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Adiabatic Theorems and Applications to the Quantum Hall Effect

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Abstract. We study an adiabatic evolution that approximates the physical dynamics and describes a natural parallel transport in spectral subspaces. Using this we prove two folk theorems about the adiabatic limit of quantum mechanics: 1. For slow time variation of the Hamiltonian, the time evolution reduces to spectral subspaces bordered by gaps. 2. The eventual tunneling out of such spectral subspaces is smaller than any inverse power of the time scale if the Hamiltonian varies infinitly smoothly over a finite interval. Except for the existence of gaps, no assumptions are made on the nature of the spectrum. We apply these results to charge transport in quantum Hall Hamiltonians and prove that the flux averaged charge transport is an integer in the adiabatic limit.

1. Introduction

The adiabatic limit is concerned with the dynamics generated by Hamiltonians that vary slowly in time: $H(t/\tau)$ in the limit that the time scale τ goes to infinity. Quantum adiabatic theorems reduce certain questions about such dynamics to the spectral analysis of a family of operators, and in particular describe the way in which the dynamics tends to follows spectral subspaces of $H(s)$. $s=t/\tau$ is the scaled time. A folk adiabatic theorem states that if $H(s)$ has energy bands bordered by gaps, as in Fig. 1.1, then a system started at $t=0$ in a state corresponding to an energy band bordered by gaps of $H(0)$, will at time t, be in a state of the corresponding energy band of $H(s)$. If, in addition $H(s)$ is infinitely differentiable and s varies over a finite interval, a second folk theorem states that asymptotically the tunneling out of such an energy band is exponentially small in τ. (For Hamiltonians that are k times differentiable, tunneling is bounded by a power of $1/\tau$.) Both folk theorems emphasize the importance of the gap condition and both make no assumptions on the nature of the spectrum in the energy bands. In contrast, most of the proofs of quantum adiabatic theorems do make additional spectral assumptions. Indeed, the earliest results were obtained for finite dimensional, self-adjoint and non-degenerate matrices. In the general case there

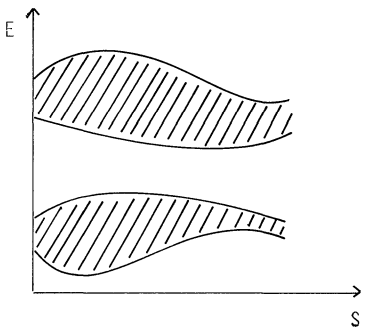

Fig. 1.1. An energy band bordered by gaps for the family of Hamiltonians $H(s)$

are two essential complications. One is that one deals, in general, with unbounded operators and the second that the spectrum need not be discrete. Unbounded operators introduce technical difficulties. More complicated spectra, on the other hand, introduce a new element.

We shall prove the adiabatic theorems in their natural "folk" setting. We shall, of course, impose technical conditions on the Hamiltonians so as to guarantee the existence of unitary time evolution. Our main tool is an "adiabatic evolution" which we introduce and study. Adiabatic evolutions define parallel transports associated with the bundle of spectral subspaces of the physical Hamiltonian. However, geometric considerations alone do not determine a unique evolution. It turns out that the requirement that the "adiabatic evolution" approximates the physical evolution "as best as possible" determines it uniquely. This is the evolution generated by:

$$H_A(s, P) \equiv H(s) + i/\tau \; [P'(s), P(s)] . \tag{1.0}$$

$P(s)$ denotes the spectral projection on the appropriate energy band of $H(s)$. Note that $H_A(s, P)$ is formally self-adjoint if $H(s)$ is. It is formally a $1/\tau$ approximant of $H(s)$. Furthermore $H_A(s, P) = H_A(s, Q)$; in particular $H_A(s, 1) = H_A(s, 0) = H(s)$.

Kato introduced the concept of adiabatic evolution in (8), where he considered the case of $H(s)$ which is completely degenerate on $P(s)$, i.e. there exists a function $E(s)$ so that $H(s)P(s) = E(s)P(s)$ for all s. In this case the generator of the adiabatic evolution, Eq. (1.0), reduces on $P(s)$ to

$$E(s)P(s) + i/\tau \; [P'(s), P(s)] .$$

The $E(s)P(s)$ term can be eliminated by a suitable choice of phase factors and so leads to the generator

$$H_K(s, P) \equiv i/\tau \; [P'(s), P(s)] ,$$

which is the generator of adiabatic evolution one commonly finds in the literature [3, 21, 25]. From a geometric point of view this is a particularly nice choice, for the holonomy associated with it is purely geometric. (This will be discussed elsewhere.)

The history of adiabatic theorems in quantum mechanics goes back to Born and Fock [4] who proved in 1928 an adiabatic theorem for $H(s)$ bounded with

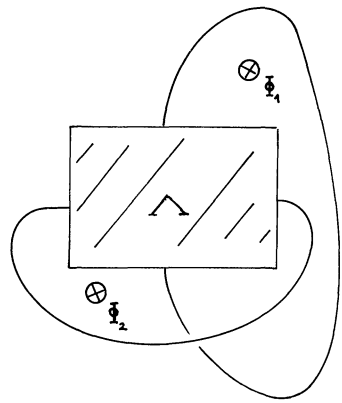

Fig. 1.2. The domain Λ with the two handles. Each handle (=loop) is threaded by a flux tube with fluxs Φ_1 and Φ_2. There is a magnetic field acting on Λ but there is no magnetic field on Λ due to the fluxes (Φ_1, Φ_2). The Hall current is defined to be the current in loop #2 due to an electromotive force acting on loop #1

purely discrete spectrum. The system is started in a nondegenerate eigenstate. In 1950 Kato [8] extended the proof to $H(s)$ that may have some continuous spectra provided the system is started in the spectral subspace of a discrete eigenvalue $E(s)$, possibly finitely degenerate. In this particular case the analysis of Sect. 2 is effectively a generalisation of Kato's to higher orders in $1/\tau$. In 1959, Lenard extended the adiabatic theorem to arbitrary high orders for Hamiltonians that are of the anharmonic oscillator type [13]. Garrido [6] and Sancho [18] have formally extended Kato's work to higher orders in τ. In 1981 Nenciu [15] proved an adiabatic theorem for bounded time dependent selfadjoint operators and announced a generalization to the case of unbounded operators. For $H(s)$ that are analytic in s, Landau and Lifshitz [11] describe formal methods to compute exponential tunneling. Such methods can and have been used also to compute the tunneling out of energy bands of continuous spectra. For example Zener tunneling of electrons in a periodic potential and constant electric field has been computed in this way.

