(sG_R) \tag{7.14}$$

and

$$V_\infty(s)\equiv \exp[is(\pi/\mathbf{k}^2)_\infty] \tag{7.15}$$

[since $V_\infty(s)$ is a strongly continuous one parameter group].

Similarly, we can obtain the *variable at infinity* $S_\infty^k(s)$, $s\in\mathbb{R}$, with

$$\omega_{M,\lambda,\boldsymbol{\lambda},v,\mathbf{v}}(S_\infty^k(s))=e^{iv_ks}, \tag{7.16}$$

by considering the $\mathbb{S}$-weak limit[3] of

$$W(sG_R^k), \qquad G_R^k=(0,R^{-3}\partial_k\Delta^{-1}f(|x|/R)),$$

with $\int f(x)dx=1$, $\widetilde{\Delta^{-1}f}=k^{-2}\tilde{f}$, so that $G_R^k\in\mathscr{T}_{\text{ext}}$. Actually, we can put

$$S_\infty^k(s)=s\text{-}\lim_{R\to\infty} W(sG_R^k) \tag{7.17}$$

and

$$S_\infty^k(s)=\exp is\left(\partial_k\left(\frac{1}{4\pi r}*\pi\right)\right)_\infty. \tag{7.18}$$

---

[3] One first proves that $\|G_R^i\|_M\to 0$ as $R\to\infty$, since

$$\|G_R^i\|_M=\int d^3k M_{22}(k)|\tilde{f}(Rk)|^2k_i^2/k^4,$$

$M_{22}(k)/k^4$ is locally integrable for $k\sim 0$ and $k_i^2|\tilde{f}(Rk)|^2\to 0$ as $R\to\infty$. Then, one has

$$\begin{aligned}\omega_{M,\lambda,\boldsymbol{\lambda},v,\mathbf{v}}(W(sG_R^i))&=\omega_M(W(sG_R^i))\exp[-ivR^{-3}\int dx\partial_k f(|x|/R)],\\ &\prod_m \exp[-iv_mR^{-3}\int dx\, x_m\partial_i f(|x|/R)]\\ &=\omega_M(W(sG_R^i))\exp i\sum_m\delta_{im}v_m\xrightarrow[R\to\infty]{}\exp i\sum_m\delta_{im}v_m\end{aligned}$$

In the analogy with the jellium model the variables at infinity have the following physical interpretation: $\varphi_\infty(x)$ is a gauge parameter, $(\partial_i\varphi)_\infty$ corresponds to the mean electron current, $(\pi/k^2)_\infty(x)=((4\pi r)^{-1}*\pi)_\infty(x)$ to the mean electric potential and $(\partial_i(1/4\pi r)*\pi)_\infty$ to the mean electric field. The effective dynamics α_Π^t, with reference to ω_M, has a non-trivial action on the above variables at infinity; explicitly one has

$$\alpha_\Pi^t\varphi_\infty=\varphi_\infty\cos\omega_0 t+\left(\frac{1}{4\pi r}*\pi\right)_\infty\omega_0\sin\omega_0 t\,, \tag{7.19}$$

$$\alpha_\Pi^t\left(\frac{1}{4\pi r}*\pi\right)_\infty=\left(\frac{1}{4\pi r}*\pi\right)_\infty\cos\omega_0 t-\varphi_\infty\sin\omega_0 t/\omega_0\,, \tag{7.20}$$

$$\alpha_\Pi^t(\partial_i\varphi)_\infty=(\partial_i\varphi)_\infty\cos\omega_0 t+\left(\partial_i\frac{1}{4\pi r}*\pi\right)_\infty\omega_0\sin\omega_0 t, \tag{7.21}$$

$$\alpha_\Pi^t\left(\partial_i\frac{1}{4\pi r}*\pi\right)_\infty=\left(\partial_i\frac{1}{4\pi r}*\pi\right)_\infty\cos\omega_0 t-(\partial_i\varphi)_\infty\sin\omega_0 t/\omega_0\,. \tag{7.22}$$

The above non-trivial "classical motion" of the variables at infinity will play an important role in the proof of a generalized Goldstone theorem, (see following section and [1] for the general strategy).

