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Renormalization Group Approach to Lattice Gauge Field Theories

I. Generation of Effective Actions in a Small Field Approximation and a Coupling Constant Renormalization in Four Dimensions

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Abstract. We study four-dimensional pure gauge field theories by the renormalization group approach. The analysis is restricted to small field approximation. In this region we construct a sequence of localized effective actions by cluster expansions in one step renormalization transformations. We construct also β-functions and we define a coupling constant renormalization by a recursive system of renormalization group equations.

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0. Introduction. Formulations of Results

In this paper we continue our study of the ultraviolet stability problem for lattice approximations of gauge field models. We consider here the renormalization group approach to four-dimensional pure gauge field theories. We restrict the study to a part of the problem, namely we want to understand how to generate the effective actions in a small field approximation, and how to perform a coupling constant renormalization by a system of recursive renormalization group equations (Callan-Symanzik equations).

The renormalization group approach we use here follows the ideas of Wilson [74, 75]. This approach was developed by the author for superrenormalizable gauge field theories in the papers [4–16], see especially [4–6] for an explanation of ideas and an introduction to techniques. The renormalization group approach was pursued by Gawedzki and Kupiainen in [39–44], and other papers, for another class of models. Some of their ideas are very important here, and we use them quite extensively.

Renormalization group ideas in rigorous results in quantum field theory already have a long history. We do not intend to give a responsible account of it, but we would like to mention here some papers and methods which were especially important for us. Some aspects of these ideas are present in early papers of Glimm and Jaffe, most clearly in [46] in the form of the phase space cell expansion. This method has been developed since then by many authors [30, 61–63, 31–34]. The most recent and interesting achievement is a control of the infrared renormalizable and asymptotically free critical $\lambda(\phi^4)_4$ model [31], and the construction of the two-dimensional Gross-Neveu model [76*], see also [77*]. The next important and decisive step towards the constructive renormalization group approach was done by Gallavotti and collaborators [19–21, 37, 38]. In these papers some fundamental ideas and techniques were introduced, which laid a basis for future developments. Especially the papers of this author were influenced by them in an essential way. The renormalization group approach was developed in numerous papers, mainly from a perturbative, or a numerical point of view. Nonperturbative and discretized versions of renormalization group equations, called Callan-Symanzik equations, play an important role in this and subsequent papers. Some fundamental papers on these equations are [45, 69, 70]. Also we mention that renormalization group methods in Wilson's form were investigated in the papers [65, 37, 38] from a perturbative point of view. The fully developed, nonperturbative methods were presented in the above mentioned papers by Gawedzki and Kupiainen.

Applications of renormalization transformations involve two kinds of problems. An integral defining the transformation is divided into two subintegrals, one determined by the "small field" region, another by a "large field" region. Controlling small field integrals is connected with understanding renormalization problems. In our case the only important renormalization is a coupling constant renormalization, but there also is a simple vacuum energy renormalization. We study the renormalization transformations under the simplifying assumption that integrals are restricted to small fields. This simplification allows the representation of renormalization group transformations by transformations acting on effective actions. Thus applying a sequence of small field renormalization transformations, we obtain a sequence of effective actions with new effective coupling constants. We use methods and results developed for gauge field theories, Abelian and non-Abelian, in the series of papers [7–16]. The fundamental technical results obtained in those papers are valid in four-dimensional theories (in fact they are dimension-independent), and they form a technical core of this paper. Not only technical results are important here, but also some ideas used previously in a simpler framework. For example the analysis of renormalization in Higgs models done in the paper [7], which seems to be restricted to superrenormalizable models, provides ideas how to represent effective actions and how to analyze them using Ward-Takahashi identities, which are essential for renormalizable models. We can say, with some exaggeration, that the method of this paper is the method of [7] for the resummed perturbative expansion. Other methods and results used in this paper are taken from the papers [41–43] of Gawedzki, Kupiainen; especially we use the idea that the effective actions should be considered as analytic functions defined on spaces of complex field configurations. In the above papers it was used as a method of proving convergence to a fixed point. Here we use it as a purely technical device, simplifying a formulation of inductive assumptions on effective actions, and simplifying many technical aspects of cluster expansions. In Gawedzki and Kupiainen's papers, scaling transformations play a fundamental role. We do not have them in the class of nonlinear models considered here; instead we analyze effective actions using regularity properties of field configurations, and symmetries. The important symmetries are given by lattice Euclidean transformations and gauge transformations. They yield a set of Ward-Takahashi identities. These identities, together with the regularity properties, allow the analysis of renormalization transformations and the reduction of them to simple transformations of effective coupling constants. More precisely, the identities determine β-functions which we use to transform the coupling constants by discrete analogs of Callan-Symanzik equations. These ideas form the core of our method, although they are rather simple from a technical point of view. The Ward-Takahashi identities are discussed in Sects. 4 and 5. The analysis of the Callan-Symanzik equations, based on perturbative calculations, will be discussed in another paper.

Let us recall now some geometric definitions. Consider a lattice as a subset of a continuous Euclidean space R^d, or a torus T obtained by the usual identification of boundary points of the cube $\{x \in R^d : -L_\mu \leqq x_\mu \leqq L_\mu, \mu = 1, \ldots, d\}$. We take $L_\mu = L^m$, where L is an odd, positive integer >11, and m is a positive integer. These spaces are divided into regular lattices of cubes with corners at points of $L^{-n}Z^d$, $n = 0, 1, \ldots$. Each lattice determines a lattice of centers of these cubes, i.e. the lattice $L^{-n}Z^d + \sum_{\mu=1}^{d} \frac{1}{2} L^{-n} e_\mu$, where e_μ denotes a unit vector of the positive μ-th axis. Field configurations are defined on these lattices. Define the initial lattice approximation on the torus

$$T_\varepsilon = \left\{ x \in \varepsilon Z^d + \sum_{\mu=1}^{d} \frac{1}{2} \varepsilon e_\mu : -L_\mu < x_\mu < L_\mu, \mu = 1, \ldots, d \right\}, \tag{0.1}$$

with a lattice spacing $\varepsilon = L^{-K}$. This torus determines a sequence of tori denoted by $T^{(k)}_{L^k\varepsilon}$ and defined by (0.1) with ε replaced by $L^k\varepsilon$, $k = 1, 2, \ldots$. A point $y \in T^{(k)}_{L^k\varepsilon}$ determines a cube of the continuous torus with a center at y and the size $L^k\varepsilon$. We denote it by $\Delta(y)$, or $\Delta_k(y)$ if it is necessary to indicate the scale. Usually we drop subscripts, or superscripts, indicating lattice spacing, or other scales, if a meaning of a symbol is clear, or if the scale is unessential. It is convenient to determine subsets of a lattice by subsets of continuous space. We consider only subsets which are unions of cubes of a division described above. Such a subset determines a set of lattice points, of a given scale, belonging to it, a set of bonds, i.e. intervals b connecting nearest neighbor points b_-, b_+, which intersect it, and a set of plaquettes, i.e. elementary squares of the lattice, which intersect it. We denote all these sets by the same symbol denoting the continuous space set.

Field configurations have values in a compact Lie group G. For definitions concerning Lie groups and algebras see [71]. We assume that G is semisimple and that it is a Lie subgroup of a group of complex unitary matrices, for example $G \subset U(N)$. (In fact a bigger part of our considerations does not depend on the semisimplicity assumption.) Gauge field configurations U are functions defined on bonds of the lattice T, with values in G. A value of a gauge field configuration U at $b = \langle x, x' \rangle$ is denoted by $U_b = U(b) = U(\langle x, x' \rangle) = U(x, x')$.

An action for gauge field models was proposed by Wilson [73], and is given by

$$A^{\varepsilon}(U) = \sum_{p \in T_{\varepsilon}} \varepsilon^{d-4}[1 - \operatorname{Re} \operatorname{tr} U(\partial p)], \tag{0.2}$$

where tr is the normalized trace, i.e. $\operatorname{tr} 1 = 1$. For $d = 4$ we drop the superscript ε from the symbol of the action. We denote also the plaquette variable by ∂U, hence $U(\partial p) = (\partial U)(p)$.

The gauge field action and its properties were discussed in many papers [22, 64, 67, 2, 27, 56, 57, 74, 75].

In this paper we are interested mainly in $d = 4$ gauge field theories. We do all considerations, and write all formulas below for this case, and only occasionally we discuss some special features of $d = 3$.

Our basic object of investigation is a sequence of actions defined recursively by applications of small field renormalization transformations, starting with the action (0.2). We apply renormalization transformations similar to those introduced by Wilson [74, 75]. To write a definition we have to recall some geometric definitions of [4, 5, 9, 10]. If y is a point of a lattice $T_{\delta}^{(n)}$ with a lattice spacing δ, then a block of k-th order, determined by the point y, is defined by

$$B^k(y) = \Delta(y) \cap T_{L^{-k}\delta}^{(n-k)} \tag{0.3}$$

[$\Delta(y)$ is a continuous space cube with a center at y and with the size δ]. Blocks of order 1 are denoted by $B(y)$. For a point $x \in B(y)$ we take a family $\mathbf{G}(y, x)$ of shortest contours with the initial point at y, and the final point at x. If $x - y = \sum_{\mu=1}^{d} \delta n_{\mu} e_{\mu}$ (δ is a lattice spacing, n_{μ} an integer such that $|n_{\mu}| \leqq L - 1/2$), then we construct contours of $\mathbf{G}(y, x)$ in the following way: take a permutation $\{\pi(1), \pi(2), \ldots\}$ of indices μ with nonzero numbers n_{μ}, next take $|n_{\pi(1)}|$ bonds in the direction sign $n_{\pi(1)} e_{\pi(1)}$ starting at y, $|n_{\pi(2)}|$ bonds in the direction sign $n_{\pi(2)} e_{\pi(2)}$ starting at $y + \delta n_{\pi(1)} e_{\pi(1)}$, and so on. We define $\mathbf{G}(y, x)$ as the family of contours generated by all such permuations. For a bond c and a point $x \in B(c_-)$, let $[x, x']$ denote a contour which is obtained by a parallel transport of c to the point x.

Renormalization transformations are defined by averaging operations. In the previous papers we have used the definition introduced in [12]. This definition has one disadvantage, it is not symmetric with respect to lattice Euclidean transformations. Preserving this symmetry is very important for the method; therefore it is necessary to modify the definition. For technical reasons we wish to establish analyticity with respect to group valued gauge fields. Therefore we also consider the complexified group G^c. Elements of this group are defined as matrices of the form $\mathbf{U} = U'U$, where $U \in G$ and $U' = \exp iA'$, $A' \in \mathbf{g}^c$, $\mathbf{g}^c$ is the complexification of the real Lie algebra $\mathbf{g}$. Let us write the simplest, immediate extension of the definition of averages in [12]. The set of contours used in this definition was too small to be lattice Euclidean symmetric, so we extend it. We define an average of a gauge field configuration $\mathbf{U}$, corresponding to a bond $c \in T^{(1)}$, by

$$\bar{\mathbf{U}}(c) = M(\mathbf{U}, c) = \exp\left[i \sum_{x \in B(c_-)} L^{-d} \sum_{\Gamma \in \mathbf{G}(c_-, x)} \frac{1}{|\mathbf{G}(c_-, x)|} \sum_{\Gamma' \in \mathbf{G}(c_+, x')} \frac{1}{|\mathbf{G}(c_+, x')|} \right.$$
$$\left. \times \frac{1}{i} \log \mathbf{U}(\Gamma \cup [\mathrm{x}, \mathrm{x}'] \cup (-\Gamma') \cup (-\mathrm{c})) \right] \mathbf{U}(c) . \tag{0.4}$$

We need also a Euclidean invariant definition of the axial gauge fixing. A natural idea, in agreement with the above definition, would be to use variables obtained by averaging the contour variables $\{\mathbf{U}(\Gamma)\}_{\Gamma \in \mathbf{G}(y,x)}$, rather than to use one contour variable $\mathbf{U}(\Gamma_{y,x})$. Thus we have to define an average of a finite set of group elements. We introduce an axiomatric definition of such an average. It is a G^c-valued function defined on sets $\{\mathbf{U}_j : j = 1, 2, \ldots, n\}$, $\mathbf{U}_j \in G^c$, with sufficiently small diameters. We denote it by $\overline{\{\mathbf{U}_j\}} = M(\{\mathbf{U}_j\})$, and we assume that it is an analytic function having the following properties:

$$M(\{\mathbf{U}_j^{-1}\}) = M(\{\mathbf{U}_j\})^{-1}; \tag{0.5}$$

$$M(\{u\mathbf{U}_j v\}) = uM(\{\mathbf{U}_j\})v ; \tag{0.6}$$

$M(\pi\{\mathbf{U}_j\}) = M(\{\mathbf{U}_j\})$ for an arbitrary permutation π of the set $\{\mathbf{U}_j\}$; (0.7)

for a set $\{\mathbf{U}_j\}$ of elements close to the identity of the group, i.e. $\mathbf{U}_j = \exp iA_j$ with A_j in a small neighborhood of 0 in $\mathbf{g}^c$, the average is close to the identity also, and

$$\frac{1}{i} \log M(\{\exp iA_j\}) = \frac{1}{n} \sum_{j=1}^{n} A_j + (\text{higher order terms}); \tag{0.8}$$

$$\text{if} \quad \mathbf{U}_j \in G, \quad \text{then} \quad M(\{\mathbf{U}_j\}) \in G \quad \text{also}. \tag{0.9}$$

There are several proposals how to define such averages, see [74, 75, 35]. Let us write a definition which is equivalent to the one given by Federbush in [35]. The average of the set $\{\mathbf{U}_j\}$ is the element $\mathbf{U} \in G^c$ such that $\mathbf{U}_j$ are in a small neighborhood of $\mathbf{U}$, and it satisfies the equation

$$\sum_{j=1}^{n} \frac{1}{i} \log \mathbf{U}_j \mathbf{U}^{-1} = 0 . \tag{0.10}$$

This definition has all the properties listed above, as it was proved by Federbush in [35 (II)].

The considerations and results of this, and previous papers, do not depend on any particular averaging operation used; they are valid universally for all averages satisfying the above properties. Having such an average we introduce the averaged contour variable corresponding to the set of contours $\mathbf{G}(y, x)$,

$$\mathbf{U}(y, x) = M(\{\mathbf{U}(\Gamma)\}_{\Gamma \in \mathbf{G}(y,x)}) . \tag{0.11}$$

In fact the only reason for introducing the group averages with the properties (0.5)–(0.10) is to define the controur variables (0.11) which are invariant with respect to permutations of the contours Γ in $\mathbf{G}(x, y)$. Using these variables simplifies a little bit the subsequent discussion of the Euclidean invariant axial gauge fixing. To match a bit better the new axial gauge fixing to the definition of the averages of gauge field configurations, as in [12], we introduce the definition

$$\bar{\mathbf{U}}(c)=\exp\left[i\sum_{x\in B(c_-)}L^{-d}\frac{1}{i}\log \mathbf{U}(c_-,x)\mathbf{U}([x,x'])\mathbf{U}(x',c_+)\mathbf{U}(-c)\right]\mathbf{U}(c)\,. \qquad (0.12)$$

This average has properties similar to the properties of the average introduced in (0.4), especially all results of the paper [12] are valid for it. The proofs are in most cases unchanged; in others only minor and obvious modifications are needed. These modifications are connected again rather with the fact that we consider G^c-valued configurations, than with the different definition of the average. Let us stress that both definitions are equally good for our purposes, in fact we may use many other definitions. It is possible to axiomatize them also, listing all essential properties, but it is not interesting enough to do it here. We mention only this possibility in case it will be convenient to use some other definitions.