We apply the results to a problem that arises in the context of the quantum Hall effect. The choice was made not so much because this best illustrates the theorems, but, because our original interest in the Hall problem has led us to reexamine the adiabatic limit.

In the Hall effect, one is interested in a current in the y direction due to an electric field in the x direction. More precisely, in the current flowing in loop #2 of Fig. 1.2, due to an electromotive force (emf) acting on loop #1. The Hall conductance is the ratio of current to electromotive force (which is switched on "adiabatically"). By Faraday-Lenz law the emf is related to the rate of change of the flux threading loop #1 and so the limit of weak emf, which is the limit of interest in linear response theory, is related to the adiabatic limit.

The renewed interest in the Hall conductance comes from a remarkable experimental discovery made in 1980 by von-Klitzing [9]. He found that at low temperatures the Hall conductance of certain two dimensional layers, was quantized to be an integer multiple of $1/2\pi$, in units of $e=\hbar=1$.

An understanding of this phenomenon is based on the observation [12, 16, 1, 2, 23] that a suitable definition of the conductance in linear response can be interpreted geometrically within the theory of characteristic classes [14]. The geometric character of the problem comes from the following structure: Hall Hamiltonians, which are defined precisely in Sect. 3, are time dependent Hamiltonians depending on one parameter, the flux Φ_2 in Fig. 1.2. The time dependence comes from the flux Φ_1 which increases with time to generate an emf. Although the two fluxes play distinct roles in the Hamiltonian, in the adiabatic limit the role played by Φ_1 reduces to that of a second parameter. One is naturally led to consider the fibre bundle over the two dimensional torus with points in the base labeled by $\Phi \equiv (\Phi_1, \Phi_2)$ and fibres which are the corresponding spectral subspaces. The Hall current turns out to be related to the holonomy associated to adiabatic evolution.

In linear response, the study of charge transport (defined below) is intimately related to the study of the conductance. In particular, integer quantization of the charge transport relates to quantization of the conductance in multiples of $1/2\pi$. By charge transport one means the following: Consider the charge transported around loop #2 as the flux generating the emf in loop #1 increases by one unit of quantum flux, which is 2π in our units. The analysis of charge transport requires control of the dynamics for finite times, and so is amenable to rigorous analysis. The analysis of the conductance is harder because it involves long times: The conductance is defined as the asymptotic (in time) ratio of current to emf, assuming a constant emf for large times. This limit introduces extra complications and shall not be considered here.

The main result which we prove in Sect. 3 is that in the adiabatic limit, the charge transport averaged over the flux in loop #2 is an integer. (For fixed flux in loop #2 we find no reason for integer charge transport.) This is shown by comparing the actual charge transport with the adiabatic charge transport. The latter, when averaged over Φ_2, is an integer for geometric reasons that were mentioned above. The two charge transports differ by two terms. The first is a tunneling term, and so, small by an application of an adiabatic theorem. The second term is not related to tunneling in any obvious way. Moreover, in general this term has no reason to be small, but as we show, its average over the flux in loop #2 is of order of at most $1/\tau$.

Motivated by an argument of Thouless and Niu [24] for the absence of power law corrections to the conductance, we have been tantalized by the prospect that there may be no finite power law corrections to integer charge transport. We did not find any obvious support for this. In [24] an implicit assumption on the Green's function is made which enables Thouless and Niu to relate the problem to that of noninteracting electrons in homogeneous fields, for which the result holds. Also, Shapiro [20] described a class of models with no corrections to linear response whatsoever.

Let us motivate our choice of adiabatic evolution, Eq. (1.0). The physical evolution, from (scaled) time s' to s, $U_\tau(s; s')$, is the solution to the initial value problem:

$$i\partial_s U_\tau(s; s') = \tau H(s) U_\tau(s; s'), \quad s \geqq s', \quad U_\tau(s'; s') = 1. \tag{1.1}$$

With $P(s)$ we associate an adiabatic evolution, $U_A(s;s',P)$, in such a way that it mimics $U_\tau(s;s')$ as closely as possible, except that it decouples $P(s)$ from its complement $Q(s)$. Decoupling means:

$$U_A(s;s',P)P(s')=P(s)U_A(s;s',P), \qquad U_A(s;s',P)Q(s')=Q(s)U_A(s;s',P). \quad (1.2)$$

We say that $U_A(s;s',P)$ has the intertwining property. The requirement that $U_A(s;s',P)$ approximates $U_\tau(s;s')$ as closely as possible, subject to intertwining, says that, under natural conditions to be specified in Sect. 2, the following relations hold:

$$\begin{aligned} P(s)U_\tau(s;s')P(s')&=U_A(s;s',P)P(s')+O((s-s')^2),\\ Q(s)U_\tau(s;s')Q(s')&=U_A(s;s',P)Q(s')+O((s-s')^2). \end{aligned} \quad (1.3)$$

The physical evolution is therefore approximated by the adiabatic evolution generated by:

$$H_A(s,P)=-i/\tau\ [\partial_s U_A(s;s',P)]U_A^*(s;s',P)|_{s'=s}.$$

Equations (1.3) give, (formally),

$$[H_A(s,P)-H(s)-i/\tau P'(s)]P(s)=0, \qquad [H_A(s,P)-H(s)-i/\tau Q'(s)]Q(s)=0. \quad (1.4)$$

Using $P(s)+Q(s)=1$ and $P^2(s)=P(s)$, one checks that (1.4) implies formally Eq. (1.0). From now on we shall use Eq. (1.0) as the definition of $H_A(s,P)$. Furthermore, $U_A(s,P)\equiv U_A(s;s'=0,P)$ is, by definition, the solution of the corresponding initial value problem. In the next section we shall prove that $H_A(s,P)$ indeed generates an intertwining unitary evolution.

