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On the Relevance of Some Constructions of Highest Weight Modules Over (Super-) Kac-Moody Algebras (In connection with a paper by Wakimoto)

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Abstract. We correct and improve results of Wakimoto on the (ir)reducibility of his construction of $A_1^{(1)}$ highest weight modules (HWM). For a very large class of (super-) Kac-Moody algebras we argue that such a HWM is most relevant when it is isomorphic to a proper factor-module of the corresponding reducible Verma module with the same highest weight. In the same situation we present a general procedure to check the reducibility of the HWM in consideration.

1.

This note has several purposes. First (in Sect. 2) we correct a statement (Theorem 2) in [1] concerning irreducibility of highest weight modules over $A_1^{(1)}$ constructed there. We also show how to obtain similar theorems for a much larger class of algebras, namely, semisimple Lie algebras, (super-) affine Lie algebras, basic classical Lie super-algebras. For completeness we give the proof of the corrected statement (Theorem A). Next (in Sect. 3) we comment on the relevance of such highest weight module (HWM) constructions in the same general setting. Our point is that unless a physical application is involved a HWM $V(\Lambda)$ with highest weight Λ is of interest when the corresponding Verma module $M(\Lambda)$ is reducible while $V(\Lambda)$ is isomorphic to a proper factor-module of $M(\Lambda)$ and especially when $V(\Lambda)$ is irreducible. We present a general procedure to check the reducibility of $V(\Lambda)$ in the cases when $M(\Lambda)$ is reducible. We apply this procedure to the HWM constructed in [1] and obtain a result (Theorem B) on the reducibility of a large class of these HWM.

2.

In [1] Wakimoto constructed a family of highest weight modules $\pi_{\mu\nu}$ over the affine Lie algebra $A_1^{(1)}$ parametrized by $(\mu, \nu) \in \mathbb{C}^2$. We shall not repeat the construction of the $\pi_{\mu\nu}$ representation spaces $V(\mu, \nu)$ from [1]. We need only the action of the $A_1^{(1)}$ Cartan subalgebra $\mathscr{h}_0$ elements α_0^v, α_1^v on the highest weight vector $\psi_{\mu\nu}$:

$$\pi_{\mu\nu}(\alpha_0^v)\psi_{\mu\nu} = -(1+\mu+\nu^2/2)\psi_{\mu\nu} = \Lambda_{\mu\nu}(\alpha_0^v)\psi_{\mu\nu}, \tag{1a}$$

$$\pi_{\mu\nu}(\alpha_1^v)\psi_{\mu\nu} = (\mu-1)\psi_{\mu\nu} = \Lambda_{\mu\nu}(\alpha_1^v)\psi_{\mu\nu}. \tag{1b}$$

Consequently the action of the central element $\hat{c} = \alpha_0^v + \alpha_1^v$ giving the level of the HWM is

$$\pi_{\mu\nu}(\hat{c})\psi_{\mu\nu} = -(2+\nu^2/2)\psi_{\mu\nu}. \tag{2}$$

Among other things Wakimoto made a statement – Theorem 2 – on the irreducibility of $V(\mu, \nu)$ which contains a mistake. The correct statement is:

Theorem A. *Let μ and ν be complex numbers satisfying the following two conditions:*

i) $\nu \neq 0$;

ii) *for any integer $n \geqq 0$, neither $\mu - n\nu^2/2$ nor $-\mu-(n+1)\nu^2/2$ is a positive integer.*

Then the $A_1^{(1)}$-module $V(\mu, \nu)$ is irreducible and isomorphic to $L(\Lambda_{\mu\nu})$, where $\Lambda_{\mu\nu} = -(1+\mu+\nu^2/2)\Lambda_0 + (\mu-1)\Lambda_1$.

In Theorem 2 of [1] instead of ii) we find:

ii)$_W$ for any integer $n \geqq 0$, neither $\mu - n(2+\nu^2/2)$ nor $-\mu - 2n - (n+1)\nu^2/2$ is a positive integer.

We are motivated to give the proof of Theorem A for two reasons. First, no proof of Theorem 2 of [1] is presented there. The second and more important reason is related to the fact that Theorem A is – as Theorem 2 should have been – an easy corollary of the "if" part of the Kac-Kazhdan [2] criterion for irreducibility of Verma modules over affine Lie algebras. We shall present this fact in a much more general situation.