Now we consider renormalization transformations, which have the general form

$$(T\varrho)(V)=\int dU t(V,U)\varrho(U)\,. \qquad (0.13)$$

Here U, V are gauge field configurations on the lattices T, $T^{(1)}$ correspondingly, and $t(V,U)$ is a gauge invariant kernel, for example see the definitions in [9, 12]. If ϱ is a gauge invariant function, then $T\varrho$ is gauge invariant also. We are interested in functions restricted to small field regions, which in this case means that the function ϱ and the integral in (0.13) are restricted to regular configurations U, i.e. configurations satisfying bounds $|U(\partial p)-1|<\varepsilon_0$, $p\in T$, with ε_0 positive and sufficiently small, and we consider $T\varrho$ on configurations V satisfying similar bounds on $T^{(1)}$. The kernel $t(V,U)$ introduces connections between fields U and V, like the equalities $\bar{U}=V$ in the definitions of [9, 16], or more generally $|\bar{U}-V|<\varepsilon_0$ on bonds of $T^{(1)}$. Even with these restrictions the underintegral expression in (0.13) is still invariant with respect to the gauge transformations u satisfying $u(y)=1$ for $y\in T^{(1)}$. We have to fix a gauge in order to remove this invariance, as in [9, 16], where the block axial gauge was used to this purpose. In order to maintain the Euclidean invariance on the lattice $T^{(1)}$ we have to use other expressions than the contour variables $U(\Gamma_{y,x})$. We use the variables $U(y,x)$ defined in (0.11). It is inconvenient to fix the gauge by the δ-functions $\delta(U(y,x))$, because unlike the axial gauge fixing in [9, 16] these functions determine complicated nonlinear restrictions on gauge fields. Instead we introduce exponential gauge fixing functions,

$$\exp\left[-\frac{1}{2\alpha}|U(y,x)-1|^2\right]=\exp\left[-\frac{1}{\alpha}[1-\operatorname{Re}\operatorname{tr}U(y,x)]\right]. \qquad (0.14)$$

It is convenient to introduce simultaneously restrictions on the variables $U(y,x)$. We have the following identities for $y\in T^{(1)}$, $x\in B(y)$, $x\neq y$,

$$\frac{1}{z}\int du(x)\exp\left[-\frac{1}{\alpha}[1-\operatorname{Re}\operatorname{tr}U(y,x)u^{-1}(x)]\right]\chi(\{|U(y,x)u^{-1}(x)-1|<\varepsilon_0\})$$
$$=\frac{1}{z}\int du(x)\exp\left[-\frac{1}{\alpha}[1-\operatorname{Re}\operatorname{tr}u(x)]\right]\chi(\{|u(x)-1|<\varepsilon_0\})=1\,, \qquad (0.15)$$

where z is defined by the last integral. We introduce (0.15) under the integral in (0.13) and we apply the usual Faddeev-Popov procedure, i.e. we change the order of integrations and apply the gauge transformation $U \to U^{u^{-1}}$ with $u(y)=1$ for $y \in T^{(1)}$. By the gauge invariance with respect to such transformations the integrand does not depend on u and the integral over u is equal to 1. Thus we get

$$(T\varrho)(V) = \int dU t(V,U) \prod_{y \in T^{(1)}} \prod_{x \in B(y), x \neq y} \frac{1}{z} \exp\left[-\frac{1}{\alpha}[1 - \operatorname{Re} \operatorname{tr} U(y,x)] \right] \times \chi(\{|U(y,x) - 1| < \varepsilon_0\}) \varrho(U). \tag{0.16}$$

The new restrictions on $U(y,x)$ together with the previously discussed restrictions determine finally a small field region in the following sense: fix a configuration U_0 in this region, then configurations U from this region satisfy $|U - U_0| < O(1)\varepsilon_0$ on T, with an absolute constant $O(1)$. In concrete situations considered in this paper there is a natural choice of the configuration U_0 as a minimal configuration of an action.

We now define the renormalization transformations. According to our point of view, connected with small field restrictions, they map actions into actions. In the first step we define

$$A_1(V) = (\mathbf{T}_0 A)(V) = \log \mathbf{N}_0^{-1} \int dU \prod_{c \in T^{(1)}} \delta(\bar{U}(c) V^{-1}(c)) \times \chi_0 \exp\left[-\frac{1}{g_0^2} \sum_{y \in T^{(1)}} \sum_{x \in B(y), x \neq y} [1 - \operatorname{Re} \operatorname{tr} U(y,x)] - \frac{1}{g_0^2} A(U) \right], \tag{0.17}$$

where the normalization factor $\mathbf{N}_0$ is given by the same integral as above, but with $V=1$.

The action A_1 determines a function $\beta_1(g_0)$ of the bare coupling constant g_0, defined on an interval $[0, \gamma]$, $\gamma > 0$. This is a function in a sequence of β-functions. In the next section we will relate them to effective actions. Now we assume the existence of the first function, and we define the new coupling constant g_1

$$\frac{1}{g_0^2} = \frac{1}{g_1^2} + \beta_1(g_0), \quad \text{or} \quad \frac{1}{g_1^2} = \frac{1}{g_0^2} - \beta_1(g_0). \tag{0.18}$$

The equation (0.18) written in the form $(1/g_1^2) - (1/g_0^2) = (d(1/g^2))_0 = -\beta_1(g_0)$ is a finite difference approximation, corresponding to two consecutive lattice spacings (on a logarithmic scale) of the differential equation $(d/ds)(1/g^2) = -\beta(g)$, or $(dg/ds) = 1/2\beta(g)g^3$. This is the usual differential renormalization group equation, an example of a Callan-Symanzik equation, considered in quantum field theory.

The k-th action A_k determines a coupling constant g_k. The $(k+1)$-st action A_{k+1} is defined by the generalization of (0.17),

$$A_{k+1}(g_{k+1}, V) = (\mathbf{T}_k A_k)(g_{k+1}, V) = \log \mathbf{N}_k^{-1} \int dU \prod_{c \in T^{(k+1)}} \delta(\bar{U}(c) V^{-1}(c)) \times \chi_k \exp\left[-\frac{1}{g_k^2} \sum_{y \in T^{(k+1)}} \sum_{x \in B(y), x \neq y} [1 - \operatorname{Re} \operatorname{tr} U(y,x)] + A_k(g_k, U) \right], \tag{0.19}$$

where the constant $\mathbf{N}_k$ is given by the integral above with $V=1$, and the coupling constant g_{k+1} is determined from the equation

$$\frac{1}{g_k^2}=\frac{1}{g_{k+1}^2}+\beta_{k+1}(g_k). \tag{0.20}$$

The characteristic function χ_k restricts the integral to small fluctuation fields.

We continue the calculation of the effective actions until we reach the unit lattice, or rather until we reach a scale which we define as the unit scale. Let us denote the corresponding index by K, hence $\varepsilon=L^{-K}$, and the sequence of actions and coupling constants is defined for $k=0,1,\ldots,K$.

Now let us describe a general form of the actions suggested by the previous work on superrenormalizable models [4–7, 16, 17, 55]. In these papers the actions were expressed as functions of external (background) fields. Each action was represented as a sum of localized contributions coming from integrations in successive renormalization transformations, i.e. see the sums over j in (2.7) [7], (41), (47) [16]. This representation is important here also. The background fields are the same as in [16]. The integrals in (0.17), (0.19) are calculated by a variation of the steepest descent method, and an important part of it is to find critical configurations of the actions, or rather of these terms in the actions, which go to ∞ if the effective coupling constants go to 0. These critical configurations are the minimal configurations investigated in [15], i.e. the minima of the functional

$$U\rightarrow A^{\eta}(U),\quad \text{on}\quad U:\bar{U}^k=M^k(U)=V, \tag{0.21}$$

where V is a gauge field configuration on the lattice $T^{(k)}$. The space of configurations U in (0.21) is invariant with respect to the gauge transformations u defined on T and satisfying the condition $u=1$ on $T^{(k)}$. These transformations form a group, and the functional (0.21) can be considered on orbits of this group. In [15] it was proved that in a space of regular orbits there exists exactly one critical orbit, which is a set of minima of the function (0.21). We denote a configuration in this orbit by $U_k(V)$, or simply by U_k. These configurations are the background fields, and the effective action A_k depends on V through these fields. We have, as in [16]

$$A_k(g_k,V)=A_k(g_k,U_k(V))=-\frac{1}{g_k^2}A^{\eta}(U_k(V))+\mathbf{E}_k(U_k(V)). \tag{0.22}$$

The function $\mathbf{E}_k$ depends also on the effective coupling constants $g_0,\ldots,g_{k-1}$. It is a sum of contributions coming from the k successive integrations in the k renormalization transformations. In the first step the integral (0.17) yields the action $A_1=-(1/g_0^2)A^{L^{-1}}+\mathbf{E}^{(1)}$, and then applying Eq. (0.18) we obtain the representation $A_1=-(1/g_1^2)A^{L^{-1}}+[-\beta_1(g_0)A^{L^{-1}}+\mathbf{E}^{(1)}]$. The expression in the square bracket is equal to $\mathbf{E}_1$. In the second step a new expression of this type is created, in the old only a background field is changed. Thus we obtain after k steps

$$\mathbf{E}_k(U_k)=\sum_{j=1}^{k}[-\beta_j(g_{j-1})A^{\eta}(U_k)+\mathbf{E}^{(j)}(U_k)]. \tag{0.23}$$

The functions $\mathbf{E}^{(j)}$ are represented as sums of localized terms. Such a representation arises in a natural way as a result of cluster expansion, see [39–44, 25, 26]. To describe it we have to introduce new geometric definitions. Let us take a nonnegative integer j, and the continuous space T scaled properly, so that it corresponds to the lattice T_ξ, $\xi = L^{-j}$. We decompose the space T into the lattice of closed cubes of a size M, where $M = L^m$, with centers at points of the lattice $T_M^{(j+m)}$. Let us notice that we have already several scales of big blocks, for example, the scale M_0 connected with the exponential decay of basic propagators, or the scale M_1 connected with properties of the variational problem, etc. In this paper we choose the size M much bigger than the previously fixed scales. We denote this family of cubes by π_j, and the cubes by $\square$, $\square'$, etc. For a cube $\square \in \pi_j$ and $n = 1, 2, \ldots$ we define $\tilde{\square}^n$ as a cube of the size $(1+2n)M$ and with a center at the center of $\square$. We introduce a notion of a localization domain. Such a domain is a union of a connected, finite family of cubes from π_j. A connected family means that for every pair $\square$, $\square'$ of cubes from the family there exists a sequence $\square, \square_1, \ldots, \square_n, \square'$ of cubes belonging to the family and such that two consecutive cubes have a common wall, i.e. their intersection is a $d-1$-dimensional cube. We denote localization domains by X, X', Y, etc. The meaning of the symbol $\tilde{X}^n$ should be obvious. The class of all these localization domains is denoted by $\mathbf{D}_j$. Domains from different classes, i.e. classes corresponding to different indices j, are connected by scaling transformations.

Thus every localization domain X is a union of continuous space cubes from π_j. Consider a class of tree graphs contained in X and intersecting all the cubes in X. A length of a shortest graph in this class, divided by M, is the linear size of X, and is denoted by $d_j(X)$. Thus we rescale the space, so that cubes from π_j become unit cubes, and we take the distance in this scale. Let us stress the fact that we consider graphs in the continuous space. For a given X usually there are many of these shortest tree graphs. It is easy to see that there are also the shortest tree graphs formed by edges of cubes in X, hence an equivalent definition can be formulated, based on such graphs only.

We assume that the functions $\mathbf{E}^{(j)}(U)$, for regular gauge field configurations U, have the following representation

$$\mathbf{E}^{(j)}(U) = \sum_{X \in \mathbf{D}_j} \mathbf{E}^{(j)}(X, U), \tag{0.24}$$

where $\mathbf{E}^{(j)}(X, U)$ are analytic and gauge invariant functions of U, depending on U restricted to X, and satisfying the inequality

$$|\mathbf{E}^{(j)}(X, U)| \leqq E_0 \exp(-\kappa d_j(X)), \tag{0.25}$$

with a sufficiently large constant κ. In the next section we will give precise definitions, now we are interested in basic properties of the above representation. Let us study implications of the inequality (0.25). We obtain

$$\begin{aligned}\left|\sum_{j=1}^{k} \mathbf{E}^{(j)}(U_k)\right| &\leqq \sum_{j=1}^{k} \sum_{X \in \mathbf{D}_j} E_0 \exp(-\kappa d_j(X)) \\ &\leqq \sum_{j=1}^{k} \sum_{\square \in \pi_j} \sum_{X \in \mathbf{D}_j, X \supset \square} E_0 \exp(-\kappa d_j(X)) \leqq \sum_{j=1}^{k} \sum_{\square \in \pi_j} E_0 O(1) \\ &= \sum_{j=1}^{k} E_0 O(1) M^{-4} |T_1^{(j)}| \leqq E_0 O(1) M^{-4} |T_1^{(k)}| \eta^{-4}, \qquad \eta = L^{-k}.\end{aligned} \tag{0.26}$$

Of course a bound of this type, with a divergence typical for a vacuum energy in the four-dimensional space, should be expected, without use of cancellations. The purpose of the terms with β-functions in (0.23) is to provide necessary cancellations, i.e. they are renormalization counterterms. In fact there is one more renormalization counterterm needed, it is a vacuum energy counterterm. It is contained in the normalization factors in the integrals (0.17), (0.19), hence we may assume that it is included already in the definition of the function $\mathbf{E}^{(j)}$, and $\mathbf{E}^{(j)}(1)=0$. Let us explain a fundamental idea behind a definition of the β-functions and the cancellations. The guiding principle here is the gauge invariance. It identifies the action $A^\eta(U_k)$, through a set of Ward-Takahashi identities, as a basic term in the effective action, the so-called marginal variable in the renormalization group language. The remaining terms sum up to a uniformly bounded expression. The gauge invariance is used here in the same way as in the analysis of the Abelian Higgs model in [7]. Let us remark that the analysis of the non-Abelian models in [16] is based on slightly different ideas. In fact we follow the method of [7] closely, as will become clear in Sects. 3–5. Now we sketch it very briefly. Consider a term in the sum (0.24). If its localization domain X is large, for example it is not contained in any cube $\tilde{\square}$, with $\square \in \pi_k$, then the inequality (0.25) ensures that

$$|\mathbf{E}^{(j)}(X, U_k)| \leqq E_0 \exp(-\kappa(L^j\eta)^{-1}) \exp(-1/2\kappa d_j(X)), \qquad (0.27)$$

and the first exponential can be bounded by an arbitrary positive power of $L^j\eta$, e.g. by $(5!/\kappa^5)(L^j\eta)^5$. This power is enough to control the sums in (0.23), (0.24). If the localization domain is contained in a cube $\tilde{\square}$, then we introduce the local coordinates constructed in Sect. F [15] for the configurations U_k on $\tilde{\square}$. This means that $U_k = \exp i\eta\mathbf{H}$ modulo a gauge transformation, where $\mathbf{H}$ is a regular, $\mathbf{g}$-valued configuration on $\tilde{\square}$, and where the cube $\tilde{\square}$ is considered as a subset of the η-lattice T_η. By the gauge invariance we have $\mathbf{E}^{(j)}(X, U_k) = \mathbf{E}^{(j)}(X, \exp i\eta\mathbf{H})$, and we expand the last expression in $\mathbf{H}$ up to the fifth order. The fifth order term can be bounded by $E_0 O((L^j\eta)^5)\exp(-\kappa d_j(X))$. The sum of lower order terms is analyzed with a help of Ward-Takahashi identities. They allow the extraction of terms, which grouped together yield an expansion of $\beta A^\eta(\exp i\eta\mathbf{H})$. The remaining terms have bounds as above. We define the function $\beta_j(g_{j-1})$ to be equal to the coefficient β. Thus we obtain

$$-\beta_j(g_{j-1})A^\eta(U_k) + \mathbf{E}^{(j)}(U_k) = \sum_{X \in \mathbf{D}_j} \mathbf{V}^{(j)}(X, U_k), \qquad (0.28)$$

where the functions $\mathbf{V}^{(j)}$ satisfy the inequalities

$$|\mathbf{V}^{(j)}(X, U_k)| \leqq O(1)(L^j\eta)^{4+\alpha} \exp(-\kappa d_j(X)), \qquad \alpha > 0, \qquad (0.29)$$

instead of (0.25). These inequalities yield a uniform bound of the sum (0.23) on the lattice T_η, i.e. the only dependence on k is through the volume $|T_\eta| = |T_1^{(k)}|$ corresponding to the scale η. Indeed, we have

$$\begin{aligned} |\mathbf{E}_k(U_k)| &\leqq \sum_{j=1}^{k} \sum_{X \in \mathbf{D}_j} O(1)(L^j\eta)^{4+\alpha} \exp(-\kappa d_j(X)) \leqq \sum_{j=1}^{k} \sum_{\square \in \pi_j} \sum_{X \in \mathbf{D}_j, X \supset \square} \\ &\quad \times O(1)(L^j\eta)^{4+\alpha} \exp(-\kappa d_j(X)) \leqq \sum_{j=1}^{k} \sum_{\square \in \pi_j} O(1)(L^j\eta)^{4+\alpha} \\ &= \sum_{j=1}^{k} O(1)(L^j\eta)^{\alpha} M^{-4} |T_1^{(k)}| \leqq O(1)(1-L^{-\alpha})^{-1} M^{-4} |T_1^{(k)}|. \end{aligned} \qquad (0.30)$$

