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Asymptotic Completeness and Multiparticle Structure in Field Theories

II. Theories with Renormalization: The Gross-Neveu Model

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Abstract. The ideas developed in Part I (ref. [1]) are applied to the recently constructed massive Gross-Neveu model. We define in this case an irreducible kernel satisfying a regularized Bethe-Salpeter equation which is convenient to derive asymptotic completeness in the 2-particle region. As in Part I, the method allows direct graphical definition of general irreducible kernels and is well suited to the analysis of asymptotic completeness and related results in more general energy regions.

A large part of the paper is devoted to a new self-contained construction (via phase space expansion) of the Gross-Neveu model. The presentation is somewhat simpler than previous ones, is more complete on some points and is best suited to our purposes.

1. Introduction

In all models that involve renormalization, a phase-space analysis [2–7] is needed, e.g. to control ultraviolet divergences. A method that has proved convenient is to introduce a suitable decomposition of momentum space into slices with, for each slice, cluster expansions with a conveniently scaled lattice. An analogue of the cluster expansion, with respect to momentum slices, is also a priori needed. However, a somewhat simpler method can be applied in fermionic theories such as the massive Gross Neveu model [0], which is a fermionic model in 2 dimensions with quartic interaction a colour number $\geqq 2$, and which is asymptotically free. In fact, in contrast to bosonic models, it is useful to expand the exponential of the interaction which is of the form

$$\exp\left[\lambda \int_{\Lambda} (\bar{\psi}\psi)^2(y)dy\right] \quad \text{as a sum} \quad \sum \frac{\lambda^n}{n!}\left[\int_{\Lambda} (\bar{\psi}\psi)^2(y)dy\right]^n .$$

Each field is decomposed into fields $\psi^{(i)}$ depending on the momentum slice i, and cluster expansions are now applied for each i to

$$\text{integrals of the form } \int \prod_v \psi^{(i)}(y_v) \prod_w \bar{\psi}^{(i)}(y_w) d\mu(\bar{\psi}^{(i)}\psi^{(i)}) .$$

Some of the vertices y_v or y_w occur in several slices and "connect" the various slices together.

Several approaches to the G.N. model, which involve slightly different renormalization procedures, can be considered. In all of them, a momentum space cut-off ϱ, which is the number of momentum slices, is first introduced and one starts from a coupling constant λ_ϱ which will tend to zero in the $\varrho \to \infty$ limit in a way such that the renormalized coupling constant λ_{ren} is different from zero (non-triviality). The construction presented below, which follows the same essential ideas but differs from earlier ones [8,9] on some technical points, seems the simplest and most convenient for our purposes.

The model is described in Sect. 2.1. The expansion of connected functions arising from cluster and Mayer expansions in each momentum slice is described in Sect. 2.2. It is shown in Sect. 2.3 to be convergent in the $\Lambda \to \infty$ (infinite volume) limit, at given momentum-space cut-off ϱ, for $|\lambda| < \lambda_\varepsilon M^{-(\varrho/2)} \varrho^{-3} m_\varrho^2$ and λ_ε sufficiently small; m_ϱ is the bare mass and $M > 1$ is the "width" of the lowest momentum slice. This is an appreciable progress with comparison to the result obtained without phase-space analysis ($\lambda \to 0$ as $M^{-2\varrho}$), but as expected, is still insufficient to control the $\varrho \to \infty$ limit, i.e. to obtain a non-trivial theory in that limit: a rearrangement of terms of the expansion, involving renormalization, is needed. It is outlined in Sect. 2.4 where convergence and decay properties of connected functions are obtained (at ϱ infinite) with λ_ϱ of the form: $[-\beta_2(\ln M)\varrho + \beta_3 \ln \varrho + D]^{-1}$, $|D|$ sufficiently large (Re $D \geqq 0$) and m_ϱ of the form $m_0\left(\dfrac{D}{-\beta_2 \ln M}\right)^\gamma \varrho^{-\gamma}, \gamma > 0$; β_2 and β_3 are the standard coefficients of the β-function. The constant λ_{ren} will be shown to be non-zero (in fact close to D^{-1} for large D) in Sect. 3; the renormalized mass m_{ren} as well as the physical mass will be of the order of m_0.

The main purpose of this article is to construct a Bethe-Salpeter (B.S.) type irreducible kernel allowing a derivation of the discontinuity formula,

$$F_+ - F_- = F_+ * F_- \ ,$$

which characterizes asymptotic completeness in the 2-particle region; here F_+ and F_- are the boundary values of the connected, amputated 4-point function F from above and below the cut [which starts at $(2m_{ph})^2$] in the physical sheet and $*$ is on mass shell convolution. [More precisely, each internal line includes, besides mass shell δ functions, the residue of the 2-point function at its pole, of the form $Z(-\not{p} + m_{ph})$ for the G.N. model.] This was achieved in previous works (see [1] and references therein) for $P(\varphi)_2$ models from the B.S. equation

$$F = G + GOF$$

where O denotes Feynman type convolution with 2-point functions. The 2-particle irreducibility of G and related analytic properties of the 2-point function needed in the derivation were obtained in [1] through a "4$^{\text{th}}$ order" cluster expansion. In a renormalizable theory like the G.N. model the operation O introduces divergences. One might first define, in a theory with cutoff α a B.S. kernel G_α satisfying:

$$F_\alpha = G_\alpha + G_\alpha O_\alpha F_\alpha \ ,$$

and obtain in turn the relation:

$$(F_\alpha)_+ - (F_\alpha)_- = (F_\alpha)_+ * (F_\alpha)_- \ .$$

But one cannot take the limit of this equation without a supplementary analysis. (One cannot e.g. exclude a priori an a la Martin pathology such as an accumulation of poles of F on the real axis in that limit.) A related approach is to introduce renormalized B.S. equations: see [10].

However, the most convenient method is to introduce a kernel G_M satisfying a regularized B.S. equation:

$$F = G_M + G_M O_M F \ ,$$

with a fixed ultraviolet cutoff in O_M. The cutoff that will be introduced will not modify the residue of the 2-point functions at their poles so that the discontinuity formula for F can still be derived.

This kernel will be constructed from a fourth order cluster expansion made only in the slice of lowest momentum that occurs in phase space analysis. In the higher slices, propagators have such an exponential fall off in euclidean space time that connectedness implies 2-particle irreducibility. Hence a usual first order cluster expansion is enough in these slices. Although it does not have a simple perturbative content, G_M will then be shown to be indeed 2-particle irreducible in the required analytic sense.

The main part of Sect. 3 is thus devoted to the introduction of this kernel. The analysis of 2-point functions is also given there.

2. Phase-Space Expansions of Connected Functions

2.1 The Model

The 4-point Green function $S_\Lambda(x_1, \ldots, x_4)$ of the G.N. model is of the form $N_\Lambda Z_\Lambda^{-1}$ with (formally):

$$(N_\Lambda)_\varrho(x_1, \ldots, x_4) = \int \psi(x_1)\psi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4) e^{\int \lambda(\bar{\psi}\psi)^2(y)dy} d\mu_\varrho(\bar{\psi}\psi) \ , \qquad (1)$$

where λ is the bare (unrenormalized) coupling constant, the measure $d\mu_\varrho$ is associated with the propagator $C_\varrho(p) = e^{-(p^2+m_\varrho^2)M^{-2\varrho}} C(p)$, $C(p) = (-\not{p} + m_\varrho) \cdot (p^2 + m_\varrho^2)^{-1}$, with $\not{p} = p_0\sigma_0 + p_1\sigma_1$: σ_0, σ_1 are 2×2 matrices such that $\sigma_0^2 = \sigma_1^2 = -1$, $\sigma_0\sigma_1 + \sigma_1\sigma_0 = 0$ (hence $\not{p}^2 = -p^2$, $p^2 = p_0^2 + p_1^2$).

By expansion of the exponential and some algebraic calculations, one gets the following form of $(N_\Lambda)_\varrho$ and an analogous form of $Z_{\Lambda,\varrho}$, which are as a matter of fact the correct definition of the model

$$N_{\Lambda,\varrho}(x_1, \ldots, x_4) = \sum_{n \geqq 0} \frac{\lambda_\varrho^n}{n!} \int_{\Lambda^n} dy_1 \ldots dy_n \begin{Bmatrix} x_1 x_2 y_1 y_1 \ldots y_n y_n \\ x_3 x_4 y_1 y_1 \ldots y_n y_n \end{Bmatrix}_\varrho \ , \qquad (2)$$

where

$$\begin{Bmatrix} u_1 \dots u_n \\ v_1 \dots v_n \end{Bmatrix}_\varrho = \det\left(C_\varrho(u_i, v_j)\right) . \tag{3}$$

The spin, respectively colour, indices of each field, which take 2 values, respectively a finite number $\geqq 2$ of values, have been left implicit.

