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Stability of Classical Solutions of Two-Dimensional Grassmannian Models

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Abstract. We show that the only finite-action solutions of the two-dimensional Grassmannian σ-model that are stable under small fluctuations are the (anti-)instanton solutions.

0. Introduction

(0.1) The two-dimensional Grassmannian σ-model is a field theory which shares many of the properties of the (more complicated) four-dimensional non-abelian gauge theories: for instance, the action is conformally invariant, there is a topological charge and the associated (anti-)instantons minimise the action among all fields with the same charge. For a survey of this theory, see [11].

(0.2) It is of interest to know whether there exist any non-instanton solutions in this model that are stable under small fluctuations. It is the purpose of this article to answer this question in the negative; thus all non-(anti-)instanton solutions are saddle points for the action. Our technique uses methods of Algebraic Geometry to ensure a sufficiently large number of non-positive modes for the fluctuation operator so that stability is only possible for (anti-)instanton solutions. These non-positive modes are essentially provided by solutions of the background fermion problem.

1. Preliminaries

(1.1) The non-linear σ-model is a field theory where the dynamical variable takes values in a Riemannian manifold (N, h). The Lagrangian density and action for this model are given by

$$L(\varphi)=h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial_\mu\varphi^\beta\,, \qquad S=\int L d^n x\,. \tag{1}$$

We are interested in finite-action solutions of the equations of motion, which are known to mathematicians as *harmonic maps* (see e.g. [3]). We shall restrict attention to the 2-dimensional Euclidean version of the model, which is of most interest to physicists since it shares a number of properties with $4d$ non-abelian gauge theories. In particular, in this case, the action is conformally invariant and, by a result of Sacks and Uhlenbeck [9], any finite-action solution of the equations of motion extends to a solution on the conformal compactification of $\mathbf{R}^2$, the Riemann sphere $S^2=\mathbf{R}^2\cup\{\infty\}$. Henceforth therefore, we shall suppose, without loss of generality, that all fields are defined on S^2 since we may then apply the methods of Algebraic Geometry.

(1.2) From now on, we take as our manifold (N, h), the complex Grassmannian $G_{r,n}$ which is the coset space $\frac{U(n)}{U(r)\times U(n-r)}$. Following Zakrzewski [11], we identify $G_{r,n}$ with the rank r projection matrices, i.e. $n\times n$ matrices φ satisfying

$$\varphi^2=\varphi\,, \qquad \varphi=\varphi^+\,, \qquad \operatorname{rank}\varphi=r\,, \tag{2}$$

where $^+$ denotes Hermitian conjugation. Differentiating (2), we see that the tangent space to $G_{r,n}$ at φ is the set of Hermitian matrices, A, satisfying

$$\varphi A\varphi=0=(1-\varphi)A(1-\varphi)\,. \tag{3}$$

The Lagrangian density (1) in this case is given by

$$L(\varphi)=\operatorname{trace}\partial_\mu\varphi\partial_\mu\varphi\,, \tag{4}$$

while the equations of motion are

$$[\varphi,\partial_\mu\partial_\mu\varphi]=0\,, \tag{5}$$

together with the constraint (2).

(1.3) $G_{r,n}$ is a Kähler manifold with Kähler 2-form ω, so that a field has a topological charge with density

$$q=i\varepsilon_{\mu\nu}\operatorname{trace}[\partial_\mu\varphi,\varphi]\partial_\nu\varphi\,, \qquad Q=\int d^2x\,q=\int_{S^2}\varphi*\omega\,. \tag{6}$$

Then $S\geqq|Q|$ with equality for the (anti-)instanton solutions of (5) which thus minimise the action over all fields with the same charge. If we replace our Euclidean co-ordinates (x_1, x_2) by holomorphic co-ordinates $x_\pm=x_1\pm ix_2$, then the instanton, respectively anti-instanton, equations are given by (a), respectively (b), below:

$$\text{(a)}\ \varphi\partial_+\varphi=0\,, \qquad \text{(b)}\ \varphi\partial_-\varphi=0\,. \tag{7}$$