For the effective action corresponding to the unit lattice, where $k=K=\log_L 1/\varepsilon$, the above bound is uniform in the lattice spacing ε. This is the essence of the ultraviolet stability concept, and the construction summarized in the representation (0.28) is, from a technical point of view, the main subject of this paper. Terms satisfying the inequality (0.29) are called irrelevant – a word, borrowed from the renormalization group language – but which should not suggest that these terms are irrelevant operators in that sense. There are no scaling transformations here, and these terms do not behave in a regular way under the renormalization transformations.

The ultraviolet stability concept has two aspects. The first concerns the effective actions generated by the renormalization group transformations, their description and the bounds they satisfy. The second concerns the behavior of the sequence of the effective coupling constants g_k generated from the bare coupling constant g_0 by the renormalization group equations (0.18), (0.20). In this paper we prove a theorem concerning the first aspect. To formulate this theorem we introduce small field domains. For the k-th action such a domain is determined by the condition $|\partial U_k(V)-1|<\varepsilon_0\eta^2$ on T_η, for ε_0 sufficiently small. Another possible definition, technically less convenient, is given by the condition $|\partial V-1|<\varepsilon_0$ on $T_1^{(k)}$. Now the main result can be formulated in a general way as follows:

Theorem 1. *If the sequence of the effective coupling constants is contained in an interval* $]0,\gamma]$ *with a sufficiently small positive* γ, *then the effective actions for small fields are given by the formulas* (0.22)–(0.24), *with terms satisfying* (0.29).

It is interesting to notice that we do not assume any special asymptotic behavior of the coupling constants, like asymptotic freedom. Hence in principle arguments could be applied to models with different behavior of running coupling constants, for example to the so-called asymptotically safe models, or finite models.

The above theorem will be reformulated more elaborately in the next section. It will be proved in this paper for an arbitrary semisimple compact Lie group $G\subset U(N)$, and for $d=4$. The proof also covers the dimension $d=3$, but in that case it is not necessary to consider the β-functions and the renormalization group equations (0.18), (0.20). The effective coupling constants are given by $g_k^2=g^2L^k\varepsilon$; hence the assumption of the above theorem is satisfied for g sufficiently small. The problem of whether the assumption is satisfied in dimension $d=4$ is addressed in the next theorem.

Theorem 2. *Let* $d=4$, $G=SU(2)$, *and let* γ *be a sufficiently small positive constant, then for a sufficiently small positive* g *there exists a bare coupling constant* $g_0=g_0(\varepsilon,g)$ *such that the sequence of the effective coupling constants* g_k *is contained in the interval* $]0,\gamma]$, *and* $g_K=g$. *Moreover, there exist constants* β, β', $0<\beta\leqq\beta'$, *such that* g_k *satisfy the inequality*

$$\frac{1}{g^2}+\beta\log(L^k\varepsilon)^{-1}\leqq\frac{1}{g_k^2}\leqq\frac{1}{g^2}+\beta'\log(L^k\varepsilon)^{-1}. \tag{0.31}$$

A proof of this theorem, based on perturbative calculations, will be given in a separate paper, where more precise asymptotic behavior will be proved. The restriction to the group $SU(2)$ is superficial and is done only to simplify calculations.

Finally let us mention that all results of this paper are valid for a certain class of two-dimensional nonlinear σ-models, the so-called nonlinear chiral models. For these models field configurations are functions U defined on sites of the lattice T, with values in the Lie group G, and the action is given by

$$A(U)=\sum_{b\subset T}[1-\operatorname{Retr}U(\partial b)]\,,\qquad U(\partial b)=(\partial U)(b)=U(b_-)U^{-1}(b_+)\,. \tag{0.32}$$

In fact all the results were obtained at first for this class of models. They are technically much easier, and many modifications and simplications are possible in this case; therefore we will discuss them separately in the future.

1. The Inductive Description of the Effective Actions

The inductive assumption is based on the results of the previous papers, e.g. see (2.7) [7], (41), (47) [16], and on the general remarks in the previous section. At first let us recall the definition of the background field configurations on which the k-th action depends. They are determined by regular gauge field configurations V given on the unit lattice $T_1^{(k)}$. The regularity means that the plaquette variables of V are small, $|V(\partial p')-1|<\varepsilon_0'$ for $p'\in T_1^{(k)}$. For such a configuration there exists exactly one regular, critical orbit of the functional

$$U\to A(U)\,,\qquad U:\bar{U}^k=M^k(U)=V\quad\text{on}\quad T^{(k)}\,, \tag{1.1}$$

and it is a minimal orbit (a set of minima). This variational problem was investigated in the paper [15], see Theorem 1 there for precise formulations. We denote by $U_k(V)$, or simply U_k, a configuration in the minimal orbit.

The k-th action $A_k(V)$ depends on V through the minimal configuration $U_k(V)$, $A_k(V)=A_k(U_k(V))$. It is defined on configurations belonging to the space $U_k(\varepsilon_0)$, i.e., $U_k(V)\in U_k(\varepsilon_0)$. The space was introduced in [14, 15], in a more general context, and here it is defined as the set of all configurations U satisfying the following regularity properties:

$$\begin{gathered}|U(\partial p)-1|=|(\partial U)(p)-1|<\varepsilon_0\eta^2\,,\qquad \eta=L^{-k}\,,\qquad p\in T\,,\\ |J|<\varepsilon_0\quad\text{on}\quad T\,,\qquad J=D_U^{\eta*}\eta^{-2}\pi\operatorname{Im}\partial U\,,\end{gathered} \tag{1.2}$$

with ε_0 sufficiently small. By Proposition 2 [12] this condition implies that $|V(\partial p')-1|<2\varepsilon_0$ for $p'\in T_1^{(k)}$, and is implied by the condition on V, with $2\varepsilon_0$ replaced by $B_3^{-1}\varepsilon_0$, see Theorem 1 [15]. The action has the following form

$$\begin{aligned}A_k(U_k)=&-\frac{1}{g_k^2}A(U_k)+\sum_{j=0}^{k-1}\{-\beta_{j+1}(g_j)A(U_k)+[\log Z^{(j)}(U_k)-\log Z^{(j)}(1)]\\&+[\mathbf{E}^{(j+1)}(g_j,U_k)-\mathbf{E}^{(j+k)}(g_j,1)]\}\,.\end{aligned} \tag{1.3}$$

We have written explicitly terms of the order 0 in the coupling constants. Let us recall that the factor $Z^{(j)}(U_k)$ is given by the Gaussian integral normalizing the Gaussian measure for a fluctuation field in j-th step integration

$$Z^{(j)}(U_k)=\int dB\,\delta(\tilde{Q}B)\exp[-\tfrac{1}{2}\langle B,\Delta^{(j)}(U_k)B\rangle]\,. \tag{1.4}$$

The quadratic form in the integral is given by

$$\langle B, \Delta^{(j)}(U_k)B\rangle = \langle H_{1,j}(U_k)B, \Delta_1(U_k)H_{1,j}(U_k)B\rangle \\ -2\langle H_{1,j}(U_k)h\tilde{C}^{(2)}(\bar{U}_k^j, B), J\rangle + \mathbf{G}^{(2)}(B), \tag{1.5}$$

where the operators $H_{1,j}$, Δ_1 are defined in Sect. D [13], and $\tilde{C}^{(2)}(\bar{U}_k^j, B)$ is the second order polynomial in the expansion of $\tilde{Q}(\bar{U}_k^j, B)$.

In many considerations it is not necessary to separate the terms of zeroth order from the remaining terms in the interaction, and then we write

$$A_k(U_k) = -\frac{1}{g_k^2}A(U_k) + \sum_{j=0}^{k-1}\{-\beta_{j+1}(g_j)A(U_k) + [\mathbf{E}^{(j+1)}(g_j, U_k) - \mathbf{E}^{(j+1)}(g_j, 1)]\}. \tag{1.6}$$

The terms in (1.3) given by explicit formulas, like (1.4), can be easily extended, by the same formulas, to a much wider class of regular gauge field configurations. For example we can extend to G^c-valued configurations satisfying the conditions (3.35)–(3.38) [13], and from results of that paper we conclude that these terms are analytic functions of the configurations. We assume that extensions of this type exist for all terms in the effective action (1.3), or (1.6).

The function $\mathbf{E}^{(j)}(g_{j-1}, U_j)$ is a result of an integration in the j-th step, after a subtraction of the previous action evaluated at U_j. We assume that it has the representation

$$\mathbf{E}^{(j)}(g_{j-1}, U_j) = \sum_{X \in \mathbf{D}_j} \mathbf{E}^{(j)}(X, g_{j-1}, U_j). \tag{1.7}$$

Again, the term corresponding to a domain X depends on U_j restricted to X. We can assume that this representation holds for the functions $\mathbf{E}^{(j)}$ in (1.3) and (1.6), because the expansion of the $\log Z^{(j-1)}(U_j)$ constructed in [16], implies, that we can represent this term in the form (1.7). Further, we assume that the terms in (1.7) are determined by functions defined on larger spaces of regular gauge field configurations. The regularity means that at least the conditions (3.35)–(3.38) [13], or (1.7)–(1.10) [14] are satisfied, i.e. first order derivatives and the special second order derivative of a configuration are small. For technical reasons it is convenient to separate the dependence on the second order derivative. From the previous papers it is clear that it appears only in the functions

$$J_j = D_{U_j}^{\xi*}\xi^{-2}\pi\,\mathrm{Im}\,\partial U_j, \tag{1.8}$$

where π denotes the projection in the space of all complex $N \times N$-matrices onto the algebra $\mathbf{g}^c$, and $\mathrm{Im}\,U = \frac{1}{2i}(U - U^{-1})$. They appear in expansions of the action, and in expressions defining propagators. We extend the terms in (1.7) introducing two variables $\mathbf{U}$, $\mathbf{J}$, the second variable replacing the functions (1.8), and the first satisfying milder regularity conditions involving first order derivatives at most. Thus we assume that there are functions $\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})$, analytic on a space of regular, complex configurations $\mathbf{U}$, $\mathbf{J}$, such that

$$\mathbf{E}^{(j)}(X, g_{j-1}, U_j) = \mathbf{E}^{(j)}(X, g_{j-1}, U_j, J_j). \tag{1.9}$$

Now we define spaces of configurations $\mathbf{U}$, $\mathbf{J}$. We define them for each j, and we are interested in local spaces determined by localization domains $X \in \mathbf{D}_j$. The gauge field configuration $\mathbf{U}$ is defined at bonds of X and has values in G^c, the configuration $\mathbf{J}$ is also defined at bonds of X and has values in $\mathbf{g}^c$. A G^c-valued gauge transformation u acts on pairs $(\mathbf{U},\mathbf{J})$ in the following way

$$(\mathbf{U},\mathbf{J})^u = (\mathbf{U}^u, R(u)\mathbf{J}) = (u_-\mathbf{U}u_+^{-1}, R(u_-)\mathbf{J}), \tag{1.10}$$

where for a bond $b = \langle b_-, b_+\rangle$ we define $u_\pm(b) = u(b_\pm)$. The orbit of the group of G^c-valued gauge transformations determined by a pair $(\mathbf{U},\mathbf{J})$ is denoted by $[(\mathbf{U},\mathbf{J})]$. We are ready to formulate the following fundamental definition.

The space $U_j^c(X,\alpha_0,\alpha_1,\gamma_0)$ *is a union of orbits* $[(\mathbf{U},\mathbf{J})]$ *determined by configurations* $\mathbf{U}$, $\mathbf{J}$ *satisfying the four conditions written below.*

(i) $\mathbf{U} = U'U$, *U has values in the group G,*

$$|\partial U - 1| < \alpha_0 \xi^2 \quad on \quad X, \tag{1.11}$$

for each cube $\square \subset X$ of a size $O(1)LM$ there exists a G-valued gauge transformation u defined on $\square$ and such, that $U^u = \exp i\xi A$,

$$|A|, \quad |\nabla^\xi A| < O(1)LMB\alpha_0 \quad on \quad \square, \tag{1.12}$$

with a sufficiently large constant B (it will be determined later).

(ii) $U' = \exp i\xi A'$, *A' has values in the algebra* $\mathbf{g}^c$,

$$|A'|, \quad |\nabla_U^\xi A'| < \alpha_1 \quad on \quad X. \tag{1.13}$$

(iii) *The configurations* $\mathbf{U}$, $\mathbf{J}$ *satisfy the bounds*

$$|\partial \mathbf{U} - 1| < \alpha_0 \xi^2, \qquad |\mathbf{J}| < \gamma_0 \quad on \quad X. \tag{1.14}$$

(iv) *We consider X as Ω_0, and we construct the sequence $\{\Omega_n\}$, $n = 1, \ldots, j$, of maximal possible domains satisfying the conditions* (1.3)–(1.6) [14] *(with j, ξ instead of k, η, $R = R_1$). For this sequence, or rather for its subsequences $\{\Omega_0, \Omega_1, \ldots, \Omega_n\}$, $n = 1, \ldots, j$, we construct the functions $U_n(V)$ in the axial gauges, for regular G^c-valued configurations V. We consider the pair*

$$(U_n(M^{\cdot}(\mathbf{U})), J_n(M^{\cdot}(\mathbf{U}))), \qquad n = 1, \ldots, j, \tag{1.15}$$

where $M^{\cdot}(\mathbf{U}, b) = M^p(\mathbf{U}, b) = \bar{U}^p(b)$ for $b \in A_p$ (see (1.5) [14]*), and $J_n(M^{\cdot}(\mathbf{U}))$ is defined by the formula* (1.8) *with $U_n(M^{\cdot}(\mathbf{U}))$ instead of U_j, L^{-n} instead of ξ. This pair satisfies the bounds.*

$$|\partial U_n(M^{\cdot}(\mathbf{U})) - 1| < \alpha_0 \xi^2, \qquad |J_n(M^{\cdot}(\mathbf{U}))| < \alpha_0 (L^n \xi)^2 \quad on \quad \tilde{X}^{-2}, \tag{1.16}$$

and the same bounds hold for U instead of $\mathbf{U}$.

