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Construction and Borel Summability of Infrared Φ_4^4 by a Phase Space Expansion

J. Feldman*, J. Magnen, V. Rivasseau, and R. Sénéor

Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau Cedex, France

Abstract. We construct the thermodynamic limit of the critical (massless) φ^4 model in 4 dimensions with an ultraviolet cutoff by means of a "partly renormalized" phase space expansion. This expansion requires in a natural way the introduction of effective or "running" constants, and the infrared asymptotic freedom of the model, i.e. the decay of the running coupling constant, plays a crucial rôle. We prove also that the correlation functions of the model are the Borel sums of their perturbation expansion.

Introduction

This paper extends the methods of constructive field theory to treat strictly renormalizable asymptotically free situations. We study the infrared behavior of massless Φ_4^4 with an ultraviolet cutoff as one of the simplest of these situations. We use an approach which has its source in the work of Glimm and Jaffe [1] on the ultraviolet limit of Φ_3^4. The basic tool of this paper is a kind of "phase space expansion" [1–5]; many of its features were already presented in [6], where it was used to control the "infrared superrenormalizable" $(\nabla\Phi)_3^4$ model. It consists of scaled cluster expansions with effective parameters. It had however to be further improved to apply to strictly renormalizable theories and this resulted also in a number of simplifications. We give to the present expansion the name PRPSE (for partially renormalized phase space expansion). The methods developed in [7] to control the large orders of perturbation theory for Φ_4^4, and in [8] to exploit rigorously asymptotic freedom at the level of Feynman graphs played an important role in the genesis of this paper; in particular they helped convince us that constructive field theory could attack non-superrenormalizable situations. In fact the results of [7] can be recovered using the present version of the phase space expansion [9, 10].

* Permanent adress: Mathematics Department, University of British Columbia, Vancouver, B.C., Canada V6T 1Y4

We also used the PRPSE to prove the ultraviolet stability of massive Gross-Neveu models, in other words of N-component massive fermionic fields in two dimensions with a quartic interaction [11, 12], and there is some hope to apply some kind of PRPSE to a 4 dimensional gauge field theory.

In fact in its present form our method seems to apply to any theory whose content is essentially perturbative (typically for which the perturbation series is Borel summable). It looks also promising for a rigorous study of the large order behavior of renormalized perturbation expansions in field theory [13–15].

The model we study will be defined precisely in the next section. It is a massless Euclidean Φ^4 theory on $\mathbb{R}^4$ with an ultraviolet cutoff of Pauli-Villars type. Other types of ultraviolet regularizations, like lattice versions could be treated equally well by our method. In contrast with [8], we consider the theory with the usual sign of the coupling constant. This model is asymptotically free in the infrared region and the renormalization group trajectories approach a Gaussian fixed point at a logarithmic rate [16, 17]. This behavior has been confirmed by numerical simulations, for instance [18]. The approach to the critical behavior is studied in [19, 20]. The control of this "infrared Φ_4^4" model has also been obtained by Gawedzki and Kupiainen, using a rigorous formulation of the renormalization group [21], and the constructions of both groups were presented in some detail at the Les Houches Summer School, August 1984 [22, 23]. The completion of this paper suffered a long delay (not solely due to the laziness of the authors!): indeed we found during the process of writing [11] a simplification in the treatment of the mass renormalization which avoids the use of fixed points to fix inductively the bare mass at its critical value, as is done in [21, 23]; and we thought that it was worth rewriting the present paper to include it.

The paper is organized as follows. In the first chapter, the model is defined and our main results are stated; a brief sketch of the proof is also presented. The notation and a precise definition of the expansion is given in Sect. II. In Sect. III the renormalization of the mass and the coupling constant are explained and lead to the computation of the effective (or "running") coupling constant in terms of the corresponding bare quantity. The estimates leading to the main results are also stated in Sect. III. These estimates are proven in Sect. IV, and Sect. V is devoted to Borel summability.

I. The Model

I.1 Definitions

The theory in a volume Λ (Λ is a volume cutoff smooth and vanishing outside a large compact box of $\mathbb{R}^4$ which will also be called Λ with some abuse of notation) is defined by the probability measure $d\sigma_\Lambda(\varphi)$:

$$d\sigma_\Lambda(\varphi)=[Z(\Lambda)]^{-1}\cdot e^{-\lambda\int\Lambda(x)\varphi^4(x)d^4x+\int\frac{\delta m^2(\Lambda,\lambda,x)}{2}\varphi^2(x)d^4x}d\mu_C^\Lambda(\varphi), \tag{I.1}$$

where by definition:

– $d\mu_C^\Lambda$ is the Gaussian measure on $S'(\Lambda)$ with mean 0 and covariance C

$$C(x-y)=\frac{1}{(2\pi)^4}\int e^{ip\cdot(x-y)}\frac{1}{p^2(p^2+1)^5}d^4p. \tag{I.2}$$

Remark. We choose our ultraviolet cutoff to be of this particular form in Fourier space because it allows better conservation of momenta and makes simpler the proof of Borel summability in Sect. V:

– $\Lambda(x)=h(x/\Lambda)$, where $h(x)$ is a smooth function of compact support

– δm^2 is the mass counterterm. It will be shown to exist, and will be chosen to renormalize all mass insertions by subtracting the zero momentum value of all one-particle irreducible two-point subgraphs in our expansion. This prescription will automatically fix the theory to its critical point, i.e. the asymptotic behaviour of the two point function will be massless, as shown in (I.4).

– $Z(\Lambda)=Z(\Lambda;\lambda,m^2(\Lambda,\lambda))$ is the normalization of the interacting measure $d\sigma$ (In the following we might often forget the Λ and λ dependence.)

Our basic objects of interest will be, apart from the pressure

$$p(\Lambda)=\frac{1}{|\Lambda|}\operatorname{Ln}Z(\Lambda)$$

the Schwinger functions

$$S_n^\Lambda(x_1,\ldots,x_n)=\langle\varphi(x_1)\ldots\varphi(x_n)\rangle_{\sigma_\Lambda}=\int\varphi(x_1)\ldots\varphi(x_n)d\sigma_\Lambda(\varphi)\,. \tag{I.3}$$

The connected Schwinger functions (defined as usual) will be denoted

$$S_n^{T,\Lambda}(x_1,\ldots,x_n)\,.$$

The unnormalized Schwinger functions with cutoff Λ is denoted $\mathbb{S}_\Lambda(x_1,\ldots,x_n)$.

I.2 The Results

Our results concern the existence of massless φ_4^4 in an infinite volume and its infrared behavior.

Theorem I.1 (Existence of massless φ_4^4). *For λ small enough one can choose $\delta m^2(\Lambda,\lambda,x)$ such that the thermodynamic limit of the model exists and is a massless field theory. More precisely the limits, as $\Lambda\to\mathbb{R}^4$, of $p(\Lambda)$, S_n^Λ and $S_n^{T,\Lambda}$ exist (they will be denoted with the superscript Λ dropped), and $S_2(x,y)$ does not decay exponentially in $|x-y|$ for large $|x-y|$.*

Theorem I.2 (Infrared behavior of massless φ_4^4). *The behavior of the correlation functions is Gaussian up to logarithmic corrections. More precisely, for λ small enough, there exists $O(\lambda)$, $C_1(\lambda,|x-y|)$, $C_2>0$, $C_3>0$ such that*

$$S_2(x,y)=\frac{1+O(\lambda)}{|x-y|^2}[1+C_1(\lambda,|x-y|)]\,, \tag{I.4}$$

$$|C_1(\lambda,|x-y|)|\leqq\frac{C_2}{1+\operatorname{Ln}[1+|x-y|]}\,, \tag{I.5}$$

$$|S_4^T(x_1,\ldots,x_4)|\leqq\int\prod_{i=1}^4 S_2(x_i,y)\frac{C_3}{1+\inf\limits_{(i,j)}\operatorname{Ln}(1+|x_i-x_j|)}\,, \tag{I.6}$$

where the infimum is taken over all pairs (i,j), $1\leqq i<j\leqq 4$.

Theorem I.3 (Borel summability). *The Schwinger functions S_n and S_n^T are Borel summable as functions of λ.*

These theorems are proved in Sects. III–V.

Remark. Equations (I.4)–(I.5) tells us that the two point function is asymptotic to the two point function of the free massless field at large distance.

Inequality (I.6) ensures that the truncated 4 point function is asymptotic to its Gaussian value, 0, at large distances, the approach being logarithmic. Similar inequalities hold for the higher truncated functions S_n^T, $n>4$, showing that they approach 0 at large distances. We do not write them for simplicity. In fact our method gives a systematic computation of the exact asymptotic behavior of S_n^T. For instance we can prove, if all the distances $|x_i - y_j|$ for $i,j=1,2,3,4$ are of order e^n, that:

$$S_4^T(x_1, \ldots, x_4) = \int \left[\prod_{i=1}^{4} S_2(x_i, y) \right] d^4y \frac{1}{\beta n} \left[1 + O\left(\frac{\mathrm{Log} n}{n} \right) \right], \tag{I.7}$$

where β is the first non-vanishing coefficient of the β-function. More generally using our method one should be able to put on a rigorous level the results already existing in the literature obtained by renormalization group computations, like those of [19, 20]. Finally we want to point out that this method can be applied to any one-parameter family of theories such that for small values of the parameter one is in the domain of attraction of a known fixed point (not necessarily gaussian). Typical examples of such situations are given by the Gross-Neveu model in $2+\varepsilon$ dimensions [35], the $-g\varphi_{4+\varepsilon}^4$ model [36] and the Gross-Neveu model in 3 dimensions and large N [37].

I.3 Outline of the Proof

To prove Theorems I and II we will expand the pressure of any Schwinger function S_n according to an improved version of the expansion described in [6]. We introduce an infrared momentum cutoff $M^{-\varrho}$, where M is a fixed integer. The convergence for ϱ fixed as $\Lambda \to \infty$ is easy. Take $\Lambda = M_{N\varrho}$ with N big enough; then $\Lambda(x) = h(|x| M^{-N\varrho})$; the infrared momentum cutoff is smooth and of scale $M^{-\varrho}$ so that the propagator has an exponential fall off: $|C(x-y)| \leq \exp[-M^{-\varrho}|x-y|]$, thus up to small corrections we can consider only distances smaller than $M^{O(1)\varrho}$ and on such distances the function $\Lambda(x)$ is almost constant up to corrections of order $M^{-(N-O(1))\varrho}$. In the core of the paper and after the expansion we shall consider only $\Lambda = \infty$. In particular the counterterms in this limit are translation invariant.

We will then prove the convergence of the expansion when $\varrho \to +\infty$, our estimates being uniform in ϱ. We use scaled cluster expansions. To converge, a single cluster expansion in d dimensions requires a summable propagator, hence at least a decrease like $1/|x-y|^{d+\varepsilon}$ for the propagator between x and y, ε being a positive number. As in [6], this is not the case here: the free massless propagator (and if the expansion converges, the full propagator) behaves as $1/|x-y|^2$. We avoid this problem by introducing scales of momenta and cluster expansions related to the different scales. The scales of momenta (in contrast with [6]) form an exponentially decreasing sequence $\{1, M^{-1}, \ldots, M^{-i}, \ldots, M^{-\varrho}\}$. The layer or slice of momenta of index i may be thought of as corresponding to momenta between $M^{-(i-1)}$ and M^{-i}. To be more precise, this partition of momenta into slices is realized in fact through the following decomposition of the Fourier transform of the covariance:

$$C(p)=\sum_{i=1}^{\infty} C^i(p), \tag{I.8}$$

$$C^i(p)=\frac{1}{p^2}\left[\frac{1}{(p^2M^{2(i-1)}+1)^5}-\frac{1}{(p^2M^{2i}+1)^5}\right]. \tag{I.9}$$

There is a corresponding decomposition of the field

$$\varphi(x)=\bigoplus_{i=1}^{\infty} \varphi^i(x) \tag{I.10}$$

as independent variables with φ^i having covariance C^i and free measure

$$d\mu=\prod_{i=1}^{\infty} d\mu_i, \qquad d\mu_i=d\mu_{C^i}. \tag{I.11}$$

To each momentum index i is also associated a cubic lattice $\mathbb{D}_i$, whose cubes, of length side αM^i, are union of M^4 cubes of $\mathbb{D}_{i-1}$ (for $i>1$); to each lattice we associate a decomposition of unity:

$$\sum_{\Delta\in\mathbb{D}_i} \Delta(x)=1, \quad \text{with} \quad \Delta(x)=\xi[(x-\Delta)/\alpha M^i], \tag{I.12}$$

where $\xi(y)=\prod_{k=1}^{4} \xi(y_k)$, $y=(y_1, y_2, y_3, y_4)$, and for each Δ, $\Delta=(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ is the center of Δ and $\xi(a_k)$ is a C^∞ function such that

$$\xi(a_k)=0 \quad \text{if} \quad |a_k|>3/4 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \xi(n+a_k)=1.$$

We decompose also the fields of the external variables; each external field in $\mathbb{S}(x_1, \ldots, x_n)$ has now an index i_j. Let

$$I_0=((i_1, x_1), \ldots, (i_n, x_n)). \tag{I.13}$$

The elements of this set will be called the (indices of the) true external variables; each cube containing one or more true external variables is called a true external cube.

For each scale of index i, $i=0,1,2,\ldots$ we will perform successively:

– a cluster expansion on $d\mu_i$ to estimate the correlations between different regions (unions of cubes of $\mathbb{D}_i$). It produces propagators C^i connecting cubes of $\mathbb{D}_i$. Technically they are third order Taylor expansions in parameters, called s, associated to pairs of cubes of the same index i, which test the coupling between these cubes through the gaussian measure $d\mu_i$.

– A momentum coupling expansion which tests successively in each cube Δ of $\mathbb{D}_i$ the coupling between "high" and "low" momentum fields (these notions of high and low are relative to the index of the scale one is looking at: "high" means "of index lower than or equal to i" and "low" means "of index bigger than i"). Technically they are fifth order Taylor expansions in parameters, called t_Δ, associated to single cubes Δ of $\mathbb{D}_i$, which test the coupling between low and high momentum fields in the cube through the exponential of the φ^4 interaction. More precisely, if

$$\varphi_h = \int_\Delta \sum_{j \leqq i} \varphi^j(x)dx \quad \text{and} \quad \varphi_l = \int_\Delta \sum_{j > i} \varphi^j(x)dx \tag{I.14}$$

are the corresponding high and low momenta fields, we write

$$\int_\Delta (\varphi_h(x) + \varphi_l(x))^4 \, dx = \int_\Delta (\varphi_h(x) + t_\Delta \varphi_l(x))^4 + (1 - t_\Delta^4)\varphi_l^4(x)|_{t_\Delta = 1}, \tag{I.15}$$

and we write the Taylor expansion with integral remainder to fifth order in t_Δ. Remark that each vertex created by a derivation in t_Δ contains both high and low momenta fields hooked to it, and that to each derivation is associated at least one low momentum field.

The final expansion is generated by applying these expansions to each momentum range starting from the highest ones. Its outcome appears as a sum of graphs made of cubes connected either by propagators (created by the cluster expansions) or by vertices (created by the momentum coupling expansion).

It is convenient to view this expansion organized as in Fig. 1. The horizontal direction represents the 4 space-time dimensions, and the vertical direction represents the scales of momenta, the high momenta being above. Propagators of the cluster expansions, which join two cubes, are represented by horizontal lines (since they belong to a given scale), ending at points localized in the two cubes. Fields can be represented by half-lines, and vertices may have fields of different indices hooked to them. Therefore they can be pictured as vertical dashed lines connecting 4 horizontal half-lines of various scales. Such a vertical line goes through all the cubes $\Delta \in \mathbb{D}_i$ such that x, the position of the vertex, belongs to Δ, and such that $j \leqq i \leqq k$, where j and k are respectively the smallest and highest scales of the 4 fields hooked to the vertex. By a very natural convention we will say that these vertices or vertical dashed lines connect precisely these cubes which contain the fields hooked to the vertex (remark that there are therefore at most 4 such cubes, which are necessarily contained one within another). In this representation our expansion can be viewed as a cluster expansion in both the horizontal and the vertical directions. (This point of view is close to the one developed in [30], using a staggered lattice.)

However there is one important difference between the horizontal and the vertical directions: by power counting (see [9, 10], or the next sections) one needs at least 5 vertical links to obtain exponential decrease in the vertical direction, while in the horizontal direction a single link is enough (since the propagators in a momentum slice have an infrared cutoff, well adapted to the summation over the cubes of corresponding size). This forces us to push the momentum expansion to fifth order in each t_Δ variable, to be sure that the remainder term has enough vertical coupling to ensure summability. We remark that it is also for this reason that renormalization enters the picture, to restore the vertical decrease when it is missing.

Finally there is a third kind of "link" between cubes in our expansion, namely we will consider that the cubes Δ of index i and Δ' of index $i+1$, $\Delta \subset \Delta'$, are linked through an "open gate" if t_Δ is nonzero. We remark that by our rule for the t_Δ

dependence (see the next sections) if the gate between Δ and Δ' is open, there are at least 5 low momentum fields attached to the vertical dashed lines crossing the gate, and if the gate is closed there are at most 4 such low momentum fields (see Fig. 1).