2.2 Phase-Space Expansion (Before Renormalization)

The propagator C_ϱ is written in momentum-space in the form:

$$C_\varrho(p) = \sum_{i=1}^{\varrho} C^{(i)}(p) , \quad C^{(i)}(p) = C(p)\eta_i(p) , \tag{4}$$

where

$$\eta_1(p) = e^{-(p^2+m_\varrho^2)M^{-2\alpha}} = e^{-V_1(p)} , \tag{5}$$

$$\eta_i(p) = e^{-V_i(p)} - e^{-V_{i-1}(p)} , \quad i > 1 , \tag{6}$$

$$V_i(p) = (p^2 + m_\varrho^2)M^{-2(\alpha+i-1)} , \tag{6'}$$

for a given integer $M > 1$, which will remain fixed; α will be chosen e.g. equal to one in Sects. 2.2 and 2.3; it is convenient to choose it equal to 2 in Sect. 2.4.

The corresponding decomposition of C_ϱ in euclidean space-time [by Fourier transformation of (4)] yields expressions of $N_{\Lambda,\varrho}$ and $Z_{\Lambda,\varrho}$ in which the determinant $(\ )_\varrho$ is replaced by a sum of products of determinants corresponding to different slices (in each of which C is replaced by $C^{(i)}$] with some vertices x or y occurring possibly in several slices. Cluster expansions are made independently in each slice for each determinant. They are obtained here by replacing each propagator $C^{(i)}(x, y)$ by $C^{(i)}(x, y; s)$, according e.g. to the procedure described in Sect. 3.1 of [1]. [Note that a product of propagators $C(u_l, v_l)$ is missing there in (20)]. Variables $s_{\Delta,\Delta'}$ are here those associated with pairs of squares containing points x or y. A Taylor expansion is then applied to the determinant. The lattice D_i will be chosen different in each slice. In fact, we note that $C^{(i)}(x)$ satisfies the bounds:

$$|C^{(1)}(x)| < \text{const } e^{-m_\varrho|x|} , \tag{7}$$

$$|C^{(i)}(x)| < \text{const } M^i e^{-\frac{1}{2}M^{2(i-1)}|x|^2} , \quad i > 1 , \tag{7'}$$

obtained by direct inspection. [In contrast to C or $C^{(1)}$, which has in p-space a pole singularity at $p^2 = -m_\varrho^2$ away from euclidean space, $C^{(i)}$ has no pole. Hence $C^{(i)}(x)$ decays faster than any exponential $e^{-\alpha|x|}$ in euclidean space-time.] The lattice spacing in slice i, $i > 1$ will be taken equal to M^{-i}: this is the optimal way of decomposing the integration domains in space-time into cells where $C^{(i)}$ is approximately constant and such that the sum over cells is controlled by the exponential fall off of $C^{(i)}$ uniformly in i:

$$\sum_{\Delta \in D_i} \exp\left[-\frac{M^{2(i-1)}}{2} d^2(0, \Delta)\right] < \text{const independent of } i.$$

Thus, in the bounds, the local and non-local aspects (in x-space) will be decoupled: the couplings between squares of the same lattice will be controlled by the (scaled) fall off of the propagators; each vertex can then be localized in a square, and bounds on each contribution follow from power counting, with factors M^i for each propagator $C^{(i)}$ and M^{-2i} for the summation of one vertex in a square of D_i (= the surface of the square).

As in Part I, we put apart the bare propagators C_ϱ linked to the external vertices $x_1, \ldots, x_4$. In view of the extension of Sect. 3, a 4th order expansion in the momentum slice 1 will be used, but it is sufficient to use 1st order expansions in all other slices, since in fact $\exp[-\frac{1}{2}M^{(2i-1)}|x|^2] < \text{const}\, e^{-4m|x|}$ if e.g. $M > 4m$. A Mayer procedure is also applied in each slice. The connected function $H^c_{\Lambda,\varrho}(z_1, \ldots, z_l)$, $l \leqq 4$, is then expressed as a sum

$$H^c_{\Lambda,\varrho}(z_1, \ldots, z_l)$$

$$= \sum_{\underline{G}} \left(\prod_{v \in \underline{G}} \frac{\lambda^{n_v}}{n_v!} \right) \sum_{\{\Delta_v\}}' \left(\prod_{i=1}^{\varrho} \frac{1}{N_i!} \right) \int_{y_\alpha \in \Delta_\alpha} \prod_{v \in \underline{G}} dy_1^{(v)} \ldots dy_{n_v}^{(v)} R(\{y_1^{(v)}, \ldots, y_{n_v}^{(v)}\}, z_1, \ldots, z_l) \,, \tag{8}$$

where the sum runs over connected graphs $\underline{G}$ defined as follows. In each slice i, $\underline{G}$ has $N_i \geqq 0$ vertices $v = 1, \ldots, N_i$ to which will be associated squares Δ_v of D_i. Each vertex v, or square Δ_v, in each slice is equipped with $n_v \geqq 0$ "original" interaction vertices and with $n'_v \geqq 0$ "derived" interaction vertices, with $n_v + n'_v \geqq 1$. Each interaction vertex has 4 legs, 2 "ψ" and 2 "$\bar{\psi}$" which are distributed between 1, 2, 3 or 4 slices; the vertex is said to belong to each of these slices. It is (by definition) an original vertex in the slice of highest index (to which it belongs). It is a derived vertex in the other slices (to which it belongs). Original and derived vertices in different slices corresponding to a common interaction vertex (to which the same point y or z will be associated) will be joined by lines ---.

Each interaction vertex $(0, r)$ corresponding to a point z_r always occurs as an original or derived vertex in slice 1 with a leg corresponding to the propagator C_ϱ to which the point z_r is attached. In each slice i, pairs of vertices V may be joined by Mayer lines, or by the propagator line (ψ) —— $(\bar{\psi})$ that runs between original or derived interaction vertices they contain, with at most p_i lines —— between any pair (v, v') of vertices of $\underline{G}$ if the order of the cluster expression in slice i is p_i. The graph $\underline{G}$ has to be connected with respect to the set of all vertices v occurring in different slices when all lines are taken into account.

For each $\underline{G}$, the sum Σ' runs over all possible sets $\{\Delta\}$ of squares varying in Λ subject to the following conditions. Two squares linked by a Mayer line must coincide and a square in a slice i must be contained in a square of a lower slice whenever corresponding vertices v are linked by a line --- (since these squares contain a common point y or z). On the other hand, if v contains an original or derived vertex $(0, r)$ then the square Δ must contain the point z_r.

For each point $(y_1^{(v)}, \ldots, y_{n_v}^{(v)})$ in the integration domain, R is is a product of factors (which are propagators and determinants) associated for each slice to each part of $\underline{G}$ connected in the slice by lines ——, of factors -1 for each Mayer line and of a further symmetry factor (smaller than one). The first factors are those arising