2. Adiabatic Theorems

We consider time dependent Schrödinger Hamiltonians, $H(s)$, $s\in I$, with I a bounded open interval of the real axis, containing the origin. $s=t/\tau$ is the scaled time of Sect. 1. $H(s)$ is assumed to satisfy conditions (i)–(iii) below. (i) and (ii) are technical conditions that guarantee the existence and regularity of a unitary time evolution operator. (iii) is a gap condition. The conditions are not optimal in order to make the proofs simple [19].

(i) $H(s)$, $s\in I$, is a family of self-adjoint linear operators on a Hilbert space X, bounded from below, with an s-independent domain D, closed with respect to the graph norm of $H(0)$.

(ii) The function $H(s)$, $s\in I$ with values in the Banach space of linear operators $L(D,X)$ is k times continuously differentiable (k will be specified).

(iii) $H(s)$ has gaps in the spectrum and $P(s)$ is the spectral projection on a finite band bordered by gaps, i.e. there are two real valued, continuous functions $g_+(s)$, $g_-(s)$, and $\varepsilon>0$ such that

$$\operatorname{dist}[(g_+(s),g_-(s));\sigma(H(s))]>\varepsilon, \qquad (s\in I).$$

[$\sigma(H)$ is the spectrum of H.]

It can be shown that (i) and (ii) imply that the resolvent of $H(s)$ is k-times continuously differentiable. Furthermore for all $k\geqq 1$,

$$H'(s)[H(s)+i]^{-1}=[H(s)+i]\partial_s[H(s)+i]^{-1},$$

and is norm continuous. Hence the theorem of Kato and Yoshida about the existence and uniqueness of the solution to the time dependent Schrödinger equation is applicable [26, 17, 22].

Theorem 2.1. *Let $k>1$, $\tau>0$, and suppose $H(s)$ satisfies* (i) *and* (ii). *Then the initial value problem*

$$i\partial_s U_\tau(s)x=\tau H(s)U_\tau(s)x\,,\qquad U_\tau(0)x=x\,,\qquad s\in I\,,$$

has a unique solution with the properties: $U_\tau(s)$ is unitary, strongly continuous in s and $U_\tau(s)x$ is continuously differentiable for all $x\in D$.

The condition (iii) that $P(s)$ is a projection on a finite energy band is not a restriction, since $H(s)$ is bounded below so either $P(s)$ or $Q(s)$ projects on a finite energy interval.

Let Γ denote a contour around the piece of the spectrum associated to $P(s)$, traversed in the clockwise sense. Then

$$P(s)=1/2\pi i\int_\Gamma R(s,z)dz\,. \tag{2.1}$$

The integral here and in the following is the Riemann integral in the strong sense. $R(s,z)$ is the resolvent: $1/[H(s)-z]$. (i)–(iii) guarantee the existence and unitarity of $U_A(s,P)$, and norm differentiability of $R(s,z)$ and $P(s)$:

Lemma 2.2. *Let $H_A(s,P)$ be given by* (1.0) *and $H(s)$ satisfying* (i)–(iii) *with $k>2$, then the adiabatic evolution $U_A(s,P)$ given by*

$$i\partial_s U_A(s,P)x=\tau H_A(s,P)U_A(s,P)x\,,\qquad (x\in D)$$

with initial condition $U_A(0,P)=1$ exists and is unitary for all s in I. It maps D into itself, is strongly continuous and $U_A(s,P)x$ is continuously differentiable for all $x\in D$.

Proof. $H_A(s,P)$ satisfies the conditions in Theorem 2.1 since $H(s)$ does, and since

$$P'(s)=-1/2\pi i\int_\Gamma R(s,z)H'(s)R(s,z)dz \tag{2.2}$$

is bounded and differentiable by (i)–(iii).

$H_A(s,P)$ was chosen so that the adiabatic evolution decouples $P(s)$ and $Q(s)$. This is intertwining property of Sect. 1:

Lemma 2.3. *Under the condition* (i)–(iii), *one has*

$$U_A(s,P)P(0)=P(s)U_A(s,P)\,,\qquad s\in I\,. \tag{2.3}$$

Proof. We use a technique due to Kato [8]: Show that both sides solve the same initial value problem. Since $U_A(0,P)=1$, (2.3) is an identity for $s=0$. For the left-hand side,

$$\partial_s U_A(s,P)P(0)=-i\tau H_A(s,P)\,(U_A(s,P)P(0))\,.$$

For the right-hand side,

$$\begin{aligned}(P(s)U_A(s,P))' &= -i\tau P(s)H_A(s,P)U_A(s,P)+P'(s)U_A(s,P)\\ &= -i\tau H(s)P(s)U_A(s,P)+P(s)[P'(s),P(s)]U_A(s,P)\\ &\quad +(P'(s)P(s)+P(s)P'(s))U_A(s,P)\\ &= -i[\tau H(s)+i[P'(s),P(s)]]P(s)U_A(s,P)\\ &= -i\tau H_A(s,P)(P(s)U_A(s,P)).\end{aligned} \tag{2.4}$$

We have used the identities

$$\begin{aligned}P(s)P'(s)P(s)&=0,\\ P'(s)&=P(s)P'(s)+P'(s)P(s).\end{aligned} \tag{2.5}$$

We now turn to comparing the adiabatic evolution with the physical evolution. The standard way to compare dynamics is to construct and examine the analog of the wave operators of scattering theory [7], $\Omega(s)$:

$$\Omega(s)=U_A^*(s,P)U_\tau(s),\qquad s\in I. \tag{2.6}$$

$\Omega(s)$ is unitary and should be close to the identity for large τ.

We introduce the following kernel

$$K_\tau(s,P)=U_A^*(s,P)[P'(s),P(s)]U_A(s,P), \tag{2.7}$$

and an expansion for $\Omega(s)$: Let

$$\Omega_0(s)=1,\qquad \Omega_j(s)=-\int_0^s K_\tau(t,P)\Omega_{j-1}(t)dt \quad \text{for} \quad j>0.$$

We can now state Theorem (2.4) which compares the two evolutions.