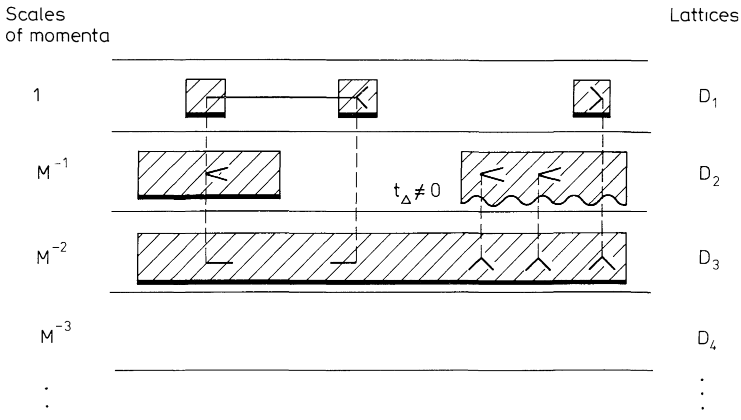

Fig. 1. A connected graph in the phase space expansion

In Fig. 1 open gates are represented by a wavy line at the bottom of Δ and closed gates are represented by corresponding fat straight lines. A technical consequence is that our horizontal cluster expansions are like those of [27–29], i.e. propagators can connect directly cubes far apart, but our vertical expansion is more like the Erice cluster expansion [38], since the gates are between cubes which are "vertical neighbors," hence have some similarity with the Dirichlet surfaces of [38].

Let us summarize the connection rules of our expansion. Two cubes Δ and Δ' of indices i and j are connected if either:

– there is a propagator (horizontal line) between them (this requires $i=j$)

– there is a vertex (dashed vertical line) and two half lines hooked to it, one in each cube (this requires $i \neq j$ and $\Delta \subset \Delta'$ or $\Delta' \subset \Delta$)

– there is an open gate (wavy line) between them (this requires $j=i+1$ and $\Delta \subset \Delta'$ or the converse, and is equivalent to the presence of at least 5 low momentum fields attached to the dashed lines crossing the interface between Δ and Δ').

A connected graph G is then defined as a set of cubes together with their fields and propagators such that any two cubes can be linked through a chain of connected cubes, and its amplitude, i.e. the associated contribution in our expansion will be denoted $A(G)$.

A general argument of the Kirkwood-Salzburg type then shows that the normalized Schwinger functions exist if the unnormalized Schwinger functions can be written as (each G_α being a connected graph):

$$S=\sum_{n} \sum_{G_1,\ldots,G_n} \prod_{\alpha=1}^{n} A(G_\alpha) \tag{I.16}$$

with

$$\sum_{\substack{\Delta \in G_\alpha \\ \|G_\alpha\|=p}} |A(G_\alpha)| < e^{-Kp}, \tag{I.17}$$

where Δ is some fixed cube, $\|G_\alpha\|$ is the number of cubes in G_α, and K is some positive constant (see Sect. III), which one ought to be able to make large enough by choosing the bare coupling constant λ small enough (to beat the combinatoric constants generated by the machinery of the cluster and Mayer expansions).

In (I.17) the sum over the graphs G_α is in fact a sum over the cubes composing the graph, and is controlled either horizontally by the exponential decrease of the propagator ($C^i(x,y) \leqq M^{-2i} \exp(-M^{-i}|x-y|)$) which allows to sum on cubes of $\mathbb{D}_i$ $\left[\text{because } \sum_{\Delta' \in \mathbb{D}_i} \exp(-M^{-i} \operatorname{dist}(\Delta, \Delta')) \leqq O(1)\right]$ or vertically by the exponential decrease associated to vertical lines by the combined use of power counting and of "useful" renormalizations [10, 11], which in turn imply the use of effective or running constants. Let us sketch this power counting analysis and this use of "partial" renormalizations, which is the core of our method.

Let us call "almost local subgraph" or in short "subgraph" of a graph G a connected component of G_j, where G_j is the subset of the cubes of G which have indices $\leqq j$. The *index* of an almost local subgraph is then defined as the index of the largest cube that it contains. Let us call "external fields" of such a subgraph the fields of index $>j$ hooked to its vertices.

The result of the power counting analysis of the next sections (see Proposition IV.7 and Theorem III.5) is similar to the standard power counting analysis explained in detail in [9]; namely to any such subgraph of scale j with e external fields, of scale $k>j$, is associated a net factor $M^{-(e-4)(k-j)}$, after the summation over internal vertices and positions of the internal cubes of the subgraph have been performed.

As in [9], if $e \geqq 5$, $e-4 \geqq e/5$ and this exponential decrease can be attributed to the external vertices, so that every corresponding vertical dashed line becomes an exponential "spring"; it can then be used to control the vertical sums in a way very similar to the way horizontal sums are controlled by the horizontal "springs" corresponding to the propagators.

Such cases correspond to the so-called irrelevant operators of the renormalization group analysis.

On the other hand subgraphs with 2 or 4 external fields do not give any vertical decrease ($e=4$) or even give exponential vertical growth ($e=2$). They correspond respectively to marginal and relevant operators, and need to be renormalized.

The renormalization will consist of a Taylor subtraction at zero momentum; for example a two point function $\int \varphi(x)\varphi(y) f(x,y)\,dx\,dy$ after renormalization becomes

$$-1/2 \int [\varphi(x)-\varphi(y)]^2 f(x,y)\,dx\,dy\,. \tag{I.18}$$

We have $\varphi(x)-\varphi(y)=(x-y)\nabla\varphi(x)+$ terms with two gradients. Each gradient increases by one the local power counting.

We will describe briefly our procedure, using the language of [10]. A "local subgraph," i.e. a connected subgraph with the frequencies of its internal lines (say j) higher than those of its external lines (say k), looks more and more "point-like" (hence its name) relative to the external scale k when $k-j$ becomes larger and larger (this is because the internal lines have high infrared cutoff, hence fast decrease). Therefore it is intuitively obvious that such subgraphs can be regularized in an efficient way by subtracting truly local ("point-like") counterterms. Moreover, fortunately, they are precisely the subgraphs in our expansion which require such a regularization when their power counting is bad ($e \leqq 4$). The counterterms corresponding to these subgraphs will be called "useful." However to preserve axiomatic requirements, in the standard theory of perturbative renormalization, one is also forced to introduce the counterterms, which we call "useless," corresponding to "non-local subgraphs," i.e. subgraphs which have some external lines higher than some internal lines. Such counterterms cannot be efficiently added to the corresponding subgraphs (since these subgraphs do not look "point-like" when seen from the external world through the "probes" of the external lines). They must be kept uncancelled in the expansion in the form of additional "insertions." Remark that "useless" counterterms are not only useless (except for axiomatic requirements), but somewhat troublesome, since in the case of the coupling constant they create renormalons [10].

Let us call "bare" parameters (like in field theory) those defined at the high scale (here $i=0$) and "renormalized" parameters those defined at the lower scale (here $i=\infty$).

In this model the renormalizations to be performed correspond to the mass, and coupling constant.

We choose to perform completely the mass renormalization (i.e. introduce the useless and useful corresponding counterterms) because this will fix the physical mass to be zero, i.e. we will be sitting at the critical point. This is possible because the corresponding useless insertions are summable (they do not create renormalons). We define a cluster and a Mayer expansion which allows the factorization of the one particle irreducible (1PI) two point functions. We can associate each mass counterterm with a graph, the consequence being that we don't need a fixed point procedure to compute the mass counterterm.

On the contrary we will use the *bare* parameter for the coupling constant, and we will add and subtract in the exponential of the interaction only the corresponding *useful* counterterms. The useful counterterms introduced will be used to compensate the corresponding almost local graphs; this will result in a net transfer of internal convergence into external convergence through gradients acting on the external fields (see Sect. IV). The effective external power counting becomes then equivalent to that of a n-point function with $n>4$. Hence the vertical exponential decay of the corresponding "springs" is restored (except for subtleties due to the fact that we do not perform the wave function renormalization). Then the useful counterterms "subtracted" in the exponential of the interaction will be absorbed in the redefinition of effective coupling constants for the lower scales. We remark that this process requires factorization of the high energy and low energy fields in the exponential of the interaction, hence it requires that the corresponding "gates" are closed. This will be precisely the case with our rules, because a proper subgraph of scale j with $e \leqq 4$ necessarily has all its t_Δ-parameters at scale j equal to 0. In perturbation theory there is a wave function renormalization, it will appear to be finite and of the order of λ; thus we choose to introduce no wave function counterterms.

Finally to achieve the bound (I.17) one has to extract a small factor per cube to control the combinatorics and to give at the end the e^{-K} factor per cube. This is obtained by taking λ small enough (depending on M) and by using:

a) *The Asymptotic Freedom of the Model.* We obtain at each scale i a "running coupling constant" λ_i which is given inductively by:

$$\lambda_{i+1}=\lambda_i+\delta\lambda_i, \tag{I.19}$$

where $\delta\lambda_i$ = sum of the useful coupling constant counterterms due to the introduction of the i^{th} range of momenta. At lowest order in λ_i, $\partial\lambda_i$ is quadratic with a negative coefficient:

$$\delta\lambda_i=-c_i\lambda_i^2+\text{higher order terms}. \tag{I.20}$$

Since $c_i\approx c$, c constant, one has:

$$\lambda_i\approx\frac{\lambda_1}{1+c\lambda_1(i-1)} \tag{I.21}$$

with $\lambda_1=\lambda$. The theory is asymptotically free because $c>0$, the consequence being that $\lambda_i\approx 1/i$.

With this running coupling constant essentially all quantities of interest are given by the lowest order contributions in the effective interaction. In particular the difference between the bare and renormalized wave function constants is bounded (and small if λ is small because $\sum\limits_{i=1}^{\infty}\lambda_i^2\approx\sum\limits_{i=1}^{\infty}i^{-2}$ is finite. This is important, since the correct asymptotic flow of the effective constants is expressed in terms of the "renormalized" rather than the "bare" wave function constant (in contrast with [11]). However (always in contrast with [11]) we do not need here to know the correct flow with great precision, and the behavior (I.21) (without even the precise value of the constant c) is sufficient for our purpose of controlling the thermodynamic limit of this critical φ_4^4 model. The value of the wave function term at scale i is of order λ_i^2 so that the wave function insertion at scale k coming from the contributions of the higher scales is of order:

$$\sum_{i=1}^{k-1}\lambda_i^2\leqq O(1)\lambda. \tag{I.22}$$

Thus the wave functions insertions are small.

b) *The Domination of the Low Momentum Fields.* We have to overcome the problem (not present in [11]) of the ordinary divergence of perturbation theory due to the large number of graphs. In our expansion this occurs when many vertices created in small, high frequency cubes have low momenta fields which accumulate in a single large low frequency cube.

The value of a graph is given by an integral over:

– the exponential of the interaction, called A,

– the high momentum fields, called B (roughly speaking a field of index i belonging to a vertex of index j is a high momentum field if $i\leqq j$, see Sect. III),

– the low momentum fields, called C (roughly speaking a field of index i belonging to a vertex of index j is a low momentum field if $i>j$, see again Sect. III).

We want to use a bound of the type:

$$\left|\int ABC\,d\mu(\varphi)\right|\leqq\left[\int B^2\,d\mu(\varphi)\right]^{1/2}\sup_{\varphi}|AC|, \tag{I.23}$$

i.e. to dominate the low momentum fields using the positivity of the interaction. This can be effective only if the low momentum fields are smeared, since the interaction in the exponential is smeared in the volume Λ. Hence we write:

$$\varphi(x)=\int \eta_i(x-y)\varphi(y)dy+\delta\varphi(x), \tag{I.24}$$

where $\eta_i(x)=(\alpha M^i)^{-4}\eta(x/\alpha M^i)$, and η is a smooth function of integral one with compact support, such that $\eta(x)=0$ for $|x|>1/4$. This defines $\delta\varphi$ which, being a fluctuation field at scale i, can be considered as a high momentum field.

Using Hölder's inequality we have:

$$M^{-4i}\int|\eta_i(x-y)\varphi(y)dy|\leqq M^{-i}\sum_{\Delta\in\mathbb{D}_i}\left[M^{-4i}\int_\Delta|\eta_i(x-y)|^{4/3}dy\right]^{3/4}\left[\int_\Delta\varphi(y)^4dy\right]^{1/4}$$

$$\leqq O(1)M^{-i}\sum_\Delta\left[\left|\int_\Delta\varphi(y)^4dy\right|\right]^{1/4}, \tag{I.25}$$

where the sum on Δ is finite (restricted to the cubes of $\mathbb{D}_i$ which are neighbours of the cube containing x). The interaction in Δ to the power $1/4$ is easily bounded by the exponential of the interaction. [This requires $\mathrm{Re}\lambda>0$ (see Sect. V). Hence the domination is related to the stability of the model and is not possible for the (ultraviolet asymptotically free) negative $\lambda\varphi^4$ theory in 4 dimensions.] The conclusion is that a low momentum field in $\mathbb{D}_i$ is of order M^{-i}.

Finally the introduction of the smearing operation with the η function forces us to add a corridor around the support of each graph, hence to augment slightly the connection rules given above.

II. The Expansion

The first part of the expansion consists of a cluster expansion on the cubes of $\mathbb{D}_i$; for each i it will control the thermodynamic limit for the part of the interaction corresponding to momenta of index i. The second part is a momentum coupling expansion which measures for each i the strength of the coupling between momenta higher than M^{-i} and lower than M^{-i}. The third part is a Mayer expansion. The fourth one is the renormalization of the two and four point Mayer graphs and the computation of the effective coupling constant.

II.1 The Cluster Expansion

For each i we define first an interpolating covariance $C^i(s,x,y)$ by

$$C^i(s,x,y)=\sum_{(\Delta,\Delta')}s_{\Delta,\Delta'}[\Delta(x)\Delta'(y)+\Delta(y)\Delta'(x)]C^i(x,y)+\sum_\Delta\Delta(x)\Delta(y)C^i(x,y), \tag{II.1}$$

where the sums are on pairs of distinct cubes or on cubes of $\mathbb{D}_i$ and where each s runs from 0 to 1. We remind the reader that we use the same symbol Δ to stand for the cube $\Delta\in\mathbb{D}_i$, the centre of the cube, the function $\Delta(x)=\xi((x-\Delta)/\alpha M^i)$ (which belongs to a partition of unity associated with $\mathbb{D}_i$) and the operator on $L^2(\mathbb{R}^4)$ consisting of multiplication by $\Delta(x)$.

Theorem II.1. *Pick any real number γ and let $\Gamma=(C^i)^\gamma$ be the γ^{th} power of C^i. For $j=1,2,3,\ldots$ let $\Delta_j\in\mathbb{D}_i$ and let $\alpha M^i n_j$ be the centre of Δ_j. Hence $n_j\in\mathbb{Z}^4$. Let k be any positive integer. Then there exists a constant K independent of i, α (for α large enough) and the Δ_j's such that*

a) *$\Delta_1\Gamma[\Gamma^{-1},\Delta_2]\Delta_3$ is a bounded operator on L^2 with operator norm*

$$\|\Delta_1\Gamma[\Gamma^{-1},\Delta_2]\Delta_3\|\leqq K\alpha^{-1}[1+|n_1-n_2|^2]^{-k}[1+|n_2-n_3|^2]^{-k}. \tag{II.2}$$

b) *$\Delta_1\Gamma\Delta_2\Gamma^{-1}\Delta_3$ is a bounded operator on L^2 with operator norm*

$$\|\Delta_1\Gamma\Delta_2\Gamma^{-1}\Delta_3\|\leqq K[1+|n_1-n_2|^2]^{-k}[1+|n_2-n_3|^2]^{-k}. \tag{II.3}$$

c) *If the operator Q obeys $\|\Delta_1 Q\Delta_2\|\leqq K'[1+|n_1-n_2|^2]^{-2-\varepsilon}$ for some $\varepsilon>0$ then $\|Q\|\leqq KK'$.*

d) *$C^i(s)$ is an operator of positive type. $\|C^i(s)^{1/2}(C^i)^{-1/2}\|\leqq K$ and $\|(C^i)^{-1/2}C^i(s)^{1/2}\|\leqq K$.*

e) *Let G be either $C^iC^i(s)^{-1}$ or $C^i(s)(C^i)^{-1}$. Then*

$$\|\Delta_1 G\Delta_2\|\leqq K[1+|n_1-n_2|^2]^{-k}.$$

Proof. We will use the symbol K to stand for many different, but all irrelevant, constants.

a) If we use C to denote $(C^1)^\gamma$ (in momentum space) and h to denote the fourier transform of the function ξ, then the kernel, in momentum space, of the operator $\Delta_1\Gamma[\Gamma^{-1},\Delta_2]\Delta_3$ is

$$\begin{aligned}B(p,q)=\int drds&[(\alpha M^i)^4h(\alpha M^i(p-r))\exp(i(p-r)\cdot\alpha M^i n_1)]\,C(M^i r)\\ &[(\alpha M^i)^4 h(\alpha M^i(r-s))\exp(i(r-s)\cdot\alpha M^i n_2)]\,[C(M^i r)^{-1}-C(M^i s)^{-1}]\\ &[(\alpha M^i)^4 h(\alpha M^i(s-q))\exp(i(s-q)\cdot\alpha M^i n_3)]\,.\end{aligned} \tag{II.4}$$

