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Time Decay of Solutions to the Cauchy Problem for Time-Dependent Schrödinger–Hartree Equations

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Abstract. We consider the time-dependent Schrödinger–Hartree equation

$$iu_t + \Delta u = \left(\frac{1}{r} * |u|^2\right)u + \lambda \frac{u}{r}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1}$$

$$u(0,x) = \phi(x) \in \Sigma^{2,2}, \quad x \in \mathbb{R}^3, \tag{2}$$

where $\lambda \geqq 0$ and $\Sigma^{2,2} = \{g \in L^2;\ \ \|g\|^2_{\Sigma^{2,2}} = \sum_{|\alpha| \leqq 2} \|D^\alpha g\|_2^2 + \sum_{|\beta| \leqq 2} \|x^\beta g\|_2^2 < \infty\}$.

We show that there exists a unique global solution u of (1) and (2) such that

$$u \in C(\mathbb{R}; H^{1,2}) \cap L^\infty(\mathbb{R}; H^{2,2}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}; \Sigma^{2,2})$$

with

$$u_t \in L^\infty(\mathbb{R}; L^2).$$

Furthermore, we show that u has the following estimates:

$$\|u(t)\|_{2,2} \leqq C, \quad \text{a.e.} \quad t \in \mathbb{R},$$

and

$$\|u(t)\|_\infty \leqq C(1+|t|)^{-1/2}, \quad \text{a.e.} \quad t \in \mathbb{R}.$$

1. Introduction and Main Results

We consider the time decay of solutions to the Cauchy problem for the equation in $L^2 = L^2(\mathbb{R}^3)$

$$iu_t + \Delta u = f(|u|^2)u + \lambda V u, \quad t \in \mathbb{R}, \tag{1.1}$$

$$u(0) = \phi, \tag{1.2}$$

where $u_t = \partial_t u$, $f(|u|^2) = |x|^{-1} * |u|^2 = \int_{\mathbb{R}^3} |u(t,y)|^2/|x-y|dy$, $\lambda \geqq 0$, $V = 1/|x|$ and ϕ

is a given initial data. Let $\Sigma^{l,m}$ be the Hilbert space defined by

$$\Sigma^{l,m}=\{g\in L^2; \|g\|^2_{\Sigma^{l,m}}=\sum_{|\alpha|\leqq l}\|D^\alpha g\|_2^2+\sum_{|\beta|\leqq m}\|x^\beta g\|_2^2<\infty\}$$

with the inner product

$$(g,g)_{\Sigma^{l,m}}=\sum_{|\alpha|\leqq l}(D^\alpha g, D^\alpha g)+\sum_{|\beta|\leqq m}(x^\beta g, x^\beta g),$$

where

$$(f,g)=\int_{\mathbb{R}^3} f\cdot\bar{g}\,dx,\quad D^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}\quad\text{and}\quad x^\beta=x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}$$

$$(|\alpha|=\alpha_1+\alpha_2+\alpha_3,\quad |\beta|=\beta_1+\beta_2+\beta_3).$$

We shall prove the following:

Theorem 1. *We assume that $\lambda\geqq 0$ and $\phi\in\Sigma^{2,2}$. Then there exists a unique global solution u of (1.1) and (1.2) such that*

$$u\in C(\mathbb{R}; H^{1,2})\cap L^\infty(\mathbb{R}; H^{2,2})\cap L^\infty_{\mathrm{loc}}(\mathbb{R}; \Sigma^{2,2})$$

with

$$u_t\in L^\infty(\mathbb{R}; L^2). \tag{1.3}$$

Furthermore, there exist positive constants C_1 depending only on $\|\phi\|_{\Sigma^{2,1}}$ and C_2 depending only on $\|\phi\|_{\Sigma^{2,2}}$ such that

$$\|u(t)\|_{2,2}\leqq C_1,\quad \text{a.e.}\quad t\in\mathbb{R}, \tag{1.4}$$

and

$$\|u(t)\|_\infty\leqq C_2(1+|t|)^{-1/2},\quad \text{a.e.}\quad t\in\mathbb{R}. \tag{1.5}$$

In what follows positive constants will be denoted by C and will change from line to line. If necessary, by $C(*,\dots,*)$ we denote positive constants depending only on the quantities appearing in parentheses.

It has been shown in [9] that any solution $u\in C(\mathbb{R};\Sigma^{2,2})$ of (1.1) and (1.2) with $\lambda=0$ satisfies

$$\|u(t)\|_{2,2}\leqq C(\|\phi\|_{\Sigma^{2,1}})\cdot(1+\log(1+|t|)), \tag{1.6}$$

and

$$\|u(t)\|_\infty\leqq C(\|\phi\|_{\Sigma^{2,2}})(1+|t|)^{-1/2}. \tag{1.7}$$

Therefore, one of our results (1.4) is an improvement of (1.6).