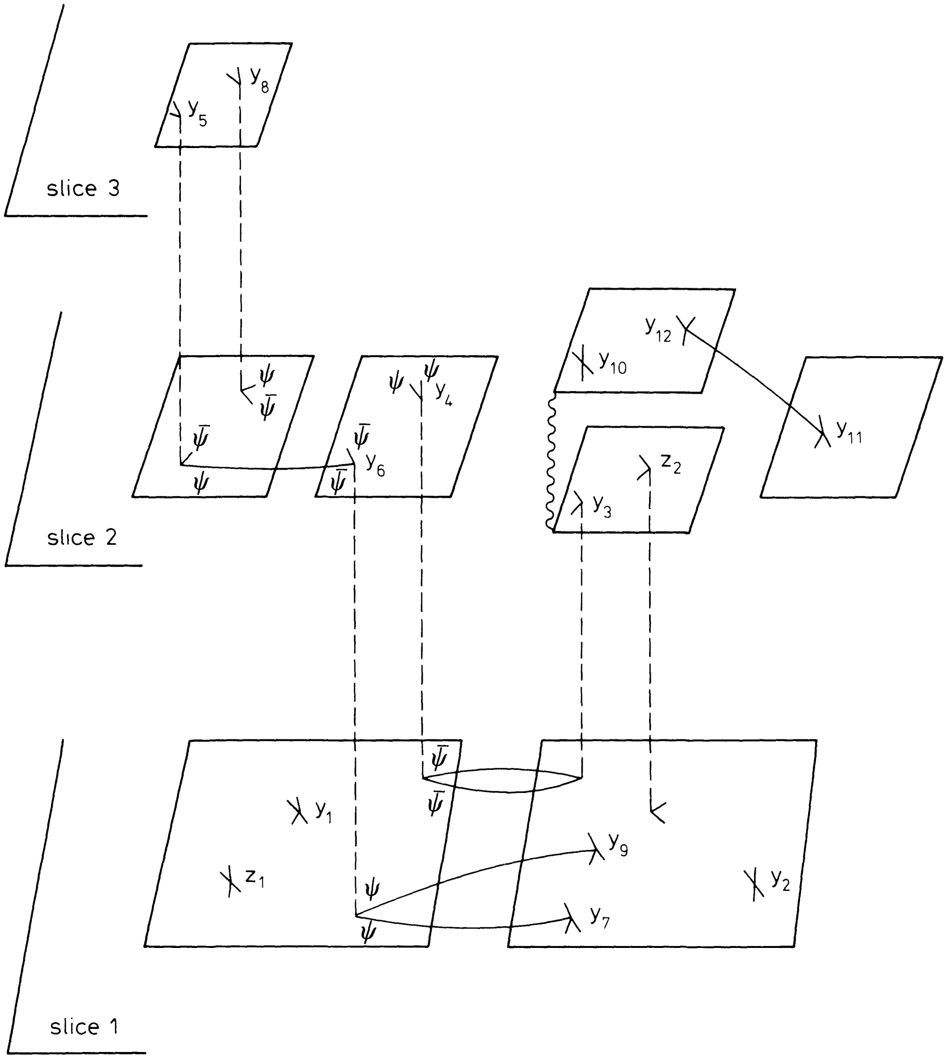


Fig. 1. A configuration of squares contributing to a connected function $H^c_{\Lambda,\varrho}$. Each vertex has two legs ψ and two legs $\bar{\psi}$, only some of which are indicated explicitly. All fields of slice i are to be labelled by i. ∿∿∿ denotes a Mayer line

from the cluster expansions. An example of a set of squares $\{\Delta\}$ for a given graph $\underline{G}$, with indication of corresponding points y_α and z_r and of lines arising from those of $\underline{G}$ is shown in Fig. 1. The factor associated e.g. to the connected part in slice 2 ($p=1$) on the left is equal, up to a sign, to

$$\int_0^1 C^{(2)}(y_5, y_6) \begin{Bmatrix} y_8 y_4 y_4 \\ y_8 y_5 y_6 \end{Bmatrix}^{(2)} (s) ds ,$$

where s is the variable associated to the pair of squares; the propagators $C^{(2)}(y_\alpha, y_\beta; s)$ involved in the determinant are equal to $C^{(2)}(y_\alpha, y_\beta)$, respectively to

$sC^{(2)}(y_\alpha, y_\beta)$, if y_α, y_β belong to the same square, respectively do not belong to the same square. In the terms { } there is one point y_α in the first line, respectively y_β in the second line, for each leg ψ, respectively $\bar{\psi}$, attached to y_α, respectively y_β, except those already linked by explicit lines ——.

We conclude with some definitions. Being given a connected graph $\underline{G}$, we shall denote by $\varepsilon_{i,k}$ its connected components obtained by removing the part of $\underline{G}$ that lies in slices $<i$, by $\underline{G}_{i,k}$ the connected components $\varepsilon_{i,k}$ that have at least one leg in slice i and by $L_{i,k}$ the restriction of $G_{i,k}$ to slice i (i.e. $L_{i,k}$ is connected when the lines —— and Mayer lines of slice i, as also possible connections via slices $>i$, are taken into account). The *external legs* of $\varepsilon_{i,k}$ or $\underline{G}_{i,k}$ are legs of $\underline{G}$ that lie in slices $<i$ and are hooked by lines --- to vertices of $\varepsilon_{i,k}$ or $\underline{G}_{i,k}$.

A graph $\varepsilon_{i,k}$ or $\underline{G}_{i,k}$ is called a r-point subgraph if it has r external legs. A *skeleton graph* $\hat{\underline{G}}$ is associated to each graph $\underline{G}$ by defining skeleton graphs for each connected component $L_{i,k}$ in the same way as in Part I except that possible connections via slices $>i$ play here the same role as the lines —— of slice i. (The skeleton is defined for each $L_{i,k}$ by starting e.g. from the lowest vertex.) Contributions that have the same skeleton graph $\hat{\underline{G}}$ can be regrouped together and R is replaced in Eq. (8) by a corresponding factor $\hat{R}$.

2.3. *Convergence ($\Lambda \to \infty$, ϱ fixed)*

We begin with some preliminary lemmas:

Lemma 1 [9]. *Given a slice i, a set if squares Δ of D_1 and in each square $l'(\Delta)$ points $y'_{j,\Delta}$, $l''(\Delta)$ points $y''_{j,\Delta}$, (some of which may coincide), with $\sum_\Delta l'(\Delta) = \sum_\Delta l''(\Delta)$, there exists a constant C independent of the number and position of the squares and of the point y', y'' in these squares such that*:

$$\left| \begin{Bmatrix} y'_{1,\Delta_1} \cdots y'_{l'(\Delta_1),\Delta_1} y'_{1,\Delta_2} \cdots y'_{l'(\Delta_2),\Delta_2} \cdots \\ y''_{1,\Delta_1} \cdots y''_{l''(\Delta_1),\Delta_1} \cdots\cdots\cdots\cdots\cdots \end{Bmatrix}^{(i)}_{(s)} \right| \leqq \prod_\Delta [CM^{i/2}]^{l(\Delta)} , \tag{9}$$

where $l(\Delta) = l'(\Delta) + l''(\Delta)$.

A stronger version of this result is given in [9]. An alternative and simpler proof is given in appendix. The following Lemmas 2 and 3 can be found in analogous forms in various previous works in constructive theory or in statistical mechanics.

Lemma 2. *Given an integer $p \geqq 1$, any set of squares $\Delta_0, \Delta_1, \ldots, \Delta_N$ of the lattice D_i, $\Delta_\alpha \neq \Delta_\beta$ if $\alpha \neq \beta$, and, in each square Δ_α, $q(\alpha)$ points $y_{1,\alpha}, \ldots, y_{q(\alpha),\alpha}$ (some of which may coincide), the following bound holds for $i > 1$ (with a constant C_2 independent of i, N, $\Delta_0, \ldots, \Delta_N$ and of the points y)*:

$$\sum_G \prod_{l \in G} (C_1 e^{-\mathrm{const}\, M^{i-1} d(l)}) < C_2^q , \tag{10}$$

if C_1 is small enough, where the sum runs over all connected or non-connected graphs G *composed of lines joining points of different squares, with at most p lines between any pair of squares, $d(l)$ is the distance between the squares linked by line l and $q=\sum_{\alpha=0}^{N} q_\alpha$.*

We give the proof for simplicity at $p=1$. The sum Σ is bounded by the sum over all ways of drawing from each point y of each square Δ_α zero, one or more lines joining y to points of squares $\Delta_\beta \neq \Delta_\alpha$, with at most one line between y and each square Δ_β. For each y, the sum is bounded by a constant independent of N, y and of the number and position of the squares if:

$$C_1 < \left[\sum_{\Delta \in D_1} e^{-\text{const}\, M^{i-1} d(\Delta_\alpha, \Delta)} \right]^{-1} .$$

Lemma 3.

$$\sum_{N \geqq 0} \frac{C'^N}{N!} \sum_{\substack{\Delta_1, \ldots, \Delta_N \in D_i \\ \Delta_\alpha \neq \Delta_\beta \text{ if } \alpha \neq \beta}} e^{-\text{const}\, M^{i-1} l(\Delta_0, \ldots, \Delta_N)} < C'' \tag{11}$$

for $i>1$, where C'' is arbitrarily small if C' is small enough; $l(\Delta_0, \ldots, \Delta_N)$ is the shortest length of all graphs joining $\Delta_0, \ldots, \Delta_N$, with possible intermediate points (or squares of D_i), and connected with respect to $\Delta_0, \ldots, \Delta_N$.

This bound can be obtained e.g. by noting that $l(\Delta_0, \ldots, \Delta_N)$ is larger than one half the minimal length of trees joining $\Delta_0, \ldots, \Delta_N$ without intermediate points or squares. The left-hand side of (11) is thus bounded by introducing for each N a sum over all trees with factors $\exp\left[-\frac{\text{const}}{2} M^{i-1}|l|\right]$ for each line of the tree. The end of the proof is then analogous to that of (46) of [1].

We now state:

Proposition 4. *For any $\varrho > 0$ and λ sufficiently small, $H^c_{\Lambda,\varrho}$ has a well defined limit H^c_ϱ when $\Lambda \to \infty$. Moreover $\forall \varepsilon$, $\varepsilon > 0$, $\exists \lambda_\varepsilon > 0$ independent of Λ and ϱ such that*

$$\int_{\substack{z_2 \in \Delta_2 \ldots z_l \in \Delta_l \\ \Delta_2, \ldots, \Delta_l \in D_1}} dz_2 \ldots dz_l |H^c_{\Lambda,\varrho}(z_1, \ldots, z_l)| < \text{const} |\lambda_\varepsilon|^l e^{-(m-\varepsilon) l(z_1, \Delta_2, \ldots, \Delta_l)}$$
$$\text{if} \quad |\lambda| < \lambda_\varepsilon M^{-\varrho/2} \varrho^{-3} m_\varrho^2 . \tag{12}$$

Remark. In this section, the lattice spacing in slice 1 can be taken of the order of m_ϱ^{-1}. Thus in slice 1, Lemmas 3 and 4 still apply (with M^{i-1} replaced by m_ϱ). The factor m_ϱ^2 in the bound on $|\lambda|$ in Proposition 4 arises from the surface of the squares of this redefined slice 1.