The norm of this integral operator on L^2 is bounded by

$$\begin{aligned}\|B\|\leqq \sup_p\int dq|B(p,q)|=&\sup_p\int dq\left|\int drds\, e^{ir\cdot(n_2-n_1)}e^{is\cdot(n_3-n_2)}h(p-r)h(r-s)h(s-q)\right.\\ &\left.\times C(r/\alpha)[C(r/\alpha)^{-1}-C(s/\alpha)^{-1}]\right|,\end{aligned}$$

where we have scaled p,q,r, and s by a factor of αM^i, using the fact that the sup over p is the same as the sup over $\alpha M^i p$. To display the decay in $|n_1-n_2|$ and $|n_2-n_3|$ we write

$$e^{ir\cdot(n_2-n_1)}=[1+|n_1-n_2|^2]^{-k}[1-\nabla^2]^k e^{ir\cdot(n_2-n_1)}, \tag{II.5}$$

and the analogous formula for $\exp[is\cdot(n_3-n_2)]$ and integrate by parts. Derivatives that act on the h's are of no importance: they merely replace one Schwartz space function by another. Any derivative that acts on a C or a C^{-1} produces a factor of $1/\alpha$ by the chain rule:

$$\partial[C(r/\alpha)]=(1/\alpha)(\partial C)(r/\alpha),\qquad \partial[C(r/\alpha)^{-1}]=(1/\alpha)[\partial(C^{-1})](r/\alpha),$$

and similarly for $C^{-1}(s/\alpha)$. Now for any n^{th} order ($n\geqq 0$) derivative ∂^n and any real number γ (positive or negative) there is a constant K_n such that

$$|\partial^n((C^1)^\gamma)(p)|\leqq K_n/[1+|p|]^{\delta\gamma+n}. \tag{II.6}$$

(For our cutoffs $\delta = 12$.) Consequently, for any $n, m \geqq 0$,

$$\begin{aligned}|(\partial^n C)(r/\alpha)\partial^m(C^{-1})(r/\alpha)| &\leqq K[1+|r/\alpha|]^{-\delta\gamma-n}[1+|r/\alpha|]^{\delta\gamma-m}\\ &= K[1+|r/\alpha|]^{-n-m} \leqq K\\ |(\partial^n C)(r/\alpha)\partial^m(C^{-1})(s/\alpha)| &\leqq K[1+|r/\alpha|]^{-\delta\gamma-n}[1+|s/\alpha|]^{\delta\gamma-m}\\ &\leqq K[1+|r/\alpha|]^{-\delta\gamma}[1+|r/\alpha|+|(r-s)/\alpha|]^{\delta\gamma}\end{aligned}$$

(assuming $\gamma > 0$; if not, interchange the roles of r and s)

$$\begin{aligned}&\leqq K[1+|(r-s)/\alpha|]^{\delta\gamma}\\ &\leqq K[1+|r-s|]^{\delta\gamma}\end{aligned}$$

and all terms in which at least one derivative acts on a C or C^{-1} obey the desired bound. In the event that no derivatives act on any C or C^{-1} we simply use (again writing explicitly only the case $\gamma > 0$)

$$|C(r/\alpha)[C(r/\alpha)^{-1} - C(s/\alpha)^{-1}| = |C(r/\alpha)\alpha^{-1}(r-s)\nabla(C^{-1})(r/\alpha + t(s-r)/\alpha)|$$

for some $0 \leqq t \leqq 1$

$$\begin{aligned}&\leqq K\alpha^{-1}|r-s|[1+|r/\alpha|]^{-\delta\gamma}[1+|r/\alpha+t(s-r)/\alpha|]^{\delta\gamma-1}\\ &\leqq K\alpha^{-1}|r-s|[1+|r/\alpha|]^{-\delta\gamma}[1+|r/\alpha|+|(s-r)/\alpha|]^{\delta\gamma}\\ &\leqq K\alpha^{-1}|r-s|[1+|s-r|]^{\delta\gamma}.\end{aligned}$$

b) This follows immediately from part a) and

$$\Delta_1\Gamma\Delta_2\Gamma^{-1}\Delta_3 = \Delta_1\Delta_2\Delta_3 - \Delta_1\Gamma[\Gamma^{-1}, \Delta_2]\Delta_3.$$

c) Let $f, g \in L^2$. Since $\sum_{\Delta} \Delta(x) = 1$,

$$\begin{aligned}|(f, Qg)| &\leqq \sum_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} |(\Delta_1 f, \Delta_2 Q\Delta_3\Delta_4 g)| \leqq \sum_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \|\Delta_1 f\|\,\|\Delta_2 Q\Delta_3\|\,\|\Delta_4 g\|\\ &\leqq \sum_{\Delta_1,\Delta_4} \|\Delta_1 f\| KK'[1+|n_1-n_4|^2]^{-2-\varepsilon}\|\Delta_4 g\| \qquad \text{(II.7)}\end{aligned}$$

by the hypothesized bound on Q and the fact that, to yield a nonzero term, we must have $|n_1 - n_2| \leqq 2$ and $|n_3 - n_4| \leqq 2$. Interpreting this sum as

$$\sum \varphi_{n_1} B(n_1, n_4)\psi_{n_4}$$

and using that $\|B\|$ is bounded by the $L^1 - L^\infty$ norm of its kernel gives

$$|(f, Qg)| \leqq [\sum \|\Delta_1 f\|^2]^{1/2} KK'[\sum \|\Delta_4 g\|^2]^{1/2} \leqq KK'\|f\|\,\|g\|,$$

since $\Delta(x)^2 \leqq \Delta(x)$.

d) We write

$$C^i(s) = C^i - (C^i)^{1/2}A(s)(C^i)^{1/2}, \qquad \text{(II.8)}$$

where

$$\begin{aligned}A(s) &= \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'})(C^i)^{-1/2}\Delta C^i \Delta'(C^i)^{-1/2}\\ &= \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'})\{\Delta\Delta' + [(C^i)^{-1/2}, \Delta]C^i\Delta'(C^i)^{-1/2} + \Delta(C^i)^{1/2}[\Delta', (C^i)^{-1/2}]\}.\end{aligned}$$

(II.9)

Now

$$\left\| \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'}) \Delta \Delta' \right\| \leqq \sup_x \sum_{\Delta \neq \Delta'} \Delta(x)\Delta'(x) = \sup_x \left[1 - \sum_{\Delta} \Delta(x)^2 \right] = 1 - 2^{-4} \tag{II.10}$$

Hence by parts a), b), and c) of the current theorem with $\Gamma = (C^i)^{1/2}$,

$$\|A(s)\| \leqq 1 - 2^{-4} + O(1/\alpha) < 1, \tag{II.11}$$

if α is large enough. Consequently

$$(f, C^i(s) f) = \|(C^i)^{1/2} f\|^2 - ((C^i)^{1/2} f, A(s)(C^i)^{1/2} f)$$

must lie between $[2^{-4} - O(1/\alpha)]$ and $[2 - 2^{-4} + O(1/\alpha)]$ times $\|(C^i)^{1/2} f\|^2$.

e) For $G = C^i(s)(C^i)^{-1}$ this is an immediate consequence of (II.3). For the other case we write (but dropping the superscript i)

$$\begin{aligned} G^{-1} &= C(s)C^{-1} = 1 - C^{1/2}A(s)C^{-1/2} = 1 - \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'}) \Delta C \Delta' C^{-1} \\ &= 1 - \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'}) \{\Delta\Delta' - \Delta C[C^{-1}, \Delta']\} = I(s) + D(s), \end{aligned} \tag{II.12}$$

with

$$I(s) = 1 - \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'}) \Delta \Delta'$$

and

$$D(s) = \sum_{\Delta \neq \Delta'} (1 - s_{\Delta,\Delta'}) \Delta C [C^{-1}, \Delta'] .$$

Since $I(s)$ is a multiplication operator whose inverse is bounded by (II.10) and

$$\|\Delta_1 D(s) \Delta_2\| \leqq K\alpha^{-1}[1 + |n_1 - n_2|^2]^{-k}$$

by (II.2) we can achieve the desired bound by expanding

$$G = \sum_{n \geqq 0} (-1)^n I(s)^{-1} [D(s) I(s)^{-1}]^n . \quad \square \tag{II.13}$$

Definition II.2. An s-dependent Schwinger function is obtained by replacing the measure $d\mu$ on the fields by $d\mu(s)$ which is the measure of covariance $C(s)$. However for a reason that will be explained in the section dealing with renormalization (Sect. II.4) we do not wish the fields belonging to the mass counterterm to depend on s. This wish may be implemented as follows. Prior to the introduction of any s-dependence we have (using [33, Lemmas V.4 and VI.2]):

$$e^{1/2 \int \delta m^2(x) \varphi(x)^2 dx} = : e^{1/2 \int dx dy \varphi(x) [1 - \delta m^2 C]^{-1} \delta m^2(y) \varphi(y)} : V \tag{II.24}$$

$$= e^{1/2 \int dx dy (C\delta/\delta\varphi)(x) [1 - \delta m^2 C]^{-1} \delta m^2(y) (\delta/\delta) \varphi(y)} V . \tag{II.25}$$

Hence it suffices to leave the C's in (II.15) independent of s. We remark that V is simply a constant (a sum of vacuum graphs) which appears in both the numerator and the denominator and hence cancels out, and that $C[1 - \delta m^2 C]^{-1} \delta m^2 C$ is the sum of all possible strings of mass counterterms $C\delta m^2 C \dots C\delta m^2 C$.

The elementary step of the cluster expansion is [for one (Δ, Δ')]

$$F(s)|_{s_{\Delta,\Delta'}=1} = (D^0_{\Delta,\Delta'} + D^1_{\Delta,\Delta'} + D^2_{\Delta,\Delta'} + I_{\Delta,\Delta'}) F(s) \tag{II.16}$$

with:

$$D^k_{\Delta,\Delta'}F(s)=\frac{1}{k!}\,[d^k/ds^k_{\Delta,\Delta'}F(s)]|_{s_{\Delta,\Delta'}=0}\,,\qquad k=0,1,2\,, \tag{II.17}$$

$$I_{\Delta,\Delta'}F(s)=1/2\int_0^1 ds_{\Delta,\Delta'}(1-s_{\Delta,\Delta'})^2\,\frac{d^3}{ds^3_{\Delta,\Delta'}}\,F(s)\,, \tag{II.18}$$

and for $\Delta,\Delta'\in\mathbb{D}_i$:

$$d/ds_{\Delta,\Delta'}\int P(\varphi)d\mu(s)=\int\Delta(x)C^i(x,y)\Delta'(y)dxdy\left[\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)}\,P(\varphi)\right]d\mu(s)\,. \tag{II.19}$$

The cluster expansion is ($s\equiv 1$ means that $s_{\Delta,\Delta'}=1$, for all $\Delta,\Delta'\in\mathbb{D}_i$ and for all i):

$$F(s\equiv 1)=\prod_i\prod_{(\Delta,\Delta')\in\mathbb{D}_i}(D^0_{\Delta,\Delta'}+D^1_{\Delta,\Delta'}+D^2_{\Delta,\Delta'}+I_{\Delta,\Delta'})F(s)\,. \tag{II.20}$$

We do not actually evaluate the $\delta/\delta\varphi$ derivatives until after the mass renormalization has been performed. The $D^0_{\Delta,\Delta'}$ operator decouples Δ from Δ', the operator $D^1_{\Delta,\Delta'}$ (respectively $D^2_{\Delta,\Delta'}$) decouples Δ from Δ' and creates one (respectively two) propagator linking Δ to Δ'. We say in this case that there is at least one derivation in Δ and one in Δ'. The reason that we push the cluster expansion to third order is that we want to analyze the mass renormalization in terms of one particle irreducible kernels. We remark that the operator $I_{\Delta,\Delta'}$ creates three propagators between Δ and Δ', hence Δ and Δ' must then belong to the same one particle irreducible kernel.

II.2 The Decoupling Expansion

The different layers of momenta are coupled through the interaction term. For each i and each cube $\Delta\in\mathbb{D}_i$ we introduce an interpolating parameter t_Δ, $0\leqq t_\Delta\leqq 1$ which tests the coupling between the high momentum fields φ^j, $j\leqq i$, and the low momentum field

$$\varphi_{l,i}=\sum_{j>i}\varphi^j\,. \tag{II.21}$$

(See, however, the remarks at the end of this section.) For the first slice, the introduction of t_Δ, $\Delta\in\mathbb{D}_1$ for the φ^4 term of the interaction is obtained by replacing:

$$\lambda\int(\varphi(x))^4d^4x$$

by

$$\lambda\int\left[\sum_\Delta[t_\Delta\Delta(x)(\varphi^1(x)+\varphi_{l,1}(x))]^4d^4x+\lambda_2\int\left[1-\left(\sum_\Delta\Delta(x)t_\Delta\right)^4(\varphi_{l,1}(x))^4d^4x\right]\right.. \tag{II.22}$$

where λ_2 is defined below.

Definition II.3. More generally, at scale i we define

$$\varphi(t,x)=\sum_j a(x,\{t\},0,j)\varphi^j(x) \tag{II.23}$$

with

$$a(x,\{t\},k,j)=\prod_{m=k+1}^{j}\left[\sum_{\Delta\in\mathbb{D}_m} t_\Delta \Delta(x)\right]. \tag{II.24}$$

Now

$$\varphi_{l,k}(t,x)=\sum_{j>k} a(x,\{t\},k,j)\varphi^j(x). \tag{II.25}$$

The quartic part of the interaction, $I_4(t,x)$ is now:

$$I_4(t,x)=\lambda\varphi(t,x)^4+\sum_{1\leqq k}\lambda_{k+1}(1-a(x,\{t\},k-1,k)^4)[\varphi_{l,k}(t,x)]^4, \tag{II.26}$$

where the λ_k are the effective coupling contants and are defined below.

Definition II.4. No t-dependence is introduced into the quadratic part of the interaction.

The reason for this is discussed in Sect. II.3. So is the t-dependence for fields downstairs.

Let $F(t)$ be the unnormalized Schwinger function with the t dependence. For each $\Delta\in\mathbb{D}_i$ we write:

$$F=F(t_\Delta)|_{t_\Delta=1}=F(0)+F'(0)+\ldots+(1/4!)F^{(4)}(0)+\int_0^1\frac{(1-t_\Delta)^4}{4!}F^{(5)}(t_\Delta)dt_\Delta, \tag{II.27}$$

where we have suppressed the dependence with respect to the other t-variables. For all the terms in the l.h.s. of (II.27) but $F(0)$ we say that there is at least one derivation in Δ.

The t dependence in (II.26) has been chosen to play two distinct roles. To display them let's suppress the x-dependence in (II.26). In other words consider

$$\lambda\left\{\sum_{j\geqq 1}\left[\prod_{1\leqq m\leqq j}t_m\right]\varphi^j\right\}^4+\sum_{k\geqq 1}\lambda_{k+1}(1-t_k^4)\left\{\sum_{j>k}\left[\prod_{k+1\leqq m\leqq j}t_m\right]\varphi^j\right\}^4. \tag{II.28}$$

To concentrate on the first role replace $t_j\varphi^j$ by ψ^j and pretend for a minute that ψ^j does not depend on t_j. Suppose that we are in the i^{th} decoupling expansion and wish to apply a t_i-derivative. Hence at this stage all t_j's with $j>i$ are still 1. Further suppose that the largest value of j for which $t_j=0$ is h. (If no such j exists set $h=0$.) Hence momentum scales above h have been decoupling from those below h. Under these conditions the only terms of (II.28) that can possibly depend on t_i are

$$\begin{aligned}\sum_{i\geqq k\geqq h}\lambda_{k+1}(1-t_k^4)\left\{\sum_{j>k}\left[\prod_{m=k+1}^{j-1}t_m\right]\psi^j\right\}^4 &= \sum_{i>k\geqq h}\lambda_{k+1}(1-t_k^4)\\ &\times\{H_k+t_iL_k\}^4+\lambda_{i+1}(1-t_i^4)L_i^4,\end{aligned}$$

where

$$H_k=\sum_{i\geqq j>k}\left[\prod_{m=k+1}^{j-1}t_m\right]\psi^j,\qquad L_k=\left[\prod_{m=k+1}^{i-1}t_m\right]\sum_{j>i}\psi^j$$

are fields which have momenta higher than and lower than scale i respectively. (We remark that high momentum fields are decomposed into their constituent scales in our expansion but low momentum fields are not. This will be made more precise in

Sect. III.) Hence still pretending ψ^i is independent of t_i the only terms that can depend on t_i are

$$\sum_{i>k\geqq h}\sum_{1\leqq m\leqq 4}\binom{4}{m}\lambda_{k+1}(1-t_k^4)H_k^{4-m}(t_iL_k)^m-\lambda_{i+1}t_i^4L_i^4$$

$$=\sum_{i>k\geqq h}\sum_{1\leqq m\leqq 3}\binom{4}{m}\lambda_{k+1}(1-t_k^4)H_k^{4-m}(t_iL_k)^m-\left\{\sum_{i>k\geqq h}(\lambda_{k+2}-\lambda_{k+1})L_k^4\right\},$$

since $t^kL_k=L_{k-1}$ and $L_{i-1}=L_i$. Terms generated by the application of derivatives with respect to t_i should be thought of as belonging to two generic types. The first is typified by $\lambda_{k+1}H_k^{4-m}L_{km}$ with $1\leqq m\leqq 3$. These vertices always have at least one high (compared to i) momentum field and one low momentum field. The coupling constant λ_{k+1} has $k+1\geqq h+1$ with $h+1$ being the momentum scale of a certain cube of the connected component of G_i in which the vertex was produced. The spatial localization of the cube coincides with that of the vertex but its momentum is as high as possible. The second type is $(\lambda_{i+1}-\lambda_i)L_i^4$ when $t_i=0$. This will be used as a counterterm for four point functions of scale i. All the other $(\lambda_{k+2}-\lambda_{k+1})L_k^4$'s should be thought of as disguised versions of the first type. Even though they are counterterms they are not needed for renormalization cancellations and may be bounded by $O(1)\lambda_k^2L_k^4$. (See the bounds immediately following Theorem III.2.)