When $\lambda>0$, Chadam and Glassey [1] showed that there exists a unique global solution u of (1.1) and (1.2) such that

$$u\in C(\mathbb{R}; H^{1,2})\cap L^\infty_{\mathrm{loc}}(\mathbb{R}; H^{2,2})$$

with

$$u_t\in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2). \tag{1.8}$$

Furthermore, they showed that

$$\|u(t)\|_{2,2} \leqq C(\|\phi\|_{2,2}) \exp[C(\|\phi\|_{2,2})|t|], \quad \text{a.e.} \quad t \in \mathbb{R}. \tag{1.9}$$

In [3] Dias and Figueira (see also Dias [2]) considered the time decay of solutions of (1.1) and (1.2) satisfying (1.8). They showed the following $L^p (2 < p \leqq 6)$ decay estimate:

$$\|u(t)\|_p \leqq C(\|\phi\|_{\Sigma^{1,1}})(1+|t|)^{-(3/2)(1/2-1/p)}. \tag{1.10}$$

Hence our results (1.3) and (1.4) are improvements of (1.8) and (1.9), and our result (1.5) is a new estimate in the case of $6 < p \leqq \infty$. Finally we put $J = x + 2it\nabla = UxU^{-1} = S2it\nabla S^{-1}$, where $U = U(t) = \exp(it\Delta)$ and $S = S(t) = \exp(i|x|^2/4t)$.

2. Proof of Theorem 1

We start with stating some useful lemmas.

Lemma 2.1. (*The Gagliardo–Nirenberg Inequality*) *Let* $1 \leqq q, r \leqq \infty$, *and let* $j, m \in \mathbb{N} \cup \{0\}$ *satisfy* $0 \leqq j < m$. *Then we have*

$$\sum_{|\beta|=j} \|D^\beta g\|_p \leqq C(m,j,q,r,a) \sum_{|\alpha|=m} \|D^\alpha g\|_r^a \|g\|_q^{1-a},$$

for any $g \in H^{m,r} \cap L^q$ *and* $1/p = j/3 + (1/r - m/3)a + (1-a)/q$, *for all* a *in the interval* $j/m \leqq a \leqq 1$, *with the following exception: if* $m - j - (3/r)$ *is a nonnegative integer, then the above inequality is asserted for* $a = j/m$.

For Lemma 2.1 see, e.g., Friedman [4].

Lemma 2.2. (a) *Let* $1 < p < q < \infty$, $0 < \delta < 3$ *and* $1/q = 1/p - \delta/3$. *Then we have*

$$\|I_\delta(g)\|_q \leqq C(\delta,p,q)\|g\|_p, \quad \textit{for any} \quad g \in L^p,$$

where $I_\delta(g)(x) = \int_{\mathbb{R}^3} g(y)/|x-y|^{3-\delta}\, dy$.

(b) $\int_{\mathbb{R}^3} |g(x)|^2/|x|^2\, dx \leqq 4\|\nabla g\|_2^2$, *for any* $g \in H^{1,2}$.

(c) $\int_{\mathbb{R}^3} |u(t,x)|^2/|x|^2 dx \leqq \|Ju(t)\|_2^2/t^2$, *for any* $Ju(t) \in C(\mathbb{R}; L^2)$.

Proof. (a) and (b) are well known results (see, e.g., Stein [10]). We only prove (c). We have by (b)

$$\int_{\mathbb{R}^3} |u(t,x)|^2/|x|^2\, dx = \int_{\mathbb{R}^3} |S^{-1}u(t,x)|^2/|x|^2\, dx \leqq 4\|\nabla S^{-1}u(t)\|_2^2$$

$$= \|2it\nabla S^{-1}u(t)\|_2^2/t^2 = \|Ju(t)\|_2^2/t^2.$$

This completes the proof.

We consider the auxiliary equation

$$i(u_n)_t + \Delta u_n = f(|u_n|^2)u_n + \lambda V_n u_n, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{2.1}$$

$$u_n(0,x) = \phi_n(x), \quad x \in \mathbb{R}^3, \tag{2.2}$$

where $\lambda \geqq 0$, $n \in \mathbb{N}$, $V_n = 1/(|x| + (1/n))$ for $\lambda > 0$, $V_n = 0$ for $\lambda = 0$ and $\{\phi_n\}$ is a sequence in the space $\mathscr{S}(\mathbb{R}^3)$ of rapidly decreasing infinitely differentiable functions such that $\phi_n \to \phi$ strongly in $\Sigma^{2,2}$ as $n \to \infty$. For the sake of brevity we suppress the subscript n of u_n for the moment.

Proposition 2.1. *For any $n \in \mathbb{N}$, the Cauchy problem (2.1) and (2.2) has a unique global solution u such that*

$$u \in C^\infty(\mathbb{R}; \mathscr{S}(\mathbb{R}^3)). \tag{2.3}$$

Furthermore, u satisfies

$$\|u(t)\|_2 = \|\phi_n\|_2, \tag{2.4}$$

$$\begin{aligned} &\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(f(|u|^2)u(t), u(t)) + \lambda(V_n u(t), u(t)) \\ &\quad = \|\nabla \phi_n\|_2^2 + \tfrac{1}{2}(f(|\phi_n|^2)\phi_n, \phi_n) + \lambda(V_n \phi_n, \phi_n), \end{aligned} \tag{2.5}$$

$$\|u(t)\|_{1,2} \leqq C(\|\phi_n\|_{1,2}), \tag{2.6}$$

and

$$\|u_t(t)\|_2, \|u(t)\|_{2,2} \leqq C(\|\phi_n\|_{2,2}) \exp[C(\|\phi_n\|_{2,2})|t|]. \tag{2.7}$$

Proof. In the same way as in the proof of Lemmas 3.1–3.3 in [1], we have (2.4)–(2.7). (For details see Chadam and Glassey [1].) We prove (2.3). M. Tsutsumi [11] showed that there exists a positive number T^* such that the Cauchy problem (2.1) and (2.2) has a unique solution $u \in C^\infty([-T^*, T^*]; \mathscr{S}(\mathbb{R}^3))$ for each ϕ_n. By the proof of Corollary 3.2 in [11], (2.3) is obtained if a priori estimates of $\|u(t)\|_{s,2}$ $(s > (3/2))$ are shown. Therefore, (2.7) gives (2.3). This completes the proof.