Theorem 2.4. *Suppose $H(s)$, $s\in I$ satisfies* (i–iii) *above then:*

(a) $\Omega(s)$ *satisfies the Volterra integral equation*

$$\Omega(s)=1-\int_0^s K_\tau(t,P)\Omega(t)dt,$$

(b)

$$\Omega(s)-\sum_{j=0}^{N}\Omega_j(s)=O(1/\tau^N), \tag{2.8}$$

and

$$\sup_I\|\Omega_j(s)\|=O(1/\tau^{j-1})\quad \textit{for}\quad j\geqq 2. \tag{2.9}$$

Remarks. (a) The expansion is not in inverse powers of τ: $\Omega_j(s)$ contains terms proportional to $(1/\tau)^{j-1+m}$ for $m>0$. However, with the integration by parts identity given below, (2.8) can be made into a systematic $1/\tau$ expansion. In fact,

$$\Omega_1(s)=-i/\tau U_A^*(s)P'(s)U_A(s)+O(1/\tau^2), \tag{2.10a}$$

$$\Omega_2(s)=-i/\tau\left[\int_0^s U_A^*(t)P'^2(t)U_A(t)dt\right](P-Q)+O(1/\tau^2). \tag{2.10b}$$

The integrand in (2.10b) is a positive operator and so the $1/\tau$ term in Ω_2 vanishes only if $(P(s)-Q(s))P'(s)(P(s)-Q(s))$ vanished identically.

(b) The even terms in the expansion map P on P and Q on Q, while the odd terms map P on Q and vice versa.

(c) $\Omega_2(s)+\Omega_2^*(s)=\Omega_1^2(s)$ by an explicit calculation.

The proof of the theorem depends on an integration by parts identity, Lemma 2.5, which is the main technical tool of this section. We defer the proof of the theorem and first prove the lemma.

To simplify the expressions below let us introduce the shorthand:

1) $P\equiv P(0)$; $Q\equiv Q(0)$,

2) $U_A(s)\equiv U_A(s,P)$; $H_A(s)\equiv H_A(s,P)$; $K_\tau(s)\equiv K_\tau(s,P)$.

Lemma 2.5. *Let $H(s)$ satisfy* (i)–(iii) *and $Y(s)$, $X(s)$ be bounded operators, continuously differentiable in the strong sense. Let*

$$X^{\sim}(s)\equiv -1/2\pi i\int_\Gamma R(s,z)X(s)R(s,z)dz\,. \tag{2.11}$$

Then,

$$\begin{aligned}&\int_0^t QU_A^*(s)X(s)U_A(s)PY(s)ds\\ &\quad=i/\tau Q[U_A^*(s)X^{\sim}(s)U_A(s)PY(s)]_0^t-\int_0^t U_A^*(s)X^{\sim}(s)U_A(s)PY(s)ds\\ &\quad-\int_0^t U_A^*(s)X^{\sim}(s)U_A(s)PY'(s)ds-\int_0^t U_A^*(s)\,[P'(s),X^{\sim}(s)]U_A(s)PY(s)ds]\,.\end{aligned} \tag{2.12}$$

Remarks. (d) The mechanism that makes the integral small (i.e. of order $1/\tau$) is the same as in the Riemann-Lebesgue lemma: for large τ, $U_A(s)$ rotates rapidly and may be thought of as $\exp i\tau E(s)$, with $E(s)$ the energy. Because P and Q flank U_A and U_A^* the phases do not cancel. This makes the integrand rapidly oscillating, roughly, of order $\exp i\tau E_g$ with E_g the energy gap, and so the integral is small.

(e) The same formula, up to an overall minus sign, holds with P and Q interchanged.

(f) The twiddle operation in (2.11) is a special case of Friedrich's Gamma operation [5].

An immediate corollary is:

Corollary 2.6. *Suppose that in addition $R(s,z)$, $Y(s)$, $X(s)$ are C^∞ and that $X(s)$ or $Y(s)$ is supported in the interior of $[0,t]\subset I$. Then*

$$\int_0^t QU_A^*(s)X(s)U_A(s)PY(s)ds=O(1/\tau^\infty)\,. \tag{2.13}$$

Proof. The boundary terms vanish by assumptions. Since $X^{\sim}$ is also supported in $[0,t]$ if $X(s)$ is, repeated applications of the lemma yields (2.13).

We now prove (2.12). First note the operator identity:

$$Q(s)X(s)P(s)=-Q(s)([H_A(s),X^{\sim}(s)]+i/\tau[P'(s),X^{\sim}(s)])P(s)\,, \tag{2.14}$$

which follows from

$$[H(s),X^{\sim}(s)]=-1/2\pi i\int_\Gamma dz[(H(s)-z),R(s,z)X(s)R(s,z)]=[P(s),X(s)]\,.$$

Now, from (1.0),

$$[X(s), P(s)] = -[H_A(s), X^{\sim}(s)] + i/\tau[[P'(s), P(s)], X^{\sim}(s)], \tag{2.15}$$

which proves (2.14). From the intertwining property and the equation of motion,

$$\begin{aligned} QU_A^*(s)X(s)U_A(s)P &= i/\tau Q(U_A'^*(s)X^{\sim}(s)U_A(s) + U_A^*(s)X^{\sim}(s)U_A'(s) \\ &\quad - U_A^*(s)[P'(s), X^{\sim}(s)]U_A(s))P \\ &= i/\tau Q[(U_A^*(s)X^{\sim}(s)U_A(s))' \\ &\quad - U_A^*(s)[X^{\sim\prime}(s) + [P'(s), X^{\sim}(s)]]U_A(s)]P. \end{aligned} \tag{2.16}$$

Integrating gives (2.12).