The domain $\tilde{X}^{-2}$ is obtained from X by taking away two layers of cubes from π_j, which are closest to the boundary of X. Thus it is a domain X' such that $\tilde{X}'^2 = X$. The first three conditions (i)–(iii) in the above definition are rather simple and natural, the conditions of this form appeared many times in the previous papers, e.g. see (3.35)–(3.38) [13]. The last condition (iv) is connected with the fact that in some important constructions in the next sections we have to substitute the functions (1.15) in place of the variables $\mathbf{U}$, $\mathbf{J}$. We have to make sure, then, that they have required properties. Usually we consider spaces with $\gamma_0 = \alpha_0$, and then we omit the constant γ_0 from the symbol denoting the space.

Let us make now a few remarks about the above definition.

At first we show that there are some simple and natural spaces contained in $U_j^c(X, \alpha_0, \alpha_1)$. Let us take the space of configurations $(\mathbf{U}, \mathbf{J})$ satisfying the conditions (i)–(iii) with constants α_0', α_1' instead of α_0, α_1. We assume that the constants α_0', α_1' are smaller than α_0, α_1 correspondingly, then obviously these three conditions are satisfied in the original formulation. Now consider the functions $U_n(M^{\cdot}(\mathbf{U}))$. We have $M^{\cdot}(\mathbf{U}) = \bar{\mathbf{U}}^p$ on Λ_p, $|\partial \bar{\mathbf{U}}^p - 1| < 2\alpha_0'(L^p\xi)^2$. From Proposition 9 [15] we obtain

$$
\begin{aligned}
|\partial U_n(M^{\cdot}(\mathbf{U})) - 1| &< B_3 2\alpha_0' L^{-2n}(L^n\xi)^2 = 2B_3\alpha_0'\xi^2\,, \\
|J_n(M^{\cdot}(\mathbf{U}))| &< B_3 2\alpha_0'(L^n\xi)^2 \quad \text{on} \quad \tilde{X}^{-2}\,.
\end{aligned}
\tag{1.17}
$$

It is obvious that for α_0' sufficiently small the above estimates imply the condition (iv), thus the configuration $(\mathbf{U}, \mathbf{J})$ belongs to the space $U_j^c(X, \alpha_0, \alpha_1)$. We specialize this example even more and consider configurations satisfying (i)–(iii) with α_0', α_1' as above, and such that $\mathbf{J} = D_{\mathbf{U}}^{\xi *}\xi^{-2}\pi\,\mathrm{Im}\,\partial\mathbf{U}$ satisfies the bound $|\mathbf{J}| < \alpha_0'$, similarly the configuration constructed for U instead of $\mathbf{U}$. Then the pairs $(\mathbf{U}, \mathbf{J})$ belong to $U_j^c(X, \alpha_0, \alpha_1)$. In particular the minimal configurations U_j satisfying the bound $|\partial U_j - 1| < \varepsilon_0\xi^2$ with ε_0 sufficiently small, satisfy the above conditions.

Let us come back to the description of the actions. We assume that the function $\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})$ is defined and analytic on the space $U_j^c(X, \alpha_0, \alpha_1)$, with some positive, absolute constants α_0, α_1 (i.e., constants independent of X and j). It depends on the configurations restricted to X, i.e. on $(\mathbf{U}, \mathbf{J})|_X$. It is a C^∞-function of $g_{j-1} \in [0, \gamma]$, (or analytic), with a positive, absolute γ. There exists a constant E_0 such that

$$
|\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})| \leqq E_0 \exp(-\kappa d_j(X))
\tag{1.18}
$$

for $M \geqq M(\kappa)$, γ sufficiently small, and for all configurations $(\mathbf{U}, \mathbf{J}) \in U_j^c(X, \alpha_0, \alpha_1)$.

Symmetries are the subject of this point in our discussion of properties of the actions. The most important is gauge invariance. We assume that all functions $\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})$ are gauge invariant with respect to the group of all gauge transformations (1.10). Explicitly

$$
\mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}^u, R(u)\mathbf{J}) = \mathbf{E}^{(j)}(X, g_{j-1}, \mathbf{U}, \mathbf{J})
\tag{1.19}
$$

for all G^c-valued gauge transformations u. The spaces $U_j^c(X, \alpha_0, \alpha_1)$ are, by the definition, gauge invariant also. This assumption is an easily verifiable statement for all explicitly defined terms in the action (1.3). These assumptions imply that the action $A_k(U)$ defined on the space $U_k(\varepsilon_0)$, which is contained in all the spaces $U_j^c(X, \alpha_0, \alpha_1)$, is gauge invariant with respect to all G-valued transformations. Other symmetries are Euclidean lattice transformations. We assume that the action (1.3) is invariant with respect to the transformations of the lattice $T^{(k)}$. More precisely, we notice that the explicitly defined expressions in the j-th term in (1.3) are invariant with respect to the Euclidean transformations of the lattice $T^{(j+1)}$, and we assume that this is true for all expressions in this term.

Now we describe the most important expressions in (1.3), (1.6), the β-functions $\beta_{j+1}(g_j)$. They are determined by the functions $\mathbf{E}^{(j+1)}(g_j, U_{j+1})$ in (1.6). Let us denote $\mathbf{E}^{(j+1)}(g_j, B)=\mathbf{E}^{(j+1)}(g_j, U_{j+1}(\exp iB))$. We define

$$\Pi^{ab}_{j+1,\mu\nu}(g_j,x,x')=\left(\frac{\delta^2}{\delta B^a_\mu(x)\delta B^b_\nu(x')}\mathbf{E}^{(j+1)}\right)(g_j,0). \tag{1.20}$$

This is the vacuum polarization tensor of the theory defined by the j-th fluctuation field integral. We will analyze this tensor thoroughly in Sect. 5; now we describe its basic properties necessary to formulate a definition of the β-functions. The function $\mathbf{E}^{(j+1)}$ is gauge invariant, hence it is invariant with respect to global transformations $V\to R(v)V$, $v\in G$, or $B\to R(v)B$. The function (1.20) can be considered as a function of μ, ν, x, x', with values in the tensor product $\mathbf{g}\otimes\mathbf{g}$, and these values are invariant with respect to the transformations $R(v)\otimes R(v)$, $v\in G$. This is possible only if they are proportional to the identity matrix, i.e. to δ^{ab}. By the Euclidean invariance of $\mathbf{E}^{(j+1)}$ the function (1.20) is Euclidean covariant. This implies

$$\begin{aligned}\Pi^{ab}_{j+1,\mu\nu}(g_j,x,x')&=\delta^{ab}\Pi_{j+1,\mu\nu}(g_j,x-x'),\\ \Pi_{j+1}(g_j,rb,rb')&=\Pi_{j+1}(g_j,b,b'),\end{aligned} \tag{1.21}$$

where $\Pi_{j+1,\mu\nu}(g_j,x)$ is a real valued function, and r is a Euclidean rotation leaving the lattice $T^{(j+1)}$ invariant. Now we take a limit of these functions as $T^{(j+1)}\nearrow Z^d$. This limit exists by the localized representation (1.7). The function $\beta_{j+1}(g_j)$ is defined by

$$\begin{aligned}\beta_{j+1}(g_j)&=-\left(\frac{\partial^2}{\partial p_1\partial p_2}\tilde{\Pi}_{j+1,12}\right)(g_j,0)\\ &=-\left(\frac{\partial^2}{\partial p_\mu\partial p_\nu}\tilde{\Pi}_{j+1,\mu\nu}\right)(g_j,0)=\sum_x \Pi_{j+1,\mu\nu}(g_j,x)x_\mu x_\nu\end{aligned} \tag{1.22}$$

for μ, ν arbitrary, $\mu\neq\nu$, where $\tilde{f}(p)$ denotes the Fourier transform of the function $f(x)$, $x\in Z^d$.

It is a smooth function defined on the interval $[0,\gamma]$, (or analytic), uniformly bounded on this interval together with all derivatives. We will investigate other properties in a separate paper.

This completes the description of the inductive assumptions. Now we can formulate a precise version of Theorem 1.

Theorem 3. *There exist positive constants κ_0, $M(\kappa)$, γ, ε_0, ε_1, α_0, α_1 such, that if $\kappa\geqq\kappa_0$, $M\geqq M(\kappa)$, $0<g_k\leqq\gamma$ for $k=0,1,\dots,K$, then the sequence of actions A_k, defined inductively by the small field renormalization transformations* (0.17)–(0.20), *satisfy all the inductive assumptions described between* (1.1)–(1.22). *The constants ε_0, ε_1, α_0, α_1 depend on M and satisfy numerous restrictions, which will become clear in the proof. The constant γ depends on all other constants.*

In the rest of the paper we will be proving this theorem.

2. Fluctuation Field Integral in $k+1$ Renormalization Transformation

We assume that after k steps we have obtained the action A_k described in the previous section, and we apply the next renormalization transformation $\mathbf{T}_k$ restricted to a small field region by a characteristic function χ_k

$$(\mathbf{T}_k A_k)(W) = \log \mathbf{N}_k^{-1} \int dV \delta(\bar{V} W^{-1}) \times \chi_k \exp\left[-\frac{1}{g_k^2} \mathbf{G}(V) - \frac{1}{g_k^2} A(U_k(V)) + \mathbf{E}_k(U_k(V)) \right]. \tag{2.1}$$

The gauge covariance of the averages implies that the δ-functions in (2.1) are invariant under the gauge transformations $V \to V^v$, $W \to W^v$, because $\delta(U)$ is invariant under the transformations $U \to R(u)U$, $u \in G$. The meaning of the function $\mathbf{E}_k$ is obvious, it is equal to $(1/g_k^2)A + A_k$. We consider the new action on regular configurations W defined on $T^{(k+1)}$. More exactly we assume that W is so regular that the minimal configurations $U_{k+1}(W)$ exist and belong to the space $U_{k+1}(\varepsilon_0)$. By Proposition 2 from the paper [12] this implies that $|W(\partial p') - 1| < 2\varepsilon_0$ for $p' \in T^{(k+1)}$. We will define the characteristic function χ_k in such a way that the domain of integration in (2.1) is restricted to configurations V for which $U_k(V) \in U_k(\varepsilon_0)$. Thus the inductive assumption of the previous section is valid for the action $A_k(U_k(V))$.

We calculate the integral (2.1) applying the saddle point method. At first we look for critical points of the function

$$V \to \mathbf{G}(V) + A(U_k(V)), \qquad V : \bar{V} = W. \tag{2.2}$$

Under the above regularity assumptions there exists the exactly one critical point, which is obtained by taking the critical orbit of the function $A(U_k(V))$ considered on the subspace, and choosing the element of the orbit satisfying the axial gauge conditions $\mathbf{G}(V) = 0$. This critical configuration, which is a minimum of the function (2.2), is denoted by $V^{(k)} = V^{(k)}(W)$, and is related to the minimal configuration $U_{k+1}(W)$ in the axial gauge by the equality

$$V^{(k)} = \bar{U}_{k+1}^k = M^k(U_{k+1}). \tag{2.3}$$

We introduce new integration variables $V' = V(V^{(k)})^{-1}$, or $V = V'V^{(k)}$, and we assume that the characteristic function χ_k and the δ-functions in (2.1) restrict the variables $B' = (1/i)\log V'$ to a sufficiently small neighborhood of 0. A gauge transformation v of V induces the gauge transformation v of $V^{(k)}$ and the transformation $B' \to R(v)B'$. The expressions in (2.1) are invariant with respect to these transformations.

As in Sect. G [15] we write $U_k(V'V^{(k)}) = U'_k(V')U_{k+1}$, and we transform at first $U'_k(V')$ into the axial gauge $Ax_k(T^{(k)}, U_{k+1})$. This does not change the expressions in (2.1) because this gauge transformation is equal to 1 on $T^{(k)}$. Next we apply the gauge transformation u_k constructed in [14] and changing the axial gauge into the Landau gauge. We obtain $U_k'^{u_k^{-1}} = \exp i\eta \mathbf{H}_k(B')$, where the $\mathbf{g}$-valued configuration $\mathbf{H}_k(B')$ is regular and of the same order as B'. Its properties are described in Sect. G [15]. The gauge transformation u_k can be expressed in terms of the configuration $\mathbf{H}_k(B')$ by the formula (106) in [12] with $U_1 = \exp i\eta \mathbf{H}_k$, and it implies that u_k is an analytic function of $\mathbf{H}_k$, hence of B' for $\mathbf{H}_k$ satisfying a boundedness condition of the form $|\mathbf{H}_k| < a_1$, with an absolute, positive constant a_1.

Let us write the expressions under the integral (2.1) in the new variables B'. The expression under the δ-function is equal to

$$M(V'V^{(k)})M(V^{(k)})^{-1} = \exp i\tilde{Q}(B'). \tag{2.4}$$

The gauge fixing term under the exponential in (2.1) is equal to $\mathbf{G}(V'V^{(k)}) = \tilde{\mathbf{G}}(V') = \mathbf{G}(B')$. The variables $\tilde{V}'(y,x) = (V'V^{(k)})\ (y,x)\ (V^{(k)}(y,x))^{-1}$ have an expansion of the form $\tilde{V}'(y,x) = 1 + \tilde{B}'(y,x) + \ldots$, where $\tilde{B}'(y,x)$ is a linear function, hence the gauge fixing expression has the representation

$$G(B') = \sum_{y \in T^{(k+1)}} \sum_{x \in B(y), x \neq y} \tfrac{1}{2}|\tilde{B}'(y,x)|^2 + G_3(B') = \tfrac{1}{2}G^{(2)}(B') + G_3(B'). \tag{2.5}$$

The function $G_3(B')$ is an analytic function of B', with an expansion beginning with third order terms, localized in blocks of the lattice $T^{(k)}$.