8. Spontaneous Symmetry Breaking and Mass Gap

We can now give a rigorous discussion of spontaneous symmetry breakings in the Stückelberg-Kibble model, by showing that the gauge symmetry β^λ (as well as $\beta^{\boldsymbol\lambda}$, γ^ν, and $\gamma^{\boldsymbol\lambda}$) is spontaneously broken in the representation Π with translationally invariant ground state Ψ_0, (corresponding to the state ω_M)[4], that the conditions of the generalized Goldstone's theorem are satisfied and that the corresponding *generalized Goldstone's bosons have a non-zero mass*

$$m=\omega_0=\sqrt{4\pi}\,e \tag{8.1}$$

(mass generation by variables at infinity).

We start by showing that β^λ, $\beta^{\boldsymbol\lambda}$, γ^μ, and $\gamma^{\boldsymbol\mu}$ are generated by local charges on $\mathscr{A}_\ell$ in the sense that $\forall A\in\mathscr{A}_\ell$,

$$\beta^\lambda(A)=\|\ \|\text{-}\lim_{R\to\infty} e^{iQ_R\lambda}Ae^{-iQ_R\lambda}\equiv\|\ \|\text{-}\lim_R\beta_R^\lambda(A)\,, \tag{8.2}$$

with

$$e^{iQ_R\lambda}=W(\lambda G_R)\,,\qquad G_R=(0,f_R)\,. \tag{8.3}$$

[4] It follows from Eq. (5.9) that $\omega_{M,\Lambda}$ is a ground state (i.e. invariant) under time translations and with positive energy spectrum iff $M_{11}=(k^2)^{-1}$, $M_{22}=k^2$, $M_{12}=0=M_{21}$. The following discussion can be done for any ground state $\omega_{M,\Lambda}$, but for simplicity we consider the case $\Lambda=0$

Clearly since the product is norm continuous[5] it suffices to prove Eq. (8.2) for the Weyl operators $A=W(F)$, $F\in\mathcal{T}_{\text{ext}}$, which generate $\mathcal{A}_\ell$. Then, by using the CCR's on $\mathcal{A}_\ell$ (see Sect. 5) we have

$$W(\lambda G_R)W(F)W(-\lambda G_R)=W(F)\exp i\langle F,\lambda G_R\rangle \tag{8.4}$$

and

$$\lim_{R\to\infty}\langle F,\lambda G_R\rangle=\lambda\tilde{f}_1(0). \tag{8.4'}$$

This shows that Eq. (8.2) holds for any A belonging to the algebra $\mathcal{A}_\ell^0$, finitely generated by the Weyl operators $W(F)$, $F\in\mathcal{T}_{\text{ext}}$. Since β_R^λ is norm preserving, this also implies Eq. (8.2) for the norm closure of $\mathcal{A}_\ell^0$ i.e. for $\mathcal{A}_\ell$.[6]

Furthermore, Eqs. (8.4), (8.4′) imply that $\forall A=W(F)$, $F\in\mathcal{T}_{\text{ext}}$, the norm derivative exists

$$\|\ \|-\frac{d}{d\lambda}\beta_R^\lambda(A)=W(F)i\int dx f_1(x)f_R(x)e^{i\langle F,\lambda G_R\rangle}, \tag{8.5}$$

and that

$$\|\ \|\text{-}\lim_{R\to\infty}\frac{d}{d\lambda}\beta_R^\lambda(W(F))=\|\ \|\frac{d}{d\lambda}\beta^\lambda(W(F)). \tag{8.6}$$