Proposition 2.2. *Let u be the solution constructed in Proposition 2.1. Then we have*

$$\begin{aligned} &\|Ju(t)\|_2^2 + 4t^2(\tfrac{1}{2}(f(|u|^2)u(t), u(t)) + \lambda(V_n u(t), u(t))) \\ &\quad = \|x\phi_n\|_2^2 + 4\int_0^t s(\tfrac{1}{2}(f(|u|^2)u(s), u(s)) + \lambda(V_n u(s), u(s)))\, ds \\ &\quad\quad + (4\lambda/n)\int_0^t s\|V_n u(s)\|_2^2\, ds, \end{aligned} \tag{2.8}$$

$$\|Ju(t)\|_2^2 \leqq C(\|\phi_n\|_{\Sigma^{1,1}})(1 + |t|), \quad \text{for } n > 4\lambda, \tag{2.9}$$

and

$$\left|\int_1^t s^{-2}\|Ju(s)\|_2^2\, ds\right| \leqq C(\|\phi_n\|_{\Sigma^{1,1}}), \quad \text{for} \quad |t| \geqq 1 \quad \text{and} \quad n > 4\lambda. \tag{2.10}$$

Proof. When $\lambda = 0$, (2.8) was shown by Ginibre and Velo [6, 7]. When $\lambda > 0$, (2.8) was shown by Dias and Figueira [3]. Therefore, we prove (2.9) and (2.10). We assume that $t \geqq 0$. The case $t \leqq 0$ can be treated similarly. Differentiating (2.8) with respect to t, we have

$$\frac{d}{dt}(\|Ju(t)\|_2^2 + 4t^2(\tfrac{1}{2}(f(|u|^2)u(t), u(t)) + \lambda(V_n u(t), u(t))))$$

$$= 4t(\tfrac{1}{2}(f(|u|^2)u(t), u(t)) + \lambda(V_n u(t), u(t))) + (4\lambda/n)t \| V_n u(t) \|_2^2. \tag{2.11}$$

We multiply (2.11) by t^{-1} and integrate with respect to t to obtain

$$\begin{aligned} t^{-1} \| Ju(t) \|_2^2 &+ 4t(\tfrac{1}{2}(f(|u|^2)u(t), u(t)) + \lambda(V_n u(t), u(t))) \\ &= \| Ju(1) \|_2^2 + 4(\tfrac{1}{2}(f(|u|^2)u(1), u(1)) + \lambda(V_n u(1), u(1))) \\ &\quad - \int_1^t s^{-2} \| Ju(s) \|_2^2 \, ds + (4\lambda/n) \int_1^t \| V_n u(s) \|_2^2 \, ds, \quad \text{for } t \geqq 1. \end{aligned} \tag{2.12}$$

By (2.5) and Lemmas 2.1–2.2 we have

$$|(f(|u|^2)u(t), u(t))| \leqq C \| u(t) \|_{12/5}^4 \leqq C(\| \phi_n \|_{1,2}), \tag{2.13}$$

$$\begin{aligned} |(V_n u(t), u(t))| &\leqq \| V_n u(t) \|_2 \| u(t) \|_2 \\ &\leqq C \| \nabla u(t) \|_2 \| u(t) \|_2 \leqq C(\| \phi_n \|_{1,2}), \end{aligned} \tag{2.14}$$

and

$$\| V_n u(t) \|_2^2 \begin{cases} \leqq 4 \| \nabla u(t) \|_2^2 \leqq C(\| \phi_n \|_{1,2}) \\ \leqq t^{-2} \| Ju(t) \|_2^2 \end{cases}. \tag{2.15}$$

We have by (2.8) and (2.13)–(2.15),

$$\begin{aligned} \| Ju(t) \|_2^2 &\leqq \| x\phi_n \|_2^2 + C \int_0^t s(|(f(|u|^2)u(s), u(s))| \\ &\quad + |(V_n u(s), u(s))|) \, ds + (4\lambda/n) \int_0^t s \| V_n u(s) \|_2^2 \, ds \\ &\leqq C(\| \phi_n \|_{\Sigma^{1,1}})(1+t)^2, \quad \text{for } t \geqq 0. \end{aligned} \tag{2.16}$$

We obtain by (2.12)–(2.16),

$$t^{-1} \| Ju(t) \|_2^2 + (1 - (4\lambda/n)) \int_1^t s^{-2} \| Ju(s) \|_2^2 \, ds \leqq C(\| \phi_n \|_{\Sigma^{1,1}}), \quad \text{for } t \geqq 1. \tag{2.17}$$

Since $(4\lambda/n) < 1$ (2.16) and (2.17) give (2.9) and (2.10). This completes the proof.

Remark 2.1 (2.10) plays an important role to improve (1.6).