Now we analyze an expansion of the action under the exponential in (2.1). Using the gauge invariance of the action, and the formulas (41), (174) [15] we have

$$\begin{aligned} A(U_k(V'V^{(k)}) &= A(\exp i\eta \mathbf{H}_k(B')U_{k+1}) = A(U_{k+1}) \\ &\quad + \langle \mathbf{H}_k(B'), J_{k+1}\rangle + \tfrac{1}{2}\langle \mathbf{H}_k(B'), \Delta\mathbf{H}_k(B')\rangle + V_0(\mathbf{H}_k(B')) \\ &= A(U_{k+1}) + \langle H_{1,k}B' + \mathbf{A}_{1,k}, J_{k+1}\rangle - \langle H_k D_k(H_{1,k}B' + \mathbf{A}_{1,k}), J_{k+1}\rangle \\ &\quad + \tfrac{1}{2}\langle H_{1,k}B' + \mathbf{A}_{1,k}, \Delta(H_{1,k}B' + \mathbf{A}_{1,k})\rangle \\ &\quad - \langle H_{1,k}B' + \mathbf{A}_{1,k}, \Delta H_k D_k(H_{1,k}B' + \mathbf{A}_{1,k})\rangle \\ &\quad + \tfrac{1}{2}\langle H_k D_k(H_{1,k}B' + \mathbf{A}_{1,k}), \Delta H_k D_k(H_{1,k}B' + \mathbf{A}_{1,k})\rangle \\ &\quad + V_0(H_{1,k}B' + \mathbf{A}_{1,k} - H_k D_k(H_{1,k}B' + \mathbf{A}_{1,k})). \end{aligned} \tag{2.6}$$

Let us omit for simplicity the subscript k. By Eq. (171) [15] we have $\langle \mathbf{A}_1, J\rangle = 0$. In the third term on the right hand side above we decompose the function D into the sum $D^{(2)} + D_3$, where $D^{(2)}$ is the second order term equal to $C^{(2)}$, and D_3 is the higher order remainder. The term with $C^{(2)}$ together with the next term on the right hand side of (2.6) yield the expression

$$\tfrac{1}{2}\langle H_1 B' + \mathbf{A}_1, \Delta_1(H_1 B' + \mathbf{A}_1)\rangle = \tfrac{1}{2}\langle H_1 B', \Delta_1 H_1 B'\rangle + \tfrac{1}{2}\langle \mathbf{A}_1, \Delta_1 \mathbf{A}_1\rangle, \tag{2.7}$$

see the definitions (3.127), (3.128) [13]. Denoting terms of at least third order in $H_1 B'$ by $V(H_1 B')$ we get

$$A(U_k(V'V^{(k)})) = A(U_{k+1}) + \langle H_1 B', J\rangle + \tfrac{1}{2}\langle H_1 B', \Delta_1 H_1 B'\rangle + V(H_1 B'). \tag{2.8}$$

Finally we can define the characteristic function χ_k

$$\chi_k = \prod_{b \in T^{(k)} \setminus \{b_0(c):\, c \in T^{(k+1)}\}} \chi(\{|B'(b)| < \varepsilon_1\}). \tag{2.9}$$

Another possibility is to take $g_k/\gamma_k \varepsilon_1$ instead of ε_1, where $\gamma_k = C\log(L^k\varepsilon)^{-1}$ with C sufficiently large. It has the advantage that the functions $\mathbf{E}^{(j)}$, β_j are analytic functions of the effective coupling constants, but it has some disadvantages in perturbative calculations also. We have formulated the implications of both possibilities in the inductive description. The above restrictions imply also restrictions on $B'(b_0(c))$, with the constant ε_1 replaced by $O(\varepsilon_1)$, because these variables can be expressed in terms of the remaining ones as in the first step. This we will discuss in detail.

Let us write the integral we obtain from (2.1) by the above described transformation

$$(2.1)=\log \mathbf{N}_k'^{-1}\int dB'\sigma(B')\delta(\tilde{Q}(B'))$$
$$\times \chi_k \exp\left[-\frac{1}{g_k^2}\mathbf{G}(B')-\frac{1}{g_k^2}A(U_{k+1})-\frac{1}{g_k^2}\langle H_1B',J\rangle\right.$$
$$\left.-\frac{1}{2g_k^2}\langle H_1B',\Delta_1H_1B'\rangle-\frac{1}{g_k^2}V(H_1B')+\mathbf{E}_k(U_k(\exp iB'V^{(k)}))\right], \quad (2.10)$$

where the factors σ_0 are included into the normalization factor $\mathbf{N}_k'$. In this integral we make a change of variables linearizing the function $\tilde{Q}(B')$. This operation was discussed several times in the previous papers, e.g. see Sect. C [15], Sect. E [14]. Here we have a particularly simple unit lattice situation.

At first we introduce an operator h. It transforms $\mathbf{g}$-valued functions B defined at bonds of the lattice $T^{(k+1)}$ into such functions defined at bonds of $T^{(k)}$. The function hB is equal to 0 everywhere, except the set $\{b_0(c):c\in T^{(k+1)}\}$. [Let us recall that for $c\in T^{(k+1)}$ the bond $b_0(c)$ is defined as the bond of the lattice $T^{(k)}$ contained in c and belonging to the corridor $B(c)=\{b\in T^{(k)}:b_-\in B(c_-),\ b_+\in B(c_+)\}$.] Furthermore, the operator h satisfies the identity $L\tilde{Q}h=I$ on $T^{(k+1)}$. Of course h is uniquely defined by these conditions, in fact it is a very simple operator given by the equality $(hB)(b_0(c))=h(c)B(c)$, where $h(c)$ is a linear operator on the Lie algebra $\mathbf{g}$, equal to an inverse of a coefficient at the variable $B'(b_0(c))$ in $(\tilde{Q}B')(c)$, multiplied by L^{-1}. We are looking for an analytic, $\mathbf{g}$-valued function $\tilde{D}(B')$, defined at bonds of $T^{(k+1)}$, and such that the transformation $B'=B-h\tilde{D}(B)$ linearizes the function $\tilde{Q}(B')$. The function $\tilde{D}(B)$ is determined by the equation

$$L\tilde{Q}B'+\tilde{C}(B')=L\tilde{Q}B-\tilde{D}(B)+\tilde{C}(B-h\tilde{D}(B))=L\tilde{Q}B\,.$$

It is easy to prove, following the proofs in the above mentioned papers, that there exists exactly one solution of this equation, and that it is an analytic function of B. From this equation we obtain also that $\tilde{D}(B)$ has an expansion beginning with quadratic terms, and $\tilde{D}^{(2)}(B)=\tilde{C}^{(2)}(B)$. The above change of variables yields the integral with the δ-function $\delta(\tilde{Q}B)$. Next we make the scaling transformation $B=g_kB'$. In the expression under the exponential the third, linear term in (2.10) vanishes now, $-(1/g_k)\langle H_1B',J\rangle=0$ because $\langle\delta A',J\rangle=0$ for all $\delta A'$ satisfying the condition $\tilde{Q}Q_k\delta A'=0$, and H_1B' satisfies it. Hence the only term with a negative power of g_k is the action evaluated at the configuration U_{k+1}. Terms of the order 0 in g_k are

$$\langle H_1h\tilde{C}^{(2)}(B'),J\rangle-\tfrac{1}{2}\mathbf{G}^{(2)}(B')-\tfrac{1}{2}\langle H_1B',\Delta_1H_1B'\rangle\,. \quad (2.11)$$

The quadratic form in B' above is equal to $-1/2\langle B',\Delta^{(k)}B'\rangle$, see the definition (3.156) [13] with the δ-function gauge fixing term replaced by the exponential one. This quadratic form defines the k-th normalization factor $Z^{(k)}(U_{k+1})$ given by the formula (1.4) with $j=k$. The quadratic form defines also a Gaussian measure. We perform the same operation as in the first step, namely using the δ-functions.

$\delta(\tilde{Q}B')$ we eliminate the variables $B'(b_0(c))$, $c\in T^{(k+1)}$. Denoting the remaining variables by B we have $B'=CB$, C is the operator determined by the configuration $V^{(k)}$, and the measure becomes a Gaussian measure in variables B, with the covariance $C^{(k)}=C^{(k)}(U_{k+1})=(C^*\Delta^{(k)}C)^{-1}$.

After these transformations we obtain the following expression for the new action:

$$\begin{aligned}A_{k+1}(U_{k+1})= &-\frac{1}{g_k^2}A(U_{k+1})+\mathbf{E}_k(U_{k+1})+[\log Z^{(k)}(U_{k+1})-\log Z^{(k)}(1)]\\ &+\log \mathbf{N}_k''^{-1}\int d\mu_{C^{(k)}}(B)\chi_k \exp\Bigg[\operatorname{Tr}\log\left(I-h\left(\frac{\delta}{\delta B}\tilde{D}\right)(g_kCB)\right)\\ &+\log\sigma(g_kCB-h\tilde{D}(g_kCB))+\frac{1}{g_k^2}\langle H_1h\tilde{D}_3(g_kCB),J\rangle-\frac{1}{g_k^2}\mathbf{G}_3(g_kB)\\ &+\frac{1}{g_k^2}\langle H_1g_kCB,\Delta_1H_1h\tilde{D}(g_kCB)\rangle\\ &-\frac{1}{g_k^2}\langle H_1h\tilde{D}(g_kCB),\Delta_1H_1h\tilde{D}(g_kCB)\rangle\\ &-\frac{1}{g_k^2}V(H_1(g_kCB-h\tilde{D}(g_kCB)))\\ &+\{\mathbf{E}_k(U_k(\exp i[g_kCB-h\tilde{D}(g_kCB)]V^{(k)}))-\mathbf{E}_k(U_k(V^{(k)}))\}\Bigg]. \end{aligned}\tag{2.12}$$

Let us notice that the normalization constant $\mathbf{N}_k''$ is equal to the integral above at $U_{k+1}=1$. The expression under the exponential is clearly a sum of two terms, one is connected with the expansion of the action $-(1/g_k^2)A(U_k(V))$ and the measure in (2.1), and we denote it by $\mathbf{P}^{(k)}(g_k,U_{k+1},B)$, another is the expression in the curly bracket $\{\ldots\}$. The integral in (2.12) defines the new term $\mathbf{E}^{(k+1)}$ in the inductive definition of the action A_{k+1} by the formula

$$\mathbf{E}^{(k+1)}(g_k,U_{k+1})=\log\int d\mu_{C^{(k)}}(B)\chi_k\exp[\mathbf{P}^{(k)}(g_k,U_{k+1},B)+\{\ldots\}].\tag{2.13}$$

Let us remark that the expression under the exponential above vanishes at $g_k=0$, and

$$\log\mathbf{N}_k''=\mathbf{E}^{(k+1)}(g_k,1).\tag{2.14}$$

Finally we perform the coupling constant renormalization

$$\frac{1}{g_k^2}=\frac{1}{g_{k+1}^2}+\beta_{k+1}(g_k)\tag{2.15}$$

with the β-function defined by the formulas (1.20), (1.22) for $j=k$. The equalities (2.12), (2.14) together with the definitions (2.13), (2.15) imply that the action A_{k+1} is given by (1.3) with $k+1$ instead of k. By the inductive assumption, and by the properties of the expressions given by explicit formulas, all terms in this representation satisfy the required properties, except possibly the last term (2.13) in the sum.

Let us discuss briefly the gauge invariance and the Euclidean invariance of this term. The gauge invariance was discussed already several times in the previous papers, so let us recall only that all the expressions in (2.12), together with the measure, are invariant with respect to the gauge transformations

$$U_{k+1} \to U^u_{k+1}, \qquad B' \to R(u)B', \qquad (R(u)B')(b) = R(u(b_-))B'(b). \tag{2.16}$$

The characteristic function is invariant with respect to the transformations of the fluctuation field B', because they are local, orthogonal transformations; therefore the expression (2.13) is gauge invariant. Now consider a Euclidean symmetry r of the torus T, preserving the torus $T^{(k+1)}$. We define generally

$$(rU)(b) = U(rb), \qquad rb = r\langle b_-, b_+\rangle = \langle rb_-, rb_+\rangle. \tag{2.17}$$

By their definitions the expressions in (2.1) are invariant with respect to these transformations. If we split the field $V = V'V^{(k)}$, and $V^{(k)}$, U_{k+1} transform as above, then the expressions are still invariant assuming that V' transforms as follows:

$$\begin{aligned} (rV')(b) &= V'(rb) \quad \text{if } rb \text{ is positively oriented,} \\ (rV')(b) &= R(V^{(k)}(rb))V'^{-1}(-(rb)) \quad \text{if } rb \text{ is negatively oriented.} \end{aligned} \tag{2.18}$$

In these formulas the bonds b are positively oriented. Let us recall that the representation $V(b) = V'(b)V^{(k)}(b)$ holds for such bonds only. The above definition secures the identity $r(V'V^{(k)}) = (rV')(rV^{(k)})$, hence the invariance also. The transformation (2.18) generates an orthogonal transformation of the fluctuation field B', hence all the remaining operations preserve the invariance for the same reasons as for the gauge invariance. Thus the effective action is invariant with respect to the Euclidean transformations (2.17) of the background field.

Thus the proof of Theorem 3 is reduced to proving the remaining properties of (2.13), i.e. to a construction of the representation (1.7) with terms having the analytic extensions satisfying the bound (1.18). We will do it in several steps, and the rest of the paper is almost completely devoted to these problems.

3. An Expansion of Terms in Fluctuation Field Integral, and Preliminary Analytic Extension

The underintegral expression in (2.12), (2.13) is a sum of two expressions. One is the function $\mathbf{P}^{(k)}(g_k, U_{k+1}, B)$, which is given by the sum of all terms under the exponential in (2.12), except the terms in the curly bracket. This function, although it has a complicated structure, is simple to understand. It is an analytic function of the fluctuation field B defined on the unit lattice $T_1^{(k)}$. It is bounded by a second order polynomial in B, with coefficients of the order $O(\varepsilon_1)$, hence it can be treated as a small perturbation of the basic quadratic form in the Gaussian measure $d\mu_{C^{(k)}}$. As a function of the gauge field configuration U_{k+1} it has a straightforwardly defined extension $\mathbf{P}^{(k)}(g_k, \mathbf{U}, \mathbf{J}, B)$ such, that

$$\mathbf{P}^{(k)}(g_k, U_{k+1}, B) = \mathbf{P}^{(k)}(g_k, U_{k+1}, J_{k+1}, B). \tag{3.1}$$

This extension is obtained by replacing the function J_{k+1} in all propagators and operators by the variable $\mathbf{J}$, and by replacing the configuration U_{k+1} in all other places by the variable $\mathbf{U}$. The function $\mathbf{P}^{(k)}(g_k, \mathbf{U}, \mathbf{J}, B)$ is defined and analytic on the space of all configurations $\mathbf{U}$, $\mathbf{J}$ satisfying the conditions (i)–(iii) in Sect. 1, for some α_0', α_1' sufficiently small, but much bigger than α_0, α_1 there. The only problem is to localize this function, i.e., to construct the representation (1.7) with terms satisfying bounds of the type (1.18), modified by factors connected with the fluctuation field B. We will analyze this problem in one of the next sections in a general case.

The problem we consider in this section is connected with terms in the curly bracket in (2.12). We want to represent them as a sum of localized, irrelevant terms, i.e. terms satisfying the bound (0.28). We make a first step to get the desired representation, we expand terms in the curly bracket, and we construct preliminary analytic extensions of terms in the expansions.

Let us take the partition π_k. We construct a cover of the space T by cubes $\square$, which are unions of 2^d neighbouring cubes from π_k. For this cover we take a partition of unity $1 = \sum_{\square} \zeta_\square$ with smooth functions $\zeta_\square$. More exactly we assume that if y is a center of the cube $\square$, then $\zeta_\square(x) = \prod_{\mu=1}^{d} \zeta(M^{-1}(x_\mu - y_\mu))$, where $\zeta \in C_0^\infty(R^1)$, $\zeta(t) = 1$ for $|t| \leqq 1/3$, $\zeta(t) = 0$ for $|t| \geqq 2/3$, ζ has derivatives up to the second order bounded by 5. Let us recall that we consider the continuous space T and all the cubes in the scale corresponding to the lattice T_η.

