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Parallel Transport in the Determinant Line Bundle: The Non-Zero Index Case

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Abstract. For a product family of Weyl operators of possibly non-zero index on a compact manifold X, we express parallel transport in the determinant line bundle in terms of the spectral asymmetry of a Dirac operator on $\mathbb{R} \times X$. This generalizes the results of [7], where we dealt only with invertible operators.

0. Introduction

Let X be a compact spin manifold of even dimension with spin bundle $S = S_+ \oplus S_- \to X$ and let $E \to X$ be a hermitian vector bundle over X. Let $\bar{S}$ and $\bar{E}$ be the pullbacks of S and E to $\mathbb{R} \times X$ with the induced inner products and let $\bar{\nabla}^E$ be a connection on $\bar{E}$. Thus $\bar{\nabla}^E = d_{\mathbb{R}} + \theta + \nabla^E_{(\cdot)}$, where $\theta \in \Omega^1(\mathbb{R}) \otimes C^\infty(X, \operatorname{End} E)$ and for each $y \in \mathbb{R}$, ∇^E_y is a connection on $E \to X$. Let ∂_y be the Weyl operators $\partial_y : L^2(X, S_+ \otimes E) \mapsto L^2(X, S_- \otimes E)$ coupled to the connection ∇^E_y and the (y-independent) metric on X.

The constructions of [5] applied to these data yield a smooth determinant line bundle $\mathscr{L}$ over $\mathbb{R}$ with a natural hermitian metric and compatible connection. If index $\partial_y = 0$, $\mathscr{L}$ has a canonical section. In [8] we assumed that for all y, $\operatorname{Ker} \partial_y = 0$ and $\operatorname{Ker} \partial_y^\dagger = 0$, and we gave a formula expressing parallel transport in $\mathscr{L}$ in terms of this section and the spectral asymmetry $\eta(H)$ of the formally self-adjoint Dirac operator H on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ coupled to the connection $\bar{\nabla}^E$ and the product metric on $\mathbb{R} \times X$.

In this paper we investigate parallel transport in the case that index ∂_y is not necessarily zero. We continue to assume that $\operatorname{Ker} \partial_y = 0$, but now weaken the assumption $\operatorname{Ker} \partial_y^\dagger = 0$ by assuming only that there exists a $V_- \subset L^2(X, S_- \otimes E)$ which is a complement to $\operatorname{Ker} \partial_y^\dagger$ for all y. Let $V_-^\perp$ be the orthogonal complement of V_- viewed as a trivial sub-bundle of the Hilbert bundle $\mathscr{H}_- = \mathbb{R} \times L^2(X, S_- \times E)$, and give V_- the connection induced by orthogonal projection

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from the connection $d_{\mathbb{R}} + \theta$ on $\mathscr{H}_-$. Let $\bar{V}$ be the translationally invariant subspace of $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ corresponding to $V = L^2(X, S_+ \otimes E) \oplus V_-$ and let $H|_{\bar{V}}$ be the operator on $\bar{V}$ obtained by composing H with orthogonal projection onto $\bar{V}$.

Since $\operatorname{Ker} \partial_y = 0$, there is a natural isomorphism of DET KER $\partial^\dagger$ with $\mathscr{L}$, where DET denotes the highest exterior power. In addition, orthogonal projection in $\mathscr{H}_-$ defines an isomorphism of DET $V_-^\perp$ with DET KER $\partial^\dagger$. Using these isomorphisms, we will express parallel transport in $\mathscr{L}$ in terms of parallel transport in $V^\perp$ and the spectral asymmetry $\eta(H|_{\bar{V}})$ of $H|_{\bar{V}}$.

The organization of this paper is as follows. In Sect. 1 we clarify the geometric setting and state our main results. In Sect. 2 we calculate the pull-backs of the inner product and connection of $\mathscr{L}$ to $V_-^\perp$. In Sect. 3 we define $\eta(H|_{\bar{V}})$, and in Sect. 4 we give a formula for the variation $\eta(H|_{\bar{V}})$ as $H|_{\bar{V}}$ is varied. Finally in Sect. 5 we use the results of Sects. 2 and 4 to prove our formulas for the curvature of $\mathscr{L}$ and parallel transport in $\mathscr{L}$.

In Appendix A we investigate the resolvent of $H|_{\bar{V}}$ and give some details of the arguments of Sects. 3 and 4. In Appendix B we present our notational conventions.

1. Statement of Results

In this section we state the main results of the paper. We work within the framework developed by Bismut and Freed [5, 6] for studying the determinant line bundle of a family of Weyl operators.

We are interested in the special case of the Bismut–Freed setting in which the geometric data have a product structure. The parameter space for our family of operators is a smooth manifold Y. When we consider parallel transport it will suffice to let Y be the real line $Y = \mathbb{R}$. Let X be a compact spin manifold of even dimension, and let $Z = Y \times X$, which we view as fibered over Y with fiber X and tangent space along the fibers $T^{\text{vert}}Z = Y \times TX$. Put a metric on X and the corresponding y-independent inner product on $T^{\text{vert}}Z$. Let $\bar{S}_\pm \to Z$ be the spin bundles associated to $T^{\text{vert}}Z$, so that $\bar{S}_\pm = Y \times S_\pm$, where $S_\pm \to X$ are the spin bundles on X. Let $\bar{E} \to Z$ be a complex vector bundle over Z which is of the form $\bar{E} = Y \times E$ for $E \to X$ a vector bundle over X. Put an Hermitian inner product on E and the corresponding y-independent inner product on $\bar{E}$. Let $\nabla^{\bar{E}}$ be a compatible connection on $\bar{E}$. Finally, choose the projection $P: TZ \mapsto T^{\text{vert}}Z = Y \times TX$ of the Bismut–Freed data to be given by the product structure.[1]

The constructions of Bismut and Freed applied to these data now yield a Hilbert bundle $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- \to Y$ with natural inner product and natural connection $\nabla^{\mathscr{H}}$, a bundle map $\partial: \mathscr{H}_+ \mapsto \mathscr{H}_-$ which is given by a Weyl operator on each fiber, and a determinant line bundle $\mathscr{L} \to Y$ with an inner product and compatible connection $\nabla^{(\mathscr{L})}$. If $\dim \operatorname{Ker} \partial_y$ is a locally constant function on Y, then KER ∂ and KER $\partial^\dagger$ are finite dimensional subbundles of $\mathscr{H}_+$ and $\mathscr{H}_-$, and there

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1 Note that for arbitrary geometric data as in [5], the space Z and the bundles $T^{\text{vert}} Z$, $\bar{S}$, $\bar{E}$ are always products locally over Y. However, our form for the inner product on $T^{\text{vert}} Z$ and the projection P do not hold in general, even locally

is a natural isomorphism of the bundles,

$$\Phi:(\mathrm{DET\ KER}\,\partial)^*\otimes(\mathrm{DET\ KER}\,\partial^\dagger)\xrightarrow{\sim}\mathscr{L}, \tag{1.1}$$

where DET denotes the highest exterior power.

In our case we can describe these structures explicitly. Using the product structure $\bar{E}=Y\times E$, write

$$\nabla^{\bar{E}}=d_Y+\theta+\nabla^E_{(\cdot)}, \tag{1.2}$$

where $\theta\in\Omega^1(Y)\otimes C^\infty(X,\mathrm{End}\,E)$, and for each $y\in Y,\nabla^E_y$ is a connection on $E\to X$. The Hilbert bundle $\mathscr{H}$ is trivial with $\mathscr{H}_\pm=Y\times L^2(X,S_\pm\otimes E)$. The inner product on $\mathscr{H}$ is y-independent and given by the inner product on $L^2(X,S\otimes E)$. The connection $\nabla^{\mathscr{H}}$ on $\mathscr{H}$ is given by $\nabla^{\mathscr{H}}=d_Y+\theta$. The operator ∂_y is identified with the Weyl operator $\partial_y:L^2(X,S_+\otimes E)\to L^2(X,S_-\otimes E)$ coupled to the inner product on TX and the connection ∇^E_y. The covariant derivative of ∂ as a section of $\mathrm{Hom}(\mathscr{H}_+,\mathscr{H}_-)$ is given by $\nabla^{\mathrm{Hom}(\mathscr{H}_+,\mathscr{H}_-)}\partial=d_Y\partial+[\theta,\partial]$. (Since the metric on X does not vary with $Y, d_Y\partial$ is in fact a zero-th order operator along the fibers: $d_Y\partial=c(d_Y\nabla^E)$, where c is induced by Clifford multiplication $T^*X\mapsto\mathrm{Hom}(S_+,S_-)$.)

Finally, if dim Ker ∂_y is locally constant, the inner product and connection on $\mathscr{L}$ satisfy

$$\Phi^*\nabla^{(\mathscr{L})}=\nabla^{(\mathrm{DET\,KER}\,\partial)^*\otimes(\mathrm{DET\,KER}\,\partial^\dagger)}+\omega, \tag{1.3}$$

$$\Phi^*\|\cdot\|^2_{\mathscr{L}}=\|\cdot\|^2_{(\mathrm{DET\,KER}\,\partial)^*\otimes(\mathrm{DET\,KER}\,\partial^\dagger)}\det\partial^\dagger\partial, \tag{1.4}$$

$$\omega=\{\text{f.p.a.z.}=0\}\,\mathrm{Tr}\,(\partial^\dagger\partial)^{-z-1}\partial^\dagger\nabla^{\mathrm{Hom}(\mathscr{H}_+,\mathscr{H}_-)}\partial, \tag{1.5}$$

$$\det\partial^\dagger\partial=\exp-\lim_{z\to0}\frac{d}{dz}\mathrm{Tr}\,(\partial^\dagger\partial)^{-z}. \tag{1.6}$$

Here the bundles KER ∂ and KER $\partial^\dagger$ have the inner products they inherit as sub-bundles of $\mathscr{H}_-$, and the compatible connections obtained by orthogonal projection from the connection on $\mathscr{H}_-$. The complex powers of $\partial^\dagger\partial$ are defined by contour integration. The notation $\lim_{z\to0}$ (respectively $\{\text{f.p.a.z}=0\}$) is understood as the value at $z=0$ (respectively the finite part at $z=0$) of the meromorphic continuation of a function which is analytic for $\mathrm{Re}\,z\gg0$. We will give the precise interpretation of these expressions in the next section.

Let $D:\mathscr{H}\mapsto\mathscr{H}$ be the family of formally self-adjoint Dirac operator corresponding to ∂, so that D_y is the operator on $L^2(X,S\otimes E)$ which relative to the decomposition $S=S_+\oplus S_-$ is given by $D_y=\begin{pmatrix} & \partial_y^\dagger\\ \partial_y & \end{pmatrix}$. Then (see Sect. 2) ω can be written

$$\omega=\tfrac{1}{2}\mathrm{d}\ln\det\partial^\dagger\partial+\sigma,\quad \sigma=\{\text{f.p.a.z}=0\}\tfrac{1}{2}\mathrm{Tr}_s(D^2)^{-z-1}D\nabla D, \tag{1.7}$$

Here Tr_s denotes the super-trace given $\mathrm{Tr}_s=\mathrm{Tr}\circ\Gamma$, where Γ is the endomorphism of S with $\Gamma=\pm1$ on $S_\pm$.