Proposition 2.3. *Let u be the solution constructed in Proposition 2.1. Then for any $n > 4\lambda$, we have*

$$\| u(t) \|_{2,2} \leqq C(\| \phi_n \|_{\Sigma^{2,1}}), \tag{2.18}$$

and

$$\| u_t(t) \|_2 \leqq C(\| \phi_n \|_{\Sigma^{2,1}}). \tag{2.19}$$

Proof. A standard argument gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| \Delta u(t) \|_2^2 &= \operatorname{Im}(i\Delta u_t(t), \Delta u(t)) = \operatorname{Im}(\Delta(f(|u|^2)u(t)), \Delta u(t)) \\ &\quad + \lambda \operatorname{Im}(\Delta(V_n u(t)), \Delta u(t)). \end{aligned} \tag{2.20}$$

We consider the second term of the right-hand side of (2.20),

$$\begin{aligned}\operatorname{Im}(\Delta(V_n u(t)), \Delta u(t)) &= \operatorname{Im}((\Delta V_n)u(t), \Delta u(t)) + 2\operatorname{Im}(\nabla V_n\cdot\nabla u(t), \Delta u(t))\\ &= \operatorname{Im}((\Delta V_n)u(t), -iu_t(t) + f(|u|^2)u(t) + \lambda V_n u(t))\\ &\quad - 2\operatorname{Im}(V_n\Delta u(t), \Delta u(t)) - 2\operatorname{Im}(V_n\nabla u(t), \nabla\Delta u(t))\\ &= \frac{1}{2}\frac{d}{dt}((\Delta V_n)u(t), u(t)) - 2\operatorname{Im}(V_n\nabla u(t), \nabla\Delta u(t)). \qquad (2.21)\end{aligned}$$

We denote the second term of the right-hand side of (2.21) by I_1. We have

$$\begin{aligned}I_1 &= -2\operatorname{Im}(V_n\nabla u(t), \nabla(-iu_t(t) + f(|u|^2)u(t) + \lambda V_n u(t)))\\ &= -\frac{d}{dt}(V_n\nabla u(t), \nabla u(t)) - 2\operatorname{Im}(V_n\nabla u(t), (\nabla f(|u|^2))u(t))\\ &\quad - \lambda\operatorname{Im}(\nabla V_n^2\cdot\nabla u(t), u(t)).\end{aligned}$$

Similarly we have

$$\begin{aligned}-\operatorname{Im}(\nabla V_n^2\cdot\nabla u(t), u(t)) &= \operatorname{Im}(V_n^2\Delta u(t), u(t))\\ &= \operatorname{Im}(V_n^2(-iu_t(t) + f(|u|^2)u(t) + V_n u(t)), u(t))\\ &= -\frac{1}{2}\frac{d}{dt}\|V_n u(t)\|_2^2.\end{aligned}$$

Collecting everything, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Delta u(t)\|_2^2 + \lambda(\lambda\|V_n u(t)\|_2^2 - ((\Delta V_n)u(t), u(t)) + 2\|V_n^{1/2}\nabla u(t)\|_2^2))\\ &\quad = \operatorname{Im}(\Delta(f(|u|^2)u(t)), \Delta u(t)) - 2\lambda\operatorname{Im}(V_n\nabla u(t), (\nabla f(|u|^2))u(t)) = I_2 + 2\lambda I_3. \qquad (2.22)\end{aligned}$$

By Appendix and Proposition 2.1 (2.6) we have

$$\begin{aligned}|I_2| &\leqq Ct^{-2}\|Ju(t)\|_2^2\|\nabla u(t)\|_2\|\Delta u(t)\|_2\\ &\leqq C(\|\phi_n\|_{1,2})t^{-2}\|Ju(t)\|_2^2\|\Delta u(t)\|_2 \quad \text{for } t\geqq 1. \qquad (2.23)\end{aligned}$$

We get by (2.4) and Lemma 2.2

$$\begin{aligned}|I_3| &\leqq C\|\nabla u(t)\|_6\|\nabla f(|u|^2)(t)\|_3\|V_n u(t)\|_2\\ &\leqq C\|\nabla u(t)\|_6\|u(t)\|_6\|u(t)\|_2\|V_n u(t)\|_2\\ &\leqq C(\|\phi_n\|_2)t^{-2}\|Ju(t)\|_2^2\|\Delta u(t)\|_2 \quad \text{for } t\geqq 1. \qquad (2.24)\end{aligned}$$

Since $\lambda\geqq 0$ and $\Delta V_n\leqq 0$, we obtain by (2.22)–(2.24)

$$\begin{aligned}\|\Delta u(t)\|_2^2 &\leqq \|\Delta u(1)\|_2^2 + \lambda(\lambda\|V_n u(1)\|_2^2 - ((\Delta V_n)u(1), u(1)) + 2\|V_n^{1/2}\nabla u(1)\|_2^2)\\ &\quad + C(\|\phi_n\|_{1,2})\int_1^t s^{-2}\|Ju(s)\|_2^2\|\Delta u(s)\|_2\,ds. \qquad (2.25)\end{aligned}$$