By the norm continuity of the product, Eqs. (8.5), (8.6) easily extend to $\mathcal{A}_\ell^0$. Hence, $\forall A\in\mathcal{A}_\ell^0$ and on a suitable dense domain in $\mathcal{H}^M$

$$\|\ \|\frac{d}{d\lambda}\beta^\lambda(A)=\|\ \|\text{-}\lim_{R\to\infty}\frac{d}{d\lambda}\beta_R^\lambda(A)=i\|\ \|\text{-}\lim_{R\to\infty}[Q_R,\beta^\lambda(A)]. \tag{8.7}$$

Thus, one of the crucial conditions for the (generalized) Goldstone's theorem is satisfied.[7]

In a similar way one shows that, on $\mathcal{A}_\ell$, γ^μ is generated by the local charge

$$\Gamma_R=\int dx f_R(x)\Delta\varphi(x), \tag{8.8}$$

that β^{λ_i} is generated by

$$G_R^i=\int dx f_R(x)x_i\pi(x), \tag{8.9}$$

and that γ^{μ_i} is generated by

$$\Gamma_R^i=\int dx f_R(x)x_i\Delta\varphi(x), \tag{8.10}$$

or, equivalently on $\mathcal{A}_\ell$, by

$$\Gamma_R'^i=3\int dx f_R(x)\partial_i\varphi(x). \tag{8.11}$$

---

[5] I.e. $A_n\xrightarrow{\|\ \|}A$, $B_n\xrightarrow{\|\ \|}B$ imply $A_nB_n\xrightarrow{\|\ \|}AB$

[6] In fact, if $A_n\xrightarrow{\|\ \|}A$, $\|\beta_R^\lambda(A)-\beta_{R'}^\lambda(A)\|\leqq\|\beta_R^\lambda(A_n)-\beta_{R'}^\lambda(A_n)\|+2\|A-A_n\|$ and the right-hand side converges to zero as R, $R'\to\infty$, and $n\to\infty$

[7] The above careful analysis shows that some of the rather pathological mechanisms, invoked in the literature [8] to explain the evasion of Goldstone's theorem in this model, actually do not apply

As a second condition for the generalized Goldstone's theorem (see [1–4]) we check that the charge density $\pi(x)$ associated to Q_R is integrable as a commutator (for more details see again [1, 3]). In fact, on the suitable dense domain in which (8.7) holds we have $\forall W(F)$, $F \in \mathcal{T}_{\mathrm{ext}}$,

$$[\pi(\mathbf{x}), W(F)] = W(F) f_1(\mathbf{x})$$

which is an integrable function of x. (Clearly this property extends to the dense subalgebra $\mathcal{A}_\ell^0$).

Since the group generated by β^λ, $\lambda \in \mathbb{R}$, and α_Π^t involves a finite number of generators (and the same holds for the covariance group of α_Π^t characterized in Sect. 7), condition β') of [1, 3] is satisfied. This means that only a finite number of charges $Q_R(t)$, $t \in \mathbb{R}$ associated to the time evolution of Q_R, are independent; (the same property actually holds for the time evolution of Q_R and G_R^i). An analogous condition is that the "classical motion" of the variables at infinity φ_∞, $(\partial_i \varphi)_\infty$, $((1/4\pi r) * \pi)_\infty$, $((\partial_i 1/4\pi r) * \pi)_\infty$ is a periodic motion (with frequency $\omega_0 = \sqrt{4\pi}\, e$), as in fact implied by Eqs. (7.19)–(7.22). In conclusion we have

Theorem 8.1. *The spontaneous breaking of the symmetry β^λ implies the existence of generalized Goldstone's bosons with finite mass $m = \sqrt{4\pi}\, e$ (mass gap associated to gauge symmetry breaking). The same conclusion holds for the breaking of each of the symmetries $\beta^{\boldsymbol{\lambda}}$, γ^μ, and $\gamma^{\boldsymbol{\mu}}$.*