Henceforth assume

1a. $\operatorname{Ker}\partial_y = 0$ *for all* $y \in Y$.
1b. *There exists a closed subspace* $V_- \subset L^2(X, S_- \otimes E)$ *which is a complement of* $\operatorname{Ker}\partial_y^\dagger$ *for all* $y \in Y$.

Given condition 1a, condition 1b will always hold locally on Y, since near $y_0 \in Y$ we can take $V_- = \operatorname{Im}\partial_{y_0}^\dagger$. Let $V_-^\perp$ denote the orthogonal complement in $L^2(X, S_- \otimes E)$ of V_-, and, for convenience, let $V_+ = L^2(X, S_+ \otimes E)$ and $V_- \oplus V_+$.

We will also denote by V_- and $V_-^\perp$ the corresponding trivial sub-bundles of $\mathscr{H}_-$. Then 1b is equivalent to $\mathscr{H}_- = V_- \oplus \operatorname{KER}\partial^\dagger$. We give these bundles the induced inner products and the compatible connections obtained by orthogonal projection from the connection on $\mathscr{H}_-$.

Let $D|_V : V \mapsto V$ be the family of operators on V induced by PDP, where P is orthogonal projection in L^2 onto V. Conditions 1a and 1b imply that $(D|_V)_y$ is invertible for all y. Define

$$\sigma_0 = \lim_{z \to 0} \tfrac{1}{2} \operatorname{Tr}_s (D|_V^2)^{-z-1} D|_V \nabla (D|_V), \tag{1.8}$$

where $\nabla D|_V = P(d_Y D|_V + [\theta, D|_V])$.

Let $Q : V_-^\perp \mapsto \operatorname{KER}\partial^\dagger$ be the bundle isomorphism which on the fiber over $y \in Y$ is induced by orthogonal projection in $L^2(X, S_- \otimes E)$ onto $\operatorname{KER}\partial_y^\dagger$. There is a corresponding isomorphism of line bundles

$$\operatorname{DET} V_-^\perp \xrightarrow{\operatorname{Det} Q} \operatorname{DET}\operatorname{KER}\partial^\dagger \xrightarrow{\Phi} \mathscr{L}. \tag{1.9}$$

Our approach will be to compare $\nabla^{(\mathscr{L})}$ and $\nabla^{\operatorname{DET} V_-^\perp}$ using this isomorphism. Note that the connection on $\operatorname{DET} V_-^\perp$ is just

$$\nabla^{\operatorname{DET} V_-^\perp} = d_Y + \operatorname{Tr}(1 - P)\theta. \tag{1.10}$$

Our first result is:

Proposition (1.11). *Assuming* 1a *and* 1b,

$$(\Phi \circ \operatorname{Det} Q)^* \nabla^{(\mathscr{L})} = \nabla^{\operatorname{DET} V_-^\perp} + \tfrac{1}{2} d \ln \|\operatorname{Det} Q\|^2 \det \partial^\dagger \partial + \sigma_0,$$
$$(\Phi \circ \operatorname{Det} Q)^* \|\cdot\|_{\mathscr{L}}^2 = \|\cdot\|_{\operatorname{DET} V_-^\perp}^2 \|\operatorname{Det} Q\|^2 \det \partial^\dagger \partial.$$

Here the norm of $\operatorname{Det} Q$ *is taken relative to the induced inner product on* $\operatorname{Hom}(\operatorname{DET} V_-^\perp, \operatorname{DET}\operatorname{KER}\partial^\dagger)$.

Using the expression for $\nabla^{(\mathscr{L})}$ given in Proposition (1.11) we next verify in our special case the curvature formula of Bismut and Freed ([15], Theorem 3.5):

Theorem (1.12). *Assuming* 1a *and* 1b, *the curvature of* $\nabla^{(\mathscr{L})}$ *is given by the two-form on* Y

$$\mathscr{F}^{\mathscr{L}} = 2\pi i \Big[\int_X \hat{A}(\mathscr{R}_g) \operatorname{ch}(\mathscr{F}_{\nabla^{\bar{E}}}) \Big]_{\{\text{two form}\}}. \tag{1.13}$$

Here g is the (Y-independent) metric on X, $\mathscr{R}_g$ is the curvature of the Levi–Civita connection of g, and $\mathscr{F}_{\nabla^{\bar{E}}}$ is the curvature of the connection $\nabla^{\bar{E}}$ on $\bar{E}$. $\hat{A}$ and ch are the polynomials

$$\hat{A}(\mathscr{R}) = \sqrt{\det\left(\frac{\mathscr{R}/4\pi}{\sinh \mathscr{R}/4\pi}\right)} \quad \mathrm{ch}(\mathscr{F}) = \mathrm{tr}\exp i\mathscr{F}/2\pi. \tag{1.14}$$

Now suppose $Y = \mathbb{R}$. Give Y the standard translation invariant metric $dy \otimes dy$ and give $Z = \mathbb{R} \times X$ the product metric. Then $\bar{S} \to Z$ is identified with the spin bundle of Z. Let $\bar{V}$ be the translation invariant subspace of $L^2(Z, \bar{S} \otimes \bar{E})$ determined by V.

Let H be the formally self-adjoint Dirac operator on $L^2(Z, \bar{S} \otimes \bar{E})$ coupled to the metric on Z and the connection $\nabla^{\bar{E}}$ on $\bar{E}$. In terms of the product structure $\bar{S} \otimes \bar{E} = Y \times (S \otimes E)$,

$$H = i\Gamma\left(\frac{\partial}{\partial y} + \theta\left(\frac{\partial}{\partial y}\right)\right) + D_{(\cdot)} = \left\{\begin{matrix} i\left[\frac{\partial}{\partial y} + \theta\left(\frac{\partial}{\partial y}\right)\right] & \partial^\dagger \\ \partial & -i\left[\frac{\partial}{\partial y} + \theta\left(\frac{\partial}{\partial y}\right)\right] \end{matrix}\right\}. \tag{1.15}$$

Let $H|_{\bar{V}}$ be the operator on $\bar{V}$ obtained by composing H with orthogonal projection onto $\bar{V}$.

In addition to 1a and 1b assume

2. *For* $|y| > 1$, $\theta = 0$ *and* $d\nabla^E/dy = 0$.

Thus for $|y| > 1$, $H|_{\bar{V}}$ is invariant under translations in the $\mathbb{R}$ direction.

Define

$$\eta(H|_{\bar{V}}) = \lim_{z \to 0} \lim_{\phi \to 1} \mathrm{Tr}\, \Phi H|_{\bar{V}} (H|_{\bar{V}}^2)^{-z-1/2}, \tag{1.16}$$

$$\xi(H|_{\bar{V}}) = \tfrac{1}{2}(\eta(H|_{\bar{V}}) + \dim \mathrm{Ker}\, H|_{\bar{V}}). \tag{1.17}$$

Here ϕ is a nonnegative smooth function on $\mathbb{R}$ of compact support acting as a multiplication operator on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$, and the limit $\phi \to 1$ is taken through a sequence of such ϕ increasing pointwise to the constant function 1. We introduce these cut-off functions in order to obtain trace class operators. The complex powers of $H|_{\bar{V}}^2$ are defined by contour integration and $\lim_{z \to 0}$ is understood in terms of analytic continuation. We will give a precise interpretation of (1.16) and (1.17) in Sect. 4.

Our main result is

Theorem (1.18). *Assuming* 1a, 1b, *and* 2, *the parallel transports* $\tau^{\mathscr{L}}_{\infty,-\infty}$ *and* $\tau^{\mathrm{DET}V^\perp_-}_{\infty,-\infty}$ *in* $\mathscr{L}$ *and* $\mathrm{DET}\, V^\perp_-$ *respectively from* $-\infty$ *to* ∞ *are related by*

$$\tau^{\mathscr{L}}_{\infty,-\infty} \circ \left(\frac{\Phi \circ \mathrm{Det}\, Q}{(\det \partial^\dagger \partial)^{1/2} \|\mathrm{Det}\, Q\|}\right)_{-\infty} = \left(\frac{\Phi \circ \mathrm{Det}\, Q}{(\det \partial^\dagger \partial)^{1/2} \|\mathrm{Det}\, Q\|}\right)_{+\infty} \circ \tau^{\mathrm{DET}V^\perp_-}_{\infty,-\infty} \exp - \int_{\mathbb{R}} \sigma_0$$

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \sigma_0 = \xi(H|_{\bar{V}}) \mod 1.$$

We remark that the isomorphism $\mathrm{Det}\, Q$ enters into Proposition (1.11) and Theorem (1.18) only through the combination $(\mathrm{Det}\, Q)/\|\mathrm{Det}\, Q\|$. In fact, as we indicate at the end of Sect. 2, although there are, in addition to $\mathrm{Det}\, Q$, various other natural isomorphisms of DET $V_-^\perp$ with DET $\partial^\dagger$ determined by the available data (i.e. the inner product on $\mathscr{H}_-$), these are all rescalings of $\operatorname{Det} Q$ by *positive* functions on Y (see (2.38) below). Thus a different choice of isomorphism will not affect Theorem (1.18).

2. The Inner Product and Connection on $\mathscr{L}$

In this section we calculate the pull-backs to DET $V_-^\perp$ of the connection and curvature of $\mathscr{L}$, and thus verify Proposition (1.11). We allow the parameter space Y to be an arbitrary smooth manifold and assume that conditions 1a and 1b of the previous section hold.

Let P_- be the orthogonal projection in $L^2(X, S_- \otimes E)$ onto V_-. Let $\partial' = P_- \partial$ and $\partial'^\dagger = \partial^\dagger P_-$. Let $\{\psi^i\}$ and $\{\chi^i\}$, $i = 1, 2, \ldots, n = \dim \operatorname{Ker} \partial^\dagger$, be smooth bases for KER $\partial^\dagger$ and $V_-^\perp$ respectively.

