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Invariants for Smooth Conjugacy of Hyperbolic Dynamical Systems II

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Abstract. We show that the eigenvalues of the derivatives at periodic points form a complete set of invariants for smooth local conjugacy of Anosov diffeomorphisms of T^2.

0 Introduction

One of the most important results about structural stability is that if f, g are C^∞ Anosov diffeomorphisms of a compact manifold M which are sufficiently C^0 close (exactly how close depends on C^1 properties of both f, g) then there exists a homeomorphism h such that we have

$$f \circ h = h \circ g. \tag{1}$$

Moreover, the homeomorphism constructed in the theorem is C^0 close to the identity if f, g are C^0 close and is unique among those satisfying conditions of proximity to the identity.

It is a natural question to ask how smooth can h be.

It is known that h is C^α for some $\alpha > 0$. (This α is related to the contractive and expansive constants of f, g, so that the best α yielded by the proof is always smaller than 1.) C^1 conjugacy is harder. Indeed, if h were differentiable, and $f^n(x_0) = x_0$, we would have $g^n(h^{-1}(x_0)) = h^{-1}(x_0)$ and

$$Df^n(x_0) = Dh(h^{-1}(x_0))Dg^n(h^{-1}(x_0))Dh^{-1}(h^{-1}(x_0)) \quad \text{so that,} \tag{2}$$

$$\text{Spectrum } (Df^n(x_0)) = \text{Spectrum } (Dg^n(h^{-1}(x_0))) \text{ whenever } f^n(x_0) = x_0 \tag{3}$$

(A slightly more careful argument would show that (3) is also a necessary condition for h being Lipschitz.)

There are examples [An] that show that in general, conditions (3) are violated so that there is no hope of getting differentiable conjugacy without extra hypothesis, and indeed such examples played a major role in the proposal of [Sm] to restrict the study to continuous conjugacy.

However, there are very natural questions:

A) Suppose that (3) is met, is h differentiable?

B) Is the set of diffeomorphisms satisfying (3) a manifold?

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A motivation for these questions comes from the strategy introduced in [G.K] for inverse spectral problems. Equality of the spectrum implies the analogue of (3) for the geodesic flow, and we would like to deduce from this that the manifolds are isometric. Since we are studying the Laplacian, the natural regularity class for the changes of variables is C^2.

Notice that if h is C^2 we would take another derivative of (3) and obtain further necessary conditions. These conditions have already been worked out in connection with the celebrated Sternberg theorem [St]. We just remark that in the case where f is a linear automorphism, they are nontrivial (since the determinant is 1, there are relations for the eigenvalues).

In that light, it is somewhat surprising that we have the following:

Main Theorem. *Suppose f and g are C^∞ Anosov diffeomorphisms of T^2 such that:*

a) *They are sufficiently close in the C^1-topology.*
b) *Condition (3) is met.*

Then, they are C^∞-conjugate.

Remarks. The eigenvalues of the derivatives at periodic points are not a minimal complete set of invariants. The periodic orbits accumulate, and if one knows the eigenvalues of derivatives for periodic orbits of high period one can also reconstruct those of small ones.

This accumulation of periodic orbits can serve as a heuristic explanation of why higher derivatives do not come into a complete set of invariants.

There have already been some results about smooth conjugacy of hyperbolic systems under conditions on the derivatives at periodic points. [Co, Su, Pr] discuss conjugacy of Julia sets. (Notice, however, that this is a one dimensional system so that the further obstructions taking derivatives of (3) are trivial.) Katok also has results about smooth conjugacy of Anosov geodesic flows.

In [M.M] it was shown that, for families of area preserving Anosov diffeomorphisms of T^2 starting on a linear automorphism, preservation of eigenvalues implies the preservation of the action invariants, which were shown to be complete in [LMM]. We generalize this result in that we require neither families nor linear. In spite of the fact that we do not work with symplectic techniques, we will draw heavily on techniques and results from [LMM].

Marco and Moriyon have also succeeded in proving that when $f: T^2 \to T^2$ is Anosov and has constant Lyapunov exponents on periodic orbits it is $C^{1+\varepsilon}$ conjugate to a linear automorphism (no proximity assumptions). By our Theorem 1 this implies C^∞ conjugacy.