Proof of Proposition 4. The function $\hat{R}$ (see end of Sect. 2.2) satisfies, for any given set $\{\Delta\}$, the bound (derived from Lemma 1):

$$|\hat{R}(\{y_1^{(v)}, \ldots, y_{n_v}^{(v)}\}, z_1, \ldots, z_l)| < \left[\prod_{i=1}^{\varrho} \prod_{f \in (i)} (C' M^{i/2}) \right]$$
$$\cdot \prod_{i=2}^{\varrho} \prod_{l \in (i)} e^{-M^{i-1}|l|} \prod_{l \in (1)} e^{-m_\varrho |l|} , \tag{13}$$

where $f \in (i)$ means a leg (= field) in slice i, the last products run over lines ——, and $|l|$ is the distance between vertices y or z joined by line l. We have used the inequality:

$$\exp\left[-\frac{M^{2(i-1)}}{2}|l|^2\right] < \text{const}\ e^{-2M^{i-1}|l|} . \tag{14}$$

It will be convenient to write, $\forall i > 1$:

$$e^{-M^{i-1}|l|} \leq \text{const}\left[\prod_{j=2}^{i} e^{-\frac{M^{j-1}}{4}|l|}\right] e^{-m_\varrho |l|} \tag{15}$$

to attribute each factor $\exp\left[-\frac{M^{j-1}}{4}|l|\right]$, respectively $e^{-m_\varrho|l|}$, to slice j, respectively to slice 1 and to replace $|l|$ in the bounds by the distance between the corresponding squares of D_j (or D_1), in each slice that contain relevant vertices y or z. On the other hand, factors $(N_i!)^{-1}$ will be replaced on the bounds by $\prod_k (N_{i,k}!)^{-1}$ with independent vertices v for each connected component (i, k). As in Part I, a common factor $e^{-(m_\varrho - \varepsilon) l(z_1, \ldots, z_l)}$ [up to a multiplicative factor $\prod_i (\text{const})^{N_i}$, which will give a constant for each square] can be extracted for all terms from parts $e^{-(m_\varrho - \varepsilon)|l|}$ of the factors $e^{-m_\varrho|l|}$ occurring either in (13) or in (15). Remaining factors $e^{-\varepsilon|l|}$ will be used for convergence.

(i) A "fixed" interaction vertex of $\underline{\hat{G}}$, i.e. correspondingly a "fixed" vertex y or z is chosen in a slice $\geqq i$ for each connected component $L_{i,k}$. This is e.g. z_1 in slice 1 (where there is only one connected component L_1). More generally, if $L_{i,k}$ contains a vertex already occurring as a fixed vertex in a slice $< i$, this vertex is again chosen. Otherwise, a vertex of $\underline{G}_{i,k}$ that has a leg in a slice $i' < i$ is chosen. One starts by summing all the non-fixed vertices of slice ϱ. In each $L_{i,k}$ there is by construction a tree of exponential fall of factors connecting the fixed vertex to all the vertices of $L_{i,k}$ which have not been already summed (in slices $\varrho, \ldots, i+1$). Each fixed vertex (except z_1) will be integrated directly in the square of the highest slice to which it belongs and where it is no longer "fixed."

(ii) (power counting)

The bounds (13) yield factors $M^{i/2}$ for each leg in each slice i, and a factor M^{-2i} is obtained (by integration in squares of D_i) for each vertex y or z of slice i, except those already integrated in higher slices and "fixed" ones. By writing, following [8] $M^{i/2} = \prod_{j=1}^{i} M^{1/2}$, $M^{-2i} = \prod_{j=1}^{i} M^{-2}$, a factor $M^{1/2}$, respectively M^{-2}, can be attributed to each leg, respectively each vertex y (or z), except one, of each connected component $\underline{\varepsilon}_{i,k}$. The product of these factors is equal,

for $i > 1$, to $\prod_{(i,k)} M_{i,k}$ where:

$$M_{i,k} = \prod_{f \in \varepsilon_{i,k}} M^{1/2} \prod_{\substack{\text{vertices } u \in \varepsilon_{i,k} \\ \text{minus one}}} M^{-2} = M^{(-1/2)(e_{i,k} - 4)} , \tag{16}$$

where $e_{i,k}$ is the number of external legs (in slices $< i$) of $\underline{\varepsilon}_{i,k}$. Since $e_{i,k} \geqq 2$, this factor is bounded by $\prod_{(i,k)} M$.

Finally, since any $\varepsilon_{i,k}$ contains at least two vertices u (y or z) and since any vertex occurs at most in ϱ connected components $\underline{G}_{i,k}$, the bound $(M^{\varrho/2})^{n+l}$ is obtained, where $n+l$ is the total number of vertices u.

(iii) For each set of vertices v (or squares Δ) in each slice and each set $\{n_v\}$, the number of all possible attributions of legs in slices $\leqq i$ for each original interaction vertex of slice i is bounded by $4i^3 \leqq 4\varrho^3$.

(iv) The further summation over all possible sets of lines —— joining interaction vertices gives, in view of Lemma 2, a further constant in the bound for each (original) interaction vertex: here, part of the decay factors of the lines —— has been used to apply Lemma 3. Remaining exponential decay factors will provide the fall-off factors $\exp[-\text{const } M^{i-1} l(\Delta_0, \ldots, \Delta_N)]$ needed to apply Lemma 4 for each connected component $L_{i,k}$. (The choice of λ_ε allows one to get sufficiently small constants.)

(v) The summation over all possible values of $n_v(=n_\Delta)$ in each square Δ then gives a constant for λ_ε sufficiently small:

$$\sum_{n_v \geqq 0} [\lambda \times \text{const } M^{\varrho/2} \varrho^3 m_\varrho^{-2}]^{n_v} < \text{const} \tag{17}$$

for $|\lambda| < \lambda_\varepsilon M^{-\varrho/2} \varrho^{-3} m_\varrho^2$.

Moreover, if the square Δ contains at least one original interaction vertex, the sum runs over $n_v \geqq 1$. The constant obtained in the right-hand side of (17) is then of the form const λ_ε. This is used in turn to give an (arbitrarily small) factor const $\lambda_\varepsilon^{1/4}$ for all squares.

(vi) Final bounds are obtained by independent summations made in the same way as in [1], now using Lemma 3 for each connected component (i, k) in each slice and all possible values of $N_{i,k}$. For each one, an arbitrarily small constant is obtained (for λ_ε small enough). The summation over all possible numbers of components (i, k) in each slice and over all slices is then possible and leads to the bounds (12). The same type of argument as in [1] shows moreover that $H^c_{\Lambda,\varrho}$ has a well defined limit H^c_ϱ when $\Lambda \to \infty$, which is analytic in λ in the region $|\lambda| < \lambda_\varepsilon M^{-\varrho/2} \varrho^{-3} m_\varrho^2$.

2.4. *Renormalization*

In contrast to Sect. 2.2, cluster and Mayer expansions are first made only in slices 2 to ϱ. Scalars $\delta\lambda_\varrho$, δm_ϱ, $\delta\zeta_\varrho$ are defined by the formulae:

$$\delta\lambda_\varrho = \int S^c_{(\varrho)}(u_1, u_2, u_3, u_4)|_{m_\varrho=0} du_2 du_3 du_4 \ , \tag{18}$$

$$\delta m_\varrho = \int S^c_{(\varrho)}(u_1, u_2) du_2 \ , \tag{19}$$

$$\gamma_\nu \delta\zeta_\varrho = \int (u_1 - u_2)_\nu S^c_{(\varrho)}(u_1, u_2) du_2 \ , \tag{20}$$

where $S^c_{(\varrho)}$ is the connected 2 or 4-point function of slice ϱ, i.e. the sum over connected graphs $\underline{G}_\varrho$ of slice ϱ with 2, respectively 4 external legs hooked at u_1, u_2, respectively $u_1, \ldots, u_4$, of corresponding functions $\underline{G}_{(\varrho)}(u_1, u_2)$, respectively $\underline{G}_{(\varrho)}(u_1, \ldots, u_4)$. [$S^c_{(\varrho)}$ is independent of the slices $<\varrho$ to which external legs belong. Two or three of these legs may on the other hand be attached to the same vertex u_α. This gives a contribution to $S^c_{(\varrho)}$ including δ-functions $\delta(u_i - u_j)$.] The fact that the functions $S^c_{(\varrho)}$ and the integrals (18), (19), (20) are well defined in the $\Lambda \to \infty$ limit is proved by the same methods as in [1] and Sect. 2.3. The functions $\underline{G}_{(\varrho)}$ and $S^c_{(\varrho)}$ are also euclidean invariant at Λ infinite. Hence $\delta\lambda_\varrho$, δm_ϱ and $\delta\zeta_\varrho$ are independent of u_1.