The topological charge admits a geometrical interpretation which will be useful below: a field φ defines vector bundles $\underline{\varphi}$, $\underline{\varphi}^\perp$ over S^2 of ranks r and $n-r$ respectively, by

$$\underline{\varphi}_x=\operatorname{Image}\varphi(x)\subset\mathbf{C}^n\,, \qquad \underline{\varphi}_x^\perp=\operatorname{Kernel}\varphi(x)\subset\mathbf{C}^n\,. \tag{8}$$

Clearly, $\underline{\varphi}\oplus\underline{\varphi}^\perp$ provides a non-trivial splitting of the trivial bundle $S^2\times\mathbf{C}^n$. Suitably normalising the volume of the Riemann sphere, we have

$$Q=-\deg(\underline{\varphi})\,, \tag{9}$$

where $\deg(\underline{\varphi})$ denotes the first Chern class of $\underline{\varphi}$ evaluated on the generator of $H_2(S^2)$.

(1.4) Now let φ be a finite action solution of the equations of motion (5) and let φ_t be a small fluctuation about $\varphi=\varphi_0$ with $\left.\frac{\partial\varphi_t}{\partial t}\right|_{t=0}=B$. Clearly, B satisfies (3) and, conversely, any field of Hermitian matrices satisfying (3) gives rise to a fluctuation by exponentiation. We say that φ is *stable* if for all fluctuations around φ we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} S(\varphi_t) \geqq 0,$$

i.e. if φ minimises the action up to second order. In terms of the infinitesimal fluctuation, B, we write this second derivative as

$$H_\varphi(B,B) = \left.\frac{d^2}{dt^2}\right|_{t=0} S(\varphi_t) = \int dx^2 \{\operatorname{tr} D_\mu B D_\mu B - \operatorname{tr}[[B, \partial_\mu \varphi], \partial_\mu \varphi] B\} \tag{10}$$

(cf. [3]), where the covariant derivative is given by

$$D_\mu = \partial_\mu + \operatorname{ad}[\varphi, \partial_\mu \varphi]. \tag{10a}$$

In fact, D_μ is just the pull-back of the Levi-Civita connection on $G_{r,n}$, while the second term in the integrand comes from the curvature of D_μ:

$$[D_\mu, D_\nu] = \operatorname{ad}[\partial_\mu \varphi, \partial_\nu \varphi]. \tag{11}$$

(1.5) Now let B_1, B_2 be two infitesimal fluctuations and consider the complex fluctuation $B_1 + iB_2$. As integration by parts using (11) gives the following formula of Moore and Micallef [8]:

$$\begin{aligned} H_\varphi(B, \bar{B}) &= H_\varphi(B_1, B_1) + H_\varphi(B_2, B_2) \\ &= 4 \int d^2x \{\operatorname{tr}(D_- B)^+ (D_- B) - \operatorname{tr} B^+ [[B, \partial_+ \varphi], \partial_- \varphi]\}. \end{aligned} \tag{12}$$

Further,

$$\operatorname{tr} B^+ [[B, \partial_+ \varphi], \partial_- \varphi] = -\operatorname{tr}[B^+, \partial_- \varphi][B, \partial_+ \varphi] = \operatorname{tr}[B, \partial_+ \varphi]^+ [B, \partial_+ \varphi] \geqq 0.$$

Thus we have

Theorem. *Let φ be a finite action stable solution of* (5) *and B a complex fluctuation with*

$$D_- B = 0, \tag{13}$$

then

$$[B, \partial_+ \varphi] = 0. \tag{14}$$

In Sect. 3, we shall find that there are sufficiently many solutions of (13) for (14) to force φ to be an (anti-)instanton solution.