We now return to the proof of Theorem 2.4. (a) is easy since $\Omega(0)=1$ and

$$\begin{aligned} \Omega'(s) &= i\tau U_A^*(s, P)(H_A(s, P) - H(s))U_\tau(s) \\ &= -U_A^*(s, P)[P'(s), P(s)]U_\tau(s) = -K_\tau(s, P)\Omega(s). \end{aligned}$$

Integration gives (2.8). To prove (b) note that by Lemma (2.5):

$$\begin{aligned} \left\| \int_0^s QK_\tau(t)Y(t)dt \right\| &= \left\| \int_0^s QK_\tau(t)PY(t)dt \right\| \\ &= \left\| \int_0^s QU_A^*(t)P'(t)U_A(t)PY(t)dt \right\| < \text{const}/\tau |||Y(s)|||, \end{aligned}$$

where

$$|||Y(t)||| \equiv \sup_{0<s<t} (\|Y(s)\|, \|Y'(s)\|),$$

and the constant is τ independent. Since there is a similar inequality with Q and P interchanged we get

$$\left\| \int_0^s K_\tau(t)Y(t)dt \right\| < \text{const}/\tau |||Y(s)|||. \tag{2.17}$$

It follows from (2.17) that

$$\|\Omega_{j+1}(s)\| < \text{Const}/\tau |||\Omega_j(s)|||.$$

But, since $\Omega_j'(s) = -K_\tau(s)\Omega_{j-1}(s)$ and K_τ is bounded, we learn

$$\|\Omega_{j+1}(t)\| < \text{Const}/\tau \sup_{0<s<t} (\|\Omega_j(s)\|, \|\Omega_{j-1}(s)\|).$$

Since $\Omega_0 = 1$ and $\|\Omega_1\| = O(1/\tau)$, we get (2.9).

Equation (2.8) is shown by noting that the remainder, $R_N(s)$, satisfies

$$R_N(s) = -\int_0^s K_\tau(t, P)R_{N-1}(t)dt.$$

This can be shown along the same lines.

The content of Theorem (2.4) is that the adiabatic dynamics approximates the physical dynamics and that it can be used to generate an expansion in powers of $1/\tau$. If $H(s)$ is smooth one can say more:

Lemma 2.7. *Suppose $H(s)$ satisfies* (i)–(iii) *for all $k>1$ and that $P'(s)$ is supported in* $[0,1]$ *and is C_0^∞, then for s outside* $[0,1]$,

$$Q\Omega(s)P = O(1/\tau^\infty).$$

Proof. Iterating the Volterra integral equation for Ω gives

$$Q\Omega(t)P = -Q\int_0^t K_\tau(s)dsP + \int_0^t ds \int_0^s ds' K_\tau(s)K_\tau(s')Q\Omega(s')P. \tag{2.18}$$

Suppose t is outside the support of $P(s)$. By Corollary (2.6), Lemma (2.5) and the boundedness of K,

$$\|Q\Omega(t)P\| < O(1/\tau^\infty) + \text{Const}/\tau \sup_{0<s<t} \|Q\Omega(s)P\|,$$

from which the result follows.

Remark. (g) This says that the odd terms in the expansion for $\Omega(s)$ are smaller than any power of $1/\tau$ for times outside the support of P'. The even terms we expect not to be small. The bound given in Theorem 2.4 is optimal for $j=2$. See Remark (a).

We can now formulate and prove the adiabatic theorem. The physical evolution takes P into $P_\tau(t)$, where

$$P_\tau(t) = U_\tau(t)PU_\tau^*(t), \tag{2.19}$$

whereas for the adiabatic evolution, by intertwining,

$$P(t) = U_A(t)PU_A^*(t). \tag{2.20}$$

The adiabatic theorem says that $P_\tau(t)$ is close to $P(t)$:

Theorem 2.8. *Let $k>2$ and suppose $H(s)$ satisfies* (i)–(iii). *Then:*

a) $P_\tau(t) - P(t) = O(1/\tau)$, $t \in I$.

b) *If furthermore $k=\infty$, then*

$$P_\tau(t) - P(t) = O(1/\tau^\infty), \qquad t \in I/\text{Supp}\, P'.$$

Proof. (a) is a consequence of Theorem 2.4. For (b) we first note the identity

$$P_\tau(t) - P(t) = U_A(t)[\Omega(t), P]U_\tau^*(t), \tag{2.21}$$

from which we get

$$\begin{aligned} Q(t)(P_\tau(t) - P(t)) &= U_A(t)(Q\Omega(t)P)U_\tau^*(t), \\ P(t)(P_\tau(t) - P(t)) &= -U_A(t)(P\Omega(t)Q)U_t^*(t). \end{aligned} \tag{2.22}$$

This shows that $P_\tau(t) - P(t)$ is of the order of $P\Omega(t)Q$. Lemma 2.7 completes the proof.

3. Corrections to Quantised Transport

We start by defining what we mean by Hall Hamiltonians. Then, we shall define charge transport and adiabatic charge transport and prove that in the adiabatic limit, the charge transport, suitably averaged, is an integer.

A Hall Hamiltonian is a one parameter family of time dependent Hamiltonians related to the two parameter family of Hamiltonians,

$$H^{\sim}(\Phi)=\sum_{i=1}^{N}(1/2(v_i-\Phi_1 a_1(x_i)-\Phi_2 a_2(x_i))^2+V(x_i))+\sum_{i<j}1/|x_i-x_j|, \qquad (3.1)$$

where $x_i \in \Lambda$ and Λ is a finite domain in R^3 with smooth boundary $\partial\Lambda$ and two handles. See Fig. 1.2. $V(x)$ is a real background (Coulombic) potential generated by fixed nuclei which make the sample Λ. N is the number of electrons in Λ, and the last term in (3.1) describes the electron-electron interaction. $v_j=-i\partial_j-A(x_j)$, where $A(x)$ is a smooth vector potential associated with a magnetic field $B(x)$ acting on Λ. $a_{1,2}(x)$ are two smooth vector potentials on Λ associated with two flux tubes that thread the two handles with fluxes $\Phi=(\Phi_1,\Phi_2)\in R^2$. The magnetic fields associated to $a_{1,2}$ vanish on Λ. $a_{1,2}$ are normalized so that their contour integral around the appropriate loop is one. Φ_1 is related to the emf acting on loop #1 and Φ_2 to measuring the current in loop #2.