We have by Lemmas 2.1–2.2 and Proposition 2.1,

$$\| V_n u(1) \|_2^2 \leqq C \| \nabla u(1) \|_2^2 \leqq C(\| \phi_n \|_{1,2}), \tag{2.26}$$

$$\| V_n^{1/2} \nabla u(1) \|_2^2 \leqq C \| \Delta u(1) \|_2 \| \nabla u(1) \|_2 \leqq C(\| \phi_n \|_{2,2}), \tag{2.27}$$

and

$$\begin{aligned} -((\Delta V_n) u(1), u(1)) &= 2\mathrm{Re}\,((\nabla V_n)\cdot \nabla u(1), u(1)) \\ &= -2\mathrm{Re}\,(V_n \Delta u(1), u(1)) - 2(V_n \nabla u(1), \nabla u(1)) \leqq C(\| \phi_n \|_{2,2}). \end{aligned} \tag{2.28}$$

We have by (2.25)–(2.28),

$$\| \Delta u(t) \|_2^2 \leqq C(\| \phi_n \|_{2,2}) \left(1 + \int_1^t s^{-2} \| J u(s) \|_2^2 \| \Delta u(s) \|_2 \, ds \right).$$

This and the Schwarz inequality give

$$\| \Delta u(t) \|_2^2 \leqq C(\| \phi_n \|_{2,2}) \left(1 + \int_1^t s^{-2} \| J u(s) \|_2^2 \, ds + \int_1^t s^{-2} \| J u(s) \|_2^2 \| \Delta u(s) \|_2^2 \, ds \right). \tag{2.29}$$

We obtain by (2.29), Proposition 2.2 (2.10) and Gronwall's inequality,

$$\| \Delta u(t) \|_2 \leqq C(\| \phi_n \|_{\Sigma^{2,1}}). \tag{2.30}$$

Proposition 2.1 and (2.30) yield (2.18). From (2.1), Lemma 2.2, Proposition 2.1 and (2.3) we have

$$\begin{aligned} \| u_t(t) \|_2 &\leqq \| \Delta u(t) \|_2 + \| f(|u|^2) u(t) \|_2 + \lambda \| V_n u(t) \|_2 \\ &\leqq C(\| \phi_n \|_{\Sigma^{2,1}}) + \| f(|u|^2)(t) \|_\infty \| u(t) \|_2 + C \| \nabla u(t) \|_2 \\ &\leqq C(\| \phi_n \|_{\Sigma^{2,1}}) + C \| u(t) \|_2^2 \| \nabla u(t) \|_2 \leqq C(\| \phi_n \|_{\Sigma^{2,1}}). \end{aligned}$$

Here we have used

$$\begin{aligned} \| f(|\phi\psi|) \|_\infty &\leqq \operatorname*{esssup}_{x\in\mathbb{R}^3} \int \frac{|\phi(y)||\psi(y)|}{|x-y|} dy \leqq \operatorname*{esssup}_{x\in\mathbb{R}^3} \left(\int \frac{|\phi(y)|^2}{|x-y|^2} dy \right)^{1/2} \| \psi \|_2 \\ &\leqq 2 \| \nabla \phi \|_2 \| \psi \|_2. \end{aligned} \tag{2.31}$$

This completes the proof.

Proposition 2.4. *Let u be the solution constructed in Proposition 2.1. Then for any $n > 4\lambda$, we have*

$$\| J^2 u(t) \|_2 \leqq C(\| \phi_n \|_{\Sigma^{1,2}})(1 + |t|)^{3/2},$$

where

$$J^2 = \sum_{j=1}^3 (x_j + 2it\partial_j)^2 = U |x|^2 U^{-1} = S(-4t^2 \Delta) S^{-1}.$$

Proof. We put $v(t) = S^{-1} u(t)$ for $t \in \mathbb{R} \setminus \{0\}$. It is easily verified that $v \in C^1(\mathbb{R} \setminus \{0\};$

$\mathscr{S}(\mathbb{R}^3)$),

$$S^{-1}\left(i\frac{d}{dt}+\Delta\right)Sv=\left(i\frac{d}{dt}+\Delta-\frac{1}{t}A\right)v,$$

where $A=(1/2i)(x\cdot\nabla+\nabla\cdot x)$. Therefore, v satisfies

$$iv_t=-\Delta v+\frac{1}{t}Av+f(|v|^2)v+\lambda V_n v,\quad t\in\mathbb{R}\setminus\{0\}. \tag{2.32}$$

Since J^2 commutes with $i(d/dt)+\Delta, J^2u(t)$ satisfies

$$i(J^2u(t))_t=-\Delta J^2u+J^2(f(|u|^2)u(t))+\lambda J^2(V_n u(t)),$$

from which it follows that

$$\frac{1}{2}\frac{d}{dt}\|J^2u(t)\|_2^2=\operatorname{Im}(J^2(f(|u|^2)u(t)),J^2u(t))+\lambda\operatorname{Im}(J^2(V_nu(t)),J^2u(t)). \tag{2.33}$$