Some extensions of the results of this paper to higher dimensions and to flows are being worked out by Marco and Moriyón and the author.

1. Proof of Main Theorem

This theorem will be a "bootstrap of regularity" argument.

The climb up from C^α $(0 < \alpha < 1)$ to C^∞ has a very clear milestone which is C^1. (Notice that the necessity of conditions (3) appears in C^1 but not in lower regularity.)

Once we get to C^1, several manipulations become possible and it is almost painless to get from there to C^∞. (For families of hamiltonian systems this $C^1 \Rightarrow C^\infty$ was already done in [LMM] Cor. 1.1.) Since the ideas involved in proving $C^1 \Rightarrow C^\infty$ are a subset of those required to prove $C^\alpha \Rightarrow C^1$, we prove it first and then, do the low regularity result. Again, the low regularity result breaks up into $C^\alpha \Rightarrow$ Lipschitz, Lipschitz $\Rightarrow C^1$.

We also remark that the proof is simpler when g is linear; then the regularity theory of [LMM] can be substituted by elementary arguments.

We will start by recalling some definitions and results of [LMM] we will use.

Let f be a C^∞ Anosov system in a compact manifold M. The stable and unstable foliations have C^∞ leaves with C^α ∞-jets (some $\alpha > 0$). Hence, the following definitions make sense.

We denote by C_s^k the space of functions which, when restricted to any leaf of the stable foliation are C^k and the k-jet of the restriction is continuous on the manifold. For $k < \infty$ those spaces can be given a Banach space structure similar to that of the usual C^k. For $k = \infty$ or ω, they can be given a Frechet space structure.

We will also talk about C_s^k diffeomorphisms.

Also denote by O_s^1 the space of first order linear operators tangent to the leaves of the stable foliation and of the form $\sum_l a_l D_l$ with $a_l \in C_s^\infty$ and D_l differential operators with C^∞ coefficients.

The operators in O_s^1 can be composed and, out of those powers, by addition, we can form O_s^k, "the space of k^{th} order operators tangent to the foliation."

An operator D in O_s^k maps C_s^{l+k} continuously into C_s^l, $k \in \mathbb{N}$, $l \in \mathbb{N}\, U\{\infty, \omega\}$.

Similar definitions and conclusions hold for the unstable manifold.

Since we will be using two Anosov diffeomorphisms and the stable and unstable foliations are different, so are the corresponding spaces. We will put in parenthesis the one we are talking about (e.g. $C_{s(f)}^k$).

Lemma 2.3 of [LMM] is

Lemma 1.

$$C_s^\infty \cap C_u^\infty = C^\infty.$$

The next lemma is the same essentially as Lemma 2.2 of [LMM]. There it was stated for flows and only when $k = \infty$, but the proof there works for any $k \in \mathbb{N}$. In that paper it was also shown that the result for flows implies the result for diffeomorphisms.

Lemma 2. *If f is as above $\psi \in C^0$ and $\eta \in C_s^k$ and $\psi \circ f - \psi = \eta$, then $\psi \in C_s^k$.*

Neither of the previous results assumes transitivity of f. Nevertheless, we will also be using Livsic fundamental result which does.

Theorem [Li] *Let f be as above and, moreover transitive, given any function $\eta \in C^\alpha$ $0 < \alpha \leqq 1$ ($\alpha = 1$ means here Lipschitz) satisfying*

$$\sum_{k=0}^{n-1} \eta(f^k x) = 0 \quad \forall x \text{ s.t. } f^n(x) = x. \tag{4}$$

Then, there exist a function $\psi \in C^\alpha$ s.t. $\psi \circ f - \psi = \eta$. Moreover, ψ is unique up to additive constants and, $\exists K(f)$ s.t.

$$\sup_{\substack{x,y\in M\\ x\neq y}} \frac{|\psi(x)-\psi(y)|}{|x-y|^a} \leqq K(f) \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|\eta(x)-\eta(y)|}{|x-y|^\alpha}.$$

We will often refer to (4) as "the compatibility conditions of Livsic theorem."