We impose Dirichlet boundary conditions[1] on $\partial\Lambda$ and consider (3.1) on the space of antisymmetric (fermionic) wave functions. The following is well known about the properties of such Hamiltonians:

(i) $H^{\sim}(\Phi)$ is essentially self-adjoint on the antisymmetric wave functions which are infinitely differentiable and have compact support.

(ii) $H^{\sim}(\Phi)$ has a Φ independent domain of self-adjointness D.

(iii) $H^{\sim}(\Phi)$ is real analytic in Φ as a map from D to the Hilbert space X.

(iv) $H^{\sim}(\Phi)$ has compact resolvent and is bounded from below for all Φ in R^2.

(v) $H^{\sim}(\Phi)$ is unitarily equivalent to $H^{\sim}(\Phi+2\pi)$ in either variable, Φ_1 or Φ_2.

We shall assume that for all $\Phi\in R^2$, $H^{\sim}(\Phi)$ has a non-degenerate ground state, and we write $P^{\sim}(\Phi)$ for the associated projection. Given that, it is a standard fact of perturbation theory that:

(vi) $P^{\sim}(\Phi)$ and the ground state energy $E(\Phi)$ are real analytic in Φ at points of non degeneracy.

In the theory of the Hall effect one considers time dependent Hamiltonians related to $H^{\sim}(\Phi)$ in the following way: Let F be a real valued, smooth and non-decreasing function of the scaled time, $s=t/\tau$, vanishing for all $s<0$. The dynamics of the N-particle system is given by the time dependent Hamiltonian τH,

$$\tau H(s,\phi)\equiv\tau H^{\sim}(\Phi_1=F(s),\Phi_2=\phi),$$

with projector valued function

$$P(s,\phi)\equiv P^{\sim}(F(s),\phi).$$

The emf acting on the first handle (loop) is $-F'(s)/\tau$ (Faraday-Lenz law) and is proportional to $1/\tau$: Large time scales correspond to weak emf.

[1] A more realistic model would be to take L^2 boundary conditions. For the bound states of this system, the wave function decays exponentially away from Λ, and so it is a good approximation to impose Dirichlet boundary conditions on $\partial\Lambda$

The emf on loop #1 generates the Hall current I around the loop #2. It is a function of the scaled time s and the flux ϕ, and is given by

$$I = \mathrm{Tr} P_\tau \partial_\phi H . \tag{3.2}$$

P_τ denotes the projector onto the physical state Ψ_τ defined below. This equation is the precise meaning of the statement made above that $\phi = \Phi_2$ is related to measuring the Hall current.

P_τ is defined in terms of the propagator U_τ for the initial value problem:

$$i\partial_s U_\tau(s,\phi) = \tau H(s,\phi) U_\tau(s,\phi), \qquad U_\tau(0,\phi) = 1, \tag{3.3}$$

$$P_\tau(s,\phi) \equiv U_\tau(s,\phi) P\tilde{}(0,\phi) U_\tau^*(s,\phi). \tag{3.4}$$

It is convenient to relate the restriction of $P\tilde{}$ to the Φ_2-axis to $P(0,0)$. For that we define an adiabatic parallel transport $U_A\tilde{}$ along that axis:

$$\begin{aligned} &i\partial_\phi U_A\tilde{}(\phi) = \tau H_A\tilde{}(\phi) U_A\tilde{}(\phi), \qquad U_A\tilde{}(0) = 1, \\ &H_A\tilde{}(\phi) \equiv H\tilde{}(0,\phi)\Psi + i/\tau \quad [\partial_\phi P\tilde{}(0,\phi), P\tilde{}(0,\phi)]. \end{aligned} \tag{3.5}$$

According to Lemma 2.3, $U_A\tilde{}$ possesses the intertwining property,

$$P\tilde{}(0,\phi) = U_A\tilde{}(\phi) P\tilde{}(0,0) U_A\tilde{}^*(\phi). \tag{3.6}$$

The physical state is defined by:

$$\Psi_\tau(s,\phi) \equiv U_\tau(s,\phi) U_A\tilde{}(\phi) \Psi_0, \tag{3.7}$$

where Ψ_0 is a normalized vector in the range of $P\tilde{}(0,0)$.

The physical state will be compared with the adiabatic state Ψ_A, defined analogously: Let U_A be the adiabatic propagator defined by the initial value problem:

$$i\partial_s U_A(s,\phi) = \tau H_A(s,\phi) U_A(s,\phi), \qquad U_A(0,\phi) = 1, \qquad H_A \equiv H + i/\tau[\partial_s P, P]. \tag{3.8}$$

Now define the adiabatic state Ψ_A by:

$$\Psi_A(s,\phi) \equiv U_A(s,\phi) U_A\tilde{}(\phi) \Psi_0. \tag{3.9}$$

In the analysis of the total charge transport it will be necessary to control the ϕ dependence of the physical time evolution U_τ. This can be done along the lines used in the theory of ordinary differential equations [10]. The key role is played by the formula (3.10) below (Duhamels principle).

Lemma 3.1. *The propagator U_τ defined by* (3.3) *is differentiable in ϕ in the following sense: There exists an operator valued function $\partial_\phi U_\tau$ from D to X such that for all $\tau > 0$ and $(s,\phi) \in R^2$,*

$$U_\tau(s,\phi') = U_\tau(s,\phi) + (\phi' - \phi)\partial_\phi U_\tau(s,\phi) + o(\phi' - \phi).$$

The differential $\partial_\phi U_\tau$ is explicitly given by the integral

$$U_\tau^*(s,\phi)\partial_\phi U_\tau(s,\phi) = -i\tau \int_0^s U_\tau^*(t,\phi)\partial_\phi H(t,\phi) U_\tau(t,\phi) dt. \tag{3.10}$$

An analogous statement holds for the propagator U_A and the generator H_A.