Appendix A

Proof of Proposition 4.1. a) Since $\beta^\lambda, \beta^{\lambda_i}$ commute with α_L^t, it is enough to discuss the strong convergence on the states corresponding to $\lambda = 0 = \lambda_i$. Then putting $F_L \equiv A_L^t F$, $\forall F, G \in \mathcal{T}$, we have

$$\|(W(F_L) - W(F_{L'}))W(G)\Psi_0^M\|$$
$$= 2\left\{1 - \mathrm{Re}\left[\exp\left(-\tfrac{1}{4}\|F_L - F_{L'}\|_M + i\langle F_L - F_{L'}, G\rangle - \frac{i}{2}\langle F_L, F_L - F_{L'}\rangle\right)\right]\right\}, \quad \text{(A.1)}$$

where

$$\|F_L - F_{L'}\|_M^2 = [F_L - F_{L'}, F_L - F_{L'}]_M \quad \text{(A.2)}$$

and ${\Psi_0}^M$ is the Fock state corresponding to ω_M.

A necessary and sufficient condition for the vanishing of the right-hand side of Eq. (A.1) is the conververgence of the sequence F_L in the norm defined by M, [Eq. (A.2)], since the symplectic form is dominated by such a norm. Now, putting

$$\|F_L - F_{L'}\|_M^2 \equiv \int B_{ij}^{LL'}(k) \bar{\tilde{f}}_i(k) \tilde{f}_j(k) d^3k\,, \quad \text{(A.3)}$$

it is enough to prove that: i) $B_{ij}^{LL'}(k)$ converge to zero pointwise as $L, L' \to \infty$ and ii) $|B_{ij}^{LL'}(k)|$ is dominated by a locally integrable function of k, which is dominated by a polynomial at infinity; so that one can apply the Lebesgue dominated convergence theorem.

Property i) follows trivially from the pointwise convergence of $\tilde{V}_L(k)$ to $\tilde{V}(k)$, $\forall k \neq 0$. For property ii) we get after some (lengthy) calculations

$$|B_{11}(k)| \leqq C(k), \qquad |B_{22}(k)| \leqq \frac{k^4}{\inf\limits_L \omega_L^2(k)} C(k),$$

$$|B_{12}(k)| \leqq \frac{k^2}{\inf\limits_L \omega_L(k)} C(k), \qquad \omega_L^2(k) \equiv k^2(1 + \tilde{V}_L(k)),$$

$$C(k) \equiv 8(1 + \alpha^2(k))^{1/2} \left[r(k)^{-1} + \left(\sup_L \omega_L^2(k) \right) r(k)/k^4 \right].$$

Furthermore, since

$$\tilde{V}_L(k) \geqq 0, \tag{A.4}$$

(this actually holds for a wide class of infrared regularizations), we have $\omega_L^2(k) \geqq k^2$ and hence

$$k^2 \Big/ \inf_L \omega_L(k) \leqq |k|,$$

so that it suffices to have the local integrability of $C(k)$. By using that

$$\tilde{V}_L(k) \leqq \text{const}\, |k|^{-2}, \quad \text{for} \quad k \to 0$$

and that $\tilde{V}_L(k)$ is regular at infinity, the local integrability of $C(k)$ follows from conditions (2.4), (4.3)[8].

Thus, we have the strong convergence of $W(F_L)$ on the dense set D_0^M of states of the form

$$\Phi = \sum_i \lambda_i W(G_i) \Psi_0^M .$$

Since $\| W(F_L) \| = 1$, we easily get the strong convergence on the whole Hilbert space $\mathscr{H}^M$; furthermore we also get the strong convergence of the norm limits

$$\| \ \|\text{-}\lim_{N \to \infty} \sum_{i=1}^{N} \alpha_i W(A_L^t F_i)$$

as $L \to \infty$, i.e. the strong convergence of $\alpha_L^t(A)$, $\forall A \in \mathscr{A}$.