Transitivity, however, is assured for all Anosov diffeomorphism of 2 and 3 dimensional manifolds (in [Fr] is also shown that the manifolds carrying them have to be T^2 or T^3)

Theorem [Ne1]. *All Anosov diffeomorphisms whose stable or unstable foliation is of dimension 1, are transitive.*

Remarks. Lemma 2.3 of [LMM] is of a local character, hence it applies word for word for any geometric object which can be reduced in a coordinate patch to a set of functions (e.g. diffeomorphisms).

Lemma 2.2 there, however, is global; to study ψ at one point we need to study η in the whole orbit, so we cannot use it for geometric objects without some trickery.

We would also like to recall that many standard proofs in hyperbolic dynamical systems can be modified to be just existence of fixed points for perturbations of hyperbolic operators in some Banach space (this is done quite systematically, e.g. in [Sh]). If one appeals to the implicit function theorem-rather than to the contraction mapping principle—one can very often obtain smooth dependence on parameters. For example, one can get smooth dependence on parameters in the conjugating homeomorphism in the structural stability theorem.

One can also get smooth dependence on parameters in the local stable manifold in a hyperbolic fixed point and, since the stable (unstable) foliation theorem can be reduced to the existence of a stable manifold of a fixed point in a Banach space of sections (see [Sh]) one can get smooth dependence of the foliations on the parameter affecting the Anosov diffeomorphisms. (It is very easy to check that the auxiliary operators depend differentiably on the Anosov diffeomorphisms.)

The C^∞ dependence is also very uniform. If we topologize the space of k-jets $k\in\mathbb{N}$ with the C^0-topology, it is a Banach manifold. If the diffeomorphism moves along a C^∞ curve in the space of diffeomorphisms topologized with a C^k+k_0 topology (some $k_0>0$ that can be figured out) then, the k-jet of the stable manifold also moves along a C^∞ curve.

The proof of the main theorem, which we will now start, will use some softer statements about smooth dependence and its uniformity.

According to the theorem of Franks and Newhouse [F, Ne1] there is one and only one fixed point for Anosov diffeomorphisms of T^2. By conjugating with appropriate translations, we can assume that the fixed points of f and g are 0. (This is done only for notational purposes, any periodic point would do as well.)

Since the stable manifolds of a fixed point can be characterized by topological conditions, as soon as $h\in C^\infty$ we have

$$h(W_0^{s(g)}) = W_0^{s(f)}, \quad h^{-1}(W_0^{s(f)}) = W_0^{s(g)}.$$

More generally, h transforms pieces of leaves of the stable foliation of g into pieces of leaf of the stable foliation of f. (This is a non-trivial amount of regularity since the leaves are C^∞ and h was only assumed to be C^0.)

It is immediate to prove for h the conjugating homeomorphism and u any function.

Proposition. *If* $h \in C^k_{s(g)}$ $U: T^2 \to \mathbb{R} \in C^k_{s(f)}$, *then* $U \circ h \in C^k_{s(g)}$.

We also have

$$f(W_0^{s(f)}) = W_0^{s(f)}, \quad g(W_0^{s(g)}) = W_0^{s(g)}.$$

Consequently the problem of conjugation can be restricted to the stable foliations of 0. Moreover, since those leaves are dense (a consequence of transitivity [Ka]) what happens on them determines what happens in the whole manifold. The fact that they are one-dimensional will give us an extra handle on them.

In order to take advantage of this one dimensionality, it will be quite important to pick an appropriate set of coordinates.

Notation. Once we pick coordinates in $W_0^{s(g)}, W_0^{s(f)}$, h, f, g all become functions $\mathbb{R} \to \mathbb{R}$ which we will denote by $\tilde{h}, \tilde{f}, \tilde{g}$ respectively.

Lemma 3. *We can find coordinates in* $W_0^{s(f)}$, $W_0^{s(g)}$ *in such a way that*

1) *Derivatives with respect to the coordinates extend to a vector field in* $O^1_{s(f)}\, O^1_{s(g)}$ *respectively.*

2) $\sup_{t \in \mathbb{R}} |\tilde{h}(t) - t| < \infty$.