Note also that there is at least one low momentum field per d/dt_i derivative. Hence from (II.27) we see that if $\Delta\in\mathbb{D}_i$ has $t_\Delta\neq 0$ at the end of the expansion there are at least 5 low momentum fields below this cube. When we reinstate the t_i-dependence in $\psi^i=t_i\varphi^i$ it is no longer true that d/dt_i must produce a low momentum field; it may produce a φ^i instead. But φ^i will result in essentially the same estimates as a low momentum field and hence may be viewed as such.

That brings us to the second role of the t-dependence in (II.26) and the reason we are using $\psi^i=t_i\varphi^i$ rather than $\psi^i=\varphi^i$. When $t_\Delta=0$ for all $\Delta\in\mathbb{D}_i$ in some region $\mathbb{Q}\subset\mathbb{R}^4$ all fields $\varphi^i(x)$ with $x\in\mathbb{Q}/\{$a strip around the boundary of $\mathbb{Q}\}$ disappear from the interaction. This is necessary for the factorization "the amplitude of a graph is the product of the amplitudes of its connected components" used in (I.6). Recall that since the functions $\Delta(x)$ are smooth the measure $d\mu_{C(s)}$ can connect fields (possibly in the exponent) of neighbouring cubes Δ and Δ' even when $s_{\Delta,\Delta'}=0$.

II.3 The Mayer Expansion

Definition II.5. Two cubes Δ and Δ' of indices i and j are connected if either:

– there is a propagator (horizontal line) between them (this requires $i=j$)

– There is a vertex (dashed vertical line) and two half lines hooked to it, one in each cube (this requires $i\neq j$ and $\Delta\subset\Delta'$ or $\Delta'\subset\Delta$).

– There is an open gate (wavy line) between them (this requires $j=i+1$ and $\Delta\subset\Delta'$ or the converse, and is equivalent to the presence of at least 5 low momentum fields attached to the dashed lines crossing the interface between Δ and Δ').

– If $i=j$, Δ and Δ' are neighbours and one of them contains a field or an "open gate." We consider in this case that Δ and Δ' are joined by a "neighbour link."

– If Δ and Δ' are joined by a Mayer link. This is explained below.

A connected graph G is then defined as a set of cubes such that any two of them can be linked through a chain of connected cubes.

After the cluster and Mayer expansions up to scale $i-1$ and the cluster expansion of scale i, we have connected components that we call polymers of scale i (or i-polymers). These objects still have disjointness constraints at scale i but not at scales $j<i$, and they do not yet have Mayer links of scale i. On these objects we perform the i-th Mayer expansion. It requires the following definition:

Definition II.6. A 1PI subgraph of index i is a subset of an i-polymer of G_i which contains at least one cube of index i and which cannot be disconnected by cutting at most one propagator. In addition cubes joined by a dashed vertical line, a "neighbour link" or a Mayer link are viewed as being irreducibly linked. The external legs of a 1PI subgraph of index i consist of fields of index $j>i$ belonging to vertices of the subgraph and of (half)propagators connecting vertices of the graph to vertices outside the subgraph.

The Mayer expansion of scale i uses the forest F of all i-polymers and all 1PI 2-point subgraphs of G_i which do not contain true external fields. The i-th support $\underline{B}_i$ of $B\in F$ is the set of the cubes of $B\cap\mathbb{D}_i$ that are not in the 1PI 2-point subgraphs of B.

Let $\mathbb{P}(F)$ be the set of all of pairs of distinct elements of F that are not pairs for which both elements are vacuum components or both have true external legs. In our expansion the i-th supports of the elements of F are disjoint. We express this condition for the pairs in $\mathbb{P}(F)$ as:

$$\prod_{(B,B')\in\mathbb{P}(F)} e^{-V(B,B')}, \tag{II.29}$$

where

$$V(B,B')=\begin{bmatrix}\infty & \text{if} & \underline{B}\cap\underline{B}'\neq 0\\ 0 & \text{if} & \underline{B}\cap\underline{B}'=\emptyset\,.\end{bmatrix} \tag{II.30}$$

The Mayer expansion is generated by expanding the disjointness constraints:

$$\prod_{(B,B')\in\mathbb{P}(F)} [J_{B,B'}+1], \tag{II.31}$$

$$J_{B,B'}=[e^{-V(B,B')}-1]=\begin{bmatrix}-1 & \text{if} & \underline{B}\cap\underline{B}'\neq\emptyset\\ 0 & \text{if} & \underline{B}\cap\underline{B}'=\emptyset\,.\end{bmatrix} \tag{II.32}$$

$J_{B,B'}$ becomes an irreducible link between B and B'. The Mayer expansion is organized inductively as follows. We select one i-polymer and expand all disjointness constraints involving this polymer and another i-polymer. This gives two terms; in the first, the polymer is no longer constrained, and is called a dressed polymer; in the other we have an intermediate object containing one or more Mayer links of scale i, called a partially dressed polymer. We repeat the operation until there are no disjointness constraints left between the intermediate object and other i-polymers, obtaining a dressed polymer. We then repeat the operation until all i-polymers have been transformed into dressed polymers. We repeat then this inductive process for all the constraints involving 1PI two point subgraphs. In each step we pick a minimal (in the sense of the forest) 1PI 2-point subgraph which is at the end of a maximal chain of such subgraphs. (Recall that since a Mayer link is 1PI each addition of a Mayer link changes the forest of 1PI 2-point subgraphs).

This defines the Mayer graphs at scale i. In particular the 1PI 2-point subgraphs are now free of constraints on their supports, even when they are not almost local. The convergence of the Mayer expansion is shown in Sect. III.2.

At this stage all t_j's with $j>i$ are 1. At later stages of the expansion they of course take values other than 1. We allow the t's associated with almost all fields downstairs to follow their natural values. However external fields of almost local 2-point functions of scale i are not given a t-dependence. In other words for them we hold $t_j=1$ for all $j>i$. This is legal because we will never dominate these fields. It is desirable because the fact that we do not perform a wave function renormalization makes it hard to control the combinatoric factors that would be associated with t-derivatives acting on these fields.

II.4 The Renormalization

The purpose of this paper is to construct a model having physical mass zero. Since our covariance has mass zero we will achieve renormalized and consequently physical mass zero if all 1PI 2-point Mayer are renormalized regardless of the relationship between their internal scales and the scales of their external fields. When we say that a 2-point Mayer graph is renormalized we mean that a mass counterterm, whose coefficient is the value of the graph at zero external momentum, is subtracted from it. We remark firstly that there is no law barring us from estimating the graph and counterterm portions of a renormalized Mayer graph separately and we shall indeed do so in the "useless counterterm" situation. We remark secondly that despite the indications of superficial power counting it is not necessary to perform a wave function renormalization: we will not have exponential decay in the vertical direction but after subtraction of the mass counterterm a 2-point Mayer graph of scale i is $O(\lambda_i^2)=O(1/i^2)$ and this is still summable.

To show that we have the right to renormalize all 1PI 2-point Mayer graphs we must show that the renormalization procedure can be implemented by counterterms, i.e. by the inclusion in the exponent of a term $\int \delta m^2(x)\varphi^2(x)dx$. (Until we take the infinite volume limit there will be a weak x-dependence in δm^2.) Since it is impossible for 1PI 2-point subgraphs to overlap this is relatively easy. $\delta m^2(x)$ will be a sum of terms of different scales

$$\delta m^2(x)=\sum_{k=1}^{\varrho}\delta m^2(x,k) \tag{II.33}$$

defined inductively on the scale k. Suppose that we are now working on scale i. By the beginning of the i^{th} step we will already know $\delta m^2(x,1),\dots,\delta m^2(x,i-1)$ and the running coupling constants $\lambda=\lambda_1,\lambda_2,\dots,\lambda_i$. During the i^{th} step we will first define $\delta m^2(x,i)$ and then λ_{i+1}. $\delta m^2(x,i)$ is itself a sum

$$\delta m^2(x,i)=\sum_{n\geqq 0}\delta m^2(x,i;n)$$

with $\delta m^2(x,i;n)$ being the value, at zero external momentum, of (minus) the sum $K_2(i,n)(x,y)$ of all 1PI 2-point Mayer graphs of scale i that have precisely n proper 1PI 2-point subgraphs. In evaluating the Mayer graph all proper 1PI 2-point subgraphs are renormalized, as are the appropiate 4-point subgraphs, but the

Mayer graph itself is of course not renormalized. To perform the evaluation we need to know $\delta m^2(x,i';n')$ for $i' \leqq i$, $n'<n$ (for the renormalization of the 2-point subgraphs) and $\lambda_{i'}$ for $i' \leqq i$. This will indeed be the case if, for each fixed i, we define $\delta m^2(x,i;n)$ by induction on n.

Having discussed the definition of δm^2 we now discuss the mechanics of the cancellation between K_2's. and δm^2's. The propagators that form the external lines of the K_2's have been generated by the cluster expansion. Hence they are of the form $d/ds_{\Delta,\Delta'}C(s)=\Delta C \Delta'$. Upon summation over the different possible values of Δ and Δ' we get simply C, not $C(s)$. Even when K_2 has external fields rather than propagators we can apply $\varphi = C\delta/\delta\varphi$ [correcting of course for Wick ordering as in (II.14)] to those external fields to get propagators that are independent of s. (Note that the external fields belong necessarily to a scale in which the cluster expansion has not yet been performed.) Hence we get strings of K_2's joined by $C(s=1)$'s and terminated by $C(s=1)\delta/\delta\varphi$'s. The same is true for the mass counterterms. See (II.15). Furthermore neither the mass counterterms nor, thanks to the Mayer expansion, the K_2's are subject to disjointness constraints ("K_2 exponentiates") and we can combine each K_2 with its corresponding counterterm to form a renormalized graph.

Once the mass renormalization has been performed we take care of the numerous $\delta/\delta\varphi$'s that have been generated by the cluster expansion etc. They are, except for those that find themselves at the ends of chains of mass insertions simply evaluated. To the latter we apply [33, Lemma VI.2]:

$$\int \prod [C\delta/\delta\varphi](x_i) F(\varphi) d\mu_{C(s)} = \int :\prod [CC(s)^{-1}\varphi](x_i): F(\varphi) d\mu_{C(s)} . \qquad \text{(II.34)}$$

This is so that the end of each chain introduces a single φ rather three of them. Normally it doesn't matter that the end of each cluster bond produces three φ's: the exponential decay of the propagator together with the disjointness constraints inherent in the cluster expansion can control any local $n!$. But the Mayer expansion has removed the disjointness constraints from the mass insertions and we have to be more careful. See Propositions IV.5 and IV.6. That the $CC(s)^{-1}$ is harmless is shown in Theorem II.1 *d*).

The definition of λ_{i+1} is given by

$$\lambda_1 = \lambda , \quad \lambda_{i+1}(x) = \lambda_i(x) + \delta\lambda_i(x) , \quad \delta\lambda_i(x) = -\int K_4(i)(x, x_1, x_2, x_3) dx_1 dx_2 dx_3 , \qquad \text{(II.35)}$$

where $K_4(i)$ is (the sum of all 4-point Mayer graphs of scale i that do not contain a $\delta\lambda_i$ vertex) divided by (the sum of all vacuum Mayer graphs of scale i). Recall that a 4-point Mayer graph may contain at most one $\delta\lambda_i$ vertex. See the discussion following (II.27).

Remark. The renormalized values of

$$\int K_2(x,y)\varphi(x)\varphi(y)dxdy$$

and

$$\int K_4(x_1,x_2,x_3,x_4)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)dx_1dx_2dx_3dx_4$$

are

$$\int K_2(x,y)\varphi(x)\varphi(y)dxdy - \int K_2(x,y)[\varphi^2(x)+\varphi^2(y)]/2dxdy$$
$$= -1/2\int K_2(x,y)[\varphi(x)-\varphi(y)]^2 dxdy$$

and

$$\int K_4(x_1,x_2,x_3,x_4)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)dx_1dx_2dx_3dx_4 - \int K_4(x_1,x_2,x_3,x_4)\varphi(x_1)^4 dx_1dx_2dx_3dx_4$$

$$= \int K_4(x_1,x_2,x_3,x_4) \sum_{j=1}^{4} \left[\prod_{i=1}^{j} \varphi(x_i)\right] [\varphi(x_{j+1})-\varphi(x_j)] \prod_{k=j+2}^{4} \varphi(x_k)dx_1dx_2dx_3dx_4 ,$$

respectively. It is the difference $[\varphi(z)-\varphi(z')]$ which is responsible for the good that renormalization does.

III. The Main Results

For clarity we prove the convergence of the expansion only in the case $\varrho=\infty$. To prove the convergence as $\varrho\to\infty$ is straightforward. Since we are now in infinite volume we can forget the weak dependence on x of the counterterms of Sect. II.4 due to finite volume effects.

III.1 The Renormalization Group

We shall prove by induction that for $k=1,\ldots,i$ the following bounds hold:

Theorem III.1.

$$0\leqq\lambda_k\leqq\lambda_1 , \tag{III.1}$$

$$0<\lambda_k^{-1}\leqq\lambda_1^{-1}+O(1)(\ln M^k), \tag{III.2}$$

$$|\delta m^2(k)|\leqq O(1)\lambda_k M^{-2k} . \tag{III.3}$$

Let us make the convention that $K_2(i,e)(\ldots)$ (respectively K_4) is the sum of all the 2-point Mayer graphs unrenormalized (respectively 4-point Mayer graphs) of index i with e external vertices, $e=1$, 2, 3, 4. The external vertices are kept fixed. We prove in Sect. III.2:

Theorem III.2. *There exists K independent of λ such that for λ small enough:*

$$|K_4(i,e)(x_1,x_2,x_3,x_4)|\leqq O(1)[\lambda_i]^e e^{-(M^{-i}/K)\left[\sum_{k\neq l}|x_k-x_l|\right]} M^{-4i(e-1)}, \tag{III.4}$$

$$|K_2(i,e)(x_1,x_2)|\leqq O(1)(\lambda_i)^e e^{-(M^{-i}/K)|x_1-x_2|} M^{-2i-4i(e-1)} . \tag{III.5}$$

The moral of these bounds is that the leading contributions come from the lowest orders of perturbation theory. An immediate consequence is that for $M\geqq 2$ and λ small enough:

$$-[O(1)/100][\lambda_i]^2\leqq\delta\lambda_i\leqq -O(1)[\lambda_i]^2 ,$$

hence

$$[\lambda_i]^{-1}+O(1)/100\leqq[\lambda_{i+1}]^{-1}\leqq[\lambda_i]^{-1}+2O(1)$$

which, by an easy recursion argument, proves the first two statements of Theorem III.1. The remaining one is an easy consequence of the second bound of Theorem III.2.

III.2 The Bound on the Subgraphs

In this section we prove Theorem III.2 and also a bound of the same kind on the sum of the subgraphs containing some true external fields.

Theorem III.3. *Let $i_1, \ldots, i_k$, be the indices of the true external fields. The sum of the Mayer graphs of index i containing them is bounded by $k!$ times*

$$M^{-i_1} \ldots M^{-i_k} M^{-(1/2)\sup[0, i-\inf(i_1, \ldots, i_k)]}. \tag{III.6}$$

This bound shows that the sum of Mayer graphs of all indices converges. The existence part of Theorem I.1 follows. The statement on the non-exponential decay of the two point function will be proven in Sect. III.3.

We perform a complete renormalization of all two point functions. But as explained at length in Sect. I.3, it is beneficial to estimate a graph and its counterterm together only in the "useful" case. To define precisely this notion we have to introduce a "reduced index" for Mayer graphs, whose definition is inductive (see [7, 10] for similar examples). Basically, the "reduced index" ignores the "useless" mass counterterms contained in the Mayer graph.

For a Mayer graph which contains no two point subgraphs, the reduced index is equal to the ordinary index. Such a graph is called "almost local" if its reduced index is strictly smaller than the indices of all its external legs. Otherwise it is called a low momentum graph. The counterterms corresponding to low momentum two point functions are called "low momentum insertions" and are not combined with these functions. By induction on the number of two point subgraphs or counterterms, we define the reduced index of a graph as the index of the graph obtained by ignoring all the low momentum insertions. Remember that in contrast, the "almost local" two point graphs are combined with their counterterms to form a renormalized graph.

Definition III.4. For one graph, let for a φ^4 or φ^2 vertex v:

- $l(v)$ be the biggest of the indices of the derivatives d/dt or d/ds of index at most i acting on v,
- $h(v)$ be the smallest of the indices of the fields attached to v,
- $i(\delta m^2)$ be the index for a low momentum mass counterterm,
- $i(\varphi)$ be the index of a field φ.

A field φ hooked to a vertex v is called a high momentum field if $i(\varphi) \leqq l(v)$.

In contrast with high momentum fields it is sometimes convenient to consider as a single low momentum field hooked to v the sum

$$\sum_{k > l(v)} \varphi^k = \varphi_{l, l(v)}\,.$$

By convention the index of such a low momentum field is $l(v)$.

Each renormalization of 2- or 4-point graphs of index j "creates" a difference of fields:

$$\varphi(x) - \varphi(y) = \int_0^1 d\alpha (x-y) \cdot \nabla\varphi[y + \alpha(x-y)]\,. \tag{III.7}$$

The factor $(x-y)$ will be called a renormalization factor of index j. The index of the gradient ∇ acting on φ is $i(\nabla) = i(\varphi)$.