b) Since α_L^t is strongly and therefore weakly convergent (with respect to the $\mathbb{F}$ topology), the

$$\lim_{L \to \infty} (\alpha_L^t)^* \equiv (\alpha^t)^* \tag{A.5}$$

exists and it defines a mapping of $\mathbb{F}$ into $\mathscr{A}'$, the dual of $\mathscr{A}$. One can actually show that $\mathbb{F}$ is $(\alpha^t)^*$ stable. To this purpose we determine the mapping $(\alpha^t)^*$, by

[8] These conditions also imply that M_{12}/k^2 is locally integrable, since in general $\det M \geqq 1$ gives

$$M_{12}^2 \leqq M_{12}^2 + 1 \leqq M_{11} M_{22},$$

and therefore $M_{12}/k^2 \leqq M_{11}^{1/2} (M_{22}/k^4)^{1/2}$

considering the $\lim_{L\to\infty}(\alpha_L^t)^*$ on the states of the form

$$\phi_{M,\Lambda}^{G_1,G_2}(\mathscr{A})=\omega_{M,\Lambda}(W(G_1)\mathscr{A}W(G_2)) \tag{A.6}$$

with $G_i\in\mathscr{T}$, $\tilde{g}_1(k)=k^2h_1$, $h_1\in\mathscr{S}$, $\tilde{g}_2(k)\in\mathscr{S}$, $\tilde{g}_2(0)=0$. The linear span of the vectors $W(G)\Psi_0^{M,\Lambda}$ is dense in $\mathscr{H}^{M,\Lambda}\equiv\{\mathscr{A}\Psi_0^{M,\Lambda}\}$, since the G's satisfying the above condition are dense in $\mathscr{F}$ in the norm $\|\ \|_M$. Therefore the (linear combinations of) states of the form (A.6) are norm dense in the set of states associated to $\mathscr{H}^{M,\Lambda}$ and, since $(\alpha_L^t)^*$ preserves the norm, weak convergence of $(\alpha_L^t)^*$ on states of the form (A.6) implies weak convergence on all the states of $\mathscr{H}^{M,\Pi}$, and therefore on $\mathbb{F}$.

Now, by simple calculations, $\forall F\in\mathscr{T}$ one gets

$$\begin{aligned}&\lim_{L\to\infty}\omega_{M,\Lambda}(W(G_1)\alpha_L^t(W(F))W(G_2))\\&=\exp\left\{-\tfrac{1}{4}\|G_1+G_2+F^t\|_M-\frac{i}{2}(\langle G_1,G_2\rangle+\langle G_1,F^t\rangle+\langle F^t,G_2\rangle)\right\}\\&\quad\times\exp\left\{i\lambda\tilde{f}_1(0)+\sum_i\lambda_i(\partial_i\tilde{f}_1)(0)\right\},\end{aligned} \tag{A.7}$$

where F^t is the limit[9] of F_L^t in the topology of $\|\ \|_M$, i.e. for $k\neq 0$, $\tilde{F}^t(k)$ is the pointwise limit of $\tilde{F}_L^t(k)$,

$$\tilde{F}^t(k)\equiv\tilde{A}^t(k)\tilde{F}(k), \tag{A.8}$$

$$\tilde{A}^t(k)\equiv\begin{pmatrix}\cos\omega t & -(k^2/\omega)\sin\omega t\\(\omega/k^2)\sin\omega t & \cos\omega t\end{pmatrix}, \tag{A.9}$$

$$\omega\equiv\omega(k)\equiv(4\pi e^2+k^2)^{1/2}. \tag{A.10}$$

This identifies F^t as an element of $\mathscr{T}_{\text{ext}}$ [its first component at $k=0$ is defined by continuity, so that

$$\tilde{f}_1^t(0)=\cos\omega_0 t\tilde{f}_1(0)\neq\lim_{L\to\infty}\tilde{f}_{1L}^t(0)=\tilde{f}_1(0)\Big].$$