Remarks. Notice that in the proof of Lemma 3 is the only place where we used in an essential way that f, g are close. In the rest of the argument we could use—with some extra steps—just that they are C^0 conjugate which, according to [Ne1, Fr] follows from conjugacy of the actions on $H_1(T^2)$.

Proof. What we do is to construct a C^∞ function $F: S^1 \times S^1 \to T^2$ such that $F(\cdot t)$ is a family of circles which are transversal to the leaves of the stable foliations $(\langle \partial/\partial s, D_{s(f)} \rangle \leqq 1/10, \langle \partial/\partial s, D_{s(g)} \rangle \leqq 1/10\ \langle D_{s(f)}, D_{s(f)} \rangle = 1)$ and $\partial/\partial t$ is almost parallel to the directions of the stable foliations $(\langle \partial/\partial t, D_{s(f)} \rangle \leqq 9/10)$.

It then follows that each stable leaf hits every transverse circle infinitely many times.

Given a point in $W_0^{s(f)}$ we call $n_f(x)$ the number of times that the manifold transverses the family of circles.

We then take coordinates in $W_0^{s(f)}$ by assigning 0 to 0 and then, increasing the coordinate at the same rate that t is increasing.

The extension of the derivatives with respect to these coordinates is just the projection of $\partial/\partial t$ onto $D_{s(f)}$ which is in O^1_s due to the uniform transversality. (Remember that since f, g are C^1 close, the 1-jets of the leaves are close.)

It is also clear that, for two points x, y on $W_0^{s(f)}$, $W_0^{s(g)}$ we have (d_{T^2} is the standard distance on T^2),

$$t(x) - t(y) \leqq K d_{T^2}(x, y) + n_f(x) - n_g(y)$$

(unless we are in some borderline cases, which we leave the reader to fix rather than burden the exposition; to prove the lemma we just need to show $n_f(x) - n_g(h(x)) = 0$).

We can do that by interpolating smoothly between f and g by a family f_λ. The stable foliation and the conjugating h_λ solving (1) depend continuously on g. We can take one function F for all the g's in a small neighborhood and then $t(h_\lambda(x))$ should be continuous. From the fact that $d_{T^2}(h_\lambda(x), x)$ is uniformly small we conclude that $n_f(x) = n_g(h_\lambda(x))$ (unless we are in the borderline cases, in which it changes in the right way)

The construction of the function F is an intermediate step in the proof of the theorem that a vector field on T^2 can be reduced to linear. (A version can be found in [CFS] p. 408ff.) The only difference is that, there, one uses smoothness of the vector field $D_{s(f)}$ but, however, obtains the vector field $\partial/\partial t$ orthogonal to $D_{s(f)}$.

To get what we need, one just applies the standard theorem to a smooth approximation of the vector field $D_{s(f)} C^0$ close to it. One needs to verify that there are smooth C^0-approximations to this vector field which still do not have any periodic orbits (so that the classical theorem applies). This can be done in many ways. One possibility is prolonging the leaves of the foliation; we obtain a return map to one transversal circle. This return mapping can be smoothed keeping the same rotation number and the smoothing can be propagated along the foliation. The irrational rotation number prevents the existence of periodic orbits.

Theorem 1. *With the same notation of the main theorem, if h solves (1) and is in $C^k_{s(g)}$ where $k \geqq 1$, then it is in $C^{k+1}_{s(g)}$.*

Proof. The derivatives along $D_{s(g)}$ of h should be vectors along $D_{s(f)}$ so that we will have

$$(Dh)(x) D_{s(g)}(x) = \tilde{h}'(x) D_{s(f)}(x)$$

for some $\tilde{h}'(x): T^2 \to \mathbb{R}^+$, where $\tilde{h}' \in C^{k-1}_s$.

(We should remember that Dh does not make sense except when acting on vectors tangent to the stable foliation of g.)

We also have

$$(Df)(x) D_{s(f)}(x) = \tilde{f}'(x) D_{s(f)}(x),$$
$$(dg)(x) D_{s(g)}(x) = \tilde{g}'(x) D_{s(g)}(x),$$

$\tilde{f}'(x) \in C^\infty_{s(f)}$, $\tilde{g}'(x) \in C^\infty_{s(g)}$ and both are bounded away from zero.