Using the fact that the two point function is translation invariant (because $\varrho = \infty$) we can transfer gradients in the mass renormalized two point function from one leg to another, hence we consider that in a local 2-point function the two gradients act on the field with the biggest index. An almost local term is then of the form $\varphi_k \tilde{K}_2 \nabla^2 \varphi_l$ with $k \leqq l$, and by definition we incorporate in $\tilde{K}$ the factors $(x-y)$ that Eq. (III.7) generates: $\tilde{K}(x,y)=(x-y)^2 K(x,y)$. The index of such a term, noted $i(K_2)$, is then defined to be the smallest of the two external indices (here k), and the biggest one (here l) is noted $l(\tilde{K}_2)$.

We start the proof of Theorem III.2. The proof is by induction on i. Let us suppose that Theorem III.2 holds up to scale $i-1$. We will deduce from this induction hypothesis that Theorem III.5 below holds for i-polymers. From it we will verify that Theorem III.2 holds at scale i. From Eq. (III.5) of the induction hypothesis we have

$$|K_2(j,e)(x_1,x_2)| \leqq O(1)(\lambda_j)^e e^{-(M^{-j}/K)|x_1-x_2|} M^{-2j-4j(e-1)}. \tag{III.8}$$

Let us suppose that $i(K_2)=k$. As a consequence of the bounds on K_4 we have: $0 \leqq \lambda_j \leqq \inf(\lambda, O(1)/j)$, for $j<i$.

Hence $\sum_{j \leqq k} (\lambda_j)^2 \leqq O(1)\lambda$. Thus because the non-vanishing renormalized K_2 have necessarily $e=2$ (the renormalized "tadpoles" with $e=1$ vanish), the bound on a local term of index $i(K_2)=k \leqq i$ is:

$$\left|\sum_{j \leqq k} |x-y|^2 K_2(j,2)(x,y)\right| \leqq O(1)\lambda \tilde{C}_k(x,y) e^{-(M^{-k}/2K)|x-y|}, \tag{III.9}$$

where
$$\tilde{C}_k(x,y) = \sum_{j \leqq k} (1/\lambda)\lambda_j^2 M^{-4j} e^{-(M^{-j}/2K)|x-y|}. \tag{III.10}$$

We observe that for $\Delta \in \mathbb{D}_k$:

$$\int_\Delta \tilde{C}_k(x,y)\,dy \leqq O(1). \tag{III.11}$$

In a graph there are φ^4 vertices, $\delta m^2 \varphi^2$ vertices (i.e. low momentum mass counterterms), and $\varphi \tilde{K}_z \nabla^2 \varphi$ vertices. In Sect. IV we prove then:

Theorem III.5. *For ε small enough, holding all its vertices fixed, a i-polymer is bounded by (if it has 2 or 4 external lines, we denote by E the set of its external vertices):*

$O(1)$	*per cube*,
$O(1)\lambda \tilde{C}_k(x,y)$	*per $\tilde{K}_2$ local term ($i(\tilde{K}_2)=k$)*,
$O(1)\lambda_{l(v)}$	*per φ^4 vertex $v \in E$*,
$O(1)(\lambda_{l(v)})^{1/4}$	*per φ^4 vertex $v \notin E$*,
$M^{\varepsilon[l(v)-h(v)]}$	*per vertex v*,
$M^{-i(\varphi)}$	*per field φ*,
$\lambda_{i(\delta m)} M^{-2i(\delta m)}$	*per low momentum mass counterterm*,
M^j	*per renormalization of index j*,
$M^{-i(\nabla)}$	*per gradient*,
$e^{-(M^{-j}/K)d(\Delta,\Delta')}$	*per propagator of index j and K_2 insertion, of index $i(K_2)=j$ between points in cubes Δ and Δ'.*

$\left(d(\Delta,\Delta') \textit{ is obviously defined as } \inf_{x\in\Delta,\, y\in\Delta'} |x-y|\right).$

From Theorem III.5 we prove Theorems III.2 and III.3 at scale i.

For a Mayer graph Γ we define the support $S(\Gamma)$ as the union of all cubes of $\mathbb{D}$ which belong to its constituent polymers (of all scales). We also define its "Mayer structure" $T(\Gamma)$ as the set $\{n(\Delta), \Delta \in S(\Gamma)\}$ where $n(\Delta)$, the multiplicity of $\Delta \in \mathbb{D}_k$, is the number of k-polymers in Γ containing Δ, together with the set of Mayer links of Γ. To bound the sum of all Mayer graphs of scale i, we first bound the sum of all Mayer graphs having a given support.

We write

$$\operatorname{td}(\Gamma) = \prod_{j \leq i} \prod_k \operatorname{td}(\Gamma_j^k), \tag{III.12}$$

where the product over k is over the j-polymers Γ_j^k which are constituents of Γ. $\operatorname{td}(\Gamma_j^k)$ is the tree decay of scale j between the cubes of $\mathbb{D}_j$ in Γ_j^k.

Let us call the sum over all Mayer graphs having a given Mayer structure T fixed as $G(T)$, and let us put $|T| = \sum n(\Delta)$. Then

Proposition III.6. *For any $\delta > 0$ there exists a constant $O(1)$ (depending on δ) such that*

$$\sum_{T/|T|=n} |G(T)| \leq \sup_T \left\{ \sup_{\Gamma, T(\Gamma)=T} \{O(1)^n \operatorname{td}(\Gamma)^{-\delta} |\Gamma|\} \right\}. \tag{III.13}$$

Proof. The Mayer expansion described in Sect. II.3 was in two parts. At the end of the part of the Mayer expansion dealing with the connected components we have a tree of Mayer links. In the dressing process of a polymer we have only to choose for each cube whether or not it is used to create a new Mayer link. Hence the corresponding sum is bounded by a factor 2 per cube of T. For the part of the Mayer expansion dealing with the 1PI 2-point subgraphs we proceed as follows: at each intermediate stage we develop the disjointness constraints between a minimal 1PI 2-point subgraph, say B, at the end of a maximal chain and the rest. Using a small fraction δ of the exponential decay of the propagator hooked to B which forms the end of the chain, we decide which constraint is developed. This results in a factor $O(1)$ per such propagator and a factor 2 per cube as before. □

The factors $O(1)^n$ and $\operatorname{td}(\Gamma)^{-\delta}$ in (III.13) are easily controlled (for δ and λ small) by the ones of Theorem III.5, hence we will forget them from now on.

Proposition III.7. *Let us consider the sum over the absolute values of all the Mayer graphs of index i containing a given fixed cube, a given total number n of cubes and having some fixed external fields or true external fields. This sum is bounded by the right-hand side respectively of (III.4), (III.5), and (III.6), depending on the number of external fields, times a factor B^n, where B is as small as we want provided λ is small enough and M is big enough. In addition the same sum for vacuum subgraphs is simply bounded by B^n.*

Proof of Proposition III.7. Each high momentum field has an index (that we sum). Let Γ_j note a Mayer graph of index j (i.e. formed of all the cubes of index lower or equal to j), and Γ_j^k its connected components. A vertex is contained in a subgraph if at least one of its legs is contained in it.

Using an induction procedure starting from the subgraphs of largest index, we will define at least one fixed vertex per subgraph:

All external vertices (true or not) are fixed vertices. If a subgraph contains no previously defined fixed vertex we pick one at random. For a graph of index i we write for the factors of Theorem III.5:

$$M^j = M^i \prod_{j<h\leqq i} M^{-1} \tag{III.14}$$

for each field other than true external fields,

$$M^{-j} = M^{-i} \prod_{j<k\leqq i} M \tag{III.15}$$

for low momentum mass counterterms and gradients,

$$e^{-2(M^{-j}/K)d} \leqq \prod_{h\geqq j} e^{-(M^{-k}/K)d} \tag{III.16}$$

for the exponential fall off of each propagator of index j. Inequality (III.16) is true provided $M>2$. From now on we also refer to the h^{th} factor on the right-hand side of (III.16) as a propagator of scale h.

For each connected subgraph (of index j), starting from the subgraphs of smallest index we integrate over all space each nonfixed vertex of the subgraphs which have not already been integrated. For a vertex which has fields of index bigger than j we use:

$$\int g(x)h(x)dx \leqq \sup_x |h(x)| \int |g(x)|dx \leqq M^{4j} \sup_x |h(x)| \sum_{\Delta\in\mathbb{D}_j} \sup_{y\in\Delta} |g(y)|, \tag{III.17}$$

where $h(x)$ is the product of the propagators of scale bigger than j and hooked to the vertex, $g(x)$ is the product of the propagators of scale less than or equal to j and hooked to the vertex. The result is a factor M^{4j} per integrated vertex of a subgraph of index j. We apply (III.14) to this factor.

Then using Theorem III.5 we rewrite the bound on a subgraph of index i as:

$$\begin{aligned} K \leqq LL' \prod_{\Delta\in K} O(1) \prod_j \prod_l \left[M^{l(j,k)} M^{-4v(j,k)} \prod_{p\in G_j^l} e^{-(M^{-j}/K)d(p)} \right] \\ \prod_{\delta m} O(1)\lambda_{i(\delta m)} \prod_{K_2} O(1)\lambda \prod_{\substack{v\notin E \\ v\ \varphi^4\ \text{vertex}}} (\lambda_{l(v)})^{1/4} \prod_{\substack{v\in E \\ v\ \varphi^4\ \text{vertex}}} \lambda_{l(v)} \prod_{\substack{v \\ v\ \varphi^4\ \text{vertex}}} M^{\varepsilon[l(v)-h(v)]}, \end{aligned} \tag{III.18}$$

where $l(j,k) = \#$ of fields of G_j^k that are not true external fields

$+\#$ of gradients acting on fields of G_j^k

$-\#$ of renormalizations internal to G_j^k

$+2\#$ of $\delta m\varphi^2$ vertices of G_j^k,

$v(j,k) = \#$ of non-fixed vertices of G_j^k,

$d(p)$, for each propagator p is the length of p,

and

$L = M^{-2i-4i(e-1)}$ for a 2-point function of index i,

$L = M^{-4i(e-1)}$ for a 4-point function of index i,

$L = O(1)M^{-i_1} \dots M^{-i_k}$ for a subgraph containing true external fields of indices $i_1, \dots, i_k$,

L' is a tree decay of scale i between the external vertices.

Each cube is associated with a derivation, a derivation acts on a vertex and a vertex v can be derived at most $3(l(v)-h(v)+1)$ times; thus the factor $O(1)$ per cube is bounded by a factor $O(1)M^{\varepsilon(l(v)-h(v))}$, for small ε, provided M is later fixed to a large (ε-dependent) value.

We define $e(j,k)=$ number of external legs of G_j^k which are not true external legs. Then

$$\sum_{j,k} e(j,k) \geqq \sum_{v} [l(v)-h(v)]\,. \tag{III.19}$$

Thus the factor $\prod_{v} M^{2\varepsilon(l(v)-h(v))}$ is bounded by $\prod_{j,k} M^{2\varepsilon e(j,k)}$.

We use the exponential decay of the propagators to bound the sum on the cubes of G_j^k arising from formula (III.17),

$$\sum_{\Delta\in\mathbb{D}_j} e^{-(M^{-j}/K)d(\Delta,\Delta')} \leqq O(1)\,. \tag{III.20}$$

Each vertex has at most 4 legs so that all the sums are bounded by $O(1)^n$ (we omit repeating such factors in the bound since they are harmless). There remains an exponential tree decay of scale i between the external vertices for the 2 and 4-point Mayer graphs of index i.

For subgraphs with $e(j,k)>0$, $v(j,k)$ is the number of vertices of G_j^k minus one (the fixed one). Hence $-4v(j,k)+l(j,k)<0$ and we have:

$$-e(j,k)+2 \leqq -4v(j,k)+l(j,k) \leqq -e(j,k)+4\,.$$

The result is that:

$$\sum_{j,k} -[4v(j,k)-l(j,k)] + \sum_{v} 2\varepsilon[l(v)-h(v)] \leqq -\sum_{j,k}(1-8\varepsilon)\,. \tag{III.21}$$

For $\varepsilon<1/100$ we obtain a small factor per $G_j^k : M^{-(1-4\varepsilon)}$.

$$\sum K \leqq LL' \sum \prod_{j,k} M^{-(1/5)e(j,k)} \prod_{\substack{v\notin E\\ v\ \varphi^4\ \text{vertex}}} \lambda_{l(v)}^{1/4} \prod_{\substack{v\in E\\ v\ \varphi^4\ \text{vertex}}} \lambda_{l(v)} \lambda^{\#(\delta m \text{ or } K_2 \text{ insertions})}\,. \tag{III.22}$$

The sum on the left is over all subgraphs containing a fixed cube and having given fixed external structure. The sum on the right is over the total number of vertices (φ^4, δm or K_2) and over all possible assignments of the $e(j,k)$'s and $l(v)$'s. This last sum is done in the following way. For each scale j we sum over the number of vertices with $h(v)=j$, using the small factor per vertex. This sum gives a constant per scale. Then, using a fraction of the exponential decay in $e(j,k)$ of (III.22) we can sum over the choices of the $l(v)$'s. Finnally the product of the accumulated constants (one per scale) is bounded by another fraction of this decay. This achieves the control of the sum in the right-hand side of (III.22). We have thus obtained Proposition III.7 up to the fact that the λ's of the external vertices do not yet have the index i. Paying with still another fraction of the exponential decay in $e(j,k)$ in Eq. (III.22) we can transform the $\lambda_{l(v)}$'s of the external vertices v into factors λ_i, which completes the proof of Proposition III.7. $\square$

To achieve the proof of Theorem III.2 (and III.3), we have only to sum over the number n of cubes and this last sum is geometrically convergent.

III.3. The Behaviour of the Two and Four Point Functions. Proof of Theorem I.1 and I.2

The two point function S_2 is equal to the free propagator $C(x-y)$ with all possible insertions of mass-renormalized 2-point subgraphs. For simplicity let us call $B(x,y)$ the sum of all mass-renormalized 1PI 2-point subgraphs. We may write S_2 as:

$$S_2 = C + C \times B \times C + \ldots + C \times B \times C \ldots \times B \times C + \ldots, \tag{III.23}$$

the product being a convolution product. To get the exact leading behavior of S_2, we must perform the wave function renormalization, hence write in Fourier space $B(p) = B_0(\lambda)p^2 + B_1(\lambda, p)$. Then we decompose (III.23) as $S_2 = S_{2,0} + S_{2,1}$, with

$$S_{2,0} = C\left\{1 + \sum_{n \geqq 1} (B_0 \times C)^n\right\}, \tag{III.24}$$

$$S_{2,1} = \left\{\sum_{n \geqq 0} (C \times B_0)^n\right\} C \left\{\sum_{n \geqq 1} (B_1 \times C)^n\right\}. \tag{III.25}$$

In the first sum (III.24) we have a geometric series. From the bounds of Theorem III.2 we have $B_0(\lambda) = O(\lambda)$ since by Theorem III.1 $\sum_i \lambda_i^2 \leqq O(\lambda)$. Hence if we denote by $U(p^2)$ the ultraviolet cutoff (in momentum space) of Sect. I.2, we have $B_0 \times C = O(\lambda)U$, and $S_{2,0} = C\{1 - O(\lambda)U\}^{-1}$. In position space, $S_{2,0}(x-y)$ behaves like $[1 + O(\lambda)]/|x-y|^2$ for large $x-y$ [the corrections, due to the ultraviolet cutoff, vanish rapidly and can be absorbed in the correction term C_1 in (I.5). Similarly the first geometric sum in (III.25) can be bounded by $O(1)$]. Hence we have only to verify that:

$$\left|\sum_{n \geqq 1} C \times (B_1 \times C)^n\right| \leqq \frac{O(1)}{|x-y|^2\{1 + \mathrm{Log}[1 + |x-y|]\}}. \tag{III.26}$$

First we observe that the effect of the mass renormalization in B_1 is to replace the C's in the interior of (III.26) by $\nabla^2 C$'s, which behave at large distances as $|x-y|^{-4}$. Similarly the two C's at the ends are changed into ∇C, which behaves like $|x-y|^{-3}$. If we do not take into account the wave function renormalization, the kernel between these differentiated propagators behaves like $|x-y|^{-4}[\mathrm{Log}|x-y|]^{-2}$. The Log^{-2} correction to ordinary power counting comes from the λ_i^2 in the bound on K_2 of Theorem III.2. Hence, were it not for the wave function renormalization, we would have to bound:

$$|x_1|^{-3} \times |y_1|^{-4}[\mathrm{Log}|y_1|]^{-2} \times |x_2|^{-4} \times \ldots \times |y_n|^{-4}[\mathrm{Log}|y_n|]^{-2} \times |x_{n+1}|^{-3}, \tag{III.27}$$

where the x's correspond to difference variables of differentiated propagators and the y's to difference variables for the kernels. This does not satisfy the desired bound (III.26). The problem is that the mass-renormalized 2-point function has dimension 0. Consequently we do not have exponential decay between the momentum scale of an almost local 2-point insertion and its external legs. However with wave function renormalization, this decay is restored. For a two point insertion $B(y-z)$ between propagators $C(x-y)$ and $C(z-t)$, this decay can be used to transform the factor $[\mathrm{Log}|y-z|]^{-2}$ into $\{\mathrm{Log}[\sup\{|x-y|, |y-z|\}] \cdot \mathrm{Log}[\sup\{|y-z|, |z-t|\}]\}^{-1}$. Then we use:

Lemma.

$$\int \frac{d^4y}{\mathrm{Log}[\sup\{|x-y|,|y-z|\}]\cdot \mathrm{Log}[\sup\{|y-z|,|z-t|\}]\,|x-y|^3|y-z|^4}$$
$$\leqq \frac{O(1)}{\mathrm{Log}[\sup\{|x-z|,|z-t|\}]\,|x-z|^3}, \tag{III.28}$$

$$\int \frac{d^4y}{\mathrm{Log}[\sup\{|x-y|,|y-z|\}]\,|x-y|^3|y-z|^3} \leqq \frac{O(1)}{[\mathrm{Log}|x-z|]\,|x-z|^2}. \tag{III.29}$$

Applying repeatedly (III.28) to the chain of convolutions in (III.27) and applying (III.29) for the last convolution, we obtain the desired bound (III.26), hence completing the proof of Theorem I.1 and of the first part of Theorem I.2 [(I.4)–(I.5)].