We consider the second term of the right-hand side of (2.33). Since $J^2=-4t^2S\Delta S^{-1}$,

$$\begin{aligned}
&\operatorname{Im}(J^2(V_nu(t)),J^2u(t))\\
&\quad=16t^4\operatorname{Im}(\Delta(V_nv(t)),\Delta v(t))\\
&\quad=16t^4\operatorname{Im}((\Delta V_n)v(t),\Delta v(t))+2(\nabla V_n\cdot\nabla v(t),\Delta v(t)))\\
&\quad=16t^4\operatorname{Im}(((\Delta V_n)v(t),\Delta v(t))-2(V_n\nabla v(t),\nabla\Delta v(t)))\\
&\quad=16t^4\operatorname{Im}((\Delta V_n)v(t),-iv_t(t)+\frac{1}{t}Av(t)+f(|v|^2)v(t)+\lambda V_nv(t))\\
&\qquad-32t^4\operatorname{Im}(V_n\nabla v(t),\nabla(-iv_t(t)+\frac{1}{t}Av(t)+f(|v|^2)v(t)+\lambda V_nv(t)))\\
&\quad=8t^4\frac{d}{dt}((\Delta V_n)v(t),v(t))+16t^3\operatorname{Im}((\Delta V_n)v(t),Av(t))\\
&\qquad-16t^4\frac{d}{dt}(V_n\nabla v(t),\nabla v(t))-32t^3\operatorname{Im}(V_n\nabla v(t),\nabla(Av(t)))\\
&\qquad-32t^4\operatorname{Im}(V_n\nabla v(t),\nabla(f(|v|^2)v(t)))-32t^4\lambda\operatorname{Im}(V_n\nabla v(t),(\nabla V_n)v(t))\\
&\quad=\frac{d}{dt}[8t^4((\Delta V_n)v(t),v(t))-16t^4(V_n\nabla v(t),\nabla v(t))]\\
&\qquad-32t^3((\Delta V_n)v(t),v(t))+64t^3(V_n\nabla v(t),\nabla v(t))\\
&\qquad+8t^3\operatorname{Im}([A,\Delta V_n]v(t),v(t))-16t^3\operatorname{Im}([A,V_n]\nabla v(t),\nabla v(t))\\
&\qquad-32t^3\operatorname{Im}(V_n\nabla v(t),[\nabla,A]v(t))-32t^4\operatorname{Im}(V_n\nabla v(t),\nabla(f(|v|^2)v(t)))\\
&\qquad-16t^4\lambda\operatorname{Im}(\nabla v(t),(\nabla V_n^2)v(t)).
\end{aligned}$$

We note that

$$[\nabla, A] = -i\nabla, \quad [A, V_n] = -i(x\cdot\nabla)V_n \quad \text{and} \quad [A, \Delta V_n] = -i(x\cdot\nabla)\Delta V_n.$$

Therefore, we have

$$\begin{aligned}
&\operatorname{Im}(J^2(V_n u(t)), J^2 u(t)) \\
&= \frac{d}{dt}[8t^4((\Delta V_n)v(t), v(t)) - 16t^4(V_n \nabla v(t), \nabla v(t))] \\
&\quad + 16t^3((2V_n + (x\cdot\nabla)V_n)\nabla v(t), \nabla v(t)) - 8t^3((4\Delta V_n + (x\cdot\nabla)\Delta V_n)v(t), v(t)) \\
&\quad - 32t^4 \operatorname{Im}(V_n \nabla v(t), \nabla(f(|v|^2)v(t))) - 16t^4\lambda \operatorname{Im}(\nabla v(t), (\nabla V_n^2)v(t)).
\end{aligned}$$

We finally note that

$$\begin{aligned}
\operatorname{Im}(\nabla v(t), (\nabla V_n^2)v(t)) &= -\operatorname{Im}(\Delta v(t), V_n^2 v(t)) = -\operatorname{Im}\left(-iv_t(t) + \frac{1}{t}Av(t), V_n^2 v(t)\right) \\
&= \frac{1}{2}\frac{d}{dt}\| V_n v(t)\|_2^2 + \frac{1}{2t}\operatorname{Im}([A, V_n^2]v(t), v(t)) \\
&= \frac{1}{2}\frac{d}{dt}\| V_n v(t)\|_2^2 - \frac{1}{2t}((x\cdot\nabla)V_n^2 v(t), v(t)).
\end{aligned}$$

Collecting everything, we obtain

$$\begin{aligned}
&\frac{d}{dt}[\tfrac{1}{2}\| J^2 u(t)\|_2^2 + 8t^4\lambda(2(V_n\nabla v(t), \nabla v(t)) - ((\Delta V_n)v(t), v(t)) + \lambda \| V_n v(t)\|_2^2)] \\
&= 8t^3\lambda[2((2V_n + (x\cdot\nabla)V_n)\nabla v(t), \nabla v(t)) \\
&\quad + ((4\lambda V_n^2 + \lambda(x\cdot\nabla)V_n^2 - 4\Delta V_n - (x\cdot\nabla)\Delta V_n)v(t), v(t))] \\
&\quad - 32t^4\lambda \operatorname{Im}(V_n\nabla v(t), (\nabla f(|v|^2))v(t)) + 16t^4 \operatorname{Im}(\Delta(f(|v|^2)v(t), \Delta v(t)) \\
&= I_4 + I_5 + I_6.
\end{aligned} \tag{2.34}$$