Define the following functions and one-forms on Y:

$$\det \partial^\dagger \partial = \exp - \lim_{z \to 0} \frac{d}{dz} \operatorname{Tr} (\partial^\dagger \partial)^{-z}, \tag{2.1}$$

$$\omega^\theta = \{\text{f.p.a.z} = 0\} \operatorname{Tr} (\partial^\dagger \partial)^{-z-1} \partial^\dagger (\nabla \partial), \tag{2.2}$$

$$\sigma^\theta = \{\text{f.p.a.z} = 0\} \tfrac{1}{2} \operatorname{Tr} (\partial^\dagger \partial)^{-z-1} (\partial^\dagger (\nabla \partial) - (\nabla \partial^\dagger) \partial), \tag{2.3}$$

$$\sigma_0^\theta = \{\text{f.p.a.z} = 0\} \tfrac{1}{2} \operatorname{Tr} (\partial'^\dagger \partial')^{-z-1} (\partial'^\dagger (\nabla \partial') - (\nabla \partial'^\dagger) \partial'), \tag{2.4}$$

$$L^\theta = \lim_{z \to 0} \operatorname{Tr} (\partial \partial^\dagger)^{-z} \theta + \operatorname{Tr} (1 - \partial (\partial^\dagger \partial)^{-1} \partial^\dagger) \theta - \lim_{z \to 0} \operatorname{Tr} (\partial^\dagger \partial)^{-z} \theta, \tag{2.5}$$

$$\begin{aligned} z_0^\theta = {} & \operatorname{tr} (\langle \psi, \chi \rangle^{-1} \langle \nabla \psi, \chi \rangle - \langle \psi, \psi \rangle^{-1} \langle \nabla \psi, \psi \rangle) \\ & + \operatorname{tr} (\langle \psi, \chi \rangle^{-1} \langle \psi, \nabla \chi \rangle - \langle \chi, \chi \rangle^{-1} \langle \chi, \nabla \chi \rangle), \end{aligned} \tag{2.6}$$

$$h_0 = |\det \langle \psi, \chi \rangle|^2 / (\det \langle \psi, \psi \rangle \det \langle \chi, \chi \rangle). \tag{2.7}$$

In (2.6) and (2.7) we have suppressed the indices on ψ and χ, so that, for example, $\langle \psi, \chi \rangle$ denotes the matrix with entries $\langle \psi, \chi \rangle^{ij} = \langle \psi^i, \chi^j \rangle$.

The operator complex powers appearing in (2.1)–(2.5) are defined by contour integration as we now briefly describe (see [13]) using $(\partial_y \partial_y^\dagger)^{-z}$ as an example. Since $\partial_y \partial_y^\dagger$ is a formally self-adjoint differential operator with strictly positive leading symbol, and since X is compact, the spectrum of $\partial_y \partial_y^\dagger$ consists of isolated points in $[0, \infty)$. In particular, there is a $\delta > 0$ such that the disk $|\lambda| < 2\delta$ intersects the spectrum at most at $\lambda = 0$. Let $\mathscr{C}$ be the clockwise oriented curve in $\mathbb{C}$ which runs from $-\infty$ to $-\delta$ directly above the negative real axis, then clockwise around the circle $|\lambda| = \delta$, and then back to $-\infty$ directly below the negative real axis. Then for $z \in \mathbb{C}$, $\operatorname{Re} z > 0$, define

$$(\partial_y \partial_y^\dagger)^{-z} = - \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} (\partial_y \partial_y^\dagger - \lambda)^{-1}, \tag{2.8}$$

where λ^{-z} is given in terms of the branch of the logarithm with cut along $[0, \infty)$ and $\log(1) = 0$. It is easy to check that (2.8) is independent of the choice of δ.

We use this definition even though $\partial_y \partial_y^\dagger$ is not invertible. Thus the zero eigenvalue must be handled separately, and some care is required in formal manipulations.

Now by the techniques of Seeley [13] as discussed in Appendix C of [8], it can be seen that in the definitions (2.1)–(2.5), the traces involving the operator complex powers are well defined and analytic for $\operatorname{Re} z$ sufficiently large. Moreover, these traces extend to meromorphic functions of z for $\operatorname{Re} z > -1$ whose only singularities are simple poles at half-integer values of z. We have used the notation $\{\text{f.p.a.z} = 0\}$ and $\lim_{z \to 0}$ to indicate the finite part at $z = 0$ of these meromorphic continuations. In particular, $\{\text{f.p.a.z} = 0\}$ means that there is a potential pole at $z = 0$, while $\lim_{z \to 0}$ means that there is no pole.

The remaining operator trace in the definition of L^θ is actually finite dimensional since $1 - \partial_y(\partial_y^\dagger \partial_y)^{-1} \partial_y^\dagger$ is the orthogonal projection onto $\operatorname{Ker} \partial_y^\dagger$.

The main results of this section are following three Propositions.

Proposition (2.9).

1. $\| \Phi \|^2 = \det \partial^\dagger \partial$.
2. $\nabla \Phi = \omega^\theta \Phi$.
3. $\| \operatorname{Det} Q \|^2 = h_0$.
4. $\nabla \operatorname{Det} Q = z_0^\theta \operatorname{Det} Q$.

Here we view $\operatorname{Det} Q$ as a section of $\operatorname{Hom}(\operatorname{DET} V_-^\perp, \operatorname{DET} \operatorname{KER} \partial^\dagger)$ and Φ as a section of $\operatorname{Hom}(\operatorname{DET} \operatorname{KER} \partial^\dagger, \mathscr{L})$. The notation $\| \cdot \|$ and ∇ without super- or sub-scripts refers to the induced inner products and connections on these bundles.

Proposition (2.10).

1. $\operatorname{Re} \omega^\theta = \frac{1}{2} d \ln \det \partial^\dagger \partial$.
2. $\operatorname{Im} \omega^\theta = \sigma^\theta$.
3. $\operatorname{Re} z_0^\theta = \frac{1}{2} d \ln h_0$.
4. $\operatorname{Im} z_0^\theta + \sigma^\theta = \sigma_0^\theta = \sigma_0^{\theta = 0} + L^\theta - \operatorname{Tr}(1 - P_-)\theta$.

Proposition (2.11).

$$L^\theta = -\int_X \hat{A}(\mathscr{R}_g) \operatorname{tr} \theta \exp i \mathscr{F}_\nabla^E / 2\pi .$$

Here $\mathscr{R}_g$ is the curvature of the Levi–Civita connection of the metric g on X, $\mathscr{F}_{\nabla^E}$ is the curvature of ∇^E, and $\hat{A}$ and ch are the polynomials given in (1.14).

Proposition (1.11) follows immediately from Propositions (2.9) and (2.10). In addition, since the connection on $\operatorname{DET} V_-^\perp$ is $d_Y + \operatorname{Tr}(1 - P_-)\theta$, it also follows that

$$\begin{aligned} (\Phi \circ \operatorname{Det} Q)^* \nabla^{(\mathscr{L})} &= \nabla^{\operatorname{DET} V_-^\perp} + \tfrac{1}{2} d \ln \| \operatorname{Det} Q \|^2 \det \partial^\dagger \partial + \sigma_0^\theta \\ &= d_Y + \tfrac{1}{2} d \ln \| \operatorname{Det} Q \|^2 \det \partial^\dagger \partial + \sigma_0^{\theta = 0} + L^\theta . \end{aligned} \tag{2.12}$$

We will use this in Sect. 5 to compute the curvature of $\nabla^{(\mathscr{L})}$.

We now turn to the proofs of Propositions (2.9)–(2.11).

Proof of Proposition (2.11). Observe that

$$L^\theta = \lim_{z \to 0} - \operatorname{Tr}_s \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} O(\lambda) - \operatorname{Tr}_s \int_{\mathscr{O}} \frac{d\lambda}{2\pi i} O(\lambda), \quad O(\lambda) = (D^2 - \lambda)^{-1} D \theta . \tag{2.13}$$

Thus by the results of Appendix C of [8], L^θ is expressible as an integral over X of a density which in any local coordinate system is given by a universal polynomial in the components of the complete symbol of $O(\lambda)$. The evaluation of this density, which is in any case well known, was discussed i.. the proof of Proposition (4.12) in [8]. □

Proposition (2.9) and the first three statements of Proposition (2.10) are straightforward, so it remains to prove statement (2.10.4). We first prove

Lemma (2.14)

$$\mathrm{tr}(\langle\psi,\chi\rangle^{-1}\langle\nabla\psi,\chi\rangle-\langle\psi,\psi\rangle^{-1}\langle\nabla\psi,\psi\rangle)=\mathrm{Tr}((\partial^\dagger P_-\partial)^{-1}\partial^\dagger P_- -(\partial^\dagger\partial)^{-1}\partial^\dagger)\nabla\partial.$$

Corollary (2.15)

$$z_0^\theta=\mathrm{Tr}((\partial^\dagger P_-\partial)^{-1}\partial^\dagger P_- -(\partial^\dagger\partial)^{-1}\partial^\dagger)\nabla\partial+\mathrm{Tr}((\partial^\dagger P_-\partial)^{-1}\partial^\dagger P_-(\nabla P_-)\partial).$$

Note that the traces appearing on the right-hand sides of these expressions are actually finite dimensional. In fact $((\partial_y^\dagger P_-\partial_y)^{-1}\partial_y^\dagger P_- -(\partial_y^\dagger\partial_y)^{-1}\partial_y^\dagger)$ vanishes on $\mathrm{Im}\,\partial_y$ while $P_-(\nabla P_-)=(\nabla P_-)(1-P_-)$ vanishes on $\mathrm{Im}\,P_-$, and both these spaces have finite codimension in $L^2(X,S_-\otimes E)$.

Proof of Lemma (*2.14*). We use the trick of varying both sides with respect to P_-. Fix $y\in Y$ and let v be a tangent vector to Y at y. Let $\tilde P_-(t), t\in[0,1]$, be a smooth family of orthogonal projections on $L^2(X,S_-\otimes E)$ such that $\mathrm{Ker}\,P_-(0)=\mathrm{Ker}\,\partial_y^\dagger$, $\tilde P_-(1)=P_-$, and for all $t\in[0,1]$, $\mathrm{Ker}\,\tilde P_-(t)$ is a complement of $\mathrm{Ker}\,\partial_y^\dagger$. Let $\{\tilde\chi^i(t)\}$, $i=1,2,\ldots,\dim\mathrm{Ker}\,\partial_y^\dagger$, $t\in[0,1]$, be a smooth family of frames in $L^2(X,S_-\otimes E)$ such that $\{\tilde\chi^i(t)\}$ is a basis for $\mathrm{Ker}\,\tilde P_-(t)$. Consider the functions of t

$$A(t)=\mathrm{tr}(\langle\psi_y,\tilde\chi(t)\rangle^{-1}\langle\nabla_v\psi,\tilde\chi(t)\rangle-\langle\psi_y,\psi_y\rangle^{-1}\langle\nabla_v\psi,\psi_y\rangle), \tag{2.16}$$

$$B(t)=\mathrm{Tr}((\partial_y^\dagger\tilde P_-(t)\partial_y)^{-1}\partial_y^\dagger\tilde P_-(t)-(\partial_y^\dagger\partial_y)^{-1}\partial_y^\dagger)\Delta_v\partial. \tag{2.17}$$

Clearly $A(0)=B(0)=0$, while the assertion of the Proposition is equivalent to $A(1)=B(1)$. Thus it suffices to check that $dA/dt=dB/dt$.