The proof of this technical lemma is based on a formula for the difference $\Delta(s,\phi')\equiv U_\tau(s,\phi')-U_\tau(s,\phi)$ which holds on D:

$$\Delta(s,\phi')=-i\tau\int_0^s ds' U_\tau(s',\phi')U_\tau^*(s',\phi)\,[H(s',\phi')-H(s',\phi)]U_\tau(s',\phi)\,.$$

It is a consequence of the differential equation (3.3) which holds for every term in Δ. For details of the argument we refer to [10, 19].

Lemma 3.1 has two useful consequences: The first is an alternative expression for the Hall current (3.2),

$$I=i/\tau\;\partial_s\langle\Psi_\tau,\partial_\phi\Psi_\tau\rangle\,. \tag{3.11}$$

To prove this formula let us start from the right-hand side of (3.11):

$$i\partial_s\langle\Psi_\tau(s,\phi),\partial_\phi U_\tau(s,\phi)\Psi_\tau(0,\phi)\rangle=i\partial_s\langle\Psi_\tau(s,\phi),(\partial_\phi U_\tau(s,\phi))\Psi_\tau(0,\phi)\rangle\,.$$

Notice that if ∂_ϕ hits $\Psi_\tau(0,\phi)$ we get a scalar product which is independent of the first variable s. Inserting Duhamels formula (3.10) for $U_\tau^*\partial_\phi U_\tau$, one gets (3.2).

There is of course a formal derivation of (3.11) too. It is useful for mnemonic purposes:

$$\tau\langle\Psi_\tau,(\partial_\phi H)\Psi_\tau\rangle=\tau\langle\Psi_\tau,\partial_\phi H\Psi_\tau\rangle-\langle\Psi_\tau H\partial_\phi\Psi_\tau\rangle=i\partial_s\langle\Psi_\tau,\partial_\phi\Psi_\tau\rangle\,.$$

The second consequence of Lemma 3.1 concerns the control of the ϕ derivative of $\Omega\equiv U_A^*U_\tau$. Notice first that according to the results in the second chapter $\Omega=1+O(1/\tau)$. This estimate holds pointwise in $(s,\phi)\in R^2$. However, it can be readily shown to hold uniformly on any compact in R_2. Now we consider the restriction of Ω to D. Due to Lemma 3.1 the ϕ derivative exists and will be denoted by $\partial_2\Omega$. It is given by:

$$\partial_\phi\Omega(s)=i\tau\int_0^s ds'[U_A^*(s')(\partial_\phi H_A(s'))U_A(s')\Omega(s)-\Omega(s)U_\tau^*(s')(\partial_\phi H(s')U_\tau(s')]\,.$$

To make the formula more transparent we did not write the second variable ϕ explicitly. Replacing on the right-hand side of (3.12) the physical time evolution U_τ by $U_A\Omega$, one gets

$$\begin{aligned}\partial_\phi\Omega(s)=i\tau\int_0^s ds'[&U_A^*(s')(\partial_\phi H_A(s'))U_A(s')\Omega(s)-\Omega(s)\\&\times\Omega^*(s')U_A^*(s')(\partial_\phi H(s'))U_A(s')\Omega(s')]\,.\end{aligned}$$

From this and the estimate $\Omega=1+O(1/\tau)$ mentioned above one gets:

$$\partial_\phi\Omega=O(1)\,. \tag{3.13}$$

We stress that $\partial_\phi\Omega$ denotes an operator from D to X and that the norm is understood accordingly. The estimate holds, again, uniformly on compacts in R^2. Equation (3.11) and the existence of $\partial_2\Omega$ (3.12) are the technically important consequences of Lemma 3.1 mentioned before.

Remarks. (a) Formula (3.11) for the Hall current is naturally interpreted as the time derivative of the transported charge, see Eqs. (3.14, 3.15) below.

(b) The same formula (3.11) proves the periodicity of the current in ϕ: $I(s, \phi) = I(s, \phi + 2\pi)$.

(c) The bound (3.13) on $\partial_\phi \Omega$ is actually not used in the proof of the main result, Theorem 3.3. Existence and continuity in the two variables is all that is ever needed.

In the following we concentrate on the total charge transported $Q(\phi)$, and its adiabatic counterpart $Q_A(\phi)$. Suppose that the switch function F is infinitely differentiable and F' is supported in $(0, 1)$. Define the total charge transport by

$$Q(\phi) \equiv \tau \int_0^1 ds\, I(s, \phi) \tag{3.14}$$

$$= i\langle \Psi_\tau(1, \phi), \partial_\phi \Psi_\tau(1, \phi)\rangle . \tag{3.15}$$

In the last step we used the equation

$$\langle \Psi_\tau(0, \phi), \partial_\phi \Psi_\tau(0, \phi)\rangle = 0 ,$$

which follows from the adiabatic transport along the Φ_2-axis given by Eq. (3.5). The total charge transported adiabatically is defined analogously:

$$Q_A(s) \equiv i\langle \Psi_A(1, \phi), \partial_\phi \Psi_A(1, \phi)\rangle .$$

Both charges are periodic in ϕ (Remark b). Because of the geometric significance of the adiabatric evolution the adiabatic charge is related to a topological invariant:

Theorem 3.2. *Suppose that the switch function F is infinitely differentiable, increasing from 0 to 2π with derivative F' supported in the open interval $(0, 1)$. Suppose furthermore that the Hall Hamiltonian $H(\Phi)$ introduced at the beginning of this section, has a nondegenerate ground state for all $\Phi \in R^2$. Then, the average of Q_A,*

$$\langle Q_A \rangle \equiv 1/2\pi \int_0^{2\pi} d\phi Q_A(\phi) ,$$

is a first Chern character associated to the line bundle of the ground state over the two dimensional torus R^2/Γ, $\Gamma = [\Phi_1 = 2\pi k, \Phi_2 = 2\pi 1;\ k, 1$ integers$]$. So $\langle Q_A \rangle$ is an integer.

Remark. (d) This result was originally shown in [1, 16, 23]. We give here an alternate derivation.