Let us denote for simplicity the expression under the exponential in $\{\dots\}$ by B', i.e.,

$$B' = g_k CB - h\tilde{D}(g_k CB). \tag{3.2}$$

The first expansion we write below depends on the index j in the sum (1.7). We write it for the function $\mathbf{E}^{(j)}$. Making use of the gauge invariance we have

$$\begin{aligned}
&\mathbf{E}^{(j)}(U_k(\exp iB'V^{(k)})) - \mathbf{E}^{(j)}(U_k(V^{(k)})) \\
&= \mathbf{E}^{(j)}(U_j(\bar{U}_k^j(\exp iB'V^{(k)}))) - \mathbf{E}^{(j)}(U_j(\bar{U}_k^j(V^{(k)}))) \\
&= \mathbf{E}^{(j)}(U_j(M^j(\exp i\eta \mathbf{H}_k(B')U_{k+1}))) - \mathbf{E}^{(j)}(U_j(M^j(U_{k+1}))) \\
&= \mathbf{E}^{(j)}(U_j(\exp iQ_j(\eta \mathbf{H}_k(B'))\bar{U}_{k+1}^j)) - \mathbf{E}^{(j)}(U_j(\bar{U}_{k+1}^j)),
\end{aligned} \tag{3.3}$$

where we have used the results of Sect. G [15], and the equality (97) [12]. The function $\mathbf{H}_k$ and the operation Q_j are dependent on the configuration U_{k+1}. Next we have

$$\begin{aligned}
&\mathbf{E}^{(j)}(U_k(\exp iB'V^{(k)})) - \mathbf{E}^{(j)}(U_k(V^{(k)})) \\
&= \int_0^1 dt \frac{d}{dt} \mathbf{E}^{(j)}(U_j(\exp iQ_j(\eta t \mathbf{H}_k(B'))\bar{U}_{k+1}^j)) \\
&= \int_0^1 dt \left\langle \left(\frac{\delta}{\delta \mathbf{H}} \mathbf{E}^{(j)} \right) (U_j(\exp iQ_j(\eta t \mathbf{H}_k(B'))\bar{U}_{k+1}^j)), \mathbf{H}_k(B') \right\rangle \\
&= \sum_{\square} \int_0^1 dt \left\langle \left(\frac{\delta}{\delta \mathbf{H}} \mathbf{E}^{(j)} \right) (U_j(\exp iQ_j(\eta t \mathbf{H}_k(B'))\bar{U}_{k+1}^j)), \zeta_\square \mathbf{H}_k(B') \right\rangle \\
&= \sum_{\square} \int_0^1 dt \frac{d}{dt_\square} \mathbf{E}^{(j)}(U_j(\exp iQ_j(\eta(t + t_\square \zeta_\square)\mathbf{H}_k(B'))\bar{U}_{k+1}^j))|_{t_\square = 0}.
\end{aligned} \tag{3.4}$$

This yields a first localization of the expression $\{\ldots\}$. We consider now one term in the above sum, the term corresponding to a cube $\square$. Decomposing $\mathbf{E}^{(j)}$ according to (1.7), we obtain a sum of terms labeled by $X \in \mathbf{D}_j$. We divide this sum into two subsums, assigning a term to one of them according to a position of its localization domain X with respect to $\square$. The division is according to the conditions

$$X \cap (\tilde{\square}^2)^c \neq \emptyset, \quad \text{or} \quad X \subset \tilde{\square}^2. \tag{3.5}$$

Consider a domain X satisfying the first condition above. Assume that $X \in \mathbf{D}_j$. There are two cases possible, either $X \cap \tilde{\square} = \emptyset$, or $X \cap \tilde{\square} \neq \emptyset$. In the first case $\operatorname{dist}(X, \square) \geqq M$, the distance is in the scale η, hence in the scale ξ $\operatorname{dist}^{(\xi)}(X, \square) \geqq M(L^j\eta)^{-1}$. In the second case X is a big domain in ξ-scale, for example $d_j(X) \geqq (L^j\eta)^{-1}$. Thus either the exponential decay of propagators corresponding to ξ-scale, or the exponential bound (1.18) should give a small factor $O((L^j\eta)^N)$ with an arbitrary power N, enough to control the sums. Terms with domains X satisfying the first condition in (3.5) are simple to deal with, we have to write them only in a form in which the exponential factors can be clearly seen. Making use of the gauge invariance, and denoting

$$Q_j(\eta(t + t_\square \zeta_\square)\mathbf{H}_k(B')) = B_\square, \qquad Q_j(\eta t \mathbf{H}_k(B')) = B(t),$$

we have

$$\mathbf{E}^{(j)}(X, U_j(\exp iB_\square \bar{U}^j_{k+1})) = \mathbf{E}^{(j)}(X, \exp i\xi \mathbf{H}_j(B_\square) U_{k+1}). \tag{3.6}$$

The function $\mathbf{H}_j$ depends on the configuration U_{k+1}. The expression on the right-hand side is differential with respect to $t_\square$, at $t_\square = 0$, and this yields

$$\begin{aligned}
&\left\langle \left(\frac{\delta}{\delta \mathbf{A}} \mathbf{E}^{(j)}\right)(X, \exp i\xi \mathbf{H}_j(B(t))U_{k+1}), \left\langle \left(\frac{\delta}{\delta B} \mathbf{H}_j\right)(B(t)), \frac{d}{dt_\square} B_\square|_{t_\square=0} \right\rangle \right\rangle \\
&= \frac{d}{dt_\square} \mathbf{E}^{(j)}\left(X, \exp i\xi \left[\mathbf{H}_j(B(t)) + t_\square \left\langle \left(\frac{\delta}{\delta B} \mathbf{H}_j\right)(B(t)), \right.\right.\right. \\
&\quad \times \left.\left.\left. \left\langle \left(\frac{\delta}{\delta \mathbf{A}} Q_j\right)(\eta t \mathbf{H}_k(B')), L^j \eta \zeta_\square \mathbf{H}_k(B') \right\rangle \right\rangle \right] U_{k+1} \right)|_{t_\square = 0}\,.
\end{aligned} \tag{3.7}$$

The derivative $\dfrac{\delta}{\delta B}\mathbf{H}_j$ is an exponentially decaying function, with the decay rate δ_0 on the ξ-scale, see Proposition 9, (190) [15]. Let us denote

$$\delta \mathbf{H}_j = \left\langle \left(\frac{\delta}{\delta B} \mathbf{H}_j\right)(B(t)), \left\langle \left(\frac{\delta}{\delta A} Q_j\right)(\eta t \mathbf{H}_k(B')), L^j \eta \zeta_\square \mathbf{H}_k(B') \right\rangle \right\rangle. \tag{3.8}$$

In the case when $\operatorname{dist}^{(\xi)}(X, \square) \geqq M(L^j\eta)^{-1}$, this function satisfies the bound

$$\begin{aligned}
|\delta \mathbf{H}_j| \leqq\, & B_3 \exp(-\tfrac{1}{2}\delta_0 \operatorname{dist}^{(\xi)}(X, \square) - \tfrac{1}{2}\delta_0 M(L^j\eta)^{-1}) \\
& \times 2L^j\eta |\zeta_\square \mathbf{H}_k(B')| \quad \text{on} \quad X,
\end{aligned} \tag{3.9}$$

and the same for derivatives and the second order operators applied to it. Using the above bound, and the analyticity properties of $\mathbf{E}^{(j)}$, we can obtain easily the required bound for (3.7). In the second, simpler case, we obtain the exponential factor $\exp(-\delta\kappa(L^j\eta)^{-1})$ directly from the bound (1.18), here δ is a small, positive number. The remaining factor $\exp(-(1-\delta)\kappa d_j(X))$ from the bound (1.18) will be used in a proof of the inductive assumptions for new terms. Thus in both cases we have irrelevant terms.

To fulfill the conditions of the inductive assumption we have to construct an analytic extension of the expression (3.7). Let us consider more generally the function $\mathbf{E}^{(j)}(X, \exp i\xi\mathbf{A}U_{k+1})$. It is obtained from the analytic function $\mathbf{E}^{(j)}(X, \mathbf{U}', \mathbf{J}')$ by the substitution

$$\mathbf{U}' = \exp i\xi\mathbf{A}U_{k+1}, \qquad \mathbf{J}' = D^{\xi *}_{\exp i\xi\mathbf{A}U_{k+1}}\xi^{-2}\pi\,\mathrm{Im}\,\partial\exp i\xi\mathbf{A}U_{k+1}. \tag{3.10}$$

We expand the expression on the right-hand side above with respect to $\mathbf{A}$ using the formulas (1.43)–(1.54) [14]. We obtain more generally, for any regular configuration U

$$D^{\xi *}_{\exp i\xi\mathbf{A}U}\xi^{-2}\pi\,\mathrm{Im}\,\partial\exp i\xi\mathbf{A}U = D^{\xi *}_{D}\xi^{-2}\pi\,\mathrm{Im}\,\partial U + D^{\xi *}_{U}D^{\xi}_{U}\mathbf{A} + \mathbf{F}(U, \mathbf{A}), \tag{3.11}$$

where $\mathbf{F}$ is a local operator depending on U, ∂U, $\mathbf{A}$, $\nabla^{\xi}_{U}\mathbf{A}$ only. It is the term $D^{\xi *}_{U}D^{\xi}_{U}\mathbf{A}$ in this formula which is a source of a trouble. For $\mathbf{A} = \mathbf{H}_j$ we have bounds for this second order operator, but only if we use the second representation for $\mathbf{H}_j$ connected with the Eq. (180) [15]. We use the fact that $\mathbf{H}_j$ satisfies the Landau gauge condition $RD^*\mathbf{H}_j = 0$, and we replace the operator D^*D by $D^*D + DRD^* = D^*D + DD^* - DPD^* = \Delta^{\xi}_{U} - P_1 +$ (lower order, local operator). In the next steps we will construct an expansion of $\mathbf{H}_j$, and for terms in this expansion we will have good bounds, including bounds for the covariant Laplace operator. Such bounds do not hold for the operator D^*D, and this is the reason why we do not formulate second order regularity conditions, as in (1.9) [14], (2) [15], but we replace the expression J_j by the new variable $\mathbf{J}$. Consider now the formula (3.11) for $U = U_{k+1}$. We have

$$D^{\xi *}_{U_{k+1}}\xi^{-2}\pi\,\mathrm{Im}\,\partial U_{k+1} = (L^{j-1}\eta)^3 J_{k+1}, \tag{3.12}$$

and we replace J_{k+1} by the variable $\mathbf{J}$, and in the remaining expressions we replace U_{k+1} by the new variable $\mathbf{U}$. Thus we obtain the following function of the variables $\mathbf{U}$, $\mathbf{J}$

$$\mathbf{E}^{(j)}(X, \exp i\xi\mathbf{A}\mathbf{U}, (L^{j-1}\eta)^3\mathbf{J} + \Delta^{\xi}_{\mathbf{U}}\mathbf{A} - P_1\mathbf{A} + \mathbf{F}_1(\mathbf{U}, \mathbf{A}))\,. \tag{3.13}$$

We consider it on the space $U^c_j(X, 1/2\alpha_0, 1/2\alpha_1, \alpha_0)$ (notice the different constant in the bound for $\mathbf{J}$). It is an analytic function on this space, and also an analytic function of $\mathbf{A}$, for $\mathbf{A}$, $\nabla^{\xi}_{\mathbf{U}}\mathbf{A}$, $\Delta^{\xi}_{\mathbf{U}}\mathbf{A}$ sufficiently small. These restrictions can be easily obtained from the definition of the spaces, and from the form of the expressions in (3.13). Thus, there exists a constant α_2, depending on α_0 and on some absolute constants, such that the function (3.13) is analytic in $\mathbf{A}$, for $\mathbf{A}$ satisfying the conditions

$$|\mathbf{A}|, \quad |P_1\mathbf{A}|, \quad |\nabla^{\xi}_{\mathbf{U}}\mathbf{A}|, \quad |\Delta^{\xi}_{\mathbf{U}}\mathbf{A}| < \alpha_2 \quad \text{on} \quad X\,. \tag{3.14}$$

Of course, on these spaces of configurations $\mathbf{U}$, $\mathbf{J}$, $\mathbf{A}$ the function (3.13) satisfies the inequality (1.18).

To get an analytic extension of (3.7) we substitute $\mathbf{A}=\mathbf{H}_j(B(t))+t_\square\langle\ldots\rangle$, with functions $\mathbf{H}_j$, $\mathbf{H}_k$ depending on $\mathbf{U}$, $\mathbf{J}$. These functions are given by (179), (180) [15], and we replace the configurations U_{k+1}, J_{k+1} occurring in operators there by the configurations $\mathbf{U}$, $\mathbf{J}$. The derivative with respect to $t_\square$, at $t_\square=0$, can be written as the Cauchy integral

$$\frac{1}{2\pi i}\int_{|t_\square|=r} dt_\square \frac{1}{t_\square^2}\mathbf{E}^{(j)}(X,\ldots,\ldots), \tag{3.15}$$

with the radius r given by the equality

$$r\max\{|\delta\mathbf{H}_j|_X, |P_1\delta\mathbf{H}_j|_X, |\nabla_{\mathbf{U}}^{\xi}\delta\mathbf{H}_j|_X, |\Delta_{\mathbf{U}}^{\xi}\delta\mathbf{H}_j|_X\}=\tfrac{1}{3}\alpha_2.$$

We assume also that ε_1 is so small that $\mathbf{H}_j(B(t))$ satisfies (3.14) with $1/3\alpha_2$ on the right-hand side. Thus we have constructed the analytic extension (3.15) of the function (3.7), defined on the space

$$\begin{aligned}&U_j^c(X,\tfrac{1}{2}\alpha_0,\tfrac{1}{2}\alpha_1,\alpha_0)\cap\{(\mathbf{U},\mathbf{J})\colon \mathbf{U},\mathbf{J}\quad \text{satisfy the conditions}\\ &\text{(i)–(iii) for } j=k+1, \text{ and with the constants } (1+\beta)\alpha_0,\ (1+\beta)\alpha_1,\alpha_0\}.\end{aligned} \tag{3.16}$$

This function has the bound

$$\begin{aligned}|(3.15)|&\leqq\frac{1}{r}E_0\exp(-\kappa d_j(X))\leqq E_0\exp(-\kappa d_j(X))6\alpha_2^{-1}L^j\eta B_0B_3\\ &\quad\times\exp(-\tfrac{1}{2}\delta_0\,\mathrm{dist}^{(\xi)}(X,\square)-\tfrac{1}{2}\delta_0M(L^j\eta)^{-1})\,|\zeta_\square\mathbf{H}_k(B')|,\end{aligned} \tag{3.17}$$

following from (3.9). Let us stress that it follows only from the fact that $\mathbf{H}_j(B(t))$ satisfies (3.14) with $1/3\alpha_2$, and $\delta\mathbf{H}_j$ satisfies (3.9). Later we will change these functions by a localization procedure, but new localized functions will satisfy the bounds (3.14), (3.9), hence (3.17) will be preserved also.

We remark that the above considerations apply almost without a change to the second case, with a big domain X. The only difference is that we do not have the exponential factors in (3.9), (3.17) connected with the decay rate of the functional derivative $\frac{\delta}{\delta B}\mathbf{H}_j$.