To prove the last part of Theorem I.2, i.e. (I.6), we have to bound the truncated four point function at large distances. We consider the contributions of lowest orders in the effective coupling constant. The biggest one is the one with only one coupling constant whose index must be equal to the smallest index of the external legs; up to smaller corrections this index is proportional to:

$$\mathrm{Log}\left[\inf_{i,j}|x_i-x_j|\right], \tag{III.30}$$

where the x's are the 4 external points. Hence we obtain the desired result:

$$|S_4^T(x_1,\dots,x_4)| \leqq \int \prod_{i=1}^{4} S_2(x_i,y)\frac{C_3 d^4y}{1+\inf\limits_{(i,j)}\mathrm{Ln}(1+|x_i-x_j|)}. \tag{III.31}$$

IV. The Bounds

We prove Theorem III.5, i.e. we bound an i-polymer.

We return to the definition of the Mayer expansion of Sect. II.3. In Sect. III.2 it was explained how to bound the combinatoric factors generated by the expansion of the disjointness constraints of the 1PI 2-point subgraphs. However a technical problem remains. In the ordinary cluster expansion, the local factorials due to the integration of the fields hooked to many propagators which can accumulate in a single square are easily bounded by the exponential decrease associated to these propagators. This is no longer the case for the Mayerized cluster expansion, because the cubes at the far ends of the propagators of the chains of 1PI 2-point subgraphs need no longer be disjoint. Therefore we reserve a special treatment to these end-of-chains propagators. Instead of evaluating the $\delta/\delta\varphi$ at the ends of these chains as in the usual cluster expansion (i.e. applying them directly on the remaining integrand, which can generate three fields per such $\delta/\delta\varphi$) we convert them into $C(s)^{-1}\varphi$.

Each such polymer is now a sum of terms that we write as:

$$\sum \int ABCP\,d\mu. \tag{IV.1}$$

A is the exponential of the interaction, B is the product of the high momentum fields, of the true external fields, of the almost local (renormalized) mass insertions and of the "useless" mass counterterms, C is the product of the low momentum fields and P is the product of the propagators produced by the expansion.

Let us introduce some definitions. Let G be a polymer, $x \in \Delta \in \mathbb{D}_i$, $\Delta \in G$, then:

$$h(x) = \inf(k|\Delta' \in \mathbb{D}_k,\ \Delta' \text{ in the support of } G,\ x \in \Delta'), \tag{IV.2}$$

$$l(i,x) = i - h(x) + 1\,, \tag{IV.3}$$

and

$$l(\Delta) = (1/\Delta) \int_\Delta M^{\varepsilon l(i,x)} dx\,, \tag{IV.4}$$

$$n(\Delta) = 3\,\#\{\Delta', \text{ such that in formula (II.16) we choose } D^1_{\Delta,\Delta'},\ D^2_{\Delta,\Delta'} \text{ or } I_{\Delta,\Delta'}\}.$$

IV.1 The Low Momentum Fields

For each low momentum field $\varphi_{l,k}$,

$$\varphi_{l,k}(x,t) = \sum_{j>k} a(x,\{t\},k,j)\,\varphi^j\,, \tag{IV.5}$$

where $a(x,\{t\},k,j)$ was defined in (II.24). We decompose

$$\varphi_{l,k} = \hat{\eta}\varphi_{l,k} + \delta\varphi_{l,k} \tag{IV.6}$$

with:

$$\hat{\eta}\varphi_{l,k}(x,t) = \sum_{j>k} a(x,\{t\},k,j) \int \eta_k(x-y)\,\varphi^j(y)\,dy\,, \tag{IV.7}$$

$$\delta\varphi_{l,k}(x,t) = \sum_{j>k} a(x,\{t\},k,j)\,\delta_k\varphi^j(x)\,, \tag{IV.8}$$

$$\delta_k\varphi^j(x) = \varphi^j(x) - \int \eta_k(x-y)\,\varphi^j(y)\,dy\,, \tag{IV.9}$$

where $\eta_k(z) = (\alpha M^k)^{-4}\eta(z/\alpha M^k)$ was defined in (I.24). Note that η is C_0^∞ and that, since η is even, $\tilde{\eta}'(0) = 0$.

We observe in (IV.7) that the $a(x,\{t\},k,j)$ and the field $\varphi^j(y)$ do not come in the combination $\varphi_{l,k}(x,t)$ needed for domination. We thus rewrite $a(x,\{t\},k,j)$ as $a(y,\{t\},k,j)$ plus corrections using

$$\Delta(x) = \Delta(y) + (x-y)\Delta'(x) + 1/2(x-y)^2\Delta''(x+\vartheta(y-x))\,, \qquad 0 \leqq \vartheta \leqq 1\,. \tag{IV.10}$$

We start from the lowest index, j, and apply (IV.10) to all the cubes of this index appearing in the function a. For the terms containing only $\Delta(y)$ factors, the correct t-dependence for domination has been restored at this scale and we move to the scale $j-1$ and so on. As soon as we encounter a term with a factor Δ' or Δ'', say at scale q, $k < q \leqq j$, we stop using (IV.10).

If the factor is a Δ'' we have two terms, one with the field φ^q, the other with the low momentum field $\varphi_{l,q}$. For the φ^q field, we contract (use gaussian integration) and for the field $\varphi_{l,q}$ we use ordinary domination in the appropriate cube of scale q [see Eq. (IV.13)]. In the first case, since there may be up to $O(1)M^{4(q-k)}$ fields contracted in this way, the $(n!)^{1/2}$ of the gaussian integration gives $M^{2(q-k)}$ per field. Moreover the propagators of scale q give a factor $M^{-2(q-k)}$ per propagator, hence $M^{-(q-k)}$ per field better than the propagator of scale k. Similarly in the second case the factor $(n!)^{1/4}$ in Eq. (IV.13) results in a factor M^{q-k} per dominated field. On the other hand the factor $|x-y|^2\Delta''$ gives us in both cases an extra factor $M^{-2(q-k)}$.

Hence in both cases the remaining factor $M^{-(q-k)}$ may be used to sum over q, and the net result is the same as if we had dominated at scale k.

If the factor is a Δ' we use the fact that

$$\int (x-y)\eta_k(x-y)d^4y=0$$

to rewrite $\int \Delta'(x)(x-y)\eta_k(x-y)\varphi^j(y)d^4y$ as

$$\begin{aligned}&\int d^4y\Delta'(x)(x-y)\eta_k(x-y)\,[(\delta_{q+1}\varphi^j(y)-\delta_{q+1}\varphi^j(x))\\&\quad+\int d^4z(\eta_{q+1}(y-z)\varphi^j(z)-\eta_{q+1}(x-z)\varphi^j(z))]\,.\end{aligned}\tag{IV.11}$$

Since $x-y$ is of order M^k because of the function η_k, and the characteristic scale of momenta of η_{q+1} and of φ^j is less than or equal to q, the two differences in (IV.11) result both in an extra factor $M^{-(q-k)}$ which combines with the $\Delta'(x)(x-y)$ to behave just like the Δ'' in the former case. The contractions and dominations are then done as above, and the conclusion is the same. Hence in the future it suffices to consider fields which have the correct t-dependence for domination, hence we replace (IV.6) by

$$\varphi_{l,k}(t,x)=\eta\varphi_{l,k}(t,x)+\delta\varphi_{l,k}(t,x)\,.\tag{IV.12}$$

The same procedure may be applied to low momentum fields to which derivatives have been applied: simply use integration by parts to move the gradients of $\eta\nabla^n\varphi$ from the φ to the η.

To prepare the low momentum fields for domination it remains to introduce a $(1-t)$ factor for every such field, since there is a corresponding factor $(1-t^4)$ in front of the φ^4 term in the exponential [see (II.22)]. We can dominate at most 15 fields which have no $(1-t)$ factors:

$$\int [(1-t_i)\lambda_{i+1}^{1/4}\varphi_{l,i}]^n[\lambda_{i+1}^{1/4}\varphi_{l,i}]^m e^{-(1-t_i^4)\lambda_{i+1}[\varphi_{l,i}]^4}(1-t_i)^4dt\leqq O(1)^n(n!)^{1/4}\,,\tag{IV.13}$$

provided that $m\leqq 15$.

The elementary step to introduce a $(1-t)$ dependence is (we suppose $t\neq 0$):

$$\varphi_{l,i}(t,x)=\sum_{\Delta\in\mathbb{D}_i}\Delta(x)(1-t_\Delta)\varphi_{l,i}(t,x)+\varphi_{l,i-1}(t,x)-\sum_{\Delta\in\mathbb{D}_i}\Delta(x)t_\Delta\varphi^i(x)\,.\tag{IV.14}$$

We do this as long as $t\neq 0$, and obtain

$$\varphi_{l,i}(t,x)=\sum_{h(x)-1\leqq j\leqq i}\ \sum_{\Delta\in\mathbb{D}_j}\Delta(x)(1-t_\Delta)\varphi_{l,j}(t,x)-\sum_{h(x)\leqq j\leqq i}\ \sum_{\Delta\in\mathbb{D}_j}\Delta(x)t_\Delta\varphi^j(x)\,.\tag{IV.15}$$

The first term of the l.h.s. of (IV.15) consists of low momentum fields while the second term consists of new high momentum fields (which will be contracted rather than dominated). We remark that by definition $t_{h(x)-1}=0$.

After completing the steps described above (reconstitution of the dependence on t and the addition of the $1-t$ factors) each low momentum field appears in the form, for $v\in\Delta_v\in\mathbb{D}_i$,

$$\begin{aligned}&\left|\int dx\eta_i(v-x)\sum_{\Delta\in\mathbb{D}_j}\Delta(x)(1-t_\Delta)\varphi_{l,j}(t,x)\right|\\&\leqq\sup_x M^{4i}|\eta_i(v-x)|\sum_{\substack{\Delta\in\mathbb{D}_j\\ \Delta\text{ neighb. of }\Delta_v}}M^{-4i}\int\Delta(x)l(i,x)(1-t_\Delta)|\varphi_{l,j,i}(t,x)|dx\\&\leqq O(1)\sup_{\Delta\text{ neighb. }\Delta_v}l(\Delta)M^{-i}|\int\Delta(x)|\varphi_{l,j,i}(t,x)|^4dx|^{1/4}\,,\end{aligned}\tag{IV.16}$$

where

$$\varphi_{l,j,i}(x) = \begin{cases} \sum_{\Delta \in \mathbb{D}_j} \Delta(x) \frac{1-t_\Delta}{(i-j+1)} \varphi_{l,j}(t,x) & \text{if } j \leqq i. \\ 0 & \text{otherwise} \end{cases} \tag{IV.17}$$

Applying the bound (IV.16) to all low momentum fields of the polymer gives:

$$\prod_\Delta [l(\Delta)]^{O(1)(1+n(\Delta))} \prod_i \prod_{\Delta \in \mathbb{D}_i} \left[\sum_{j \leqq i} \left| \int \Delta(x) |\varphi_{l,j,i}(t,x)|^4 dx \right| \right]^{O(1)(1+n(\Delta))} \tag{IV.18}$$

times a factor M^{-i} per low momentum field $\eta_i \varphi_{l,j}$.

IV.2 Proof of Theorem III.5

The sum over all the derivations producing a given graph is controlled by the following proposition.

Proposition IV.1. *Postponing the summations on the vertices, each subdiagram G is a sum of terms T (the value is noted also T) that we bound by*

$$|\sum T| \leqq \sup_T a(T)|T| \quad \textit{for} \quad \sum_T a(T)^{-1} \leqq 1. \tag{IV.19}$$

The a(T) are given by

$$a(T) = \prod_{\Delta \in G} O(1) \prod_{\text{vertex}} O(1) M^{\varepsilon[l(v)-h(x(v))+1]} \prod_{\text{propagator}} e^{(M^{-i}/10K) d(\Delta,\Delta')}, \tag{IV.20}$$

where i is the index of the propagator and x(v) is the localization point of the vertex.

Proof. Each d/dt_i and $\delta/\delta\varphi^i$ derivative can act either on the exponent or on fields that are already downstairs. A combinatoric factor of 2 allows us to decide which of these two possibilities occurs. In the former case a factor of $O(1)(i-h(x)+1)$ allows us to decide on which term of the exponent and on which field in the term the derivative acts. We assign to each vertex localized at x a factor of $O(1)$ per momentum scale between $h(v) \geqq h(x(v))$ and $l(v)$ to control the number of derivatives of that scale acting on that vertex. Finally, given that a $\delta/\delta\varphi$ acts on a given field, an exponential in $d(\Delta,\Delta')$ is used to decide which $\delta/\delta\varphi$ it is. □

To estimate $|T|$ we want to use a bound of type (I.23),

$$|\int ABC\, d\mu| \leqq [\int B^2 d\mu]^{1/2} \sup_\varphi |AC|, \tag{IV.21}$$

i.e. we shall dominate the low momentum fields and integrate the high momentum fields.

To apply (IV.21) some work has to be done. Indeed the coupling constant $\lambda(v)$ of the vertex is different from $\lambda_{l(v)+1}$. By induction, we know that Theorem III.1 holds for $k<i$. Hence we know that $O(1)(1/2k) \leqq \lambda_k \leqq O(1)(1/k)$ if λ is small enough. Therefore for each vertex v at position x

$$\lambda(v) \leqq \lambda_{h(x)} \leqq O(1) \lambda_{l(v)+1}(l(v)-h(x)+1). \tag{IV.22}$$

Combining the bounds generated by (IV.15), (IV.16), and (IV.22) we get for a vertex of index $h(v)$ at position x in a cube $\Delta \in \mathbb{D}_{l(v)}$:

$$O(1) \lambda_{l(v)+1}^{1/4} (l(v)-h(x)+1)^{O(1)} l(\Delta)^3 \tag{IV.23}$$

times a product of fields which are hooked to the vertex. They are:

– a product of absolute values of high momentum fields or of fluctuation fields $\delta\varphi$ fields

– a product of smeared low momentum fields, each accompanied with a factor $\lambda_{l(v)+1}^{1/4}$.

For each high momentum field or fluctuation field ψ (ψ_j is either a high momentum field φ^j or a fluctuation field $\delta_j\varphi^k$) we write

$$|\psi_j(x)| \leqq 2[M^{-j} + M^j \psi_j(x)^2]. \tag{IV.24}$$

We have now to perform the integration of the field. The integral is of the form

$$\int AB'B''EFL'MP'd\mu,$$

where

A is the exponential of the interaction,

B' is the product of the real high momentum fields [i.e. which are not $\delta\varphi$ fields and are not coming from the use of (IV.15)] and of fields which belong to the external vertices,

B'' is the product of all the almost local renormalized two point insertions,

E is the product of the high momentum fields coming from the decomposition (IV.15) of the low momentum fields,

F is the product of the $\delta\varphi$ fields,

L' is the product of the low momentum fields $\varphi_{l,j,i}$ of (IV.16),

M is the product of the factors $C(s)^{-1}\varphi$ at the ends of the chains of the i-th Mayer expansion,

P' is the product of the propagators introduced by the cluster expansion (with possibly gradients acting on them).