For $t\in[0,1]$ let $\tilde P_\partial(t)$ be the (non-orthogonal) projection operator on $L^2(X,S_-\otimes E)$ with kernel $\mathrm{Ker}\,\tilde P_-(t)$ and image $\mathrm{Im}\,\partial_y$. Thus

$$\tilde P_\partial(t)=\partial_y(\partial_y^\dagger\tilde P_-(t)\partial_y)^{-1}\partial_y^\dagger\tilde P_-(t)=1-\sum_{i,j}\tilde\chi^i(t)(\langle\psi_y,\tilde\chi(t)\rangle^{-1})^{ij}\langle\psi_y^j,\cdot\rangle, \tag{2.18}$$

and for any linear operator A on $L^2(X,S_-\otimes E)$,

$$\mathrm{Tr}(1-\tilde P_\partial(t))A=\mathrm{tr}\langle\psi_y,\tilde\chi(t)\rangle^{-1}\langle\psi_y,A\tilde\chi(t)\rangle. \tag{2.19}$$

Observe that

$$\begin{aligned}\frac{dA}{dt}&=\mathrm{tr}\left(\langle\psi_y,\tilde\chi\rangle^{-1}\left\langle\nabla_v\psi,\frac{d\tilde\chi}{dt}\right\rangle-\langle\psi_y,\tilde\chi\rangle^{-1}\left\langle\psi_y,\frac{d\tilde\chi}{dt}\right\rangle\langle\psi_y,\tilde\chi\rangle^{-1}\langle\nabla_v\psi,\tilde\chi\rangle\right)\\&=\mathrm{tr}\left(\langle\psi_y,\tilde\chi\rangle^{-1}\left\langle\nabla_v\psi,\tilde P_\partial\right\rangle\frac{d\tilde\chi}{dt}\right),\end{aligned} \tag{2.20}$$

$$\frac{dB}{dt} = \mathrm{Tr}\left((\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}_-}{dt} - (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}_-}{dt} \partial_y (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \tilde{P}_- \right) \nabla_v \partial$$

$$= \mathrm{Tr}\, (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}}{dt} (1 - \tilde{P}_\partial) \nabla_v \partial. \tag{2.21}$$

On the other hand,

$$(\nabla_v \partial^\dagger) \psi_y + \partial_y^\dagger (\nabla_v \psi) = 0, \tag{2.22}$$

$$\partial_y (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial^\dagger \frac{d\tilde{P}_-}{dt} \tilde{\chi} + (1 - \tilde{P}_\partial) \frac{d\tilde{\chi}}{dt} = 0, \tag{2.23}$$

which follow by applying ∇_v and (d/dt) respectively to the equations $\partial^\dagger \psi = 0$ and $\partial_y^\dagger \tilde{P}_- \tilde{\chi} = 0$. Using (2.22), (2.23), in (2.20) we obtain

$$\frac{dA}{dt} = -\,\mathrm{tr} \langle \psi_y, \tilde{\chi} \rangle^{-1} \left\langle \nabla_v \psi, \partial_y (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}_-}{dt} \tilde{\chi} \right\rangle$$

$$= -\,\mathrm{tr} \langle \psi_y, \tilde{\chi} \rangle^{-1} \left\langle \partial_y^\dagger \nabla_y \psi, (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial^\dagger \frac{d\tilde{P}_-}{dt} \tilde{\chi} \right\rangle$$

$$= +\,\mathrm{tr} \langle \psi_y, \tilde{\chi} \rangle^{-1} \left\langle (\nabla_v \partial^\dagger) \psi_y, (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}_-}{dt} \tilde{\chi} \right\rangle$$

$$= \mathrm{tr} \langle \psi_y, \tilde{\chi} \rangle^{-1} \left\langle \psi_y, (\nabla_v \partial) (\partial_y^\dagger \tilde{P}_- \partial_y)^{-1} \partial_y^\dagger \frac{d\tilde{P}_-}{dt} \tilde{\chi} \right\rangle, \tag{2.24}$$

which by (2.19) equals the right-hand side (2.21). □

Proof of Statement (*2.10.4*). We use the previous corollary and Proposition (C6) of Appendix C of [8].

To prove the first equality, observe that

$$\sigma_0^\theta - \sigma^\theta = \mathrm{Im}\, \{\text{f.p.a.} z = 0\}\, \mathrm{Tr}\, ((\partial'^\dagger \partial')^{-z-1} \partial'^\dagger (\nabla \partial') - (\partial^\dagger \partial)^{-z-1} \partial^\dagger (\nabla \partial)), \tag{2.25}$$

$$= \mathrm{Im}\, \{\text{f.p.a.} z = 0\}\, \mathrm{Tr}\, ((\partial'^\dagger \partial')^{-z-1} \partial'^\dagger - (\partial^\dagger \partial)^{-z-1} \partial^\dagger) \nabla \partial$$

$$+ \mathrm{Im}\, \{\text{f.p.a.} z = 0\}\, \mathrm{Tr}\, (\partial'^\dagger \partial')^{-z-1} \partial'^\dagger (\nabla P_-) \partial, \tag{2.26}$$

$$= -\,\mathrm{Im}\, \{\text{f.p.a.} z = 0\}\, \mathrm{Tr} \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} O_1(\lambda)$$

$$- \mathrm{Im}\, \{\text{f.p.a.} z = 0\}\, \mathrm{Tr} \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} O_2(\lambda), \tag{2.27}$$

$$O_1(\lambda) = \lambda^{-1} ((\partial'^\dagger \partial' - \lambda)^{-1} \partial'^\dagger - (\partial^\dagger \partial - \lambda)^{-1} \partial^\dagger) \nabla \partial, \tag{2.28}$$

$$O_2(\lambda) = \lambda^{-1} (\partial'^\dagger \partial' - \lambda)^{-1} \partial'^\dagger (\nabla P_-) \partial, \tag{2.29}$$

where we have used $P_- \nabla \partial' = P_- (\nabla P_-) \partial + P_- (\nabla \partial)$. On the other hand, by

Corollary (2.15),

$$z_0^\theta = \operatorname{Tr} \int_{\mathcal{O}} \frac{d\lambda}{2\pi i} O_1(\lambda) + \operatorname{Tr} \int_{\mathcal{O}} \frac{d\lambda}{2\pi i} O_2(\lambda), \tag{2.30}$$

where $\mathcal{O}$ is the small counterclockwise circle $\{|\lambda| = \delta\}$ contained in $\mathscr{C}$.

Now since ∂_y and $\partial_y' = P_- \partial_y$ differ by an operator of infinite negative order, $O_1(\lambda)$ and $O_2(\lambda)$ have vanishing homogeneous symbol expansions. Hence by Proposition (C.6) of Appendix C of [8],

$$\{\text{f.p.a.} z = 0\} \operatorname{Tr} \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} O_i(\lambda) + \operatorname{Tr} \int_{\mathcal{O}} \frac{d\lambda}{2\pi i} O_i(\lambda) = 0. \tag{2.31}$$

The first equality of (2.10.4). Follows from (2.27), (2.30), and (2.31). To prove the second equality, observe that

$$\sigma_0^\theta - \sigma_0^{\theta=0} = \operatorname{Im} \lim_{z \to 0} \operatorname{Tr} (\partial'^\dagger \partial')^{-z-1} \partial'^\dagger [\theta, \partial'], \tag{2.32}$$

$$L^\theta = \operatorname{Im} \lim_{z \to 0} \operatorname{Tr} (\partial^\dagger \partial)^{-z-1} \partial^\dagger [\theta, \partial] - \operatorname{Tr} (\partial^\dagger \partial)^{-1} \partial^\dagger [\theta, \partial], \tag{2.33}$$

and so

$$\sigma_0^\theta - \sigma^{\theta=0} - L^\theta = \operatorname{Tr} (\partial^\dagger \partial)^{-1} \partial^\dagger [\theta, \partial] - \operatorname{Im} \lim_{z \to 0} \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} \operatorname{Tr} O_3(\lambda), \tag{2.34}$$

$$O_3(\lambda) = \lambda^{-1}((\partial'^\dagger \partial' - \lambda)^{-1} \partial'^\dagger [\theta, \partial'] - (\partial^\dagger \partial - \lambda)^{-1} \partial^\dagger [\theta, \partial]). \tag{2.35}$$

Since $O_3(\lambda)$ has a vanishing homogeneous, symbol expansion,

$$\lim_{z \to 0} - \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z} \operatorname{Tr} O_3(\lambda) = \int_{\mathcal{O}} \frac{d\lambda}{2\pi i} \operatorname{Tr} O_3(\lambda) = \operatorname{Tr} (\partial'^\dagger \partial')^{-1} \partial' [\theta, \partial'] - \operatorname{Tr} (\partial^\dagger \partial)^{-1} \partial [\theta, \partial]. \tag{2.36}$$

Combining (2.34) and (2.36), we obtain the desired

$$\sigma_0^\theta - \sigma^{\theta=0} - L^\theta = \operatorname{Tr} (\partial'^\dagger \partial')^{-1} \partial' [\theta, \partial'] = -\operatorname{Tr} (1 - \partial' (\partial'^\dagger \partial')^{-1} \partial'^\dagger) \theta = -\operatorname{Tr} (1 - P_-) \theta. \tag{2.37}$$

□

We close this section with a simple observation. In addition to Q there are several other natural bundle maps from $V_-^\perp$ to $\operatorname{KER} \partial^\dagger$ determined by the available data (i.e. the inner product on $\mathscr{H}_-$). For example, let $Q' : \operatorname{KER} \partial^\dagger \to V_-^\perp$ denote the bundle map which on the fiber over $y \in Y$ is induced by orthogonal projection in $L^2(X, S_- \otimes E)$ onto $V_-^\perp$. Then Q, $(Q^\dagger)^{-1}$, $(Q')^{-1}$ are all isomorphisms from $V_-^\perp$ to $\operatorname{KER} \partial^\dagger$. (Here † denotes the adjoint with respect to the induced inner products on $V_-^\perp$ and $\operatorname{KER} \partial^\dagger$.) However, it is easy to check that $Q' = Q^\dagger$ and $\|\operatorname{Det} Q\|^2 \operatorname{Det} (Q^\dagger)^{-1} = \operatorname{Det} Q$. In particular,

$$\frac{\operatorname{Det} Q}{\|\operatorname{Det} Q\|} = \frac{\operatorname{Det} (Q^\dagger)^{-1}}{\|\operatorname{Det} (Q^\dagger)^{-1}\|} = \frac{\operatorname{Det} (Q')^\dagger}{\|\operatorname{Det} (Q')^\dagger\|} = \frac{\operatorname{Det} (Q')^{-1}}{\|\operatorname{Det} (Q')^{-1}\|}. \tag{2.38}$$

Thus Theorem (1.18) is unchanged if Q is replaced by $(Q^\dagger)^{-1}$, $(Q')^{-1}$ or $(Q')^\dagger$.

3. The Eta Invariant of $H|_{\bar{V}}$

In this section we explain definition (1.16) of the η-invariant of $H|_{\bar{V}}$. We now assume that $Y = \mathbb{R}$ and that the geometric data satisfy conditions 1a, 1b, and 2 of Sect. 1.