Now we consider the fundamental case, the case when the second condition in (3.5) is satisfied, i.e. when $X\subset\tilde{\square}^2$. This case includes all localization domains of small sizes, and with small distances to $\square$, hence without obvious small factors. The real renormalization problem is connected exactly with this case, and we have to analyze it carefully. To do the analysis we transform the terms in (3.4). We take the function U_j constructed for the cube $\square_0=\tilde{\square}^5$ as in the condition (iv) in the definition of spaces U_j^c. Thus we construct a sequence of cubes $\{\square_n\}$, $n=0,1,\ldots,k+1$, satisfying the conditions (1.3)–(1.6) [14] (with $k+1$, $L^{-1}\eta$ instead of k, η, $R=R_1$), hence $\tilde{\square}^4\subset\square_{k+1}$. For this sequence we construct the functions U_j, $j=1,2,\ldots,k+1$, which we denote by $U_j(\square_0)$. The configuration in terms of (3.4) with localization domains X satisfying $X\subset\tilde{\square}^2$ can be written as

$$U_j(\exp iB_\square\bar{U}_{k+1}^j)=U_j(\square_0,M^{\cdot}(U_j(\exp iB_\square\bar{U}_{k+1}^j))). \tag{3.18}$$

Using the gauge invariance, and similar transformations as in (3.3), we obtain

$$\begin{aligned}&\mathbf{E}^{(j)}(X, U_j(\exp iB_{\square}\bar{U}^j_{k+1}))\\ &\quad = \mathbf{E}^{(j)}(X, U_j(\square_0, \exp iQ(\xi\mathbf{H}_j(B_{\square}))M^{\cdot}(U_{k+1}))) .\end{aligned} \tag{3.19}$$

For the expression under the exponential function above we have

$$Q(\xi\mathbf{H}_j(B_{\square})) = Q_j(\xi\mathbf{H}_j(B_{\square})) = B_{\square} \text{ on a neighborhood of } \tilde{\square}^4 . \tag{3.20}$$

Later it will be important to have the configuration $B_{\square}$ only, instead of the expression on the left hand side, so we remove from (3.19) a part of this expression localized outside $\tilde{\square}^3$. Take a function $\tilde{\zeta}_{\square} \in C_0^{\infty}(\square_0)$, $\tilde{\zeta}_{\square} = 1$ on $\tilde{\square}^3$, $\tilde{\zeta}_{\square} = 0$ outside $\tilde{\square}^4$. We have

$$\begin{aligned}&\mathbf{E}^{(j)}(X, U_j(\square_0, \exp iQ(\xi\mathbf{H}_j(B_{\square}))M^{\cdot}(U_{k+1})))\\ &= \mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tilde{\zeta}_{\square}B_{\square}M^{\cdot}(U_{k+1})))\\ &\quad + \int_0^1 d\tilde{t}\frac{d}{d\tilde{t}}\mathbf{E}^{(j)}(X, \exp i\xi\mathbf{H}_j(\square_0, (\tilde{t}(1-\tilde{\zeta}_{\square}) + \tilde{\zeta}_{\square})Q(\xi\mathbf{H}_j(B_{\square})))U_{k+1})\\ &= \mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tilde{\zeta}_{\square}B_{\square}M^{\cdot}(U_{k+1})))\\ &\quad + \int_0^1 d\tilde{t}\frac{d}{d\tilde{t}_{\square}}\mathbf{E}^{(j)}\Bigg(X, \exp i\xi\Bigg[\mathbf{H}_j(\square_0, (\tilde{t}(1-\tilde{\zeta}_{\square}) + \tilde{\zeta}_{\square})Q(\xi\mathbf{H}_j(B_{\square})))\\ &\quad + \tilde{t}_{\square}\left\langle\left(\frac{\delta}{\delta B}\mathbf{H}_j\right)(\square_0, (\tilde{t}(1-\tilde{\zeta}_{\square}) + \tilde{\zeta}_{\square})Q(\xi\mathbf{H}_j(B_{\square}))),\right.\\ &\quad \left.\times(1-\tilde{\zeta}_{\square})Q(\xi\mathbf{H}_j(B_{\square}))\right\rangle\Bigg]U_{k+1}\Bigg)\Bigg|_{\tilde{t}_{\square}=0} .\end{aligned} \tag{3.21}$$

Again, from the exponential decay of the derivative $(\delta/\delta B)\mathbf{H}_j$, and from the condition $\mathrm{dist}^{(\xi)}(X, \mathrm{supp}(1-\tilde{\zeta}_{\square})) \geq M(L^j\eta)^{-1}$, we obtain a bound of the type (3.9) for the expression $\langle\ldots\rangle$ under the exponential. Thus a bound for the second term above has the small factor $O((L^j\eta)^N)$, and we treat this term in exactly the same way as the terms (3.7) before, i.e. constructing the analytic extensions and proving the bounds (3.17).

To simplify some formulas in the future we change the first term on the right-hand side of the last equality in (3.21). Taking into account the definition of $B_{\square}$, we write

$$\begin{aligned}\tilde{\zeta}_{\square}B_{\square} &= [\tilde{\zeta}_{\square}Q_j(\eta(t+t_{\square}\zeta_{\square})\mathbf{H}_k(B')) - Q_j(\eta(t\tilde{\zeta}_{\square}+t_{\square}\zeta_{\square})\mathbf{H}_k(B'))] + \tilde{B}_{\square} ,\\ \tilde{B}_{\square} &= Q_j(\eta(t\tilde{\zeta}_{\square}+t_{\square}\zeta_{\square})\mathbf{H}_k(B')) .\end{aligned} \tag{3.22}$$

We remove the expression in the square brackets [...] repeating the procedure in (3.21). The expression is localized in $\tilde{\square}^4\backslash\tilde{\square}^3$, and the term corresponding to the second term in (3.21) has a very similar form and the same properties, so we apply the same considerations as before.

Thus we consider the expression

$$\begin{aligned}&\mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tilde{B}_{\square}M^{\cdot}(U_{k+1})))\\ &\quad = \mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tilde{B}_{\square}M^{\cdot}(U_{k+1}(\square_0, M^{\cdot}(U_{k+1}))))) .\end{aligned} \tag{3.23}$$

This equality is connected with the way we introduce the variables $\mathbf{U}$, $\mathbf{J}$, namely in the interior averages $M^{\cdot}(U_{k+1})$ we replace the configuration U_{k+1} by $\mathbf{U}$. The function $\mathbf{H}_k$ and the averaging operations Q_j in the definition of $\tilde{B}_\square$ depend on U_{k+1}. Now we introduce the variables $\mathbf{U}$, $\mathbf{J}$ in these expressions in a slightly different way. The variable $\mathbf{J}$ replaces the function J_{k+1} as usual, but the configuration U_{k+1} is replaced by the configuration $U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))$, extended as equal to $\mathbf{U}$ outside $\square_0$. We do it in order to get better properties with respect to gauge transformations of this configuration, as it will become clear in the future. Let us remark that $U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))$ has approximately the same regularity properties as $\mathbf{U}$, hence all necessary theorems are valid for operations with this configuration. Thus we consider the function

$$\mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tilde{B}_\square M^{\cdot}(U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))))) . \tag{3.24}$$

In fact we should replace the function $\mathbf{E}^{(j)}(X, U_j)$ by $\mathbf{E}^{(j)}(X, U_j, J_j)$ determined by the analytic function $\mathbf{E}^{(j)}(X, \mathbf{U}, \mathbf{J})$, as in (3.13), but this would make the formulas even much more complicated, so we will keep the above notation. Taking into account the definition (3.22) of $\tilde{B}_\square$ we can write the expression (3.24) as

$$\mathbf{E}^{(j)}(X, U_j(\square_0, M^{\cdot}(\exp i\eta(t\tilde{\zeta}_\square + t_\square\zeta_\square)\mathbf{H}_k(B')U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))))) . \tag{3.25}$$

This is obviously a well defined and analytic function of the variables $\mathbf{U}$, $\mathbf{J}$ in a sufficiently small domain. We will prove that it is analytic on the space $U_{k+1}'^{c}(\square_0, (1+2\beta)\alpha_0, (1+2\beta)\alpha_1, \alpha_0)$. This space is defined by the same conditions (i)–(iv), only the configurations $\mathbf{U}$, $\mathbf{J}$ are defined and satisfy (i)–(iii) on the whole lattice T_η. Let us assume this and let us transform further the above function. We repeat the constructions of Sect. F [15], and we introduce the same generalized axial gauge for the configuration $M^{\cdot}(\mathbf{U})$. This is achieved by a gauge transformation v, and we have

$$M^{\cdot}(\mathbf{U}) = V^v , \qquad |V(b)-1| < O(1)M\alpha_0 , \qquad |v-1| < O(1)M\alpha_1 . \tag{3.26}$$

The variables V can be expressed simply in terms of the contour variables introduced in (77) [12], e.g. in the simplest situation for a bond $b \in \tilde{\square}^4 \cap T^{(k+1)}$ we have $V(b) = \overline{\mathbf{U}}^{k+1}(\Gamma_{y_0,b_-} \cup b \cup \Gamma_{b_+,y_0})$. We assume that α_0, α_1 are chosen so that $M\alpha_0$, $M\alpha_1$ are still small. We make the next gauge transformation u_{k+1} changing the axial gauge into Landau gauge for the configuration $U_{k+1}(\square_0, V)$, hence

$$\begin{gathered}
U_{k+1}(\square_0, V) = \left(\exp iL^{-1}\eta\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right)^{u_{k+1}} , \\
\left|\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right| , \qquad \left|\nabla^{L^{-1}\eta}\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right| , \\
\left\|\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right\|_{1,\beta} < B_3 O(1)M\alpha_0 \quad \text{on} \quad \tilde{\square}^4, \quad \text{for} \quad 0 \leqq \beta \leqq \beta_0 < 1 , \\
|u_{k+1} - 1| < B_3 O(1)M\alpha_0 \quad \text{on} \quad \square_0 .
\end{gathered} \tag{3.27}$$

Performing these gauge transformations in (3.25) we obtain

$$\mathbf{E}^{(j)}\left(X, U_j\left(\square_0, M^{\cdot}\left(\exp i\eta(t\tilde{\zeta}_\square + t_\square\zeta_\square)\mathbf{H}_k(B')\exp iL^{-1}\eta\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right)\right)\right)$$

$$= \mathbf{E}^{(j)}\left(X, U_j\left(\square_0, \exp iQ\left(\frac{1}{i}\log\exp i\eta(t\tilde{\zeta}_\square + t_\square\zeta_\square)\mathbf{H}_k(B')\right.\right.\right.$$

$$\left.\left.\left.\times \exp iL^{-1}\eta\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)\right)\right)\right). \tag{3.28}$$

Making use of the gauge invariance of the function (3.25) it is necessary to apply the gauge transformations to the variables $\mathbf{J}$, B also, not only to the gauge field configurations in (3.26), (3.27). Thus in the function $\mathbf{H}_k(B')$ the variables $\mathbf{J}$, B are replaced by $R(u_{k+1}^{-1})R(v^{-1})\mathbf{J}$, $R(u_{k+1}^{-1})R(v^{-1})B$. This is a complication, but not a very serious one for the following reasons. First, there are the bounds in (3.26), (3.27) for v, u_{k+1}. They imply that the above expressions have almost the same bounds as $\mathbf{J}$, B. Second, these gauge transformations are explicitly given analytic functions of $\mathbf{U}$, depending on $\mathbf{U}$ restricted to the cube $\square_0$. This will be important later on in a final localization.

The functions (3.25), (3.26) are gauge invariant with respect to the simultaneous gauge transformations

$$\mathbf{U}\to\mathbf{U}^u, \quad \mathbf{J}\to R(u)\mathbf{J}, \quad B\to R(u)B, \tag{3.29}$$

for G^c-valued transformations u in a sufficiently small neighborhood of G-valued transformations, so that the configurations after the transformations belong to proper spaces also. The expressions in (3.28) transform in a very simple way under (3.29), namely by $R(u(y_0))$ (where y_0 is the center of the cube $\square$). Of course this is connected with the fact that the contour variables V transform this way.

In the final localization the function $\mathbf{H}_k(B')$ will be transformed into many different functions, hence it is important to understand properties of the expression obtained from (3.28) by replacing $(t\tilde{\zeta}_\square + t_\square\zeta_\square)\mathbf{H}_k(B')$ by a $\mathbf{g}^c$-valued variable $\mathbf{A}$. Let us denote

$$\begin{gathered} A = \frac{1}{i\eta}\log\exp i\eta\mathbf{A}\exp iL^{-1}\eta\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right), \\ B = Q(\eta A) \quad \text{on} \quad \square_0 . \end{gathered} \tag{3.30}$$

We hope that the last function will not be confused with the fluctuation field variable B defined on the lattice $T_1^{(k)}$. The function B above is defined on the set of bonds determining $U_j(\square_0)$, i.e. on $\left(\bigcup_{n=0}^{j-1} \Lambda_n\right) \cup \square_j^{(j)}$. Now we define a class of functions $\mathbf{A}$. We assume that $\mathbf{A}$ is regular and satisfies the bounds

$$|\mathbf{A}|, \quad |\nabla^\eta\mathbf{A}|, \quad \|\mathbf{A}\|_{1,\beta} < \alpha_2 \quad \text{on} \quad \square_0, \tag{3.31}$$

for $0 \leqq \beta \leqq \beta_0 < 1$. Obviously such bounds are satisfied by the function $(t\tilde{\zeta}_\square + t_\square\zeta_\square)\mathbf{H}_k(B')$ for ε_1 sufficiently small. We have similar bounds for the function $\mathbf{H}_{k+1}\left(\square_0, \frac{1}{i}\log V\right)$, with α_2 replaced by $B_3 O(1) M\alpha_0$, and $\square_0$ by $\tilde{\square}^4$, see (3.27).

The assumption (3.31) implies that A satisfies

$$|A|, \quad |\nabla^{\eta}A|, \quad \|A\|_{1,\beta} < 2(\alpha_2 + B_3 O(1) M\alpha_0) \quad \text{on} \quad \tilde{\square}^4. \tag{3.32}$$

These regularity conditions are basic for the further analysis.

Thus we consider the function

$$\mathbf{E}^{(j)}(X, U_j(\square_0, \exp iQ(\eta A))) = \mathbf{E}^{(j)}(X, U_j(\square_0, \exp iB)), \tag{3.33}$$

and we expand it in B up to the fifth order

$$\begin{aligned} &\mathbf{E}^{(j)}(X, U_j(\square_0, \exp iB)) \\ &= \sum_{n=0}^{4} \frac{1}{n!} \left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \bigotimes^n B \right\rangle \\ &\quad + \int_0^1 d\tau \frac{(1-\tau)^4}{4!} \left\langle \frac{\delta^5}{\delta B^5} \mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tau B)), \bigotimes^5 B \right\rangle. \end{aligned} \tag{3.34}$$

This is the fundamental expansion for the analysis of renormalization. Such expansions with respect to background fields were used many times for this purpose, on perturbative and non-perturbative levels for example see [72, 54, 68, 27, 7, 16, 41–44]. Here we will follow the methods of the papers [7, 43].

Let us consider the last term in the expansion (3.34). By the definiton of B and by the inequalities (3.32) we have $|B| < O(1)L^j\eta$, hence we expect that this term can be bounded by

$$O(1)\exp(-\kappa d_j(X))\,(O(1)L^j\eta)^5. \tag{3.35}$$

To prove this statement we have to go into rather lengthy considerations. The main problem is to investigate analyticity properties of this term on proper spaces.

We consider functions in (3.28), or (3.34), on the space $U_{k+1}^{\prime c}(\square_0, (1+2\beta)\alpha_0, (1+2\beta)\alpha_1, \alpha_0)$ of configurations $\mathbf{U}, \mathbf{J}$, and on the space of fields A satisfying (3.31). In (3.34), which is of main interest for us, we have configurations $\mathbf{U}$, $\mathbf{A}$ only. We have to prove that

$$(U_j(\square_0, \exp i\tau B), J_j(\square_0, \exp i\tau B))|_X \in U_j^c(X, \alpha_0, \alpha_1) \tag{3.36}$$

for all $\mathbf{U}, \mathbf{J}, \mathbf{A}$ in the above spaces, and for the parameters τ in the interval [0, 1]. We will prove a stronger statement, which will allow us eventually to get the bound (3.35). The analyticity of the functions in (3.34) follows from the analyticity of the two functions in (3.36), and the assumed analyticity of $\mathbf{E}^{(j)}(X, \mathbf{U}, \mathbf{J})$ on the space $U_j^c(X, \alpha_0, \alpha_1)$.