It is quite important to observe that even if $\tilde{f}(x), \tilde{g}(x)$ are defined only on the stable manifold of 0, their derivatives extend to the whole of T^2 and (if we disregard some nitpicking about whether they are defined on the manifold or the coordinates of the manifold) the extensions agree with $\tilde{f}'(x), \tilde{g}'(x)$ so that this notation is sensible.

This fact that what happens in the (dense in T^2) stable manifold extends will give us enough control in the $C^\alpha \to$ Lipschitz part of the argument.

Applying the chain rule to derivatives along $D_{s(g)}$ of (1), we have

$$\tilde{f}'(h(x)) \tilde{h}'(x) = \tilde{h}'(g(x)) \tilde{g}'(x)$$

and, taking logarithms

$$\log \tilde{f}'(h(x)) - \log \tilde{g}'(x) = -\log \tilde{h}'(x) + \log \tilde{h}'(g(x)). \tag{5}$$

According to proposition 1, the L.H.S. is in $C^k_{s(g)}$ so that, according to Lemma 2.2 of [L.M.M.] we have $\log \tilde{h}'(x) \in C^k_{s(g)}$, so that $h \in C^{k+1}_{s(g)}$.

Theorem 2. *With the same notation of the main theorem assume h solves (1) and* $\in C^\alpha$ $(0 < \alpha)$. *Assume moreover that f, g satisfy condition b) of the main theorem.*

Then, h (the expression of $\tilde{h}$ *in the coordinates picked in Lemma 3) is uniformly Lipschitz on the real line.*

Proof of Theorem 2. Equation (1) when expressed in the coordinates constructed in Lemma 3 becomes

$$\tilde{f} \delta \tilde{h} = \tilde{h} \circ \tilde{g}. \tag{6}$$

We have some a priori information about the $\tilde{h}$ because it is the restriction of the unique h close to the identity in T^2.

For example, we know it has an inverse and that

$$\tilde{h}, \tilde{h}^{-1} \in \mathrm{Id} + L^\infty.$$

(Notice that $\mathrm{Id} + L^\infty$ is closed under composition.)

The standard expansiveness argument performed with due care will give that the only solution of (6) in $\mathrm{Id} + L^\infty$ (not necessarily invertible) is precisely $\tilde{h}$.

In order to prove Theorem 2 we will study

$$\tilde{h}_n = \tilde{f}^n \circ \tilde{g}^{-n}(x)$$

and will show that $\tilde{h}_n$ converges to $\tilde{h}(x)$ uniformly on the real line and that under the hypothesis of the theorem all the $\tilde{h}_n$'s are uniformly Lipschitz. Hence $\tilde{h}$ is Lipschitz.

The first result of the previous paragraph is quite standard (it is just one of the ingredients in one of the usual proofs of the shadowing lemma) and is true in great generality. The second one, on the contrary, uses quite essentially the compatibility conditions (3); it is not true for one dimensional systems in general, but uses that our one-dimensional system is the restriction of the two dimensional one.

Proof of Theorem 2. Call $\tilde{h}(x) = x + \hat{h}$ and $\tilde{h}_n(x) = x + \hat{h}_n$ (all h are in L^∞). From (6) we have

$$\hat{h}(x) = \tilde{f}^n \circ \tilde{h} \circ \tilde{g}^{-n}(x) = \tilde{f}^n(\tilde{g}^{-n}(x) + \hat{h}(\tilde{g}^{-n}(x))).$$

Now $\tilde{f}^n$ is a contraction uniformly on the whole real line so that

$$\tilde{h}(x) - \tilde{h}_n(x) \leqq |\mathrm{lip}\, \tilde{f}|^n \sup_{x \in \mathbb{R}} \hat{h}(\tilde{g}^{-n}(x)) = |\mathrm{lip}\, \tilde{f}|^n \sup_{x \in \mathbb{R}} |\hat{h}(x)|.$$

To prove the second statement we will estimate

$$\left| \frac{d}{dx} \tilde{f}^n \circ \tilde{g}^{-n}(x) \right|,$$

uniformly in n and x. For that, we will use that there is a C^α $\alpha > 0$ function ψ in T^2 such that

$$\log \tilde{f}'(h(x)) - \log \tilde{g}'(x) = -\psi + \psi \circ g(x).$$

The existence of such a function follows from the existence part of the Livsic theorem. Transitivity of g (even if transitivity is used in other places to ensure density of the leaves, I hope this could be removed from the argument—this seems to be the only place where transitivity is essential) is ensured by the Frank and Newhouse theorem and the compatibility conditions will follow from the assumption b) of the theorem.