Our definition of the η-invariant is modelled on the definition given in [8]. In that paper we defined $\eta(H)$ by an expression involving the complex powers of H^2. These complex powers were defined by contour integration as in the definitions (2.1)–(2.5) of the previous section, and an important fact was that the essential spectrum of H^2 was contained in $[\lambda_0, \infty)$, where $\lambda_0 > 0$ was a lower bound on the spectra of $D^2_{\pm\infty}$. In the setting of this paper, $D^2_{\pm\infty}$ may have zero eigenvalues (if $\operatorname{Ker} \partial^\dagger_{\pm\infty} \neq 0$) and so the essential spectrum of H^2 extends down to 0. Thus the definition of the complex powers of H^2 is problematic. On the other hand, $D_{\pm\infty}|_V$ are invertible and hence the essential spectrum of $H|^2_{\bar{V}}$ on $\bar{V}$ is bounded away from 0 (see Proposition (A.2) of Appendix A). It is thus possible to define $\eta(H|_{\bar{V}})$ by the procedure we used in [8] to define $\eta(H)$.

Specifically, there is a $\delta > 0$ sufficiently small such that the disk $\{|\lambda| < 2\delta\}$ intersects the spectrum of $H|^2_{\bar{V}}$ in only a finite number of eigenvalues of finite multiplicity, and is disjoint from the spectra of $D_{\pm\infty}|^2_{\bar{V}}$. As in the previous section, let $\mathscr{C}$ be the contour in $\mathbb{C}$ which goes from $-\infty$ to $-\delta$ directly above the negative real axis, then clockwise around the circle $|\lambda| = \delta$, then back to $-\infty$ directly below the negative real axis. In analogy to the definitions in [8] define

$$\begin{aligned}\eta^\delta_\phi(H|_{\bar{V}})(z) &= \operatorname{Tr} \phi H|_{\bar{V}}(H|^2_{\bar{V}})^{-z-1/2} \\ &= -\operatorname{Tr} \int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} H|_{\bar{V}}(H|^2_{\bar{V}} - \lambda)^{-1}, \operatorname{Re} z > 0,\end{aligned} \tag{3.1}$$

$$\eta^\delta(H|_{\bar{V}})(z) = \lim_{\phi \to 1} \eta^\delta_\phi(H|_{\bar{V}})(z), \tag{3.2}$$

$$\eta(H|_{\bar{V}}) = \lim_{z \to 0} \eta^\delta(H|_{\bar{V}})(z) + \sum_{0<|\lambda|<\delta} \operatorname{sign}(\lambda) \operatorname{Tr} Q_\lambda, \tag{3.3}$$

$$\xi(H|_{\bar{V}}) = \tfrac{1}{2}(\eta(H|_{\bar{V}}) + \dim \operatorname{Ker}(H|_{\bar{V}})). \tag{3.4}$$

In (3.1) ϕ is a non-negative smooth function on $\mathbb{R}$ of compact support, acting as a multiplication operator on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ and in (1.2) the limit $\phi \to 1$ is taken through a sequence of such ϕ increasing pointwise to the constant function 1. As in [8], the reason for introducing these cut-off functions is that in general the operator $H|_{\bar{V}}(H|^2_{\bar{V}})^{-z-1/2}$ is not trace class. In (3.3) $\lim_{z \to 0}$ is understood in terms of analytic continuation as explained in Proposition (3.5) below. Q_λ is the orthogonal projection onto the (finite dimensional) eigenspace of $H|^2_{\bar{V}}$ corresponding to λ, and $\operatorname{sign}(\lambda)$ equals -1 if $\lambda < 0$ and $+1$ otherwise.

We will prove the following analog of Proposition (3.6) of [8].

Proposition (3.5)

1. $\eta^\delta_\phi(H|_{\bar{V}})(z)$ *extends to an analytic function of z for* $\operatorname{Re} z > -\frac{1}{2}$.
2. *As* $\phi \to 1$, $\eta^\delta_\phi(H|_{\bar{V}})(z)$ *converge uniformly on compact sets to an analytic function* $\eta^\delta(H|_{\bar{V}})(z)$ *for* $\operatorname{Re} z > -\frac{1}{2}$.
3. $\eta^\delta(H)(z)$ *depends smoothly on* $H|_{\bar{V}}$ *and z for* $\operatorname{Re} z > -\frac{1}{2}$ *and sufficiently small*

variations of $H|_{\bar{V}}$. $\eta(H|_{\bar{V}})$ mod 1 *and* $\xi(H|_{\bar{V}})$ mod 1 *are independent of* δ *and depend smoothly on* H.

Since the proof of the Proposition is modelled on the proof of Proposition (3.6) of [8] we will only sketch the main points. Further details are contained in Appendix A.

The idea is to investigate the resolvent of $H|_{\bar{V}}^2$ on $\bar{V}$ using the pseudo-differential operators which approximate the parametrix of $H^2 - \lambda$ on all of $L_0^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$. For this we extend the resolvent of $H|_{\bar{V}}^2$ to $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ by defining, for $\lambda \notin \operatorname{Spec} H|_{\bar{V}}^2$,

$$R(\lambda) = \begin{cases} ((H|_{\bar{V}})^2 - \lambda)^{-1} & \text{on } \bar{V} \\ 0 & \text{on } \bar{V}^{\perp} \end{cases}, \tag{3.6}$$

so that on $C_0^\infty(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$,

$$((\bar{H})^2 - \lambda) \circ R(\lambda) = R(\lambda) \circ ((\bar{H})^2 - \lambda) = \bar{P}. \tag{3.7}$$

Let $B(\lambda)$ be a λ dependent inverse to $H^2 - \lambda$ up to terms of order $-N-1$, $N > \dim(\mathbb{R} \times X)$, constructed from the complete symbol of H^2 as in [13, 14].

By comparing $R(\lambda)$ and $B(\lambda)$ we can show (Proposition (A.2.3)) that $R(\lambda)$ has order -2 as a operator between Sobolev spaces, and that its norm as a bounded operator on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ decays as $|\lambda|^{-1}$ for large $|\lambda|$. In particular, the contour integral in (3.1) makes sense for $\operatorname{Re} z > 0$ and defines a bounded operator on V. It also follows that the kernel of $R(\lambda)$ on L^2 is smooth off the diagonal of $Z \times Z$, and for sufficiently large positive integers n, the kernel of $HR(\lambda)^n$ in continuous everywhere.

In addition, $D_{\pm\infty}|_{\bar{V}}^2$ are invertible, it follows that $|R(\lambda)(y, x; y', x')|$ decays exponentially in $|y - y'|$, uniformly in λ (Proposition (A.2.4)).

Combining these observations, we can show as in Proposition (3.7) of [8] that for sufficiently large $\operatorname{Re} z, \phi H|_{\bar{V}}(H|_{\bar{V}}^2)^{-z-1/2}$ is trace class and $\eta_\phi^\delta(H|_{\bar{V}})$ is analytic in z with

$$\eta_\phi^\delta(H|_{\bar{V}})(z) = \int_{\mathbb{R} \times X} dy\,|dx| \left[-\int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} \phi H R(\lambda) \right](y, x; y, x). \tag{3.8}$$

To investigate the analytic continuation of $\eta_\phi^\delta(H|_{\bar{V}})(z)$, we write

$$\eta_\phi^\delta(H|_{\bar{V}})(z) = \int_{\mathbb{R} \times X} dy\,|dx|\,\phi(y) \left(\sum_{i=1}^{4} \operatorname{tr} K_i(z; y, x, y, x) \right),$$

$$K_1 = -\int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} H B(\lambda), \quad K_2 = -\int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} H(R(\lambda) - \bar{P} B(\lambda) \bar{P}), \tag{3.9}$$

$$K_3 = -\int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} H(1 - \bar{P}) B(\lambda), \quad K_4 = -\int_{\mathscr{C}} \frac{d\lambda}{2\pi i} \lambda^{-z-1/2} H \bar{P} B(\lambda)(1 - \bar{P}).$$

The singularities of each of the terms $K_i(z)$ is analyzed in detail in Proposition (A.2) of Appendix A, so here we only summarize the results. First, the singularities in the kernel of $K_1(z)$ are given by the same local expressions in the complete symbol of H which arise for the η-invariant for a Dirac operator on a compact manifold. A well-known argument using invariance theory then shows that in fact $\operatorname{tr} K_1(z;y,x,y,x)$ is analytic for $\operatorname{Re} z > -\frac{1}{2}$. Next the singularities of the kernel of $K_2(z)$ can be analyzed in the $\mathbb{R}$ and X directions separately, and we show in Proposition. (A.2) that this kernel is continuous in y, x and analytic in z for $\operatorname{Re} z > -\frac{1}{2}$. Finally, although the kernels of $K_3(z)$ and $K_4(z)$ are analytic only for $\operatorname{Re} z > \frac{1}{2}$, by using an explicit approximation (A.4) for $B(\lambda)$ we show in Lemma (A.2) that in fact the local traces $\operatorname{tr} K_3(z; y, x, y, x)$ and $\operatorname{tr} K_4(z; y, x, y, x)$ are analytic all the way to $\operatorname{Re} z > -\frac{1}{2}$.

These observations complete the proof of (3.5.1). To prove (3.5.2) we proceed as in the proof of Proposition (3.6) of [8]. First, if $H|_{\bar{V}}$ is invariant under translations in the $\mathbb{R}$ direction, then $R(\lambda)$ can be given explicitly (Proposition (A.1.5)). A calculation analogous to that of Lemma (3.13) of [8] then shows that in this case $\eta_\phi^\delta(H|_{\bar{V}})(z)$ vanishes. Next, for arbitrary $H|_{\bar{V}}, H|_{\bar{V}}(H|_{\bar{V}}^2 - \lambda)^{-1}$ can be compared with the translation invariant operators $(H_{\pm\infty}|_{\bar{V}})((H|_{\pm\infty}|_{\bar{V}})^2 - \lambda)^{-1}$, where $H_{\pm\infty} = i\Gamma(d/dy) + D_{\pm\infty}$. The exponential decay in $|y - y'|$ of $R(\lambda)(y, x; y', x')$ allows us to bound the difference of the kernels of these operators as in Lemma (3.19) of [8], and then reasoning as in Propositions (3.6) of [8] we can show that the limit $\phi \to 1$ in (1.2) exists uniformly on compact sets for $\operatorname{Re} z > -\frac{1}{2}$.

Statement (3.5.2) follows. Finally (3.5.3) is proved as in Proposition (3.6) of [8].

4. The Variation of $\eta(H|_{\bar{V}})$

We next investigate the variation of $\eta(H|_{\bar{V}})$ as H is varied. Let $T = \mathbb{R}$ and let $\{\bar{\nabla}^t\}$, $t \in T$ be a smooth family of connections on $\bar{E} \to Z$ parametrized by T, and satisfying conditions 1a, 1b, and 2 for all t. We then obtain a one parameter family $\{H^t\}, t \in T$ of operators on $C^\infty(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$, a two parameter family $\{\nabla_y^t\}, (t, y) \in T \times Y$, of connections on $E \to X$ with $\bar{\nabla}^t = dy(\partial/\partial y) + \theta^t + \nabla_{(\cdot)}^t$, and a two parameter family $\{D_y^t\}$, $(t, y) \in T \times Y$ of Dirac operators on $C^\infty(X, S \otimes E)$ coupled to ∇_y^t.