Let $\mathfrak{G}$ be either a semisimple Lie algebra, or a (super-) affine Lie algebra, or a basic classical Lie superalgebra [3, 4]. Let $V(\Lambda)$ be a HWM over $\mathfrak{G}$ with highest weight Λ. Then $V(\Lambda)$ is isomorphic to $M(\Lambda)/I(\Lambda)$, where $M(\Lambda)$ is the Verma module over $\mathfrak{G}$ with the same highest weight Λ, $I(\Lambda)$ is a submodule of $M(\Lambda)$. [We shall give $I(\Lambda)$ more explicitly in Sect. 3.] Recall also that the irreducible HWM with highest weight Λ which is denoted by $L(\Lambda)$ is isomorphic to $M(\Lambda)/I_m(\Lambda)$, where $I_m(\Lambda)$ is the maximal proper submodule of $M(\Lambda)$ [5, 2, 6]. If $M(\Lambda)$ is irreducible, i.e. $I_m(\Lambda) = \{0\}$, then $V(\Lambda) \cong M(\Lambda) \cong L(\Lambda)$.

Thus if we have constructed a HWM $V(\Lambda)$ over $\mathfrak{G}$ we can immediately state for which Λ it is certainly irreducible using the Verma module irreducibility criterions of Bernstein-Gel'fand-Gel'fand [5] (for semisimple Lie algebras), of Kac-Kazhdan [2] (for affine Lie algebras), and of Kac [6] [1] (for basic classical Lie superalgebras). (A combination of the Kac-Kazhdan and Kac criterions gives the result for super-affine Lie algebras – cf. [7].) We note also that all these criterions have one and the same form [cf. the Kac-Kazhdan one below in formula (3)].

So to finish the proof of Theorem A it remains to translate the Kac-Kazhdan criterion to the situation at hand in order to obtain i) and ii). According to that

[1] This criterion is proved only as a sufficient condition for irreducibility (cf. [6])

criterion the Verma module $M(\Lambda)$ is irreducible if and only if

$$2(\Lambda+\varrho,\beta)\neq k(\beta,\beta) \tag{3}$$

for any positive root β and any positive integer k; $\varrho\in\mathfrak{h}^*$ is defined by $(\varrho,\beta_i)=(\beta_i,\beta_i)/2$ for any simple root β_i.

We recall that any $A_1^{(1)}$ positive root β can be written as

$$\beta=p\alpha_0+(p+\kappa)\alpha_1,\quad p\in\mathbb{Z}_+,\kappa=0,\pm 1,2p+\kappa>0. \tag{4}$$

The roots with $\kappa=0$ are called imaginary roots and those with $\kappa\neq 0$ – real roots; $\alpha_1=\beta|_{p=0,\kappa=1}$ is the sl(2, $\mathbb{C}$) positive root. In this notation and choosing normalization for α_1 we have

$$(\beta,\beta')=\kappa\kappa'(\alpha_1,\alpha_1)=2\kappa\kappa'. \tag{5}$$

Thus we can rewrite (3) as

$$p(\Lambda+\varrho,\alpha_0)+(p+\kappa)(\Lambda+\varrho,\alpha_1)\neq k\kappa^2. \tag{3'}$$

We note [cf. (5)] that $(\varrho,\alpha_0)=(\varrho,\alpha_1)=1$. Writing $\Lambda=r_0\Lambda_0+r_1\Lambda_1$, $\Lambda_i(\alpha_j^\vee)=\delta_{ij}$, we have

$$p(r_0+1)+(p+\kappa)(r_1+1)\neq k\kappa^2\quad\text{or}\quad p(c+2)+\kappa(r_1+1)\neq k\kappa^2, \tag{3''}$$

where the value of the central charge is $c=r_0+r_1$. For $\Lambda=\Lambda_{\mu\nu}$, $r_1=\mu-1$, $c=-(2+\nu^2/2)$, and we have

$$-p\nu^2/2+\mu\kappa\neq k\kappa^2. \tag{3'''}$$

Now in the formulation of Theorem A we first require $\nu\neq 0$; then the case $\kappa=0$ (i.e. imaginary roots) can be dropped because it gives no restriction. Indeed for $\kappa=0$ the resulting condition $p\nu^2\neq 0$ is always fulfilled because of the requirement $2p+\kappa>0$ [cf. (4)]. Thus (3) can finally be rewritten as (for $\kappa=+1,-1$, respectively)

$$-n\nu^2/2+\mu\neq k,\ -(n+1)\nu^2/2-\mu\neq k, n\geqq 0, k\geqq 1, (n=p-\delta_{\kappa,-1}), \tag{3^{IV}}$$

which concludes the proof of Theorem A.