We bound this integral in the following way.

$$\begin{aligned}&\int AB'B''EFL'MP'd\mu\\&\leqq [\int (B'^6)d\mu \int (B''^6)d\mu \int (E^6)d\mu \int (F^6)d\mu \int (M^6)d\mu]^{1/6} \sup_{\varphi} |AL'|\,|P'|.\end{aligned} \tag{IV.25}$$

Anticipating the integral over the position of the vertices that will be performed in Proposition III.7 we can treat the factors $(l(v)-h(x)+1)^{O(1)}$ of (IV.23) as follows:

$$\left|\int_{\Delta} dx(l(v)-h(x)+1)^{O(1)} f(x)\right| \leqq l(\Delta) \int_{\Delta} dx \sup_{x\in\Delta} |f(x)|. \tag{IV.26}$$

Proposition IV.2

$$\sup_{\varphi} |AL'| \leqq \prod_{\varphi\in L'} M^{-i(\varphi)} \prod_{\Delta\in G} O(1) l(\Delta)^{O(1)(1+n(\Delta))} [n(\Delta)!]^{O(1)}. \tag{IV.27}$$

Proof. From the preparation of the low momentum fields in Sect. IV.1, L' is equal to $\prod_{\Delta} O(1) l(\Delta)^{O(1)(1+n(\Delta))}$ times:

$$\prod_{\varphi\in L'} M^{-i(\varphi)} \prod_i \prod_{\Delta\in\mathbb{D}_i} \left[\sum_{j\leqq i} \left|\omega_j \int \Delta(x) |\varphi_{l,j,i}(t,x)|^4 dx\right|\right]^{O(1)(1+n(\Delta))}. \tag{IV.28}$$

Indeed recall that each low momentum field was accompanied by a factor $\omega_{l(v)+1}^{1/4} \leqq \omega_j^{1/4}$. Moreover from (IV.17),

$$L' \leqq \prod_{\varphi \in L'} M^{-i(\varphi)} \prod_{\Delta} O(1)[n(\Delta)!]^{O(1)}$$

$$\times \exp\left[\sum_{i,j} \sum_{\Delta \in \mathbb{D}_i} \frac{\lambda_j/O(1)}{(i-j+1)^4} \int_{\Delta} [1-(t_j(x))^4][\varphi_{l,j}(x)]^4 dx\right]. \qquad \text{(IV.29)}$$

Thus because $\sum_{j \leqq i} 1/(i-j+1)^4 \leqq O(1)$

$$L' \leqq \prod_{\varphi \in L'} M^{-i(\varphi)} \prod_{\Delta} O(1)[n(\Delta)!]^{O(1)}[A]^{O(1)}. \qquad \text{(IV.30)}$$

The counterterms being exactly cancelled, there is no mass term in the exponential. Hence the exponent is negative and $0 \leqq A \leqq 1$. □

In the next proposition we recall the standard bounds on the covariances (I.9).

Proposition IV.3.

$$|P'| \leqq \prod_{i} \prod_{\substack{p \in P' \\ p \text{ of index } i}} [M^{-i(2+\#(\text{gradients acting on } p))} e^{-4(M^{-i}/K)d(p)}], \qquad \text{(IV.31)}$$

where $d(p)$ is the distance between the end points of the propagator p.

We now bound the integration over the fields. Such an integration over a product of fields K (K stands for B'^6, B''^6, E^6, F^6 or M^6) is a sum over all the possible contraction schemes (S); let $K(S)$ be the value associated with such a contraction scheme S.

If $a(K,S)>0$ is such that $\sum [a(K,S)]^{-1} \leqq 1$, then:

$$\int \prod_{\text{vertices}} dx \int K d\mu = \sum_{S} K(S) \leqq \sup_{S} a(K,S)\,|K(S)|. \qquad \text{(IV.32)}$$

Proposition IV.4. *A convenient choice for $a(K,S)$, $K \neq M^6$, $K \neq B''^6$, is*

$\exp[(O(1)M^{-i})d(p)]$	*per propagator of index i,*
$O(1)[n(\Delta)!]^{O(1)}$	*per cube $\Delta \in G$,*
$M^{\varepsilon(i-j)}$	*per field $\eta_i \varphi^j$,*
$M^{3(j-i)}$	*per field $\delta_i \varphi^j$.*

Proof. We bound the number of terms generated by the contractions. The fields can contract in an arbitrary order so we make the convention of contracting first the fields which have the smallest indices.

– *Bound on the contractions of B'.* Let us consider a field $\varphi^i(x)$, $x \in \Delta_0 \in \mathbb{D}_i$, which contracts with a field $\varphi^i(y)$, $y \in \Delta_1 \in \mathbb{D}_i$.

The choice of Δ_1 is controlled by $O(1)e^{(M^{-i}/K)\operatorname{dist}(\Delta_0,\Delta_1)}$ because

$$\sum_{\Delta_1 \in \mathbb{D}_j} e^{-(M^{-i}/K)\operatorname{dist}(\Delta_0,\Delta_1)} \leqq O(1). \qquad \text{(IV.33)}$$

A factor $O(1)+O(1)n(\Delta_1)$ suffices to control the number of choices of $\varphi^i(y)$ in Δ_1. Collecting these factors we obtain the bounds on $a'(B'^6,S)$.

– *Bound on the contractions of E.* Let us consider a field $\eta_i\varphi^j(x)$, $x\in\Delta_0\in\mathbb{D}_j$, which contracts with a field $\eta_l\varphi^j(y)$ $y\in\Delta_1\in\mathbb{D}_j$, $y\in\Delta'\in\mathbb{D}_l$.

The choice of Δ_1 is controlled by $O(1)e^{(M^{-j}/K)\,\mathrm{dist}(\Delta_0,\Delta_1)}$. The choice of l is controlled by $O(1)M^{\varepsilon(l-j)}$. A factor $O(1)+O(1)n(\Delta')$ suffices to control the number of choices of $\eta_l\varphi^j(y)$ in Δ'. Collecting these factors we obtain the bounds on $a'(E^6,S)$.

– *Bound on the contractions of F.* Let us consider a field $\delta_i\varphi^j(x)$, $x\in\Delta_0\in\mathbb{D}_j$, $x\in\Delta\in\mathbb{D}_i$ which contracts with a field $\delta_l\varphi^j(y)$, $y\in\Delta_1\in\mathbb{D}_j$, $y\in\Delta'\in\mathbb{D}_l$. Due to the order of the contractions we have: $j>l\geqq i$. The combinatorics are as above but for the choice of l and Δ'. We choose l with a factor $O(1)M^{j-l}$. We choose Δ', $\Delta'\subset\Delta_1$, with a factor $\Delta_1/\Delta'=M^{4(j-l)}\leqq M^{2(j-l)}M^{2(j-i)}$. Collecting the factors we obtain the bound on $a'(F^6,S)$. $\square$

To bound the sum over contraction schemes of the M^6 and B''^6 terms is more delicate. Let us define $p(\Delta,\Delta')$ as the number of 2-point almost local insertions or of chains of such insertions (called here "chains" for simplicity) which have one of their ends in Δ and the other in Δ' (note that Δ and Δ' need not be of the same scale).

Proposition IV.5.

$$[\textstyle\int M^6 \int B''^6]\leqq \prod_{\Delta,\Delta'} p(\Delta,\Delta')!\, e^{M^{-j(\Delta,\Delta')}d(\Delta,\Delta')p(\Delta,\Delta')}M^{-4j(\Delta,\Delta')p(\Delta,\Delta')}, \qquad (\text{IV.34})$$

where $j(\Delta,\Delta')$ is the smallest of the indices of the lattices of Δ and Δ'.

Proof. Let $p_1(\Delta)$ be the number of chains with their end field of highest index in Δ and $p_2(\Delta)$ the number of chains with their end field of lowest index in Δ. We have $p(\Delta)=p_1(\Delta)+p_2(\Delta)=2p(\Delta,\Delta)+\sum_{\Delta\neq\Delta'}p(\Delta,\Delta')$. If we contract naively the $p(\Delta)$ legs in Δ, starting with the cubes with $p(\Delta)$ maximal, we obtain, using a piece of the exponential decay of the propagators, a factor $[p(\Delta)!]^{1/2}$ for the Wick contractions. Moreover the propagators generated by this process have a power counting which has to be taken into account. The highest leg of a blob, say of index j, has a power counting M^{-j}, and the lowest leg, say of index i, bears two ∇ from the mass renormalization, hence a power counting M^{-3i}. We write the total factor from the propagators as $M^{-4j}\cdot M^{-3(i-j)}$. The factors M^{-4j} create exactly the factors $M^{-4j(\Delta,\Delta')p(\Delta,\Delta')}$ in (IV.34). It remains to show that

$$\prod_{\Delta}(p_1(\Delta)!)^{1/2}(p_2(\Delta)!)^{1/2}\leqq \prod_{\Delta,\Delta'}[p(\Delta,\Delta')M^{(2+2\varepsilon)(i-j)}e^{\varepsilon M^{-j}d(\Delta,\Delta')}]^{p(\Delta,\Delta')}, (\text{IV.35})$$

where in the r.h.s. of (IV.35) i and j are the indices of Δ and Δ' and $i\geqq j$. Let us sketch the proof of (IV.35). We have first:

$$(p_1(\Delta)!)^{1/2}\leqq \prod_{\Delta,\Delta'} p(\Delta,\Delta')^{p(\Delta,\Delta')/2}[M^{\varepsilon(i-j)}e^{\varepsilon M^{-j}d(\Delta,\Delta')}]^{p(\Delta,\Delta')}, \qquad (\text{IV.36})$$

Indeed paying the small exponential increase in $(i-j)$ in (IV.36) we can choose the index i knowing j, and then the exponential increase in the spatial distance in (IV.35) (which of course will be later controlled by the exponential decrease of the chains) pays for the choice of Δ' knowing Δ. In this way we can transform the full factorial $p_1(\Delta)!$ into local ones, and taking square roots give (IV.35). Similarly we can transform the full factorial $p_2(\Delta)!$ into local ones. This requires however to pay an extra factor $M^{4(i-j)}$ to choose the small cube of index j into a large one of index i. Thus (IV.34) holds.

Proposition IV.6. *If the summations over the "chains" of Proposition IV.5 are made independent, as will be assumed in the way we perform them later, an extra factor* $\prod_{\Delta,\Delta'} p(\Delta,\Delta')!$ *arises due to overcounting, which compensates for the one in (IV.34).*

For a detailed analysis of combinatoric problems similar to those of Propositions IV.5 and IV.6, we refer to the appendix of [15]. In particular a nice picture of the exact overcounting factors above is given in terms of the coordination numbers of the trees used in [15] and [32].

Proposition IV.7. *A bound on B'^6, $E^6(S)$, $F^6(S)$ is given by the product of:*

$e^{-(M^{-i}/K)d(p)}$	*per propagator p of index i,*
M^{-i}	*per field of D or F of index i hooked to a φ^4 vertex,*
$M^{-i}M^{-\varepsilon(i-j)}$	*per field $\eta_i\varphi^j$ of E,*
$M^{-3(j-i)}$	*per field $\delta_i\varphi^j$, $j>i$,*
M^{-i}	*per gradient acting on a function η_i,*
$(\lambda_{l(v)+1})^{1/4}$	*by vertex v,*
$M^{-2i(\delta m)}$	*per low momentum mass counterterm.*

Proof. Were it not for the $\delta\varphi$ and $\eta_i\varphi^j$ fields it would suffice to gather together the factors arising from the bound on the propagators, the pieces of coupling constant set aside and the bound on the mass counterterm.

The Fourier transform of a $\delta_i\varphi^j$ field is (see Sect. IV.1):

$$[1-\tilde{\eta}_i(p)]\tilde{\varphi}^j(p)=[1-\tilde{\eta}_i(pM^i)]\tilde{\varphi}^j(p), \tag{IV.37}$$

and we have that $\tilde{\eta}(0)=1$ and that $\tilde{\eta}'(0)=0$ so that

$$[1-\tilde{\eta}(pM^i)]\tilde{\varphi}^j(p)=\left[\int_0^1 da(1-a)\tilde{\eta}''(apM^i)\right]M^{2i}p^2\tilde{\varphi}^j(p). \tag{IV.38}$$

Using formula (IV.38) we can consider the propagator coming from the contraction of two fields $\delta_i\varphi^j$ and $\delta_{i'}\varphi^j$ as a propagator with 4 gradients (plus possibly the gradients coming from the renormalization) and multiplied by a factor $M^{2i+2i'}$. Hence using Proposition IV.3 one obtains the bound on the fields of F.

Let us consider the contribution arising from the contraction of $\eta_i\varphi^j(x)$ and $\eta_{i'}\varphi^j(y)$.

$$M^{-4i}M^{-4i'}\int dtdt'\eta[(x-t)M^{-i}]C^j(t-t')\eta[(t'-y)M^{-i'}]. \tag{IV.39}$$

After extracting part of the exponential decay of $C^j(t-t')$ we still have essentially

$$M^{-4i}M^{-4i'}\int dtdt'\eta[(x-t')M^{-i}]C^j(t-t')\eta[(t'-y)M^{-i'}] \tag{IV.40}$$

$$\leq M^{2j}M^{-4i}M^{-4i'}\int dt\eta[(x-t)M^{-i}]\eta[(t-y)M^{-i'}]$$

$$\leq M^{-i}M^{-(i-j)}M^{-i'}M^{-(i'-j)}. \quad \square \tag{IV.41}$$

We have to bound the $l(\Delta)$ factors. There are such factors associated with each vertex, field and derivation in Δ. The number of vertices or derivations is bounded by construction, the number of fields is bounded by

$$O(1) \sum_{\Delta' \text{ neighbour of } \Delta} n(\Delta') . \tag{IV.42}$$

Proposition IV.8. *For any* $N(\Delta)$

$$\prod_{\Delta \in G} [l(\Delta)]^{N(\Delta)} \leqq \prod_{\Delta \in G} O(1)[N(\Delta)!] . \tag{IV.43}$$

Thus, by (IV.42), *we obtain that:*

$$\prod l(\Delta) \leqq \prod_{\Delta \in G} O(1)[n(\Delta)!]^{O(1)} . \tag{IV.44}$$

Proof. For $\Delta \in \mathbb{D}_i$

$$l(\Delta) \leqq M^{-4i} \int_{\Delta} \frac{M^{4i-4h(x)}}{[i-h(x)+1]^2} dx , \tag{IV.45}$$

so that:

$$[l(\Delta)]^{N(\Delta)} \leqq N(\Delta)! \exp\left[\int_{\Delta} \frac{M^{-4h(x)}}{[i-h(x)+1]^2} dx\right] . \tag{IV.46}$$

The product over all the cubes of the polymer G is a product over i and over the $\Delta \in \mathbb{D}_i$, $\Delta \in G$.

$$\prod_{\Delta \in G} [l(\Delta)]^{N(\Delta)} \leqq \left[\prod_{\Delta \in G} N(\Delta)!\right] \exp\left[\sum_j \sum_i \sum_{\Delta \in \mathbb{D}_i, \Delta \in G} \int_{[x \in \Delta, h(x)=j]} \frac{M^{-4j}}{[i-j+1]^2} dx\right] . \tag{IV.47}$$

Now for j given: $\sum_i a_i \leqq O(1) \sup_i [i-j+1]^2 a_i$. Moreover, we can bound the integral over x, such that $h(x)=j$, as the sum over the cubes of $\mathbb{D}_j \subset G$ which contain x

$$\sum_j \sum_i \sum_{\Delta \in \mathbb{D}_j, \Delta \in G} \int_{[x \in \Delta, h(x)=j]} \frac{M^{-4j}}{[i-j+1]^2} dx \leqq \sum_j \sum_{\Delta \in \mathbb{D}_j, \Delta \in G} \int_{\Delta} M^{-4j} dx \leqq \sum_{\Delta \in G} O(1) . \tag{IV.48}$$

This together with (IV.47) completes the proof. □

We control the $n(\Delta)!$ per cube using the exponential decrease of the cluster propagators

Proposition IV.9. *For any* i.

$$\prod_{p \in P',\, p \text{ of index } i} e^{-(M^{-i}/K)d(p)} \leqq \prod_{\Delta \in G, \Delta \in \mathbb{D}_i} O(1)[n(\Delta)!]^{-O(1)} . \tag{IV.49}$$

Proof. Let

$$C(\Delta) = \{\Delta', \text{ such that in formula (II.16) we choose } D^1_{\Delta,\Delta'}, D^2_{\Delta,\Delta'}, I_{\Delta,\Delta'}\} .$$

There are, for $n(\Delta)$ large enough, at least $n(\Delta)/2$ cubes of $C(\Delta)$ such that $\text{dist}(\Delta, \Delta') > M^i/100 n(\Delta)^{1/4}$ thus:

$$\prod_{p \in P',\, p \text{ of index } i} e^{-(M^{-i}/K)d(p)} \leqq \prod_{\Delta \in G, \Delta \in \mathbb{D}_i} O(1) e^{-(1/200)n(\Delta)^{5/4}} \tag{IV.50}$$

$$\leqq \prod_{\Delta \in G, \Delta \in \mathbb{D}_i} O(1)[n(\Delta)!]^{-O(1)} , \tag{IV.51}$$

which proves the proposition. □

Theorem III.5 is now a consequence of Propositions IV.1–IV.9.

V. Borel Summability

In this section we prove Theorem I.3, i.e. the Schwinger functions with bare coupling constant λ and bare propagator $1/p^2$ (constructed in the previous chapters) are Borel summable functions of λ.

V.1 Introduction

Let us first recall what Borel summability is [31]:

If for some $\alpha > 0$, we have that in the disk $D_\alpha \equiv \{\lambda | (\operatorname{Im}\lambda)^2 + (\operatorname{Re}\lambda - \alpha)^2 < \alpha^2\}$:

1) $S(\lambda)$ is an analytic function
2) The following "uniform" Taylor bound holds:

$$\left| S(\lambda) - \sum_{k=0}^{n} \frac{\lambda^k}{k!} S^{(k)}(0) \right| \leqq O(1)^n |\lambda|^{n+1} n! \tag{V.1}$$

then $S(\lambda)$ is Borel summable. It can then be recovered from its perturbation expansion through its Borel transform. More precisely if we define:

$$B(b) = \sum_{k=0}^{\infty} \frac{b^k}{(k!)^2} S^{(k)}(0). \tag{V.2}$$

$B(b)$ has a nonzero radius of convergence and can be analytically continued to a strip containing the positive real axis. Moreover:

$$S(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{-b/\lambda} B(b)\, db. \tag{V.3}$$

To prove that the Schwinger functions are Borel summable we have to show that for $\alpha > 0$ small enough:

1) The Schwinger functions are analytic in the domain D_α. This, the hard part of the proof, is given in Sects. V.2 and V.3.

2) The bound (V.1) holds. This is easier and proven in Sect. V.4.