Proof. The argument resembles the one used to get from (3.14) to (3.15) but this time it runs in the opposite direction. Due to Lemma 3.1 everything in sight is continuously differentiable in s and ϕ, so we may proceed by formal computations. To make the equations more transparent we do not write the variables of the operator valued functions explicitly. By definition and the argument used to go from 3.14 to 3.15, we get

$$Q_A = i \int_0^1 ds\, \partial_s \langle \Psi_A, \partial_\phi \Psi_A \rangle .$$

Analysing the integrand one finds:

$$\begin{aligned} i\partial_s\langle\Psi_A,\partial_\phi\Psi_A\rangle &= i\langle\partial_s\Psi_A,\partial_\phi\Psi_A\rangle+\tau\langle\Psi_A,\partial_\phi H_A\Psi_A\rangle \\ &= i\langle\partial_s\ \Psi_A,\partial_\phi\Psi_A\rangle+\tau\partial_\phi\langle\Psi_A,H_A\Psi_A\rangle-\tau\langle\partial_\phi\Psi_A,H_A\Psi_A\rangle \\ &= i[\langle\partial_s\Psi_A,\partial_\phi\Psi_A\rangle-\langle\partial_\phi\Psi_A,\partial_s\Psi_A\rangle]+\tau\partial_\phi\langle\Psi_A,H_A\Psi_A\rangle\,. \end{aligned}$$

The last term can be computed:

$$\langle\Psi_A,H_A\Psi_A\rangle=E+i/\tau\ \langle\Psi_A,[\partial_s P,P]\Psi_A\rangle=E$$

by the adiabatic transport (use $P\partial_s PP=0$). E is periodic in both variables (s,ϕ) due to (v) above. Hence, it does not contribute to the average $\langle Q_A\rangle$. We are left with

$$\langle Q_A\rangle=i/2\pi\int\langle d\Psi_A,d\Psi_A\rangle\,.$$

The integration is over $[0,1]\times[0,2\pi]$. $d\Psi$ is a one form wave function, i.e. $d\Psi=\partial_\phi\Psi d\phi+\partial_s\Psi ds$, and ds and $d\phi$ are elements of a Grassmann algebra. The right-hand side is quantised being the first Chern character of the line bundle defined by the ground state over the torus [see e.g. (1) for a sketch of proof].

Remark. (e) Notice that one can not say anything about quantisation of Q_A for a fixed value of Φ_2; the average only is quantized. It is, however known that in certain cases Q_A is Φ_2 independent in the thermodynamic limit [16].

We come now to the central question of this section which is to what extent Q is approximated by Q_A. We shall see that the two are in general not close, their averages however are.

Theorem 3.3. *Under the conditions of the previous theorem one gets*

$$Q(\phi)-Q_A(\phi)=-\partial_\phi\,\mathrm{Tr}\,P(0,\phi)\Omega(1,\phi)+O(1/\tau)\,, \tag{3.16}$$

uniformly in ϕ.

Because the $\mathrm{Tr}\,P(0,\phi)\Omega(1,\phi)$ is periodic in ϕ, we get

Corollary.

$$\langle Q\rangle=\langle Q_A\rangle+O(1/\tau)\,.$$

Proof of the Theorem. In the first step we derive an appropriate expression, (3.17) for the difference $\Delta\equiv-i(Q-Q_A)$. To make the formulas more transparent we do not write the variable ϕ explicitly. The first variable s is always taken at the point $s=1$ unless otherwise stated.

By definition we get

$$\Delta=\mathrm{Tr}[P(0)U_\tau^*\partial_\phi U_\tau-P(0)U_A^*\partial_\phi U_A]\,.$$

Substituting U_τ for $U_A\Omega$, one finds

$$\Delta=\mathrm{Tr}[P_\tau(\partial_\phi U_A\Omega)\Omega^*U_A^*-P(\partial_\phi U_A)U_A^*]\,.$$

Differentiation in the first term on the right-hand side yields:

$$\Delta=\mathrm{Tr}(P_\tau-P)(\partial_\phi U_A)U_A^*+\mathrm{Tr}(P(0)\Omega^*\partial_\phi\Omega)\,. \tag{3.17}$$

In this form Δ can be estimated as follows: The first term is smaller than any power of $1/\tau$. In fact by (2.22),

$$\mathrm{Tr}(P_\tau - P)(\partial_\phi U_A)^* U_A = \mathrm{Tr}(Q\Omega P U_\tau^* \partial_\phi U_A) - \mathrm{Tr}(P\Omega Q U_\tau^* \partial_\phi U_A).$$

Both terms can be estimated separately. Consider the first one:

$$\mathrm{Tr}(Q\Omega P U_\tau^* \partial_\phi U_A) = \mathrm{Tr}(Q\Omega P)\,\mathrm{Tr}(P U_\tau^* \partial_\phi U_A).$$

The first factor is smaller than any power of $1/\tau$, by Lemma 2.7. The second is bounded uniformly in ϕ by Lemma 3.1.

We are left with the second term on the right-hand side of (3.17). From Lemma 2.7 one gets:

$$\mathrm{Tr} P(0)\Omega^* \partial_\phi \Omega = \mathrm{Tr} P(0)\Omega^* P(0)\partial_\phi \Omega + O(1/\tau^\infty).$$

The first term on the right-hand side can again be factorized:

$$\mathrm{Tr} P(0)\Omega^* P(0)\partial_\phi \Omega = \mathrm{Tr}(P(0)\Omega^* P(0))\,\mathrm{Tr}(P(0)\partial_\phi \Omega) \tag{3.18}$$

$$= \mathrm{Tr}(P(0)\Omega^* P(0)\partial_\phi\,\mathrm{Tr}(P(0)\Omega) + O(1/\tau^\infty). \tag{3.19}$$

In the last step we used the equation

$$\mathrm{Tr}\,\partial_\phi P(0)\Omega = \mathrm{Tr}[Q(0)(\partial_\phi P(0))P(0)\Omega] + \mathrm{Tr}[P(0)(\partial_\phi P(0))Q(0)\Omega].$$

Every term is smaller than any power of $1/\tau$, by Lemma 2.7. Since the first factor on the right-hand side of (3.19) is $1 + O(1/\tau)$, the second term on the right-hand side of (3.17) produces the right-hand side of (3.16).

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