We then define the "regularized" functions:

$$\begin{aligned} S^c_{(\varrho)\,\mathrm{reg}}(u_1,\ldots,u_4) &= S^c_{(\varrho)}(u_1,\ldots,u_4) - \delta\lambda_\varrho \delta(u_1-u_2)\delta(u_1-u_3)\delta(u_1-u_4) \\ &= [S^c_{(\varrho)}(u_1,\ldots,u_4) - S^c_{(\varrho)}(u_1,\ldots,u_4)|_{m_\varrho=0}] \\ &\quad + [S^c_{(\varrho)}(u_1,\ldots,u_4)|_{m_\varrho=0} - \delta\lambda_\varrho\delta(u_1-u_2)\delta(u_1-u_3)\delta(u_1-u_4)] \\ &= \sum \underline{G}_{(\varrho)\,\mathrm{reg}}(u_1,\ldots,u_4) \end{aligned} \tag{21}$$

$$\begin{aligned} S^c_{(\varrho)\,\mathrm{reg}}(u_1,u_2) &= S^c_{(\varrho)}(u_1,u_2) - \delta m_\varrho \delta(u_1-u_2) - \delta\zeta_\varrho \not{\partial}\delta(u_1-u_2) \\ &= \sum \underline{G}_{(\varrho)\,\mathrm{reg}}(u_1,u_2)\ , \end{aligned} \tag{22}$$

where the functions $\underline{G}_{(\varrho)\,\mathrm{reg}}$ are defined in the same way as $S^c_{(\varrho)\,\mathrm{reg}}$. (The functions $S^c_{(\varrho)}$ are replaced by functions $\underline{G}_{(\varrho)}$, and $\delta\lambda_\varrho$, $\delta\lambda_\varrho$ $\delta\zeta_\varrho$ are replaced by the corresponding quantities.)

We note for later estimates that:

$$\begin{aligned} &\int \underline{G}_{(\varrho)\,\mathrm{reg}}(u_1,\ldots,u_4)\psi(u_1)\psi(u_2)\psi(u_3)\psi(u_4)du_1\ldots du_4 \\ &= \int \underline{G}_{(\varrho)}(u_1,\ldots,u_4)|_{m_\varrho=0}\{\psi(u_1)[\psi(u_2)-\psi(u_1)]\psi(u_3)\psi(u_4) \\ &\quad + \psi(u_1)\psi(u_1)[\psi(u_3)-\psi(u_1)]\psi(u_4) \\ &\quad + \psi(u_1)\psi(u_1)\psi(u_1)[\psi(u_4)-\psi(u_1)]\} \\ &\quad + \int_0^{m_\varrho} \frac{d}{dm'}\underline{G}_{(\varrho)}(u_1,\ldots,u_4)|_{m'}\psi(u_1)\ldots\psi(u_4)du_1,\ldots du_4\ , \end{aligned} \tag{23}$$

where the indices $i_1 \ldots i_4$ ($<\varrho$) of the external fields and indications of fields $\bar{\psi}$ are left implicit. Each factor $\psi(u_j)-\psi(u_1)$ can be written:

$$\psi(u_j)-\psi(u_1) = (u_j-u_1)\cdot\nabla\psi(u_1+\Theta(u_j-u_1))\ , \qquad 0 \leqq \Theta \leqq 1\ . \tag{24}$$

The factor u_j-u_1 will be associated to $\underline{G}_{(\varrho)}$ and will be called an "internal regularization" factor while the gradient is applied to an external field. Equation (24) can on the other hand be symmetrized with respect to $u_1,\ldots,u_4$.

Similarly:

$$\begin{aligned} &\int \underline{G}_{(\varrho)\,\mathrm{reg}}(u_1,u_2)\psi(u_1)\psi(u_2)du_1\,du_2 \\ &= \int (u_1-u_2)_\mu(u_1-u_2)_\nu \underline{G}_{(\varrho)}(u_1,u_2)\psi(u_1)\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\nu}\psi(u_1+\Theta(u_2-u_1))du_1du_2\ . \end{aligned} \tag{25}$$

This yields again internal regularization factors and external gradients. The quantity δm_ϱ can be written on the other hand in the form:

$$\delta m_\varrho (\equiv \Sigma \underline{G}_\varrho) = \Sigma \delta' \underline{G}_\varrho\ , \tag{26}$$

where

$$\delta' \underline{G}_\varrho = \int_0^{m_\varrho} \frac{d}{dm'}\delta\underline{G}_\varrho|_{m'}dm' \tag{27}$$

and

$$\delta \underline{G}_\varrho = \int \underline{G}_{(\varrho)}(u_1, u_2) du_2 \ .$$

Formula (26) follows from the relation:

$$\Sigma \delta \underline{G}_\varrho|_{m=0} = 0 \ , \tag{28}$$

which is due to the fact that the 2-point functions $S_{(\varrho)}$ contains an odd number of propagators: the propagator $C^{(\varrho)}(p)|_{m=0}$ in p-space is an odd function of p and thus $S_{(\varrho)}(p)|_{m=0} = 0$.

The derivative $\frac{d}{dm'} \delta \underline{G}_\varrho(m')$ is a sum of terms obtained from the action of $\frac{d}{dm'}$ on one of the propagators involved in the expansion of δG_ϱ, with a total number of terms bounded by 2 times the number of vertices u of the graph $\underline{G}_\varrho \left(\leqq \prod_{\text{vertices } u} 2 \right)$, while:

$$\left| \frac{d}{dm'} C^{(\varrho)}(u_1, u_2; m') \right| < \text{const} \exp \left[-\frac{M^{2(\varrho-1)}}{2} |u_1 - u_2|^2 \right] , \tag{29}$$

i.e. a factor M^ϱ has been gained in comparison with (7′): each derivative thus plays in power counting the role of an internal regularization factor. (A similar modification holds in Lemma 1, as can be seen e.g. from the proof given in the Appendix.) We remark that no external gradients are associated to these internal regularizations. These remarks apply in a similar way to the derivatives d/dm' in the last term of (23).

From the above rearrangements of terms, it can be checked that an expansion analogous to (8) is obtained in slices 2 to ϱ, with the following differences:

(i) The coupling constant attached to each interaction vertex is either $\lambda_\varrho \equiv \lambda$ if this vertex has a leg of index ϱ (i.e. is an origianl vertex of slice ϱ) or is equal to

$$\lambda_{\varrho-1} = \lambda_\varrho + \delta\lambda_\varrho \tag{30}$$

otherwise.

(ii) A new class of vertices with 2 legs in slices $i, j \leqq \varrho - 1$, of the type $\delta m_\varrho \bar{\psi}_i \psi_j$ or $\delta \zeta_\varrho \bar{\psi}_i \not{\partial} \psi_j$, is introduced.

(iii) The class of graphs $\underline{G}$ is thus replaced by a new class of graphs $\underline{G}^{(\varrho)}$ which may include links between interaction vertices in slice ϱ (that will correspond to internal regularization factors $u_1 - u_2$ or d/dm'), and attributions of gradients to some legs in slices $< \varrho$.

Contributions associated with each new graph $\underline{G}^{(\varrho)}$ are those corresponding to the rules introduced above. They involve some displacements of the points to which external fields are attached. This will lead only to unessential changes in the proofs of convergence and decay.

We next define

$$\delta\lambda_{\varrho-1} = \int S^{c(\varrho)}_{(\varrho-1)}(u_1, \ldots, u_4)|_{m_\varrho = 0} du_2 du_3 du_4 \ , \tag{31}$$

where $S^{c(\varrho)}_{(\varrho-1)}$ is the connected 4-point function of slice $\varrho - 1$, i.e. the sum, over connected graphs $\underline{G}^{(\varrho)}_{\varrho-1}$ with 4 external legs and no external gradient in slices $< \varrho - 1$, of functions $G^{(\varrho)}_{(\varrho-1)}(u_1, \ldots, u_4)$. The quantities $\delta m_{\varrho-1}$ and $\delta\zeta_{\varrho-1}$ are defined similarly. This procedure is pursued up to slice 2 (included). This gives an expansion in which:

(i) The coupling constant attached to an interaction vertex of highest slice i is

$$\lambda_i = \lambda_{i+1} + \delta\lambda_{i+1} = \lambda_\varrho + \delta\lambda_\varrho + \delta\lambda_{\varrho-1} + \ldots + \delta\lambda_{i+1} \ , \tag{32}$$

where $\delta\lambda_j = \int S_{(j)}^{c(j+1)}(u_1, \ldots, u_4) du_2 du_3 du_4$ and $S_{(j)}^{c(j+1)}$ is a sum over 4-point subgraphs $\underline{G}_j^{(j+1)}$ with no external gradient (in slices $<j$).