Since $\xi(H|_{\bar{V}}) \bmod 1$ depends smoothly on $H|_{\bar{V}}$, the assignment $t \to \xi(H|_{\bar{V}}^t)$ defines a smooth map $T \mapsto \mathbb{R}/\mathbb{Z}$. With a little abuse of notation, let $(d\xi/dt)dt$ denote the pullback by this map of the unit normalized volume one-form on $\mathbb{R}/\mathbb{Z}$.

We will prove the following analog of Proposition (4.1) of [8].

Proposition (4.1). *As one-forms on T,*

$$dt\frac{d}{dt}\left(\xi(H|_{\bar{V}}) - \frac{1}{2\pi i}\int_Y \operatorname{Tr}_s(1-P)\theta\right) = \left[\int_Z \hat{A}(\mathcal{R}_g)\operatorname{ch}(\mathcal{F}_{dt(\partial/\partial t)+\bar{\nabla}})\right]_{\text{one-form part}}$$

$$+\frac{1}{2\pi i}\sigma\left(D_\infty|_V, \frac{dD_\infty|_V}{dt}dt\right) - \frac{1}{2\pi i}\sigma\left(D_{-\infty}|_V, \frac{dD_{-\infty}|_V}{dt}dt\right).$$

We have used the notation

$$\sigma(A, v)(z) = \operatorname{Tr}_s vA(A^2)^{-z-1} = -\operatorname{Tr}_s \int_{\mathscr{C}} \frac{d\lambda}{2\pi i}\lambda^{-z-1} vA(A^2 - \lambda)^{-1},$$

$$\sigma(A, v) = \lim_{z \to 0} \sigma(A, v)(z), \tag{4.2}$$

where Tr_s denotes the super-trace. It is part of the Proposition that these expressions make sense for $A = D|_V$ and $v = dD|_V/dt$. $\mathscr{R}_g$ is the Levi–Cevita curvature of the metric g on X, and $\mathscr{F}_{dt(\partial/\partial t)+\bar{\nabla}}$ is the curvature of $dt(\partial/\partial t) + \bar{\nabla}$ viewed as a connection on the pull-back of $\bar{E}$ to the bundle $T \times \bar{E}$ over $T \times Z$. $\hat{A}$ and ch are the polynomials given in (1.14).

Since the proof of Proposition (4.1) is modelled on the proof of Proposition (4.1) of [8] we only sketch the main points. Further details may be found in Appendix A.

As in [8], we first show that for $\delta > 0$ and $\operatorname{Re} z$ sufficiently large,

$$\frac{d}{dt}\eta_\phi^\delta(H|_{\bar{V}})(z) = 2\mathscr{V}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z) + 2\mathscr{S}_{d\phi/dy}^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z), \tag{4.3}$$

where

$$\mathscr{V}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z) = z\operatorname{Tr}\phi\int_{\mathscr{C}}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}\frac{dH|_{\bar{V}}}{dt}(H|_{\bar{V}}^2 - \lambda)^{-1}, \tag{4.4}$$

$$\begin{aligned}\mathscr{S}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z) = -\tfrac{1}{2}\operatorname{Tr}\phi\int_{\mathscr{C}}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}\Gamma\Bigg((H|_{\bar{V}}^2 - \lambda)^{-1}\frac{dH|_{\bar{V}}}{dt}H|_{\bar{V}}(H|_{\bar{V}}^2 - \lambda)^{-1}\\ - \left\{H|_{\bar{V}}, (H|_{\bar{V}}^2 - \lambda)^{-1}\frac{dH|_{\bar{V}}}{dt}(H|_{\bar{V}}^2 + \lambda)(H|_{\bar{V}}^2 - \lambda)^{-2}\right\}\Bigg).\end{aligned} \tag{4.5}$$

We proved a similar formula for the variation of $\eta(H)$ in Proposition (4.6) of [8], and the arguments given there apply in the current situation with only minor modifications. Here the relevant technical points are handled using the properties of $R(\lambda)$ given in Proposition (A.1) of Appendix A. For example, since $R(\lambda)$ has order -2 and its kernel decays exponentially in $|y - y'|$, it follows as in Proposition (3.7) of [8] that $\phi R(\lambda)^n$ is trace class for sufficiently large integers n. Then $\phi(H|_{\bar{V}}^2 - \lambda)^{-M}$ is also trace class, and so as in the proof of (4.6) of [8], an integration by parts in λ shows that in deriving (4.3) we are allowed to cyclically permute operators under trace.

We next show that $\mathscr{V}_\phi^\delta(H|_{\bar{V}}, (dH|_{\bar{V}}/dt))(z)$ extends to an analytic function for $\operatorname{Re} z > -\frac{1}{2}$ with

$$\mathscr{V}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(0)dt = \int_Z \hat{A}(\mathscr{R}_g)\operatorname{ch}(\mathscr{F}_{dt(\partial/\partial t)+\bar{\nabla}})]_{\text{one form part}} + \frac{1}{2\pi i}\int_Y \operatorname{Tr}_s\frac{d\theta}{dt}(1 - P)dt. \tag{4.6}$$

For this we proceed as in the proof of Proposition (3.5) of the previous section by writing, for $\operatorname{Re} z$ large,

$$\mathscr{V}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z) = z\int_{\mathbb{R}\times X} dy|dx|\phi(y)\left(\sum_{i=1}^4 K_i'(z; y, x, y, x)\right), \tag{4.7}$$

where $K_i'(z)$ is defined by submitting dH/dt for H in the expression (4.9) defining $K_i(z)$. The contribution of each of the terms $K_i'(z)$ is analyzed in detail in Proposition (A.2) of Appendix A, so here we only summarize the results. First, the singularities in the kernel of $K_1'(z)$ are given by the same local expressions in the complete symbols of H and dH/dt which arise for the variation of the η-invariant on a compact manifold. It then follow as usual (see Sect. 4 of [8]) that $z\,\mathrm{tr}\,K_1'(z,y,x,y,x)$ extends to an analytic function of z for $\mathrm{Re}\,z > -\frac{1}{2}$ with

$$\lim_{z\to 0} -z\,\mathrm{tr}\,K_1'(z;y,x,y,x)|dx|dt = [\hat{A}(\mathscr{R}_g)\mathrm{ch}(\mathscr{F}_{dt(\partial/\partial t)+\bar{\nabla}})]_{\text{one form part}}. \tag{4.8}$$

Next, the singularities in the kernel of $K_2'(z)$ can be analyzed in the $\mathbb{R}$ and X directions separately, and we show in Proposition (A.2) that this kernel is analytic for $\mathrm{Re}\,z > -\frac{1}{2}$, so that $\lim_{z\to 0} z\,\mathrm{tr}\,K_2'(z;y,x,y,x) = 0$. Finally, by using an explicit approximation (A.4) for $B(\lambda)$ we show in Proposition (A.2) that $z\,\mathrm{tr}\,K_3'(z;y,x,y,x)$ and $z\,\mathrm{tr}\,K_4'(z;y,x,y,x)$ are analytic for $\mathrm{Re}\,z > -\frac{1}{2}$ with

$$\lim_{z\to 0} -zK_3'(z;y,x,y,x) = \left[\Gamma\frac{d\theta}{dt}\left(\frac{\partial}{\partial y}\right)(1-P)\right](x,x), \quad \lim_{z\to 0} zK_4'(z;y,x,y,x) = 0. \tag{4.9}$$

These observations prove (4.6).

Continuing as in Proposition (4.6) of [8], we next calculate that if H^t is invariant under translations in the Y direction, then for $\mathrm{Re}\,z$ sufficiently large,

$$\mathscr{S}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(z) = \left(\int_Y \phi\,dy\right)\frac{1}{2\pi i}\frac{\Gamma(\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\Gamma(z+1)\sigma\left(\frac{dD|_V}{dt}, D|_V\right)(z). \tag{4.10}$$

In fact, the calculation of [8] applies here if we now use the expression (3.6) for $R(\lambda)$.

Next $\sigma(dD|_V/dt, D|_V)(z)$ can be expressed as a trace over all of $L^2(X, S\otimes E)$ if we substitute PDP for $D|_V$. As in Sect. 2 of [8], we may then apply standard pseudo-differential operator methods to show that the resulting expression extends to a meromorphic function of z for $\mathrm{Re}\,z > -\frac{1}{2}$ with possible simple poles whose residues are determined by the complete symbol of PDP. Using the generalized Gilkey Theorem as in Proposition (B.1) of [8] we can show that these residues vanish. Hence $\sigma(D|_V, (dD|_V/dt))(z)$ extends to an analytic function for $\mathrm{Re}\,z > -\frac{1}{2}$.

Finally, as in Proposition (4.6) of [8], the proof of Proposition (4.1) is completed by taking the limit $\phi\to 1$ in (2.3). Specifically, the exponential decay in $|y-y'|$ of $R(\lambda)(y,x;y',x')$ allows us to take the limit in (4.3) to deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\eta^\delta(H|_{\bar{V}})(0) = \lim_{\phi\to 1}\Bigg[&\mathscr{V}_\phi^\delta\left(H|_{\bar{V}}, \frac{dH|_{\bar{V}}}{dt}\right)(0)\\ &+\mathscr{S}^\delta_{d\phi_+/dy}\left(H_\infty|_V, \frac{dH_\infty|_V}{dt}\right)(0)\\ &+\mathscr{S}^\delta_{d\phi_-/dy}\left(H_{-\infty}|_V, \frac{dH_{-\infty}|_V}{dt}\right)(0)\Bigg].\end{aligned} \tag{4.11}$$

Here $\phi = \phi_+ + \phi_-$ with ϕ_+ supported in $[0,\infty)$ and ϕ_- supported in $(-\infty, 0]$. Proposition (4.1) now follows from (4.6), (4.10), and (4.11).

5. Parallel Transport and Curvature of $\nabla^{(\mathscr{L})}$

In this section we prove Theorems (1.11) and (1.12). These theorems are generalizations of Theorems (1.6) and (1.11) of our paper [8], and using the machinery we now developed, the proofs given there require only minor enhancements. Whereas in [8] the basic ingredient was Proposition (4.1) of that paper, we now use the corresponding generalization, Proposition (4.1).