Since the stable manifolds are tangent to the eigenspaces of the derivatives at periodic points we should have

$$\tilde{g}^{n'}(x_0) = \tilde{f}^n(h(x_0)) \quad \text{when} \quad g^n(x_0) = x_0.$$

Using the chain rule we have

$$\tilde{g}^{n'}(x_0) = \tilde{g}'(x_0)\tilde{g}'(g(x_0)) \cdots \tilde{g}'(g^{n-1}(x_0)),$$
$$\tilde{f}^{n'}(x_0) = \tilde{f}(h(x_0))\tilde{f}'(f \circ h(x_0)) \cdots \tilde{f}'(f^{n-1} o h(x_0)),$$

and, using (1)

$$\tilde{f}^{n'}(x_0) = \tilde{f}'(h(x_0))\tilde{f}'(\tilde{h} \circ g(x_0)) \cdots \tilde{f}'(h \circ g^{n-1}(x_0)).$$

Taking logarithms, this reduces to the compatibility conditions for the existence of ψ.

We will call $\tilde{\psi}$ the restriction of ψ to $W_0^{s(g)}$ expressed in the coordinate system constructed in Lemma 3.

Now we try to write $\log(\tilde{f}^n \tilde{g}^{-n})$ in such a way that we can obtain the uniform bounds.

We have

$$\begin{aligned}
\log(\tilde{f}^n \tilde{g}^{-n'})(x) &= (\log \tilde{f}') \circ \tilde{f}^{n-1} \circ \tilde{g}^{-n}(x) + (\log \tilde{f}') \circ \tilde{f}^{n-2} \circ \tilde{g}^{-n}(x) t \cdots \\
&= (\log \tilde{f}') \circ \tilde{f}^{n-1} \circ \tilde{g}^{-n}(x) + (\log \tilde{f}') \circ \tilde{f}^{n-2} \circ \tilde{g}^{-n}(x) t \cdots \\
&\quad + \log \tilde{f}' \circ \tilde{g}^{-n}(x) - (\log \tilde{g}')\tilde{g}^{-n}(x) - (\log \tilde{g}') \circ \tilde{g}^{-n+1}(x) - \cdots \\
&\quad - (\log \tilde{g}')\tilde{g}^{-1}(x) = \left\{ \sum_{k=0}^{n-1} ((\log \tilde{f}') \circ \tilde{f}^k - (\log \tilde{g}') \circ \tilde{g}^k) \right\} \circ \tilde{g}^{-n}(x).
\end{aligned}$$

To estimate the sup in x, it suffices to estimate the term in brackets. We start by estimating a very similar sum.

Out of the Livsic theorem and (6) we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1} (\log \tilde{f}') \circ \tilde{f}^k \circ \tilde{h}(x) - (\log \tilde{g}') \circ \tilde{g}^k(x) \\
&\quad = \sum_{k=0}^{n-1} (\log \tilde{f}') \circ \tilde{h} \circ g^k(x) - (\log \tilde{g}') \circ \tilde{g}^k(x) = \tilde{\psi} \circ g^n(x) - \tilde{\psi}(x),
\end{aligned}$$

so that these auxiliary sums are bounded uniformly in n, x.

Comparing this bound with the one we want, we see that it suffices to bound

$$\left| \sum_{k=0}^{n-1} \log \tilde{f}' \circ \tilde{f}^k \circ \tilde{h}(x) - \log \tilde{f}' \circ \tilde{f}^k(x) \right|$$

independently of n, and x.