(ii) The coefficient attached to a 2-leg vertex of highest slice i is $\sum_{j=i+1}^{\varrho} \delta m_j (= m_\varrho - m_i)$, respectively $\sum_{j=i+1}^{\varrho} \delta\zeta_j (= 1 - \zeta_i)$.

(iii) The class of graphs $\underline{G}^{(2)}$ is obtained from the inductive procedure. In particular, all 4-point subgraphs $\underline{G}_{i,k}^{(2)}$ of the graphs $\underline{G}^{(2)}$, respectively all 2-point subgraphs, have at least one external gradient attached to one of their external legs, respectively at least two external gradients.

Finally, in the last slice, we first resum in the propagator (at sufficiently small λ_ϱ, hence small δm_i, $\delta\zeta_i$) all 2-point insertions. This gives (together with an analytic continuation in λ, as needed for later purposes) an effective propagator C_1:

$$C_1(p) = \frac{C^{(1)}(p)}{1 - \varphi_1 C^{(1)}(p)} = e^{-V_1} [\not{p}(1 - \sum_{i=2}^{\varrho} \delta\zeta_i e^{-V_1}) + m_\varrho - \sum_{i=2}^{\varrho} \delta m_i e^{-V_1}$$

$$+ O(p^2) \times O(\mathrm{Sup}\,(M^{-1}, \lambda_1))]^{-1} \ , \tag{33}$$

where $\varphi_1(p)$ is the sum of 2-point insertions and $V_1 = (p^2 + m_\varrho^2) M^{-4}$, and where the expression in the right-hand side of (33) will follow from Eqs. (34), (35), (36) below. It will be shown later that $1 - \sum_{i=2}^{\varrho} \delta\zeta_i e^{-V_1}$ is close to 1 and that $m_\varrho - \sum_{i=2}^{\varrho} \delta m_i e^{-V_1}$ is close to $m_\varrho + \Sigma |\delta m_i|$. By choosing M^{-1} and λ_1 small enough (i.e. $|D|$ large enough), $C_1(p)$ will have a unique pole at a mass of the order of m_0.

A cluster expansion of order 4 (relative to C_1) and a Mayer procedure are then applied in slice 1. The result is a final expansion of H_ϱ^c of the type already described in slice 2 to ϱ. Concerning slice 1, the propagator is C_1 and on the other hand there is no 2-point function with 2 external legs in slice 1.

We now put:

$$\lambda_\varrho = [(-\beta_2 \ln M)\varrho + \beta_3 \ln \varrho + D]^{-1} \ , \tag{34}$$

$$m_\varrho = m\varrho^{-\gamma} \ , \qquad m = m_0 \left(\frac{D}{-\beta_2 \ln M} \right)^\gamma \ , \tag{35}$$

$$\zeta_\varrho = 1 \ , \tag{36}$$

where $\gamma = \lim_{i\to\infty} \lim_{\varrho\to\infty} \frac{\delta m_i}{m_i \lambda_i \beta_2}$, and outline the proof of convergence and decay [for $|D|$ sufficiently large depending on ε in Eq. (37) below] of H_ϱ^c in the $\varrho \to \infty$ limit. More precisely, the following bound is obtained:

$$\left| \int_{\substack{z_2 \in \Delta_2, \ldots, z_l \in \Delta_l \\ \Delta_2, \ldots, \Delta_l \in D_1}} H^c_\varrho(z_1, \ldots, z_l) dz_2 \ldots dz_l \right| < \text{const}\,(\varepsilon)\, |\lambda_1|\, e^{-(m-\varepsilon) l(z_1, \Delta_2, \ldots, \Delta_l)} \quad (37)$$

with moreover $\lambda_1 \neq 0$ in the $\varrho \to \infty$ limit. (For fixed ϱ, convergence and decay are obtained similarly for bare coupling constants whose absolute values are smaller than $|\lambda_\varrho|$, where λ_ϱ is defined in (34), but λ_1 will tend in general to zero in the $\varrho \to \infty$ limit for choices of bare coupling constants different from (34).)

To obtain (37), it is proved by induction that

$$\delta\lambda_i = a\lambda_i^2 + b\lambda_i^3 + O(\lambda_i^4) + \lambda_i^2 [O(e^{-(\varrho-i)}, + O(e^{-i})] \,, \quad (38)$$

$$\delta m_i = \left(\lambda_i \sum_{i+1}^{\varrho} \delta m_j C(j-i) \right) [1 + O(\lambda_i) + O(1) e^{-(\varrho-i)}] + O(1) e^{-i} \,, \quad (39)$$

$$\delta\zeta_i = \text{const}\, \lambda_i^2 (1 + \mathrm{O}(\lambda_i)) + \lambda_i^2 \mathrm{O}(\mathrm{e}^{-(\varrho-i)}) + \mathrm{O}(1)\mathrm{e}^{-i} \,, \quad (40)$$

where $a = -\beta_2 \ln M$, $b = -\beta_2 \ln M[-\beta_2 \ln M + \beta_3]$, and

$$C(k) = \int \frac{d^2p}{p^2} (e^{-V_2} - e^{-V_1})(e^{-V_{k+1}} - e^{-V_1}) \,. \quad (41)$$

These formulae will entail in turn that:

$$\lambda_i = [(-\beta_2 \ln M) i + \beta_3 \ln i + D + f(i)]^{-1} \,, \quad (42)$$

$$m_i = m i^{-\gamma} (1 + g(i)) \,, \quad i \gtrsim \frac{D}{-\beta_2 \ln M} \,, \quad (43)$$

$$m_j \leqq \text{const}\, m_0 \,, \quad j \lesssim \frac{D}{-\beta_2 \ln M} \,, \quad (43')$$

$$\left| \sum_{j=i+1}^{\varrho} \delta\zeta_j \right| \leqq O(1) \operatorname{Sup}(\lambda_i, M^{-2i}) \quad (44)$$

with $|f(i)| < 1$, $|g(i)| < 1/2$, $3/2 > |\zeta_i| > 1/2$.

Outline of the Proof of Eqs. (38), (39), (40)

(i) (Power counting) Each regularization factor, respectively each gradient acting on a field of index j, in a subgraph $\underline{G}_{i,k}$ gives a factor M^{-i}, respectively M^j. This leads to replace in the bounds the product $\prod_{(i,k)} M_{i,k}$ by $\prod_{(i,k)} M'_{i,k}$, where:

$$M'_{i,k} = M^{-1/2(e'_{i,k} - 4 + 2(\text{number of internal vertices of type } \bar{\psi}\psi)} \,, \quad (45)$$

$$e'_{i,k} = e_{i,k} + 2\,(\text{number of internal regularizations})$$
$$-2\,(\text{number of (internal) gradients}) \,. \quad (45')$$

As a consequence $e'_{i,k} \geqq \operatorname{Sup}(e_{i,k}, 6)$.

Equation (45) is true because of our definition of the first slice: the contribution to the power counting of a vertex $\bar{\psi}\psi$ in slice $i > 1$, is $M^{-2(i+\alpha-2)} \cdot M^{(i+\alpha-1)} = M^{-i-\alpha+3} \leqq M^{-1}$ for $i > 1$, and e.g. $\alpha = 2$.

In view of this power counting, the bound

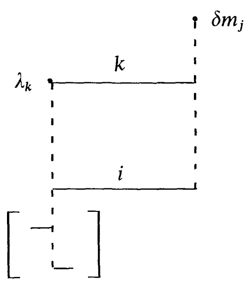
$$\prod_{i,k} M'_{i,k} \leqq \prod_{\text{vertices } u} M^{-\frac{h(u)}{6}} \tag{46}$$

holds; $h(u)$ is the difference between extremal slices containing legs attached to u. This factor allows one to sum over possible attributions of momentum slices to each leg of each vertex u.

(ii) Coefficients in λ_i^2 and λ_i^3 of $\delta\lambda_i$ in (38) are obtained by explicit computation of first order graphs. The remainder term of $\delta\lambda_i$ is then treated by the same methods as in Sect. 2.3, now applied to functions $S^c_{(i)}$ of slice i.

(iii) Formula (39) for δm_i is obtained by explicit computation of leading contributions, which come from graphs of the form

$$\sum_{\varrho \geqq j > k \geqq i}$$



including only one insertion (Note that $\lambda_k - \lambda_i \simeq \sum_{j=i+1}^{k} O(\lambda_j^2) \leqq (k-i)\ O(\lambda_i^2)$ and that, for a given k, a factor $M^{-(k-i)}$, which will control the factor $k-i$, arises from power counting of this diagram.)