We begin with Theorem (1.11). Thus let $Y = \mathbb{R}$, let $\nabla^{\bar{E}}$ be a connection on $\bar{E} \to Y \times X$, and assume that the geometric data satisfies conditions 1a, 1b, and 2 of Sect. 1. The first statement of the theorem follows immediately from the expression for the pull-back of $\nabla^{(\mathscr{L})}$ given in Proposition (1.11), so it remains to prove

$$\frac{1}{2\pi i}\int_Y \sigma_0^\theta(\nabla^{\bar{E}}) = \xi(H_{\nabla^{\bar{E}}}) \mod 1. \tag{5.1}$$

Following the proof of (1.11) in [8], we integrate the formula of Theorem (4.1) over a suitable family of connections on $\bar{E}$ interpolating from $dy(\partial/\partial y) + \nabla^E_{-\infty}$ to $dy(\partial/\partial y) + \nabla^E_{(\cdot)}$ to $\nabla^{\bar{E}} = dy(\partial/\partial y) + \theta + \nabla^E_{(\cdot)}$. The only new feature is the presence of the term $d/dt\,\mathrm{Tr}_s\theta(1-P)$, but this is easily handled. We obtain

$$\xi(H_{dy(\partial/\partial y)+\nabla^E_{(\cdot)}}|_{\bar{V}}) = \frac{1}{2\pi i}\int_Y \sigma\left(D|_V, \frac{dD|_V}{dy}dy\right) \mod 1, \tag{5.2}$$

$$\xi(H_{\nabla^{\bar{E}}}|_{\bar{V}}) - \frac{1}{2\pi i}\int_Y \mathrm{Tr}_s\theta(1-P) - \xi(H_{dy(\partial/\partial y)+\nabla^E_{(\cdot)}}|_{\bar{V}}) = \int_Y \Big[\int_X \hat{A}(\mathscr{R}_g)\operatorname{tr}\theta \exp i\mathscr{F}_{\nabla^E}/2\pi\Big] \mod 1. \tag{5.3}$$

The integrand on the right-hand side of (5.2) is $\sigma_0^{\theta=0}(\nabla^{\bar{E}})$, while by Proposition (4.11) the integrand in brackets on the right-hand side of (5.3) is $L^\theta(\nabla^{\bar{E}})$. (The notation is meant to indicate that these quantities are to be computed in terms of the original connection $\nabla^{\bar{E}}$ on $\bar{E}$). Now by Proposition (4.10.4), $\sigma_0^\theta = \sigma_0^{\theta=0} + L^\theta - \mathrm{Tr}\,\theta(1-P_-)$. Thus by adding (5.2) and (5.3), and observing that $\mathrm{Tr}_s\theta(1-P) = -\mathrm{Tr}\,\theta(1-P_-)$ we obtain (5.1).

We next indicate the proof of Theorem (1.12). We allow the parameter manifold Y to arbitrary, but continue to assume that conditions 1a and 1b hold. By (4.12)

$$(\Phi \circ \operatorname{Det} Q)^* \nabla^{(\mathscr{L})} = \nabla^{\mathrm{DET}\,V^\perp_-} + \sigma_0^\theta + \{\text{exact}\} \tag{5.4}$$

$$= d_Y + \sigma_0^{\theta=0} + L^\theta + \{\text{exact}\}, \tag{5.5}$$

and so the curvature of $\nabla^{(\mathscr{L})}$ is the two-form $d_Y(\sigma_0^{\theta=0} + L^\theta)$.

By Proposition (4.11)

$$d_Y L^\theta = -d_Y \int_X \hat{A}(\mathscr{R}_g)\operatorname{tr}\theta \exp i\mathscr{F}_\nabla/2\pi. \tag{5.6}$$

On the other hand, as in the proof of Theorem (1.6) of [8], we can use Proposition (4.1) to show

$$d_Y \sigma_0^{\theta=0} = 2\pi i\Big[\int_X \hat{A}(\mathscr{R}_g)\operatorname{ch}(\mathscr{F}_{d_Y+\nabla})\Big]_{\{\text{two-form}\}}. \tag{5.7}$$

In particular, the additional term $d_T \int_Y \mathrm{Tr}_s\theta(1-P)$ in the formula of Theorem (4.1) is exact and hence it does not affect the argument. The sum of the right-hand sides of (5.6) and (5.7) is precisely the desired expression for the curvature, as we verified in the proof of Theorem (1.6) of [8].

A. Spectrum and Resolvent of $H|_{\bar{V}}^2$. Technical Details of Sects. 3 and 4

The main results of this Appendix are the following two propositions which are needed in the proofs of Theorems (1.5) and (2.1). We use the notation of the body of the paper, Appendix B, and Appendix D of [8].

Proposition (A.1).

1. $(H|_{\bar{V}})^2$ *is self-adjoint on* $L_0^2|_V$.
2. *The essential spectrum of* $(H|_{\bar{V}})^2$ *is contained in* $[\lambda_0, \infty)$, *where* $\lambda_0 > 0$ *is a lower bound for the spectra of* $(PD_{\pm\infty}P)^2$ *on* $L_0^2|_V$.
3. *Let* $\mathscr{S}$ *be any closed sector not intersecting the positive real axis, and let* $\mathscr{S}' = \mathscr{S} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. *For any* $k \in \mathbb{Z}$, $(\alpha, l) \in \mathbb{Z} \times \mathbb{Z}$, $j \in \mathbb{Z}$, $(\beta, m) \in \mathbb{R} \times \mathbb{Z}$, *with* $0 \leqq j \leqq 2$, $0 \leqq \beta + m \leqq 2$, $\beta, m \geqq 0$, *and* α *and* l *of the same sign, the following statements hold*:
   a. *For any compact* $\mathscr{F} \subset \mathbb{C} \backslash \operatorname{Spec}((H|_{\bar{V}})^2)$ *there exists a* $c > 0$ *such that for* $\lambda \in \mathscr{F}$,
   $$\| R(\lambda) \|_{k+j,k} < c \qquad \| R(\lambda) \|_{(\alpha,l)+(\beta,m),(\alpha,l)} < c;$$
   b. *There exist* $c, r > 0$ *such that if* $\lambda \in \mathscr{S}$ *and* $|\lambda| \geqq r$, *then* $\lambda \notin \operatorname{Spec}((H|_{\bar{V}})^2)$, *and*
   $$\| R(\lambda) \|_{k+j,k} < c(1 + |\lambda|)^{-1+j/2} \qquad \| R(\lambda) \|_{(\alpha,l)+(\beta,m),(\alpha,l)} < c(1 + |\lambda|)^{-1+(\beta+m)/2};$$
   c. *For any closed* $\mathscr{F} \subset \mathscr{S}' \backslash \operatorname{Spec}((H|_{\bar{V}})^2)$, *there exists a* $c > 0$ *such that for* $\lambda \in \mathscr{F}$,
   $$\| R(\lambda) - \bar{P} B(\lambda) \bar{P} \|_{(\alpha,l)+(\beta,N),(\alpha,l)} < c(1 + |\lambda|)^{-2+\beta/2}.$$

   In particular, $R(\lambda)$ *extends to a map from* $L^2_{-\infty}$ *to* $L^2_{-\infty}$ *and* $R(\lambda)|_V$ *is the resolvent of* $(H|_{\bar{V}})^2$ *on* $L_0^2|_V$.
4. *For any closed* $\mathscr{F} \subset \{\lambda : \operatorname{Re} \lambda < \lambda_0\} \backslash \operatorname{Spec}((H|_{\bar{V}})^2)$, *there exist constants* $\rho_0 > 0$ *and* $c > 0$ *such that for* $|\rho| < \rho_0, \lambda \in \mathscr{F}, |y - y'| > 1, x, x' \in X$,
   $$\| M_{e^{-\rho y}} \circ R(\lambda) \circ M_{e^{\rho y}} \|_{0,0} < c \qquad |R(\lambda)(y, x; y', x')| < c e^{-|\rho||y - y'|}.$$
5. *If* H *is invariant under translations of* $\mathbb{R}$, *then for* $\lambda \in [\lambda_0, \infty)$, $y, y' \in \mathbb{R}, x, x' \in X$,
   $$R(\lambda)(y, x; y', x') = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \bar{P}(E^2 + (PDP)^2 - \lambda)^{-1} \bar{P}(x, x') e^{iE(y - y')},$$
   $$|R(\lambda)(y, x; y', x')| < c e^{-|\lambda - \lambda_0|^{1/2}|y - y'|}.$$

Here $R(\lambda)$ is the resolvent of $H|_{\bar{V}}$ extended to $C_0^\infty(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ as in (3.6) and $B(\lambda)$ is a λ-dependent inverse for $H^2 - \lambda$ up to terms of order $-N - 1$, $N > \dim(\mathbb{R} \times X)$. The ordinary and "mixed" Sobolev norms in (A.1) are defined in Appendix B. (A.1.3) can be proved using the methods of [13]. (A.1.1) then follows from the existence of $R(\lambda)$ for $|\lambda|$ large. (A.1.2) and (A.1.4) are consequences of the invertibility of $D_{\pm\infty}|_V^2$, and can be proved using the methods of the proofs of Propositions (A.2) and (A.6) respectively of Appendix A of [8]. (A.1.5) is obvious.

Proposition (A.2). $\operatorname{tr} K_i(z; y, x, y, x)$ *and* $z \operatorname{tr} K_i'(z; y, x, y, x)$ *extend to analytic functions for* $\operatorname{Re} z > -\frac{1}{2}$ *which are continuous in* y *and* x *and*

1. $\lim\limits_{z \to 0} - z \operatorname{tr} K_1'(z; y, x, y, x) dt = \hat{A}(\mathscr{R}_g) \operatorname{ch}(\mathscr{F}_{dt(d/dt) + \bar{\nabla}^t})]_{\text{one form part}}$,

2. $\lim_{z\to 0} -z\,\mathrm{tr}\,K_2'(z;y,x,y,x)=0$,

3. $\lim_{z\to 0} -z\,\mathrm{tr}\,K_3'(z;y,x,y,x)=\frac{1}{2\pi i}\mathrm{tr}\left[\Gamma\frac{d}{dt}\theta\left(\frac{\partial}{\partial y}\right)(1-P)\right](x,x)$,

4. $\lim_{z\to 0} -z\,\mathrm{tr}\,K_4'(z;y,x,y,x)=0$.

The operators $K_i(z)$ and $K_i'(z)$ have been defined in (3.8) and (4.7).

To prove Proposition (A.2) we need an explicit approximation to $B(\lambda)$. Choose $b^{\mathbb{R}}_{-n}\in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{C})\otimes C^\infty(X,\mathrm{End}(S\otimes E))$, $n=2,3$ such that for $E^2+|\lambda|>1$,

$$b^{\mathbb{R}}_{-2}(y,E;\lambda)=(E^2-\lambda)^{-1},\quad b^{\mathbb{R}}_{-3}(y,E;\lambda)=-2i\theta E(E^2-\lambda)^{-2}. \tag{A.3}$$

Define operators $B^{\mathbb{R}}_{-2}(\lambda)$ and $B^{\mathbb{R}}_{-3}(\lambda)$ on $C^\infty(\mathbb{R}\times X,\bar{S}\otimes\bar{E})$ by

$$B^{\mathbb{R}}_{-n}(\lambda)f(y)=\int_{\mathbb{R}}\frac{dE}{2\pi}e^{iEy}b^{\mathbb{R}}_{-n-2}(y,E;\lambda)\int_{\mathbb{R}}dy'e^{-iEy'}f(y') \tag{A.4}$$

for $f\in C_0^\infty(\mathbb{R},C^\infty(X,S\otimes E))$. These operators can be interpreted as resulting from a formal application of Seeley's construction of a λ dependent parametrix to the operator H^2 viewed as a differential operator on the infinite dimensional bundle $C_0^\infty(\mathbb{R}\times X,\bar{S}\otimes\bar{E})$ over $\mathbb{R}$. The following Lemma, which can be proved using the methods of [13], shows that they are useful approximations to $B(\lambda)$.