(1.6) *Remark.* Equation (13) is a Cauchy-Riemann equation (see below) but also admits another interpretation: the Kähler structure of S^2 endows it with a $\mathrm{Spin}^{\mathbf{c}}(2)$ structure and hence a Dirac operator. If we consider fermion fields in the background of φ then the $-i$ eigenstates of γ_5 are precisely the complex flucatuations considered above and the Dirac-like equation in the background of φ is just (13).

Warning. This $\mathrm{Spin}^{\mathbf{c}}(2)$ structure is *not* induced by the Spin(2) structure of S^2; in particular, one of the $\frac{1}{2}$-spin bundles in our case is trivial (which is why we may identify fermions with fluctuations).

Solutions of (13) have also been used by Zakrzewski [11] to provide negative modes of fluctuation for certain solutions of (5).

2. Algebraic Geometry and Vector Bundles

(2.1) Given a field φ, the infinitesimal complex fluctuations about φ are sections of the vector bundle $\varphi^{-1}TG_{r,n}\otimes\mathbf{C}$. To study solutions of (13), we apply a theorem of Koszul-Malgrange to convert the problem into one of Algebraic Geometry:

Theorem [6]. *Let $E\to M^2$ be a complex vector bundle over a Riemann surface with covariant derivative D. Then there is a unique holomorphic structure on E for which the local holomorphic sections are precisely the solutions of the equation*

$$D_{-}\sigma=0.$$

Thus we are guaranteed a sufficiently large supply of *local* solutions of (13) that they span each fibre of $\varphi^{-1}TG_{r,n}\otimes\mathbf{C}$. However, we require globally defined solutions of (13) and for this we need more structure.

(2.2) The simplest holomorphic vector bundles are the line bundles, i.e. those with one-dimensional fibres. Concerning these we have the following useful proposition (see, for instance, the book of Griffiths and Harris [4]).

Proposition. *Let $L\to M^2$ be a holomorphic line bundle of positive degree. Then for each $x\in M^2$ there is a global holomorphic section σ with $\sigma(x)\neq 0$.*

(2.3) To reduce our situation to that of (2.2), we have recourse to the factorization theorem of Birkhoff-Grothendieck [5]:

Theorem. *Let $E\to S^2$ be a holomorphic vector bundle over the Riemann sphere. Then there is an essentially unique decomposition of E as a sum of holomorphic line sub-bundles*

$$E=L_1\oplus\ldots\oplus L_r$$

with $\deg(L_i)\geqq\deg(L_{i+1})$.

Remark. This theorem only applies to S^2 and is in fact the only point in our arguments where we require that our fields be defined on S^2 rather than satisfy some other (e.g. periodic) boundary condition.

(2.4) Let us now apply the foregoing theory to the sub-bundles $\underline{\varphi}$, $\underline{\varphi}^{\perp}$ of $S^2\times\mathbf{C}^n$ defined in (1.3). First, observe that the covariant derivative

$$D_\mu=\partial_\mu+[\varphi,\partial_\mu\varphi] \tag{15}$$

satisfies

$$(1-\varphi)\circ D_\mu\circ\varphi=\varphi\circ D_\mu\circ(1-\varphi)=0, \tag{16}$$

using (3) applied to $\partial_\mu\varphi$. Thus D_μ preserves sections of $\underline{\varphi}$ and $\underline{\varphi}^{\perp}$ and so is a covariant derivative there. Now, applying Theorem (2.1), $\underline{\varphi}$ and $\underline{\varphi}^{\perp}$ become holomorphic vector bundles to which Theorem (2.3) may be applied. Thus

$$\underline{\varphi}=L_1\oplus\ldots\oplus L_r,\qquad \underline{\varphi}^{\perp}=M_1\oplus\ldots\oplus M_{n-r} \tag{17}$$

with each L_i, M_j a holomorphic line bundle and $\deg(L_i) \geqq \deg(L_{i+1})$, $\deg(M_j) \geqq \deg(M_{j+1})$. If φ_i denotes orthogonal projection onto L_i, then the condition that L_i is holomorphic is just

$$(1-\varphi_i) \circ D_- \varphi_i = 0 . \tag{18}$$

Each φ_i is rank-one projection matrix, i.e. a field with values in $G_{1,n}$ which is just complex projective space $\mathbf{C}P^{n-1}$. Thus we see that any $G_{r,n}$-field splits as a sum of $\mathbf{C}P^{n-1}$-fields satisfying (18). This observation should be of some use in studying the moduli-problem for Grassmannian models.