Furthermore, the above conditions on the G_i's guarantee that $G_i^t\equiv A^tG_i\in\mathscr{T}$ and A^t preserves the symplectic form so that

$$\begin{gathered}\|G_1+G_2+F^t\|_M=\|G_1^{-t}+G_2^{-t}+F\|_{M^t},\\M^t\equiv(A^t)^\dagger MA^t,\\\langle G_1,F^t\rangle=\langle G_1^{-t},F\rangle,\quad\text{etc}.\end{gathered} \tag{A.11}$$

Hence,

$$(\alpha^t)^*\phi_{M,\Lambda}^{G_1,G_2}(W(F))\equiv\lim_{L\to\infty}\phi_{M,\Lambda}^{G_1,G_2}(\alpha_L^tW(F))=\phi_{M^t,\Lambda}^{G_1^{-t},G_2^{-t}}(W(F)). \tag{A.12}$$

and the state on the right-hand side is still an element of $\mathbb{F}$, since one can prove that M^t satisfies the same conditions (4.3) as M. Therefore, $(\alpha^t)^*$ maps a dense subset $\mathbb{F}_0$ (that corresponding to the vectors of D) of $\mathbb{F}$, into $\mathbb{F}$, and since $(\alpha^t)^*$ is norm

---

[9] This follows from $B_{ij}^L(k)\to 0$ pointwise and condition (4.3)

continuous (and strongly convergent vectors yield norm-convergent states) $(\alpha^t)^*$ maps $\mathbb{F}$ = the norm closure of $\mathbb{F}_0$, into $\mathbb{F}$.

c) The group properties of $(\alpha^t)^*$ follow easily from the group properties of A^t, for the linear span of the states of the form (4.10), which are norm dense in the set of states associated to the vectors of $\mathscr{H}^{M,\Lambda}$. Again, since $(\alpha^t)^*$ is continuous in norm, the equation

$$(\alpha^t)^*(\alpha^s)^* = (\alpha^{t+s})^*$$

extends to all the states of $\mathbb{F}$, and $(\alpha^t)^*$ defines a one-parameter group of mappings of $\mathbb{F}$ into $\mathbb{F}$.

This in turn implies that $\alpha^t = s - \lim \alpha_L^t$ defines a one parameter group of automorphisms of $\mathscr{M} = \bar{\mathscr{A}}$, the weak closure of $\mathscr{A}$ with respect to the topology induced by $\mathbb{F}$ on $\mathscr{A}$, see [1, 3].

Appendix B

Proof of Proposition 5.1. 1) *(States on the Extended Algebra).* We start by noticing that the states in $\mathbb{F}$ are automatically states on $\mathscr{A}_\ell \subset \mathscr{M}$, and we have

$$\omega_{M,\lambda,\lambda}(W(F)) = \exp\{-\tfrac{1}{4}[F,F]_M\} \exp i\lambda \tilde{f}_1(0) \exp \sum_i \lambda_i (\partial_i \tilde{f}_1)(0). \quad \text{(B.1)}$$

for any $M \in \mathbb{M}$, $F \in \mathscr{T}_{\text{ext}}$ [the proof uses the remark following Eq. (5.1)].

2) *(Dynamics on the Extended Algebra).* With the help of the algebra $\mathscr{A}_\ell$ and Eq. (B.1) we can now write the explicit form of α^t on the states (A.6); $\forall F \in \mathscr{T}$, $F^t \in \mathscr{T}_{\text{ext}}$ and Eq. (A.7) can be written as ($W_i \equiv W(G_i)$, $G_i \in \mathscr{T}$)

$$\begin{aligned}&\omega_{M,\lambda,\lambda}(W(G_1)\alpha^t(W(F))W(G_2))\\ &\quad = \omega_{M,\lambda,\lambda}(W_1 W(F^t) W_2) \exp i[\tilde{f}_1(0) - \tilde{f}_1^t(0)] \exp \sum_i \lambda_i [(\partial_i \tilde{f}_1)(0) - \partial_i \tilde{f}_1^t(0)]\,. \quad \text{(B.2)}\end{aligned}$$