We start by explaining why the analyticity in D_α is not an easy generalization of the previous construction. The problem resides in the domination. On the circle bordering D_α we have $\operatorname{Re}\lambda = |\lambda|^2/2\alpha$. The domination argument can only use the positive part $(\operatorname{Re}\lambda)\varphi^4$ of the interaction in the exponential. Each domination of a low momentum field costs therefore a factor $(\operatorname{Re}\lambda)^{-1/4} \approx (2\alpha)^{1/4} (\operatorname{Im}\lambda)^{-1/2}$. A vertex produced by our derivation rules comes equipped with a factor $\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$ and has at most 3 low momentum fields. The $\operatorname{Re}\lambda$ part is no problem, but by the cost remark above, in the $\operatorname{Im}\lambda$ part we can dominate at most 2 fields.

The natural way out is to remark that a vertex with one high momentum field and three low momentum fields violates conservation of momentum. However to see this non-conservation this vertex must be integrated over a region of size corresponding to the low momenta. (Recall that to have exact conservation of momentum at a vertex it must be integrated over all space.) This is not the case under our previous expansion rules, since a derived vertex is integrated over its localization cube and under the old rules the localization cube is often of the scale of its highest momentum leg. The requirement that the interaction be positive (even at intermediate values of the interpolating parameters t) determined rather rigidly these rules. However since we are now dealing with the imaginary part of the interaction, we no longer have this requirement, and we can therefore alter the rules. This is done in Sect. V.2.

Another related problem is that the cutoffs (I.9) which define our slices do not have good momentum conservation properties: they do not vanish at $p^2=0$. Hence we must change the cutoffs. However the momentum-conserving cutoffs have a strong 0 at $p^2=0$, and consequently cannot be inverted as is required in Theorem II.1. This forces us to change and indeed complicate the cluster and Mayer expansions. These complications (which are described in Sect. V.3) would have appeared unnatural and obscured even further the convergence proofs of the previous sections. Therefore we preferred to delay them until this chapter.

V.2 The Modified Momentum Coupling Expansion

In the proof of the convergence of the expansion there are only two things which are sensitive to λ being complex: the fact that $|\lambda_i| \approx 1/i$ and, as discussed above, the domination of the low momentum fields by $\exp[-\mathrm{Re}\lambda\varphi^4]$.

Let us first verify that:

$$\mathrm{Re}\lambda>0 \Rightarrow |\lambda_i| \leqq \inf[|\lambda|, O(1)/i]\,. \tag{V.4}$$

This is a consequence of the fact that the recursion relation $\lambda_{i+1}=\lambda_i - c_i\lambda_i^2 + O(\lambda_i^3)$, with $c_i>0$ almost constant in i, still holds for λ complex.

Let us now consider the domination problem. We decompose at each scale each vertex into its real and imaginary parts.

$$\lambda_i\varphi^4 = (\mathrm{Re}\lambda_i)\varphi^4 + i(\mathrm{Im}\lambda_i)\varphi^4\,.$$

For the real vertices we can dominate three fields per vertex.

$$|(\mathrm{Re}\lambda_i)^{-3/4}\mathrm{Re}\lambda_i| = |(\mathrm{Re}\lambda_i)^{1/4}| \leqq (2\alpha)^{1/4}\,.$$

For the imaginary vertices we can dominate only 2 fields per vertex and still obtain a small factor $(2\alpha)^{1/2}$ per vertex:

$$|(\mathrm{Re}\lambda)^{-2/4}\mathrm{Im}\lambda| \leqq (2\alpha)^{1/2}\,.$$

We modify the expansion such that it is convergent with the above restriction for domination. To exploit translation invariance, we have to create imaginary vertices localized in cubes corresponding to the scale of the leg of second highest momentum. To do so we must change the t-dependence in the imaginary part of the interaction. (As remarked above we have no reason to preserve the positivity of the imaginary part of the interaction.) For a vertex whose frequencies are $i_1 \leqq i_2 \leqq i_3 \leqq i_4$ the ordinary t-dependence, apart from an inessential combinatoric factor and apart from the x-dependence through the functions a is just (see II.28):

$$\sum_{j=1}^{i_1-1} (1-t_{j-1}^4)\lambda_j \prod_{k=j}^{i_1} (t_k)^4\varphi^{i_1} \prod_{k=i_1+1}^{i_2} (t_k)^3\varphi^{i_2} \prod_{k=i_2+1}^{i_3} (t_k)^2\varphi^{i_3} \prod_{k=i_3+1}^{i_4} t_k\varphi^{i_4}\,. \tag{V.5}$$

In the real part of the interaction we leave this t-dependence unchanged. For the imaginary part of the interaction we replace all the t_k's with $i_1 \leqq k \leqq i_2$ by a single parameter $t(i_1, i_2)$ which couples therefore a cube of scale i_2 to the set of all the cubes of scale i_1 contained in it:

$$\sum_{j=1}^{i_1-1} (1-t_{j-1}^4)\lambda_j \prod_{k=j}^{i_1-1} (t_k)^4 \varphi^{i_1} t(i_1,i_2)\varphi^{i_2} \prod_{k=i_2+1}^{i_3} (t_k)^2\varphi^{i_3} \prod_{k=i_3+1}^{i_4} t_k\varphi^{i_4}. \tag{V.6}$$

This dependence is as good as the one of Sect. II.2 to perturb the coupling between fields of different momenta, but it doesn't preserve the positivity.

Since the t-dependence above scale i_1 is unchanged, renormalization works just as before and the flow of λ is unchanged.

The t-expansion in the ordinary parameters t_i is performed as before (up to fifth order); for each new parameter $t(i_1, i_2)$ we will perform a Taylor expansion up to order $M^{4(i_2-i_1+1)}$. This is done so as to ensure that we still have a small factor for each cube (even those of scale i_1) in the support of a Mayer graph. This Taylor expansion is performed over all scales before the other decoupling and cluster expansions. In the Taylor remainder [at $t(i_1, i_2)\neq 0$] we have a new type of link called T-link (compare to the "open gates" of Sect. I). This T-link is pictured in Fig. 2 and joins a cube of frequency i_2 to all the cubes of frequency i_1 contained in it. It is then natural to consider all these cubes of frequency i_1 as a single region when the cluster expansion of scale i_1 is applied. The main difference between this expansion and the old one is the following. While the number of vertices produced is roughly the same as before, they have been produced in a cube of scale i_2, and consequently they are no longer constrained to belong to different cubes of scale i_1. This restores conservation of momentum down to scale i_2. We shall briefly show that when $i_2 \gg i_1$, the very small factors resulting from the violation of momentum conservation make up for the loss due to the bad domination (only 2 fields per vertex) and to sloppy estimates for the gaussian integration in scale i_1.

The first remark is that with the dependence (V.6), imaginary vertices created by derivations with respect to the ordinary t_k's are o.k. Indeed in (V.6) there remain only two classes of ordinary t's: the ones with scale $k > i_2$ (differentiation with respect to these t's give vertices having at most 2 badly localized legs to be dominated) and the ones with scale $k < i_1$ (differentiation with respect to these t's give only "counterterms" hence vertices with coupling constant $\delta\lambda_k \approx \lambda_k^2$, for which domination of all the legs is straightforward).

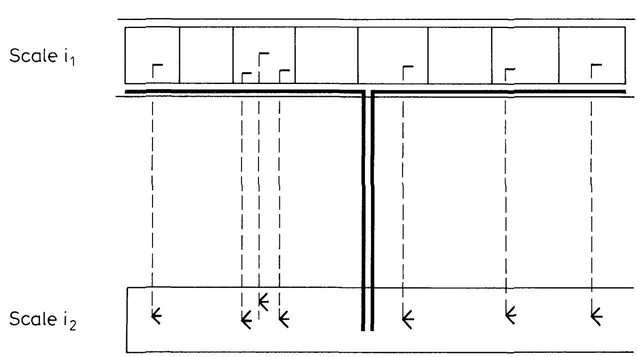

Fig. 2. A "T" link of Sect. V

It remains to bound the vertices created by the $t(i_1, i_2)$ derivations. They can be isolated from the others through a Schwarz inequality. Let us call $n \equiv M^{4(i_2-i_1+1)}$. Using in a standard way the exponential decrease of the propagator in a slice the factorials generated by the cluster expansion, the gaussian integration and the domination of fields hooked to these vertices may be bounded by n^k where k is some fixed power.

Let us use momentum conservation to compensate this bad factor. For a cube Δ of scale i_2:

$$(\tilde{\Delta})(-p_1-p_2-p_3-p_4)\tilde{\varphi}^{i_1}(p_1)\tilde{\varphi}^{i_2}(p_2)\tilde{\varphi}^{i_3}(p_3)\tilde{\varphi}^{i_4}(p_4)$$
$$=|p_1|^{4k}(\tilde{\Delta})(-p_1-\ldots-p_4)\tilde{\varphi}^{i_2}(p_2)\tilde{\varphi}^{i_3}(p_3)\tilde{\varphi}^{i_4}(p_4)\frac{\tilde{\varphi}^{i_1}(p_1)}{|p_1|^{4k}}. \tag{V.7}$$

Assuming that one uses a slice cutoff which vanishes at $p^2=0$ as p^{8k} it is easy to extract from (V.7) the desired factor $M^{-4k(i_2-i_1+1)}$. Indeed when we contract the field φ^{i_1} the factor $|p_1|^{4k}$ in the denominator of (V.7) leads to a factor M^{4i_1} in the bound. Multiplied by $\tilde{\Delta}$, the factor $|p_1|^{4k}$ in the numerator leads to a factor M^{-4ki_2}.

V.3 The Modified Cluster and Mayer Expansions

We have first to make a choice for the momentum-conserving cutoffs. A good choice is:

$$C^i(p)=\frac{1}{p^2}\left[\frac{1}{p^{10k}M^{10k(i-1)}+1}-\frac{1}{p^{10k}M^{10ki}+1}\right] \tag{V.8}$$

which obviously vanishes at $p=0$ at least as fast as p^{8k}. Since the covariance $C^i(s)^{-1}$ does not satisfy Theorem II.1, we replace the simplified cluster expansion of the previous sections by the inductive version explained in [28]. However to preserve the analysis of 1PI 2-point subgraphs of Sects. III and IV, we have to push this expansion until the one particle irreducible structure between cubes has been made entirely visible. This generalization of the cluster expansion is explained in full detail in [39], so we do not repeat it here. The price to pay is that in the ordinary Mayer expansion the s dependence of the mass counterterms does not match that of the 1PI 2-point graphs. This problem was resolved using the inverse covariance to restore the s-dependence. Such a cheap solution is no longer available, so we modify the Mayer expansion itself. We will describe these modifications and give a consistent chain of arguments to solve all difficulties, but we do not present the full corresponding formalism.

The development generates terms, each of which consists of 2PI kernels, linked through chains of 1PI 2-point subgraphs and mass counterterms, none of which can overlap. In a first stage we show how to define the portion of the mass counterterm required to renormalize 1PI 2 point subgraphs which do not themselves contain any other 2-point insertions (subgraph or counterterm). After that the rest of the computation of the mass counterterm is inductive and follows exactly Sect. II.4.

In Fig. 3 we show a typical 1PI subgraph, and a mass counterterm. To perform the cancellation, we use the following steps. First we add and subtract the local

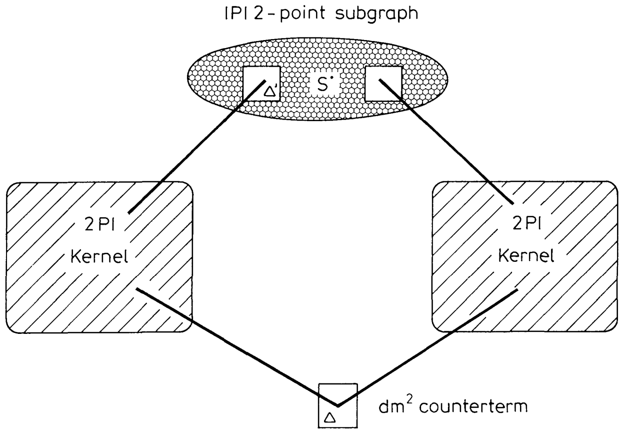

Fig. 3. The 1 particle irreducible structure

part of the 1PI subgraph, i.e. we write:

$$\int K_2(x,y)\varphi(x)\varphi(y)dxdy = \int K_2(x,y)[\varphi^2(x)+\varphi^2(y)]/2\,dxdy - 1/2\int K_2(x,y)[\varphi(x)-\varphi(y)]^2 dxdy. \tag{V.9}$$

The second term is renormalized as desired. The first term has its two external legs localized in a single cube and must be cancelled against the counterterm. However the counterterm is supported in a single cube Δ, whereas the local part in (V.9) may still have a large support $S=S'\cup\Delta'$, where Δ' contains the two external legs. To cancel these two terms when $\Delta=\Delta'$, we need to remove the non-overlap constraints involving S'. This is done with an ordinary Mayer expansion on S'. We remark that since there are still disjointness constraints between all the squares of type Δ and Δ', the local factorials remain controlled by the exponential decay of the propagators.

By this procedure we have cancelled all the mass counterterms that are not contained inside 2PI kernels. To compensate the remaining counterterms inside these kernels, we have to remove the disjointness constraints between the squares of type Δ' and these kernels. It is here that we meet the announced difficulty that the naive s-dependence of the two propagators exiting from Δ' is not the same as the s-dependence of propagators inside the kernels. This is an artificial problem because we are not bound to keep the naive s-dependence in the terms which we create by removing the disjointness constraints. Typically instead of writing $\exp[-V(B,B')]=\{\exp[-V(B,B')]-1\}+1$ as in (II.31)–(II.32) we can equally well write

$$\exp[-V(B,B')]=\{\exp[-V(B,B')]-g(B,B',s)\}+g(B,B',s), \tag{V.10}$$

where

$$g(B,B',s)\equiv f(s) \text{ if } B\cap B'\neq\emptyset; \qquad g(B,B',s)\equiv 1 \text{ if } B\cap B'=\emptyset. \tag{V.11}$$

By cleverly choosing the function $f(s)$ we can restore the desired s-dependence in all cases. All the counterterms inside a given 2PI kernel are generated either by explicit cluster derivations or by expansion of the exponential $\exp(\delta m^2\varphi^2)$. In the first case the s-dependence is somewhat messy, in particular because we use here the inductive cluster expansion of [28] in which the s parameters are not assigned to specific pairs of cubes. This introduces complicated (although straightforward) problems of notation. Hence as an example we will only describe what to do in the second case. There we have to remove the disjointness constraint between the support Σ of a 2 PI kernel K and the cube Δ' of a 1 PI subgraph with external legs $C(x, y_1)$ and $C(x, y_2)$ (hence $x \in \Delta'$ and $y_1 \in \Sigma$, $y_2 \in \Sigma$). But in the cluster expansion of [28], the s-dependence of a propagator $C(x, y; s)$ is just a product of s parameters which depends on the cubes to which x and y belong, and which we call $h(x, y; s)$. In the formula (V.11) we can then simply take $f(s) \equiv h(x, y_1; s)h(x, y_2; s)$. The reader can check that one can now use (II.34) and achieve the desired cancellations. The rest of the bounds follow as in Sect. III and IV.

V.4 The "Uniform Taylor Remainder" Bound

We will be brief since large order estimates are now relatively standard [7, 9, 10, 15, 32].

Using repeatedly the formula (II.35):

$$\lambda_{i+1} = \lambda_i + \delta\lambda_i$$

we can "undo" effective coupling constants, which means writing

$$\lambda_{i+1} = \lambda + \sum_{j \leqq i} \delta\lambda_j . \tag{V.12}$$

We continue until either all coupling constants are λ or the total number of coupling constants λ explicitly appearing is $n+1$. Note that the coupling constants that we "undo" in priority are the coupling constants of vertices or counterterms produced by the expansion; if these coupling constants are exhausted before order $n+1$ is reached, we write a Taylor expansion of the exponential of the interaction to first order, and in the remainder term we "undo" the coupling constant of the created vertex. We repeat this until order $n+1$ in λ is reached.

After subtracting the perturbative expansion in λ up to order n, the Taylor remainder is given by a big expansion similar to the one of previous chapters, and is bounded in the same way. The only difference is that a certain number p of new vertices have been created and a certain number q of coupling constant renormalizations are missing. Since we stopped our development as soon as order $n+1$ in λ is reached, we must have $p+q \leqq n+1$.

The q missing renormalizations create q logarithmic divergences; in a graph with lowest momentum scale i these divergences give at most an additional factor i^q to the usual bounds. But there is at least one global exponential decrease $M^{-i/2}$ in any term of the development (see Theorem III.3). Then we use

$$\sum_i i^q M^{-i/2} \leqq O(1)^q q! \tag{V.13}$$

This is a factorial of "renormalon" type [12].

Finally the p additional vertices are no longer produced according to the cluster and momentum expansion rules. They can therefore accumulate in small regions of phase space, hence create factorials of "instanton" type [12–15]. To bound them, we remark that if $p=\sum_{\Delta} p(\Delta)$, where $p(\Delta)$ is the number of vertices localized in Δ, we have a factor $\left[\prod_{\Delta} p(\Delta)!\right]^{-1}$ from the Taylor formula. Moreover these additional vertices give an additional contribution $O(1)^p$ times $\left[\prod_{\Delta} p(\Delta)!\right]^2$ to the bounds. Clearly this results in a factor at most $O(1)^p p!$. Combining this factor with the one of (V.13) we get (V.1).

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