Lemma (A.5). *$B^{\mathbb{R}}_{-n}(\lambda)$ extend to operators on $L^2_{-\infty}(\mathbb{R}\times X,\bar{S}\otimes\bar{E})$ and for all $k\in\mathbb{Z}$, $(\alpha,l)\in\mathbb{Z}\times\mathbb{Z}$, $\beta\in\mathbb{R}$ with α and l of the same sign and $0\leqq\beta\leqq 2$, there exists $c>0$ and $M\in\mathbb{Z}$ such that*

1. $\|B(\lambda)-B^{\mathbb{R}}_{-2}(\lambda)\|_{(\alpha,l)+(1+\beta,M),(\alpha,l)}<c(1+|\lambda|)^{-1+\beta/2}$,
2. $\|B(\lambda)-B^{\mathbb{R}}_{-2}(\lambda)-B^{\mathbb{R}}_{-3}(\lambda)\|_{(\alpha,l)+(2+\beta,M),(\alpha,l)}<c(1+|\lambda|)^{-1+\beta/2}$.

Using Lemma (A.5) we can give the

Proof of Proposition (A.2). We have already studied the operators $K_1(z)$ and $K_1'(z)$ in our analysis of $\eta(H)$ and $d\eta/dt(H)$ in Sects. 3 and 4 of [8]. There we showed that $\mathrm{tr}\,K_1(z;y,x,y,x)$ and $\mathrm{tr}\,K_1'(z;y,x,y,x)$ extend to meromorphic functions of z for $\mathrm{Re}\,z>-\frac{1}{2}$ with possible poles at half integer values of z whose residues are expressible in terms of local expressions in the complete symbols of H and dH/dt. By invariance arguments and Gilkey's theorem we then showed that the residues of $\mathrm{tr}\,K_1$ vanish for $\mathrm{Re}\,z>-\frac{1}{2}$ while those of $\mathrm{tr}\,K_1'$ vanish for $z>0$. Finally, we showed that the residue of $\mathrm{tr}\,K_1'$ at $z=0$ must agree with the well known formula (A.2.1) for the variation of the η-invariant for the Dirac operator over a *compact* manifold. These observations establish the claims concerning K_1 and K_1'.

We next investigate K_2 and K_2'. First note that from (A.1.2.c), there exists a $c>0$ such that for all $(\alpha,l),\beta$ as in Proposition (A.1.2), and $\lambda\in\mathscr{C}$,

$$\|H(R(\lambda)-\bar{P}B(\lambda)\bar{P})\|_{(\alpha,l)+(\beta,N),(\alpha,l)}<c(1+|\lambda|)^{-3/2+\beta/2}. \tag{A.6}$$

Consequently, since N has been chosen such that $N>\dim(\mathbb{R}\times X)$, Lemma (B.3) of Appendix B implies that $H(R(\lambda)-\bar{P}B(\lambda)\bar{P})$ has a continuous kernel and that there exists a $c>0$ such that for $\lambda\in\mathscr{C}, \varepsilon>0$, and y,x,y',x',

$$|H(R(\lambda)-\bar{P}B(\lambda)\bar{P})(y,x;y',x')|<c(1+|\lambda|)^{-1+\varepsilon}. \tag{A.7}$$

It follows that $K_2(z)$ has a continuous kernel which is analytic for $\operatorname{Re} z>\frac{1}{2}$. By a similar argument, the identical result holds for $K_2'(z)$. The claims concerning $K_2(z;y,x,y,x)$ and $K_2'(z;y,x,y,x)$ follow.

To investigate $K_3(z)$, write.

$$K_3(z)=L_1(z)+L_2(z), \tag{A.8.a}$$

$$L_2(z)=-\int_{\mathscr{C}}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}H(1-\bar{P})(B(\lambda)-B^{\mathbb{R}}_{-2}(\lambda)-B^{\mathbb{R}}_{-3}(\lambda)), \tag{A.8.b}$$

$$L_2(z)=-\int_{\mathscr{C}}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}H(1-\bar{P})(B^{\mathbb{R}}_{-2}(\lambda)+B^{\mathbb{R}}_{-3}(\lambda)). \tag{A.8.c}$$

The bounds (A.5.2) imply that for (α,l), β as in (A.5) and $\lambda\in\mathscr{C}$,

$$\|H(1-\bar{P})(B(\lambda)-B^{\mathbb{R}}_{-2}(\lambda)-B^{\mathbb{R}}_{-3}(\lambda))\|_{(\alpha,l)+(\beta,\dim X+1),(\alpha,l)}<c(1+|\lambda|)^{-3/2+\beta/2}, \tag{A.9}$$

and thus $L_1(z)$ has a continuous kernel which is analytic for $\operatorname{Re} z>-\frac{1}{2}$. On the other hand, for $\operatorname{Re} z$ large,

$$\begin{aligned}
&-L_1(z;y,x,y,x)\\
&=\int_{\mathbb{R}}\frac{dE}{2\pi}\int_{\mathscr{C}}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}\\
&\cdot\left[\left(i\Gamma\left(iE+\theta\left(\frac{\partial}{\partial y}\right)\right)+D_y\right)(1-P)(b^{\mathbb{R}}_{-2}(\lambda;y,E)+b^{\mathbb{R}}_{-3})(\lambda;y,E)\right](x,x).
\end{aligned} \tag{A.10}$$

The singularities of the continuation to $z=0$ arise only from the terms $D_y(1-P)b_{-2}$, $-\Gamma E(1-P)b_{-3}$ and $i\Gamma\theta(1-P)b_{-3}$ and only from the integration region $E^2+|\lambda|\geqq 1$. Using the explicit expressions (A.3) for b_{-2} and b_{-3} we see that the local trace of the first of these terms vanishes since $\operatorname{tr}[(1-P)D](x,x)=0$, while the singularities in the local traces of the second and third terms cancel. (This cancellation is precisely the cancellation responsible for the finiteness at $z=0$ of the η invariant of the Dirac operator on S^1.)

By a similar argument we can show that $z\operatorname{tr}K_3'(z;y,x,y,x)$ is analytic for $\operatorname{Re} z>-\frac{1}{2}$, and moreover

$$\begin{aligned}
&\lim_{z\to 0}-z\operatorname{tr}K_3'(z;y,x,y,x)\\
&=\lim_{z\to 0}z\operatorname{tr}\left[\frac{dH}{dt}(y)(1-P)\int_{E^2+|\lambda|\geqq 1}\frac{dE}{2\pi}\frac{d\lambda}{2\pi i}\lambda^{-z-1/2}b^{\mathbb{R}}_{-2}(\lambda;y,E)\right](x,x)\\
&=\lim_{z\to 0}-z\frac{\Gamma(z)}{2\pi^{1/2}\Gamma(z+\frac{1}{2})}\operatorname{tr}\left[\Gamma\frac{dH}{dt}(y)(1-P)\right](x,x)\\
&=\frac{1}{2\pi i}\operatorname{tr}\left[\Gamma\frac{d}{dt}\theta\left(\frac{\partial}{\partial y}\right)(1-P)\right](x,x).
\end{aligned} \tag{A.11}$$

To get the last equality, note that $dH/dt = i\Gamma(d/dt)\theta(\partial/\partial y) + (dD/dt)$ and use $\operatorname{tr}(dD/dt)(1-P) = 0$. This proves (A.2.3).

The operators K_4 and K_4' can be investigated in a similar manner. In particular, we observe that $\lim_{z\to 0} zK_4'(z)$ can be calculated by replacing $B(\lambda)$ by $B^{\mathbb{R}}_{-2}(\lambda)$, but this substitution leads to the operator $\bar{P}B^{\mathbb{R}}_{-2}(\lambda)(1-\bar{P}) = 0$.

This completes the proof of (A.2). □

B. Notational Conventions

Let $W = X$ and $F = S \otimes E$ or $W = \mathbb{R} \times X$ and $F = \bar{S} \otimes \bar{E}$. Put a metric on W and a fiberwise hermitian inner product and compatible connection on F. For $W = \mathbb{R} \times X$ we will impose the additional requirement that the connections and inner products be invariant under translations in the $\mathbb{R}$ direction. This is a reasonable condition since we are assuming that our operators H are translation invariant for large $|y|, y \in \mathbb{R}$.

For integer k, let $\|\cdot\|_k$ denote the $k-th$ Sobolev norms on $C_0^\infty(W,F)$ determined by the metric, inner product and connection [13], and let $L_k^2(W,F)$ denote the Sobolev spaces obtained by completing $C_0^\infty(W,F)$ relative to these norms. Set $L^2_{-\infty}(W,F) = \bigcup_{k\in\mathbb{Z}} L_k^2(W,F)$.

In the case $W = \mathbb{R} \times X$, we define "mixed" Sobolev norms on sections of F as follows. For $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, and $f \in C_0^\infty(W,F)$ define the (α,k) Sobolev norm of f by

$$\|f\|^2_{(\alpha,k)} = \int_{\mathbb{R}} \frac{d\omega}{2\pi}(1+\omega^2)^\alpha \|\hat{f}(\omega,\cdot)\|_k^2 \quad \hat{f}(\omega,x) = \int_{\mathbb{R}} dy\, f(y,x)e^{i\omega y}, \tag{B.1}$$

where $\|\cdot\|_k$ denotes the k-th Sobolev norm on $C^\infty(X,F)$.

As with the usual Sobolev norms, the norms $\|\cdot\|_{(\alpha,k)}$ extended real valued functions $\|\cdot\|_{(\alpha,k)}: L^2_{-\infty}(W,F) \to [0,\infty]$. For a linear operator $O: L^2_{-\infty}(W,F) \to L^2_{-\infty}(W,F)$ and $j,k \in \mathbb{Z}$, $\alpha,\beta \in \mathbb{R}$ we set

$$\|O\|_{(j,\beta),(k,\alpha)} = \sup_{f \in L^2_{-\infty}(W,F)} \frac{\|Of\|_{(j,\beta)}}{\|f\|_{(k,\alpha)}}. \tag{B.2}$$

We will use these mixed Sobolev norms to measure the singularities in kernels of operators whose order in the $\mathbb{R}$ and X directions are different.

Lemma (B.3). *Let $O: L^2_{-\infty}(W,F) \mapsto L^2_{-\infty}(W,F)$. Suppose there exists $k > \frac{1}{2}\dim X$, $\alpha > \frac{1}{2}$, and $c > 0$ such that $\|O\|_{(\alpha,k),(-\alpha,-k)} < c$. Then as a bounded operator on $L_0^2(W,F)$, O has a continuous kernel and $|O(y,x;y',x')| < cc'(X)$, where $c'(X) > 0$ depends on X but not O.*

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