We assume that $t \geqq 0$. The case $t \leqq 0$ can be proved similarly. In view of the fact that $(x\cdot\nabla)V_n \leqq 0$, $(x\cdot\nabla)V_n^2 \leqq 0$, $-(x\cdot\nabla)\Delta V_n \leqq 0$ and $\lambda \geqq 0$, we have

$$\begin{aligned}
|I_4| &\leqq 32t^3\lambda[(V_n\nabla v(t), \nabla v(t)) + \lambda \| V_n v(t)\|_2^2 - (\Delta V_n v(t), v(t))] \\
&= 32t^3\lambda[-(V_n\nabla v(t), \nabla v(t)) + \lambda \| V_n v(t)\|_2^2 - 2\operatorname{Re}(V_n \Delta v(t), v(t))] \\
&\leqq Ct^3[\| V_n v(t)\|_2^2 - 2\operatorname{Re}(V_n\Delta v(t), v(t))] \\
&\leqq Ct^3(\|\nabla v(t)\|_2^2 + \| V_n v(t)\|_2 \|\Delta v(t)\|_2) \\
&\leqq Ct\| Ju(t)\|_2^2 + C\| Ju(t)\|_2 \| J^2 u(t)\|_2.
\end{aligned} \tag{2.35}$$

Here we have used Lemma 2.2. We obtain by Lemma 2.2, Proposition 2.2

$$|I_5| \leqq 32t^4\lambda \|\nabla v(t)\|_2 \| V_n v(t)\|_2 \| f(2\operatorname{Re} v\overline{\nabla v})\|_\infty \leqq Ct^4\lambda \|\nabla v(t)\|_2^2 \| f(|v\overline{\nabla v}|)\|_\infty. \tag{2.36}$$

In the same way as in the proof of (2.31) we have

$$\|f(|v\overline{\nabla v}|)\|_\infty \leqq 2\|\nabla v\|_2^2. \tag{2.37}$$

This and (2.36) yield

$$|I_5| \leqq Ct^4\|\nabla v(t)\|_2^4 \leqq C\|Ju(t)\|_2^4. \tag{2.38}$$

By Appendix we have

$$|I_6| \begin{cases} \leqq Ct^{-1}\|Ju(t)\|_2^3\|J^2u(t)\|_2, \\ \leqq C\|\nabla u(t)\|_2\|Ju(t)\|_2^2\|J^2u(t)\|_2. \end{cases} \tag{2.39}$$

Now we put

$$\alpha(t) = \tfrac{1}{2}\|J^2u(t)\|_2^2 + 8t^4\lambda(2(V_n\nabla v(t), \nabla v(t)) - ((\Delta V_n)v(t), v(t)) + \lambda\|V_nv(t)\|_2^2).$$

We note that $\frac{1}{2}\|J^2u(t)\|_2^2 \leqq \alpha(t)$. We obtain from (2.34), (2.35), (2.38), (2.39) and Proposition 2.2,

$$\frac{d}{dt}\alpha(t) \leqq C(\|\phi_n\|_{\Sigma^{1,1}})((1+t)^2 + (1+t)^{1/2}\alpha(t))$$

$$\leqq \alpha(t)(1+t)^{-1} + C(\|\phi_n\|_{\Sigma^{1,1}})(1+t)^2.$$

This gives

$$\frac{d}{dt}(\alpha(t)(1+t)^{-1}) \leqq C(\|\phi_n\|_{\Sigma^{1,1}})(1+t),$$

from which we get the desired result.

Proof of Theorem 1. A simple calculation gives

$$\|x^2u_n(t)\|_2 \leqq C\|u_n(t)\|_{2,2}(1+t^2) + C\|J^2u_n(t)\|_2. \tag{2.40}$$

By Proposition 2.1–2.3, (2.40) and a standard argument we conclude that there exists a unique function u satisfying (1.3) and (1.4) such that as $n \to \infty$,

$$u_n \to u \quad \text{weakly star in } L^\infty(\mathbb{R}; H^{2,2}) \cap L^\infty_{\text{loc}}(\mathbb{R}; \Sigma^{2,2}),$$
$$u_n \to u \quad \text{strongly in } C(\mathbb{R}; H^{1,2}),$$

and

$$(u_n)_t \to u_t \quad \text{weakly star in } L^\infty(\mathbb{R}; L^2).$$

It is easily seen that u solves the Cauchy problem (1.1), (1.2) in the distribution sense. Next we show that u satisfies (1.5). From Proposition 2.3 we see that u satisfies

$$\|u(t)\|_{2,2} \leqq C(\|\phi\|_{\Sigma^{2,1}}), \quad \text{a.e.} \quad t \in \mathbb{R}. \tag{2.41}$$

Proposition 2.2 and Proposition 2.4 give

$$\|Ju(t)\|_2 \leqq C(\|\phi\|_{\Sigma^{1,1}})(1+|t|)^{1/2}, \quad \text{a.e.} \quad t \in \mathbb{R}, \tag{2.42}$$

and

$$\|J^2u(t)\|_2 \leqq C(\|\phi\|_{\Sigma^{1,2}})(1+|t|)^{3/2}, \quad \text{a.e.} \quad t \in \mathbb{R}. \tag{2.43}$$

We have by using Lemma 2.1 and (2.41)–(2.43)

$$\begin{aligned}\|u(t)\|_\infty &\leqq C(1+|t|)^{-1/2}(\|Ju(t)\|_2+\|\nabla u(t)\|_2)^{1/2}\\ &\quad\times(1+|t|)^{-1}(\|J^2u(t)\|_2+\|\Delta u(t)\|_2)^{1/2}\\ &\leqq C(\|\phi\|_{\Sigma^{2,2}})(1+|t|)^{-1/2}, \quad \text{a.e.} \quad t\in\mathbb{R}.\end{aligned}$$

This completes the proof.