(iv) Formula (40) of $\delta\zeta_i$ is obtained similarly by inspection from the leading contributions: the latter have two vertices which are either of the type $(\bar\psi\psi)^2$, $\bar\psi\psi$ or $\bar\psi\not\partial\psi$.

(v) It will be convenient to write $\lambda_i = K(\lambda_i K^{-1})$.

Convergence properties in slice 2 to ϱ (including the control of remainder terms) are ensured by choosing K small enough, by choosing D such that $|\lambda_i K^{-1}| \leqq 1$, $\forall i$, and by considering M^{-1} small enough. Values of K and M^{-1} are here independent of ε and such that the vertices $K(\bar\psi\psi)^2$, $(\Sigma\delta\zeta_j)\bar\psi\not\partial\psi$, $(\Sigma\delta m_j)\bar\psi\psi$ are small enough. By this we mean that the product of the coefficient times the contribution of the vertices to the power counting is small. For the mass term the coefficient is bounded [see (43′)], and the contribution to the power counting is bounded by M^{-1} [see (45)]. For the $(\bar\psi\psi)^2$ the power counting is zero and the coefficient is bounded by $1/D$ and small for D large. Finally for $\bar\psi\not\partial\psi$ the power counting is also zero, for i large $|\delta\zeta_i| \leqq \text{const}\ \lambda_i^2$ and for $i=2$ $|\delta\zeta_2| \leqq \text{const}\ M^{-4}$; more precisely, (44) holds and shows that the coefficient is small for M^{-1} small.

In slice 1, convergence is ensured for any given ε by choosing $|D|$ large enough (depending on ε) so that all coefficients $\lambda_i K^{-1}$ be sufficiently small. We note as a matter of fact that a factor $(\lambda_i K^{-1})^{1/4}$ can be attached to each field of slice 1. This is obvious if this field is attached to a vertex $\lambda_i(\bar\psi\psi)^2$. Other fields are external legs of connected graphs which have $r \geqq 4$ external legs. One checks that these graphs made of $(\bar\psi\psi)^2$, $\bar\psi\psi$, $\bar\psi\not\partial\psi$ vertices contain at least $r/4$ vertices $(\bar\psi\psi)^2$.

3. Irreducible Kernels

As announced, we wish in this section (i) to reobtain the non-triviality of the theory ($\lambda_{\text{ren}} \neq 0$) (ii) to establish the analytic structure of the 2-point function (analyticity up to $(4m_{ph})^2 - \varepsilon$ apart from the pole at $p^2 + m_{ph}^2 = 0$, whose position defines the physical mass) and (iii) to define the 2-particle irreducible kernel G_M satisfying the regularized B.S. equation:

$$F = G_M + FO_M G_M . \tag{47}$$

The proof of (ii) and (iii) is close to that given in [1] with, however, some complications: in particular, 2-point functions are not directly exhibited in the graphical expansion of the 4-point connected (non-amputated) function. This is due to the fact that the particle analysis was made only in slice one. We have thus to reconstruct a B.S. like equation. The result is the regularized equation (47), where O_M is the convolution with modified 2-point functions that have now a sufficient decrease in euclidean directions. Their residue at the pole is the same as that of the 2-point function (as required to derive asymptotic completeness: see introduction). On the other hand, we note that the quantities such as C_ϱ and φ are matrices and not scalars. However they do commute (since by invariance properties they can be expressed in terms of the matrices $\not{p}$ and 1). This justifies the formulae below.

In all the following, irreducibility properties are to be understood with respect to the lines —— of slice 1: a graph is r-particle irreducible in a given channel if it cannot be divided into two parts by cutting r lines —— of slice 1 or less. As in [1], the expansions of the functions $H^c(z_1, \ldots, z_l)$ give a corresponding expansion of the 4-point connected function $S^c(x_1, x_2, x_3, x_4)$ which, by direct graphical inspection can be written:

$$S^c(x_1, \ldots, x_4) = \int \left[\prod_{k=1}^{2} S'(x_k, u_k)\right] F'(u_1, \ldots, u_4) \left[\prod_{k=3}^{4} S'(u_k, x_k)\right] du_1 \ldots du_4 \tag{48}$$

with (in momentum space):

$$S'(p) = C_\varrho(p) + C_\varrho(\varphi_1 + \varphi_2) S''(p) , \tag{49}$$

$$\begin{aligned} S''(p) &= C_1(p) + C_1 \varphi_2 C_1 + C_1 \varphi_2 C_1 \varphi_2 C_1 + \ldots \\ &= \frac{C_1(p)}{1 - \varphi_2 C_1(p)} = \frac{C^{(1)}(p)}{1 - (\varphi_1 + \varphi_2) C^{(1)}(p)} = \frac{e^{-V_1}}{\not{p} + m_\varrho - \varphi e^{-V_1}} , \end{aligned} \tag{49'}$$

where φ_2 is the 2-point 1-particle irreducible function and $\varphi = \varphi_1 + \varphi_2$ [φ_1 was introduced in (33)]. F' is defined by an expansion analogous to that of S^c except that it is restricted to graphs that are 1-particle irreducible in all $1 \to 3$ channels. (The propagators C_ϱ are those attached to the points $z_1, \ldots, z_l$.) The last equalities in (49) follows from (33) and (4).

We note from (48) that:

$$S'(p) = e^{-(V_\varrho - V_1)} S''(p) = \frac{e^{-V_\varrho}}{\not{p} + m_\varrho - \varphi e^{-V_1}} . \tag{50}$$

This implies that S' is symmetric like S'' so that in (48) the factor S' for the in and outgoing particles is the same.

The 2-point function is equal, by direct inspection, to

$$S = C_\varrho + C_\varrho \varphi C_\varrho + C_\varrho \varphi C_1 \varphi C_\varrho + C_\varrho \varphi C_1 \varphi_2 C_1 \varphi C_\varrho + \ldots = C_\varrho + S' \varphi C_\varrho \tag{51}$$

and satisfies the relation:

$$S = AS' \ , \tag{52}$$

$$A = 1 + \varphi C_\varrho [1 - e^{(V_\varrho - V_1)}] \ . \tag{53}$$

For any $\varepsilon > 0$ and $|D| > D_0$, D_0 sufficiently large (depending on ε), φ is known by the methods of Sect. 2 to be analytic and bounded in modulus by $m_0 + \text{const}\ (\varepsilon)|\lambda_1|$ in the region $s < (3m_0)^2 - \varepsilon$: in fact $\varphi_1 = -m_0 + O(\lambda_1)$ and $\varphi_2 = O(\lambda_1)$. On the other hand

$$C_\varrho(1 - e^{(V_\varrho - V_1)}) \equiv (-\not{p} + m_\varrho)^{-1} [1 - e^{(p^2 + m_\varrho^2)(M^{-2\varrho} - M^{-2})}] \text{ is analytic}$$

(without pole) and bounded in that region. More precisely, one sees easily that, for λ_1 sufficiently small and M large enough (depending on m_0), the functions A and A^{-1} are analytic and close to 1. Thus S' and S have a common (unique) pole at the physical mass m_{ph} defined by the equation:

$$p^2 [1 - ae^{-V_1}]^2 + [m_\varrho - be^{-V_1}]^2 = 0 \ , \tag{54}$$

where we have put:

$$\varphi(p) = a(p^2)\not{p} + b(p^2) \ ,$$

and by an easy calculation S is shown to be of the form

$$\frac{Z(-\not{p} + m_{ph})}{p^2 + m_{ph}^2} + K(p) \ ,$$

where K is analytic up to $(4m_{ph})^2 - \varepsilon$ (and bounded in euclidean directions). We have used the equality at $p^2 + m_{ph}^2 = 0$:

$$B(-\not{p} + m_{ph}) = B|_{\not{p} \to -m_{ph}}(-\not{p} + m_{ph}) \ ,$$

where B is any matrix of the form $b_1\not{p} + b_2$. From (54) we note that:

$$[\not{p} + m_\varrho - \varphi e^{-V_1}]\Big|_{\substack{\not{p} \to -m_{ph} \\ p^2 + m_{ph}^2 = 0}} = 0 \ , \tag{54'}$$

where the left-hand side is by definition $-m_{ph} + m_\varrho - (-am_{ph} + b)e^{-V_1}$ taken at $p^2 = -m_{ph}^2$.