3.

Let $\mathfrak{G}$ and $V(\Lambda)$ be as above. We would like to make the general remark that unless $V(\Lambda)$ has arisen in some physical application it is as relevant as the corresponding Verma module $M(\Lambda)$ if the latter is irreducible [since then $V(\Lambda)\cong M(\Lambda)$]. The HWM $V(\Lambda)$ is of great interest when $M(\Lambda)$ is reducible and $V(\Lambda)\cong L(\Lambda)\cong M(\Lambda)/I_m(\Lambda)$. The case of reducible $V(\Lambda)=M(\Lambda)/I(\Lambda)$ with $\{0\}\neq I(\Lambda)\neq I_m(\Lambda)$ is also of some interest.

Thus in order to appraise a particular HWM construction $V(\Lambda)$ (e.g. $V(\mu,\nu)$ in [1]) one should examine the reducibility of $V(\Lambda)$ if the corresponding Verma module $M(\Lambda)$ is reducible. The general procedure for this is as follows. Let

$$2(\Lambda+\varrho,\beta)=k(\beta,\beta) \tag{6}$$

hold for some positive root β and some $k \in \mathbb{N}$. Then $M(\Lambda)$ is reducible with invariant submodule $I^{\beta}(\Lambda)$ which is isomorphic to $M(\Lambda - k\beta)$ when $\mathfrak{G}$ is a semisimple Lie algebra [5], or an affine Lie algebra [2] [$k=1$ for β imaginary, i.e. $(\beta, \beta)=0$], or a basic classical Lie super-algebra and β or 2β – even root [6]. In the latter case if β is an odd root and 2β is not an even root the submodule $I^{\beta}(\Lambda)$ is isomorphic to an invariant embedding of $M(\Lambda - \beta)$ with kernel which will become clear below. These isomorphisms are realized with the help of the socalled singular vectors. The singular vector $v_\beta \in M(\Lambda)$ differs from the highest weight vector v_0 of $M(\Lambda)$ and has the properties of the highest weight vector of $M(\Lambda - k\beta)$, i.e.

$$X v_\beta = 0, \quad \text{for} \quad X \in \mathfrak{G}^+, \qquad X v_\beta = (\Lambda - k\beta)(X) \cdot v_\beta, \quad \text{for} \quad X \in \mathscr{h}, \tag{7}$$

where we have invoked the standard decomposition $\mathfrak{G} = \mathfrak{G}^+ \oplus \mathscr{h} \oplus \mathfrak{G}^-$, $\mathscr{h}$ is the Cartan subalgebra of $\mathfrak{G}$, $\mathfrak{G}^+$ (respectively $\mathfrak{G}^-$) is comprised from the positive (respectively negative) root spaces. We recall that $M(\Lambda) \cong U(\mathfrak{G}^-)v_0$, where $U(\mathfrak{G}^-)$ is the universal enveloping algebra of $\mathfrak{G}^-$. Then we have [8–12]:

$$I^{\beta}(\Lambda) = U(\mathfrak{G}^-) v_\beta . \tag{8}$$

Furthermore the singular vector v_β can be realized as [8–13]

$$v_\beta = \mathscr{P}_k^{\beta}(f_1, \ldots, f_\ell) v_0 , \tag{9}$$

where $\mathscr{P}_k^{\beta}$ is a homogeneous polynomial of degrees $kn_1, \ldots, kn_\ell$, $n_i \geqq 0$ are from $\beta = \sum_{i=1}^{\ell} n_i \alpha_i$, α_i are the simple roots, $\ell = \operatorname{rank} \mathfrak{G}$; $f_i \in \mathfrak{G}^-$ are the negative canonical generators: $[e_i, f_j] = \delta_{ij} h_i$, $i, j = 1, \ldots, \ell$, $e_i \in \mathfrak{G}^+$, $h_i \in \mathscr{h}$, e_i (respectively f_i) spans the α_i (respectively $-\alpha_i$) root space.