In fact, one can use Eq. (B.1) for the extended Weyl operator $W_1 W(F^t) W_2$ and rewrite the right-hand side of Eq. (A.7) in the form (B.2). Alternatively, one can reduce the discussion of Eq. (B.2) to the states $\omega_{M,\lambda=0,\lambda_i=0} \equiv \omega_M$ on which $\alpha_L^t W(F)$ converges simply to $W(F^t)$. In fact, putting $\omega_{M,\lambda,\lambda_i} \equiv \beta^* \omega_M$, $\Lambda \cdot \mathbf{c} \equiv \tilde{f}_1(0) - i \sum_i \lambda_i (\partial_i \tilde{f}_1)(0)$, and using $\alpha_L^t \beta = \beta \alpha_L^t$ one has

$$\begin{aligned}\lim_{L\to\infty} \beta^* \omega_M(W_1 \alpha_L^t(F) W_2) &= \lim_{L\to\infty} \omega_M(\beta(W_1)\alpha_L^t(W(F))\beta(W_2)) \exp I\Lambda \cdot \mathbf{c}\\ &= \beta^* \omega_M(W_1 \beta^{-1}(W(F^t)) W_2) \exp i\Lambda \cdot \mathbf{c}\\ &= \exp[i(\Lambda \cdot \mathbf{c} - \Lambda \cdot \mathbf{c}^t] \beta^* \omega_M(W_1 W(F^t) W_2)\,.\end{aligned}$$

Furthermore, Eq. (B.2) holds for $W(F)$, $F \in \mathscr{T}_{\text{ext}}$. In fact, $\mathscr{T}$ is dense in $\mathscr{T}_{\text{ext}}$ in the topology given by $\| \;\|_M$, and therefore if

$$F_n = (f_1, f_{2n}) \xrightarrow{\| \;\|_M} F(f_1, f_2) \in \mathscr{T}_{\text{ext}}, \quad \text{(B.3)}$$

then, by Eq. (B.1), (in each $\mathscr{H}^{M,\Lambda}$)

$$s\text{-}\lim W(F_n) = W(F)\,. \quad \text{(B.4)}$$

Since α^t is weakly continuous, the left-hand side of Eq. (B.2) with $F \in \mathcal{T}_{\text{ext}}$ can be obtained as the limit when $F_n \to F$ as in (B.3). In the right-hand side, $\forall F \in \mathcal{T}_{\text{ext}}, F^t$, defined by Eqs. (A.8), (A.9) (and continuity of $\tilde{f}_1$ at $k=0$), belongs to $\mathcal{T}_{\text{ext}}$; then, using (B.1), by explicit computation one can show that the right-hand side can be obtained as the limit when $F_n \to F$ (the three terms are not separately continuous when $F_n \to F$!). In conclusion, in the representation given by the states $\omega_{M,\lambda,\lambda_i}$, $\forall F \in \mathcal{T}_{\text{ext}}$, one has[10]

$$\alpha^t(W(F)) = W(F^t) e^{i\lambda(\tilde{f}_1(0) - \tilde{f}_1^t(0))} \exp \sum_i \lambda_i (\partial_i \tilde{f}_1(0) - \partial_i \tilde{f}_1^t(0)). \tag{B.5}$$

3) *(Algebraic Dynamics).* With the help of the above results, we can now write the action of the algebraic dynamics α^t with no reference to a specific representation and obtain Eq. (5.9). Such an equation clearly shows that the algebraic dynamics α^t maps $\mathcal{A}_\ell$ into the algebra $\mathcal{G}(\mathcal{A}_\ell \cup \mathcal{A}_\infty)$ generated by $\mathcal{A}_\ell$ and the variables at infinity $U_\infty(\alpha)$, $T^i_\infty(s)$, through finite linear combinations and norm closures.