In view of (48) and (53), the connected, amputated 4-point function F is equal to:

$$F(p_1, \ldots, p_4) = \left[\prod_{k=1}^{4} A(p_k)\right]^{-1} F'(p_1, \ldots, p_4) \ . \tag{55}$$

F', and hence F, are known by the methods of Sect. 2 to be analytic and bounded in euclidean space (and in a strip around it). Similarly λ_{ren}, i.e. the value of F at zero momentum is itself shown to be equal to λ_1 at first order in λ_1 and is then as announced different from zero at small λ_1 (non-triviality).

Finally, let G'_M be defined as F' except that the sum is restricted to graphs that are 3-particle irreducible in the $2\to 2$ channel considered and let G_M be defined as:

$$G_M(p_1,\ldots,p_4)=\left[\prod_{k=1}^{4} A(p_k)\right]^{-1} G'(p_1,\ldots,p_4) \ . \tag{56}$$

By direct inspection, G'_M satisfies the equation

$$F'=G'_M+G'_M O''_M F' \ , \tag{57}$$

where O''_M is defined with functions S'' on internal lines. This shows in turn that:

$$F=G_M+G_M O_M F \tag{58}$$

with O_M now defined with functions $SAe^{V_\varrho-V_1}$ on internal lines, i.e. functions SAe^{-V_1} in the $\varrho\to\infty$ limit.

Since, in view of (53) and (54′), $Ae^{V_\varrho-V_1}=1$ at $p^2=-m_{ph}^2$ and when $\not{p}$ is replaced by $-m_{ph}$, the residue of $SAe^{V_\varrho-V_1}$ does coincide with the residue $Z(-\not{p}+m_{ph})$ of S at the pole.

Irreducibility properties of G'_M [and in turn of G_M in view of (56)] are established by combining methods of Sect. 2 and of [1]. (See some more details in [10].)

Appendix (with the collaboration of J. Feldman)

In this appendix we give a simplified proof of the determinant bound of [8, Appendix 1]. To be precise we consider the determinant of the matrix whose (i,j) matrix elements are $A_{i,j}=C(x_i,y_j)$ with $x_i, y_j, \in \mathbb{R}^d$. We will assume only that C and its lowest order derivatives $D^{\mathbf{m}}C$ obey:

$$|D^{\mathbf{m}}C(x,y)|\leqq K(\mathbf{m})L^{\delta+|\mathbf{m}|}e^{-L|x-y|} \ . \tag{A.1}$$

This is true for the propagators we use in the main body of this paper with $d=2$, $\delta=1$ and $L=M^i$. C may of course be equipped with spinor indices, but these play absolutely no role in the bound and we suppress them from the notation. If D is a paving of $\mathbb{R}^d$ by cubes of side L^{-1} we define n_Δ and $\bar{n}_\Delta$ to be the number of x_i's and y_j's respectively in $\Delta\in D$. We also define $n=\sum_\Delta n_\Delta=\sum_\Delta \bar{n}_\Delta$.

The most naive bound on $|A|$ is gotten simply by expanding the determinant and ignoring potential cancellations between terms. If P_n is the set of permutations of $\{1,2,\ldots,n\}$,

$$|A|\leqq B\equiv \sum_{\pi\in P^n}\prod_{i=1}^{n} K(\mathbf{0})L^\delta e^{-L|x_i-y_{\pi(j)}|} \ . \tag{A.2}$$

Using a small amount of the exponential decay to turn the global $n!$ into local n_Δ!'s in the usual way we have, for any $\zeta>1$,

$$B\leqq K_1(\zeta)^n L^{\delta n}\prod_\Delta (n_\Delta!)^{1/2}(\bar{n}_\Delta!)^{1/2}\sup_{\pi\in P_n}\prod_{i=1}^{n} \mathrm{e}^{-(L/\zeta)|x_i-y_{\pi(j)}|} \ . \tag{A.3}$$

The improved bound is

Theorem. *Let r be any positive integer. If C obeys (A.1) for all* $\mathbf{m}$ *with* $|\mathbf{m}| \leqq 4dr$, *then*:

$$|A| \leqq BK^n \prod_{\Delta} \frac{1}{(n_\Delta!)^r (\bar{n}_\Delta!)r} . \tag{A.4}$$

Remark. K is an unimportant constant which depends on r, the $K(\mathbf{m})$'s, and the number of values the suppressed spinor indices may take.

Proof. Let σ be the number of values spinor indices may take.

Divide each cube $\Delta \in D$ into $\frac{n_\Delta}{2\sigma d^p}$ cubes each of side $\left[\frac{2\sigma d^p}{n_\Delta}\right]^{1/d} L^{-1}$.

The integer p will be chosen later. We will denote the new smaller cubes Δ_α, their centers z_α and the number of x's they contain n_α. Then we may write the i^{th} row of A

$$C(x_i, \cdot) = C(z_\alpha, \cdot) + [(x_i - z_\alpha) \cdot D] C(z_\alpha, \cdot) + \cdot + \frac{1}{(p-1)!} [(x_i - z_\alpha) \cdot D]^{p-1} C(z_\alpha, \cdot)$$

$$+ \frac{1}{(p-1)!} \int_0^1 dt (1-t)^{p-1} [(x_i - z_\alpha) \cdot D]^p C[x_i + (1-t)(z_i - x_i)] , \tag{A.5}$$

where Δ_α is chosen to contain x_i. Expanding out the dot products $(x_i - z_\alpha) \cdot D$, the i^{th} row is written as a sum of $1 + d + d^2 + \ldots + d^p \leqq d^{p+1}$ vectors. Thus, by the row-wise multilinearity of the determinant we get a sum of at most $(d^{p+1})^n$ determinants. Consider any one of these new determinants which is non-zero. Of its n_α rows having $x_i \in \Delta_\alpha$, all but at most σd^p must be p^{th} order derivative terms from (A.5). Furthermore, by (A.1) and the fact that every component of $x_i - z_\alpha$ is bounded by $(2\sigma d^p / n_\Delta)^{1/d} L^{-1}$, we have that every matrix element containing a q^{th} order derivative is bounded by a constant times $(1/n_\Delta)^{q/d} L^\delta \exp[-L|x_i - y_j|]$. Hence, expanding each nonzero new determinant as in (A.3),

$$|A| \leqq (d^{p+1})^n K_2^n \left[\prod_\Delta \prod_\alpha (1/n_\Delta)^{(n_\alpha - \sigma d^p)p/d}\right] B .$$

Since

$$\sum_{\alpha \text{ s.t. } \Delta_\alpha \subset \Delta} (n_\alpha - \sigma d^p) = n_\Delta - \sigma d^p \left(\frac{n_\Delta}{2\sigma d^p}\right) = n_\Delta / 2 ,$$

$$|A| \leqq K^n \prod_\Delta \{[1/n_\Delta]^{n_\Delta (p/2d)}\} B . \tag{A.6}$$

Interchanging the roles of rows and columns and taking the geometric mean of the resulting bound and (A.6) yields the theorem.

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References

0. Mitter, P., Weisz, P.H.: Phys. Rev. D**8**, 4410 (1973)
   Gross, D., Neveu, A.: Phys. Rev. D**10**, 3235 (1974)
1. Iagolnitzer, D., Magnen, J.: Asymptotic completeness and multiparticle structure in field theories. Preprint (Ecole Polytechnique) 1986

2. Glimm, J., Jaffe, A.: Positivity of the φ_3^4 Hamiltonian. Fort. Phys. **21**, 327 (1973)
3. Benfatto, G., Cassandro, M., Gallavotti, G., Nicolò, F., Presutti, E., Scaciatelli, E.: Ultraviolet stability in Euclidean scalar field theories. Commun. Math. Phys. **71**, 95 (1980)
4. Magnen, J., Sénéor, R.: The infrared behaviour of $(\nabla\varphi)_3^4$. Ann. Phys. **152**, 130 (1984)
5. Gawedzki, K., Kupiainen, A.: Lattice dipole gas and $(\nabla\varphi)^4$ models at long distances: decay of correlations and scaling limit. Commun. Math. Phys. **92**, 531 (1984)
6. Gawedzki, K., Kupiainen, A.: Massless lattice φ_4^4 theory: rigorous control of renormalizable asymptotically free model. Proceedings of Les Houches Summer School (1984); Phys. Rev. Lett. **54**, 92 (1985), Commun. Math. Phys. **99**, 197 (1985)
7. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Proceedings of Les Houches Summer School (1984), and Construction and Borel summability of infrared φ_4^4 by a phase space expansion. Preprint Ecole Polytechnique (1986)
8. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Massive Gross-Neveu model: a rigorous perturbative construction. Phys. Rev. Lett. **54**, 1479 (1985); A renormalizable field theory: the massive Gross-Neveu model in two dimensions. Commun. Math. Phys. **103**, 67 (1986)
9. Gawedzki, K., Kupiainen, A.: Gross-Neveu model through convergent perturbation expansion. Commun. Math. Phys. **102**, 1 (1985)
10. Bros, J., Iagolnitzer, D., Magnen, J.: (In preparation)

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