Remark 2.2. We can apply our method used in this paper to the following system:

$$i(u_j)_t+\Delta u_j=\sum_{k=1}^{N}(u_jv_{k,k}-u_kv_{j,k})+\lambda u_j/r, \quad t\in\mathbb{R},$$

$$u_j(0)=\phi_j\in\Sigma^{2,2},$$

where $j=1,2,\dots,N$, $v_{j,k}=r^{-1}*u_j\bar{u}_k$, and $\lambda>0$.

Especially the equality (2.34) in the proof of Proposition 2.4 is useful to investigate the decay properties of solutions for the linear Schrödinger equations,

$$iu_t+\Delta u=Vu, \quad t\in\mathbb{R}, \quad u(0)=\phi,$$

where $V=V(x)$ is real-valued function satisfying some additional conditions.

Appendix

Lemma A. *Let* $f(\phi)(x)=\int_{\mathbb{R}^3}\phi(y)|x-y|^{-1}\,dy$. *Then we have*

$$|\mathrm{Im}\,(\Delta(f(|\phi|^2)\phi),\Delta\phi)|$$
$$\leqq Ct^{-2}\|J\phi\|_2^2\|\nabla\phi\|_2\|\Delta\phi\|_2, \quad \textit{for } \phi\in\Sigma^{2,1} \quad \textit{and} \quad t\in\mathbb{R}\setminus\{0\}. \tag{A.1}$$

$$|t^4\mathrm{Im}\,(\Delta(f(|\phi|^2)S^{-1}\phi),\Delta S^{-1}\phi)|$$
$$\begin{cases}\leqq C|t|^{-1}\|J\phi\|_2^3\|J^2\phi\|_2\\ \leqq C\|\nabla\phi\|_2\|J\phi\|_2^2\|J^2\phi\|_2, \quad \textit{for } \phi\in\Sigma^{2,2} \quad \textit{and} \quad t\in\mathbb{R}\setminus\{0\}.\end{cases} \tag{A.2}$$

Proof. (See also [9]). We put $\phi(t)=S^{-1}\phi$ and

$$f_j(\psi)(x)=\int_{\mathbb{R}^3}(x_j-y_j)\psi(y)/|x-y|^3\,dy, \quad 1\leqq j\leqq 3.$$

A simple calculation gives

$$\begin{aligned}t^4\mathrm{Im}\,(\Delta(f(|\phi(t)|^2)\phi(t)),\Delta\phi(t)) &= -2t^4\mathrm{Im}\sum_{j=1}^{3}(f_j(\mathrm{Re}\,\overline{\phi(t)}\partial_j\phi(t))\phi(t),\Delta\phi(t))\\ &\quad+4t^4\mathrm{Im}\,(f(\mathrm{Re}\,\overline{\phi(t)}\nabla\phi(t))\nabla\phi(t),\Delta\phi(t)).\end{aligned}$$

We have by Hölder's inequality and Lemmas 2.1–2.2

$$\begin{aligned}&\|f_j(\mathrm{Re}\,\overline{\phi(t)}\partial_j\phi(t))\phi(t)\|_2\\ &\leqq\|f_j(\mathrm{Re}\,\overline{\phi(t)}\partial_j\phi(t))\|_3\|\phi(t)\|_6\\ &\leqq C\|\overline{\phi(t)}\partial_j\phi(t)\|_{3/2}\|\phi(t)\|_6\leqq C\|\phi(t)\|_6^2\|\partial_j\phi(t)\|_2\\ &\leqq C\|\nabla\phi(t)\|_2^3\leqq C|t|^{-3}\|J\phi\|_2^3.\end{aligned}$$

From (2.31) it follows that

$$\begin{aligned}\| f(\operatorname{Re}\overline{\phi(t)}\nabla\phi(t))\nabla\phi(t)\|_2 &\leqq \| f(|\overline{\phi(t)}\nabla\phi(t)|)\|_\infty \|\nabla\phi(t)\|_2 \\ &\leqq C\|\nabla\phi(t)\|_2^3 \leqq C|t|^{-3}\|J\phi\|_2^3.\end{aligned}$$

Therefore, we have the first inequality of (A.2). Similarly, we have

$$\begin{aligned}t^4\operatorname{Im}(\Delta(f(|\phi|^2)\phi(t)),\Delta\phi(t)) = &-2t^4\operatorname{Im}\sum_{j=1}^{3}(f_j(\operatorname{Re}\bar{\phi}\partial_j\phi)\phi(t),\Delta\phi(t)) \\ &+4t^4\operatorname{Im}(f(\operatorname{Re}\bar{\phi}\nabla\phi)\nabla\phi(t),\Delta\phi(t)).\end{aligned}$$

The second inequality of (A.2) follows from

$$\| f_j(\operatorname{Re}\bar{\phi}\,\partial_j\phi)\|_3 \leqq C\|\phi(t)\|_6\|\partial_j\phi\|_2 \leqq C|t|^{-1}\|J\phi\|_2\|\nabla\phi\|_2$$

and

$$\| f(\operatorname{Re}\bar{\phi}\,\nabla\phi)\|_\infty \leqq C|t|^{-1}\|J\phi\|_2\|\nabla\phi\|_2.$$

Inequality (A.1) is obtained in the same way as in the preceding argument.

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