Each of the terms in the sum can be bounded by

$$\sup_x |(\log \tilde{f}' \circ \tilde{f}^{k'})(x)| \sup_x |\tilde{h}(x) - x|$$

but, applying again the chain rule we see that those are bounded by the terms of a geometric series. $\left(\text{Notice that } \sup_x |\tilde{f}'(x)| < 1, \text{ hence } \sup_x |\tilde{f}^{k'}(x)| \leqq \left(\sup_x |\tilde{f}(x)|\right)^k.\right)$ This completes the proof of Theorem 2.

Theorem 3. *If $\tilde{h}$ is Lipschitz and* b) *is satisfied, then $h \in C^1_{s(g)}$.*

Proof. By the density of the stable manifold $W_0^{s(g)}$ we obtain that it is Lipschitz in every leaf of the stable foliation and with the same Lipschitz constant.

By the fundamental theorem of calculus, on each leaf there is a set of full Lebesgue measure on which there is a derivative for $\tilde{h}$.

The sense in which this derivative exists is strong enough to guarantee that the chain rule applies and, hence, (5) is fulfilled.

So, we have

$$(\log \tilde{h}'(x) - \psi(x)) - (\log \tilde{h}' - \psi) \circ g(x) = 0$$

almost everywhere on each leaf.

"Almost everywhere on each leaf" is an extremely strong property. Because of the absolute continuity of the foliation it implies almost everywhere in the two dimensional Lebesgue measure of the torus. More importantly, it implies almost everywhere for the backwards Bowen–Ruelle measure because the Bowen–Ruelle measure can be projected onto the unstable manifolds to give a measure with smooth density (in the literature, e.g. [Ne2], p. 64 one finds C^1 density, but probably one can do better; C^1 is more than enough for present purposes).

Since Bowen–Ruelle measure is ergodic, we have

$$\log \tilde{h}'(x) - \psi(x) = \text{cte a.e. } B - R.$$

Therefore, there exists at least one leaf in which

$$\log \tilde{h}'(x) - \psi(x) = \text{cte a.e. in the leaf.}$$

But, for a transitive Anosov system, all stable leafs are dense in the manifold (see e.g. [Ka]). Since the leaf is dense in T^2 and ψ is continuous on all T^2, it follows that $\exp[\psi(x) - \text{cte}]$ is the true derivative (see [LMM] for more details on the argument which is nevertheless standard) and, hence $h \in C^1_{s(g)}$.

This completes the proof of Theorem 3.

Putting together Theorems 1, 2, 3 we get that $h \in C^\infty_{s(g)}$.

We now observe that

$$f^{-1} \circ h = h \circ g^{-1}$$

so that applying the previous result we also have $h \in C^{\infty}_{s(g^{-1})} = C^\infty_{u(g)}$.

Invoking Lemma 2.3. of [LMM] finishes the proof of the theorem.

Remark. The conclusion $h \in C^1 \Rightarrow h \in C^\infty$ does not require proximity assumptions for f and g.

If $f = h \circ g \circ h^{-1}$ and $h \in C^1$, we can find another $\bar{h} \subset C^\infty$ and sufficiently C^1 close to h ([Hi] Thm 2.7) so that our local theorem can be applied to f and $\bar{f} = \bar{h} \circ g \circ \bar{h}^{-1}$. Uniqueness of local conjugations shows $h \in C^\infty$.

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They, as well as P. Collet and G. Gallavotti deserve some credit for asking continuously about the removal of families. The question of whether the eigenvalues of the derivatives at periodic points are a complete set of invariants was asked to me by S. Newhouse in Erice 1983.

Note. During the time ellapsed from the submission to revision, there have been some developments worth mentioning.

Journé has found an alternative proof to Lemma 2.3 of [L.M.M] using different regularity properties of the foliation. [Jo]

Marco and Moriyón have shown that for Anosov systems in T^2, constant lyapunov exponents at periodic orbits imply smooth conjugacy to a linear automorphism, and they have also shown that the periods and the lyapunov exponents are complete sets of invariants for families of Anosov flows on 3-manifolds. This later result involves study of vector cohomology equations. [MM2]

Moriyón and myself have also shown that equality of lyapunov exponents at periodic orbits for two Anosov systems on T^2 implies C^∞ conjugacy without proximity assumptions and removed the use of families in the result about flows.

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