3. Stability of Classical Solutions

(3.1) Let φ be a solution of (5). We use the description (17) to construct fluctuations satisfying (13). Consider the line bundle $L_i^* \otimes M_j$; a section, B, of this bundle is a field of rank-one matrices satisfying (3) and so is a complex fluctuation about φ. Further, B satisfies

$$\psi_j B \varphi_i = B \tag{19}$$

with φ_i as above and ψ_j denoting orthogonal projection onto M_j. Lastly, the condition that B be holomorphic is easily seen to be Eq. (13) by comparing the covariant derivatives (10a) and (15).

(3.2) We can now state and prove our main theorem:

Theorem. *Let φ be a finite-action classical solution of the Grassmannian σ-model. Then φ is stable if and only if it is an (anti-)instanton solution.*

Proof. First we assume that $\deg(L_r) \leqq \deg(M_{n-r})$. Then, for $1 \leqq j \leqq n-r$, we have

$$\deg(L_r^* \otimes M_j) = \deg(L_r^*) + \deg(M_j) = \deg(M_j) - \deg(L_r) \geqq 0 ,$$

using elementary properties of the degree (cf. [4]). Now fix $x \in S^2$, by (2.2) and (3.1) we have a fluctuation B_j satisfying (13) and

$$\psi_j B_j \varphi_r = B_j$$

with $B_j(x) \neq 0$. Further, by Theorem (1.5), we have

$$[B_j, \partial_+ \varphi] = 0 ,$$

that is

$$(\partial_+ \varphi) \psi_j B_j \varphi_r = \psi_j B_j \varphi_r (\partial_+ \varphi) .$$

Multiplying both sides by φ gives

$$\varphi(\partial_+ \varphi) \psi_j B_j \varphi_r = 0 ,$$

since $\varphi \psi_j$ vanishes and so at x, where $B_j \neq 0$, we have

$$\varphi(\partial_+ \varphi) \psi_j = 0 .$$

Then, summing over j gives

$$\varphi(\partial_+ \varphi)(1-\varphi) = 0 ,$$

while from (3) we have

$$\varphi(\partial_+ \varphi) \varphi = 0 ,$$

whence

$$\varphi \partial_+ \varphi = 0$$

at x. Since x was arbitrary we have shown that φ is an instanton solution. If $\deg(L_r) \geqq \deg(M_{n-r})$, a similar argument using $M_{n-r}^* \otimes L_i$ shows that φ is an anti-instanton solution. □

4. Remarks and Extensions

(4.1) In case that $r=1$, i.e. the $\mathbf{C}P^{n-1}$-model, Theorem (3.2) has been proved by both mathematicians and physicists [10, 11], also by considering solutions of (13); although in this case the analysis is simplified by the definiteness of the curvature term in (12). Moreover, Zakrzewski [11] has proved (3.2) for certain special solutions of the general Grassmannian model.

(4.2) It is of interest to consider whether similar results are available for other non-linear σ-models. By using a considerable refinement of the above techniques, Burstall et al. [1] have shown

Theorem. *Let φ be a finite-action stable classical solution in a non-linear σ-model with φ taking values in a Riemannian manifold N. Then*

i) *if N is a compact irreducible Hermitian symmetric space, φ is an (anti-)instanton solution (i.e. a $\pm$ holomorphic map);*

ii) *if N is a compact symmetric space with $\pi_2(N)=0$, then φ is constant.*

In particular, taking $N=S^n$ [the $O(n)$ σ-model] or N a compact semi-simple Lie group (the principal chiral model) we see that there are no non-trivial stable solutions. For $N=S^n$ this was well-known, [7, 2].

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