Finally, to get the stability of $\mathcal{G}(\mathcal{A}_\ell \cup \mathcal{A}_\infty)$, under α^t we have to show that α^t maps $\mathcal{A}_\infty$ into itself. In fact, by the weak continuity of α^t, we have

$$\begin{aligned} \alpha^t U_\infty(s) &= \alpha^t \lim_{R\to\infty} W(sF_R) = \lim_{R\to\infty} \alpha^t W(sF_R) \\ &= \lim_R W(sF_R^t) \exp[is\varphi_\infty(f_1(0) - \tilde{f}_1^t(0))] \\ &= U_\infty(s\cos\omega_0 t) U_\infty(s(1-\cos\omega_0 t)) = U_\infty(s). \end{aligned} \tag{B.6}$$

Similarly, by using Eqs. (5.7) we have

$$\begin{aligned} \alpha^t T^i_\infty(s) &= \lim_{R\to\infty} \alpha^t W(sF^i_R) \\ &= \lim_R W(sF_R^{i,t}) \exp\left[s\sum_j (\partial_j\varphi)_\infty [(\partial_j \tilde{f}_1^i)(0) - (\partial_j \tilde{f}_1^{i,t})(0)\right] \\ &= T^i_\infty(s\cos\omega_0 t) \prod_j T^j_\infty(s\delta_{ij}(1-\cos\omega_0 t)) = T^i_\infty(s). \end{aligned}$$

In conclusion the algebra at infinity $\mathcal{A}_\infty$ generated by $U_\infty(s)$ and $T^i_\infty(s)$ is pointwise invariant[11] under α^t.

References

1. Morchio, G., Strocchi, F.: Spontaneous symmetry breaking and energy gap generated by variables at infinity. Commun. Math. Phys. **99**, 153 (1985)
2. Morchio, G., Strocchi, F.: ISAS Report 35/84/E.P
3. Morchio, G., Strocchi, F.: Mathematical structures for long range dynamics and symmetry breaking. J. Math. Phys. **28**, 622 (1987)
4. Morchio, G., Strocchi, F.: Infrared problem in QED, Higgs phenomenon and long-range interactions, lectures at the Erice school on: Fundamental problems in gauge field theory. Velo, G., Wightman, A.S. (eds.). New York: Plenum Press 1986

---

[10] It is worthwhile to remark that $\tilde{f}_1(0) - \tilde{f}_1^t(0) = \tilde{f}_1(0)(1-\cos\omega_0 t)$ for $F \in \mathcal{T}$, whereas for $F \in \mathcal{T}_{\text{ext}}$, one has in addition the term $-\omega_0^{-1} \sin\omega_0 t(k^2 \tilde{f}_2)$ $(k=0)$

[11] This property holds in general if the set of states does not give rise to factorial representations in which α^t is broken

5. Morchio, G., Strocchi, F.: Spontaneous breaking of the Galilei group and the plasmon energy gap. Ann. Phys. (NY) **170**, 310 (1986)
6. Morchio, G., Strocchi, F.: Long range dynamics and broken symmetries in gauge models. The Schwinger model. J. Math. Phys. (in press)
7. Kibble, T.W.: Proc. Int. Conf. Elementary Particles. Oxford: Oxford University Press 1965
8. Guralnik, G.S., Hagen, C.R., Kibble, T.W.: In: Advances in particle physics, Vol. 2. Cool, R.L., Marshak, R.E. (eds.). New York: Interscience 1986
9. Streater, R.F.: Goldstone's theorem and related topics. In: Many degrees of freedom in field theory. Streit, L. (ed.). New York: Plenum Press 1978
10. Brattelli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics, Vol. II. Berlin, Heidelberg, New York: Springer 1981
11. Haag, R., Kadison, R.V., Kastler, D.: Nets of C^*-Algebras and classification of states. Commun. Math. Phys. **6**, 81 (1970)

Communicated by K. Osterwalder

Received December 4, 1986