We can explain now the peculiarities for basic classical Lie superalgebras. If β is an odd root and 2β is not an even root then if (6) holds for some highest weight Λ, it holds also for $\Lambda - k\beta$, $k \in \mathbb{Z}$; moreover $M_k \equiv M(\Lambda - k\beta)$ is reducible[2] and is invariantly embedded in M_{k-1}. Thus we have an infinite chain of embedding maps. We can use the Grassmannian properties of the odd generators to show that this infinite chain is exact. Thus the kernel of the embedding of M_k in M_{k-1} is $I^{\beta}(\Lambda - (k+1)\beta)$.

Returning to the general case we note that v_β may be identically zero in some concrete HWM $V(\Lambda)$. More precisely, if we write as above $V(\Lambda) \cong M(\Lambda)/I(\Lambda)$ then we have:

$$I(\Lambda) = \bigoplus_{\beta} U(\mathfrak{G}^-) v_\beta , \tag{10}$$

where the sum is over the positive roots for which (6) holds *and* the corresponding v_β is equal to zero in the HWM $V(\Lambda)$. If v_β is zero for *all* β for which (6) holds, then $I(\Lambda) = I_m(\Lambda)$ and $V(\Lambda) \cong L(\Lambda)$. In particular, the standard (= integrable [3]) representations of the affine Lie algebras are obtained in this way when (6) holds for all simple roots and hence for all positive real roots; however it is enough to

[2] Here we suppose that v_β is not zero which is not proved in this situation although we do not know any counterexample to this conjecture. If it is proved the Kac criterion [6] would become also a necessary condition for irreducibility (cf. [13] for su(2, 2/N))

take the sum in (10) over the simple roots. (The highest weights of the standard representations are dominant integral weights [3].)

We go back to the HWM construction of [1]. The question of reducibility of $V(\mu, \nu)$ is discussed only in the case $\nu=0$ without the procedure outlined above. The author expects that $V(\mu, 0)$ is irreducible for every $\mu \notin \mathbb{Z}\backslash\{0\}$. The latter is true with respect to the real roots. Indeed for $\nu=0$ we have from (3‴)

$$\kappa\mu \neq \kappa^2 k, \quad k \geqq 1, \tag{11a}$$

which for real roots, i.e. $\kappa = \pm 1$ gives

$$\mu \neq \pm k \quad \text{or} \quad \mu \notin \mathbb{Z}\backslash\{0\}. \tag{11b}$$

However one should note that (11 a) can not be fulfilled for imaginary roots, i.e. for $\kappa=0$. This means that the Verma module $M(\Lambda_{\mu 0})$ is reducible with respect to any imaginary root. So we should examine whether $V(\mu, 0)$ is reducible with respect to the imaginary roots.

We start with the simplest imaginary root $\beta=\bar{d}=\alpha_0+\alpha_1$. The corresponding singular vector (cf. (37) of [11]) is:

$$v_d = ((\mu+1) f_0 f_1 + (1-\mu) f_1 f_0) v_0, \ \mu \in \mathbb{C}, \ e_k v_d = 0. \tag{12}$$

We substitute f_0, f_1 with $Y(-1)$, $X(0)$, respectively from [1] and v_0 with $\psi_{\mu 0}=1$ to obtain:

$$v_d(V(\mu, 0)) = 0. \tag{13}$$

Thus we have proven:

Theorem B. *Let* $\nu=0$ *and* $\mu \in \mathbb{C}$. *Then the representation* $V(\mu, 0)$ *of* [1] *is irreducible with respect to the imaginary root* $\beta=\bar{d}$.

We should stress that we do not claim that $V(\mu, 0)$ is irreducible. Since $M(\Lambda_{\mu 0})$ is reducible with respect to any imaginary root $\beta=q\bar{d}=q(\alpha_0+\alpha_1)$, $q \in \mathbb{N}$, one should find the explicit expressions for the singular vectors $v_{qd}=\mathscr{P}_q(f_0, f_1)v_0$, $q \geqq 2$, where $\mathscr{P}_q$ is a homogeneous polynomial of degree q in both f_0 and f_1. $V(\mu, 0)$ will be irreducible if and only if $v_{qd}=0$ for all $q \in \mathbb{N}$. In our opinion this is a very interesting problem.

Note added. We have just read a very interesting paper [14] on the singular vectors of Verma modules over affine Lie algebras. The singular vectors v_β for β real are found for arbitrary Kac-Moody algebras. For imaginary roots v_β are given explicitly only for $A_1^{(1)}$. Thus the problem we posed above may be attacked now more directly.

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