At first let us take $\mathbf{A} = 0$. We have

$$\begin{gathered} U_j(\square_0, \exp i\tau Q(L^{-1}\eta \mathbf{H}_{k+1})) = (\exp i\xi \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1})))^{u_j}, \\ |\mathbf{H}_j(\square_0, \tau Q(\ldots))|, \quad |\nabla^{\xi}\mathbf{H}_j(\square_0, \tau Q(\ldots))| < B_3^2 O(1) M\alpha_0 L^{j-1}\eta \quad \text{on} \quad \tilde{\square}^3, \\ |u_j - 1| < B_3^2 O(1) M\alpha_0. \end{gathered} \tag{3.37}$$

Assume now that $B_3^2 O(1) M\alpha_0 < 1/2\alpha_1$, then the orbit of the configuration $U_j(\square_0, \ldots)$ above contains the configuration $\exp i\xi \mathbf{H}_j(\ldots)$ satisfying the conditions (i), (ii) on the cube $\tilde{\square}^3$, hence on X. We will prove that this configuration satisfies all the conditions (i)–(iv). The following identity holds:

$$\begin{aligned}
&U_n(X, M^{\cdot}(\exp i\xi \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1}))))\\
&\quad = U_n(X, M^{\cdot}(U_j(\square_0, \exp i\tau Q(L^{-1}\eta \mathbf{H}_{k+1}))))^{(\bar{u}_j)^{-1}}\\
&\quad = U_j(\square_0, \exp i\tau Q(L^{-1}\eta \mathbf{H}_{k+1}))^{(\bar{u}_j)^{-1}}\\
&\quad = (\exp i\xi \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1})))^{u_j(\bar{u}_j)^{-1}}, \qquad (3.38)
\end{aligned}$$

where $\bar{u}_j$ is a gauge transformation constant on blocks naturally connected with the function $U_n(X, \cdot)$, and equal to u_j at centers of the blocks. This identity and the bound on u_j in (3.37) imply that the condition (iv) is a consequence of the condition (iii), with a bit better constant, for the pair of configurations

$$\exp i\xi \mathbf{H}_j(\ldots), \quad D^{\xi *}_{\exp i\xi \mathbf{H}_j(\ldots)} \xi^{-2}\pi \operatorname{Im} \partial \exp i\xi \mathbf{H}_j(\ldots).$$

Using (97) [12], (3.27), (3.26) we obtain for $\tau = 1$

$$\begin{aligned}
U_j(\square_0, \exp iQ(L^{-1}\eta \mathbf{H}_{k+1})) &= U_j(\square_0, M^{\cdot}(\exp iL^{-1}\eta \mathbf{H}_{k+1})^{v_j^{-1}})\\
&= U_j(\square_0, M^{\cdot}(U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))))^{v_j^{-1}(\bar{u}_{k+1})^{-1}v^{-1}}\\
&= U_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))^{(v\bar{u}_{k+1}v_j)^{-1}}. \qquad (3.39)
\end{aligned}$$

The assumption that the configuration $\mathbf{U}$ belongs to the space $U'^{c}_{k+1}(\square_0, (1+2\beta)\alpha_0, (1+\beta)\alpha_1, \alpha_0)$ implies in particular the inequalities

$$\begin{aligned}
&|\partial U_{k+1}(\square_0, M^{\cdot}(\mathbf{U})) - 1| < (1+2\beta)\alpha_0 (L^{-1}\eta)^2,\\
&|J_{k+1}(\square_0, M^{\cdot}(\mathbf{U}))| < (1+2\beta)\alpha_0 \quad \text{on} \quad \tilde{\square}^3.
\end{aligned} \qquad (3.40)$$

The first inequality, and the identities (3.39), (3.37) imply

$$\begin{aligned}
&|\partial \exp i\xi \mathbf{H}_j(\square_0, Q(L^{-1}\eta \mathbf{H}_{k+1})) - 1|\\
&\quad = |R((v\bar{u}_{k+1}v_j u_j)^{-1})(\partial U_{k+1}(\square_0, M^{\cdot}(\mathbf{U})) - 1)|\\
&\quad < \exp B_3^2 O(1) M\alpha_0 \exp B_3 O(1) M\alpha_0 \exp B_3 O(1) M\alpha_0\\
&\qquad \times \exp O(1) M\alpha_1 (1+2\beta)\alpha_0 (L^{-1}\eta)^2\\
&\quad < (1+3\beta)\alpha_0 (L^{j-1}\eta)^2 \xi^2 \quad \text{on} \quad \tilde{\square}^3, \qquad (3.41)
\end{aligned}$$

for α_1 sufficiently small, e.g. $O(1)M\alpha_1 \leqq \beta$. A similar inequality holds for the expression replacing the variable $\mathbf{J}$

$$\begin{aligned}
|D^{\xi *}_{\exp i\xi \mathbf{H}_j(\ldots)} \xi^{-2}\pi \operatorname{Im} \partial \exp i\xi \mathbf{H}_j(\ldots)| &= |R((v\bar{u}_{k+1}v_j u_j)^{-1})(L^{j-1}\eta)^3 J_{k+1}(M^{\cdot}(\mathbf{U}))|\\
&< (1+3\beta)\alpha_0 (L^{j-1}\eta)^3 \text{ on } \tilde{\square}^3. \qquad (3.42)
\end{aligned}$$

From the bounds in (3.37), and an elementary inequality, we get

$$\begin{aligned}
&|\partial \exp i\xi \mathbf{H}_j(\ldots) - 1 - i\xi^2(\partial^{\xi}\mathbf{H}_j(\ldots)| \leqq \tfrac{1}{2}(\partial|\mathbf{H}_j(\ldots)|)^2 \xi^2 \exp \partial\xi |\mathbf{H}_j(\ldots)|\\
&\quad < 8(B_3^2 O(1) M\alpha_0 L^{j-1}\eta)^2 \xi^2 \exp 4B_3^2 O(1) M\alpha_0 L^{-1}\eta\\
&\quad < (4B_3^2 O(1) M)^2 \alpha_0^2 (L^{j-1}\eta)^2 \xi^2 \quad \text{on} \quad \tilde{\square}^3. \qquad (3.43)
\end{aligned}$$

We assume that $(4B_3^2 O(1) M)^2 \alpha_0 \leqq \beta$. This second restriction is not essentially stronger than the first one, because we have already assumed the restriction

$O(1)M\alpha_1 \leqq \beta$ on α_1. The above inequality and (3.41) yield

$$|\partial^\xi \mathbf{H}_j(\square_0, Q(L^{-1}\eta \mathbf{H}_{k+1}))| < (1+4\beta)\alpha_0(L^{j-1}\eta)^2 \quad \text{on} \quad \tilde{\square}^3. \tag{3.44}$$

Furthermore, the function $\mathbf{H}_j$ is given by (174) [15], i.e. $\mathbf{H}_j(\square_0, B) = H_{1,j}B + \mathbf{A}_{1,j}(B) - H_j D_j(H_{1,j}B + \mathbf{A}_{1,j}(B))$, where the functions on the righ-hand side are at least of second order, except the first term. This implies

$$|\partial^\xi \mathbf{H}_j(\square_0, Q(L^{-1}\eta \mathbf{H}_{k+1})) - \partial^\xi H_{1,j} Q(L^{-1}\eta \mathbf{H}_{k+1})| < B_3(B_3 O(1) M\alpha_0 L^{j-1}\eta)^2 < \beta\alpha_0(L^{j-1}\eta)^2 \quad \text{on} \quad \tilde{\square}^3. \tag{3.45}$$

Of course we have used also the exponential decay properties of all the functions which appeared above. Thus finally we obtain

$$|\partial^\xi H_{1,j} Q(L^{-1}\eta \mathbf{H}_{k+1})| < (1+5\beta)\alpha_0(L^{j-1}\eta)^2 \quad \text{on} \quad \tilde{\square}^3. \tag{3.46}$$

This inequality is linear in $Q(\ldots)$, hence it is valid also for this expression replaced by $\tau Q(\ldots)$, $\tau \in [0, 1]$. From the above considerations it is obvious that we can reverse the arguments, thus we obtain (3.44) with $\tau Q(\ldots)$, and the factor $1+6\beta$ on the right-hand side. Similarly, the inequality (3.43) yields

$$|\partial \exp i\xi \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1}))| < (1+7\beta)\alpha_0(L^{j-1}\eta)^2 \xi^2 \quad \text{on} \quad \tilde{\square}^3. \tag{3.47}$$

Let us notice that $j \leqq k$, hence $L^j\eta \leqq 1$ and $(1+7\beta)L^{-2} \leqq 1$ for β not too large. The above inequality is the required first inequality in the condition (iii). The second inequality in this condition is obtained in the same way. We start with (3.42) and we use again the formulas and the bounds (1.43)–(1.54) [14]. They give a bound of the type (3.44), but for the second order operator $\partial^{\xi *}\partial^\xi$. Then the same reasoning as between (3.44)–(3.47) gives the required second inequality in (iii).

Now we consider the general case, with $\mathbf{A}$ satisfying (3.31). From (3.30) we have

$$B = Q(\eta A) = Q(L^{-1}\eta \mathbf{H}_{k+1}) + Q',$$

$$Q' = \int_0^1 dt_1 \left\langle \left(\frac{\delta}{\delta A} Q\right)(\eta A(t_1 \mathbf{A}, L^{-1}\mathbf{H}_{k+1})), \eta \left\langle \left(\frac{\delta}{\delta \mathbf{A}} A\right)(t_1 \mathbf{A}, L^{-1}\mathbf{H}_{k+1}), \mathbf{A} \right\rangle \right\rangle, \tag{3.48}$$

$$A(\mathbf{A}, L^{-1}\mathbf{H}_{k+1}) = \frac{1}{i\eta} \log \exp i\eta \mathbf{A} \exp iL^{-1}\eta \mathbf{H}_{k+1}.$$

The function Q' satisfies the inequality

$$|Q'| \leqq O(1) L^j \eta |\mathbf{A}| < O(1)\alpha_2 L^j \eta \quad \text{on} \quad \square_0, \tag{3.49}$$

with an absolute constant $O(1)$. It is an analytic and almost local function of $\mathbf{A}$ and $\mathbf{H}_{k+1}$. We can prove that for α_2 sufficiently small we still have (3.36). Instead we will prove a more general result. We replace $\tau Q'$ by the variable B' with values in $\mathbf{g}^c$, and we prove (3.36) for B' satisfying $|B'| < \alpha_3$, with a sufficiently small constant α_3. To prove the conditions (i)–(iv) we write

$$\begin{aligned} &\mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1}) + B') \\ &= \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1})) + \int_0^1 dt_2 \left\langle \left(\frac{\delta}{\delta B}\mathbf{H}_j\right)(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1}) + t_2 B'), B' \right\rangle \\ &= \mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta \mathbf{H}_{k+1})) + \mathbf{A}_2, \end{aligned} \tag{3.50}$$

where the last equality is a definition of $\mathbf{A}_2$. It implies

$$|\mathbf{A}_2|, \quad |\nabla^{\xi}\mathbf{A}_2| \leqq B_3|B'| < B_3\alpha_3,$$

and if $B_3\alpha_3 < 1/2\alpha_1$, then this and the inequality in (3.37) imply the conditions (i), (ii). Consider now the derivative

$$\begin{aligned}&|\partial^{\xi}\mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta\mathbf{H}_{k+1}) + B')|\\ &\leqq |\partial^{\xi}\mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta\mathbf{H}_{k+1}))| + |\partial^{\xi}\mathbf{A}_2| < (1+6\beta)\alpha_0(L^{j-1}\eta)^2 + 2B_3\alpha_3\\ &\leqq (1+6\beta))L^{-2}\alpha_0 + 2B_3\alpha_3 \leqq (1+7\beta)L^{-2}\alpha_0 \quad \text{on} \quad \tilde{\square}^3, \end{aligned} \tag{3.51}$$

where we have assumed $2B_3\alpha_3 \leqq \beta L^{-2}\alpha_0$. The above inequality, and the inequality (3.43) slightly modified for the present situation, give the bound

$$|\partial \exp i\xi\mathbf{H}_j(\square_0, \tau Q(L^{-1}\eta\mathbf{H}_{k+1}) + B') - 1| < (1+8\beta)L^{-2}\alpha_0\xi^2 \quad \text{on} \quad \tilde{\square}^3. \tag{3.52}$$

This implies the first inequality in the condition (iii). The second is proved in the same way, as it was already discussed. The condition (iv) follows from the corresponding identity (3.38). Thus we have proved (3.36) in several versions. Let us formulate the one which will be used to bound the terms in (3.34).

Lemma 4. *For* $\mathbf{U} \in U_{k+1}^{\prime c}(\square_0, (1+2\beta)\alpha_0, (1+2\beta)\alpha_1)$, $\mathbf{A}$ *defined on* $\square_0$ *and satisfying* (3.31), B' *defined on the set of bonds connected with the definition of* $U_j(\square_0, \cdot)$ *and satisfying* $|B'| < \alpha_3$, *we have*

$$(U_j(\square_0, \exp i(\tau B + B')), J_j(\square_0, \exp i(\tau B + B')))|_X \in U_j^c(X, \alpha_0, \alpha_1) \tag{3.53}$$

for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ *sufficiently small and satisfying all the restrictions. The functions in* (3.53) *are analytic on the above spaces.*

Let us consider the last term in (3.34). We want to bound it by an expression of the form (3.35). We have

$$\begin{aligned}&\left|\int_0^1 d\tau \frac{(1-\tau)^4}{4!}\left\langle \frac{\delta^5}{\delta B^5}\mathbf{E}^{(j)}(X, U_j(\square_0, \exp i\tau B)), \bigotimes^5 B\right\rangle\right|\\ &\leqq \sup_{r\in[0,1]}\left|\int_{|\sigma|=r} d\sigma \frac{1}{\sigma^6}\mathbf{E}^{(j)}(X, U_j(\square_0, \exp i(\tau B + \sigma B)))\right|\\ &\leqq E_0\alpha_3^{-5}|B|^5\exp(-\kappa d_j(X)). \end{aligned} \tag{3.54}$$

Because $|B| < O(1)L^j\eta$, e.g. $|B| < \alpha_1 L^j\eta$, hence we have the required bound. In fact we have to be more careful, because this bound holds on $\tilde{\square}^4$ only. We should localize the inner product in domains $\tilde{\square}^4, (\tilde{\square}^4)^c$, and if at least one factor has a localization in $(\tilde{\square}^4)^c$, then the expression can be estimated by an arbitrary power of $L^j\eta$ using the exponential decay properties. The considerations are similar to those done on several occasions, e.g. (3.6), (3.21), and we do not repeat them. Let us make a final remark about the constant in the inequality (3.54). It is much bigger than E_0, and further bounds and summations will make it worse. Eventually we have to recover the constant E_0 in the inductive assumption (1.18), and the mechanism for it is provided by the differentiation with respect to $t_{\square}$, at $t_{\square} = 0$. This differentiation yields a factor $O(1)\alpha_2^{-1}\varepsilon_1$, and taking ε_1 sufficiently small we can get the required constant.

The considerations of this section gave a full control over the terms of the expansion of the difference $\mathbf{E}_k(U_k(\exp iB'V^{(k)})) - \mathbf{E}_k(U_k(V^{(k)}))$, except the terms in the sum on the right-hand side of (3.34). These terms are connected with the renormalization, and we will analyze them in the next section. The remaining terms are irrelevant according to our terminology, i.e. they satisfy bounds of the form (0.28) on their domains of analyticity.

4. Ward-Takahashi Identities and Their Consequences

Let us consider the sum on the right-hand side of (3.34). We can still separate some terms which are very small due to the exponential decay. It is convenient to localize the configuration B in a neighborhood of the cube $\tilde{\square}^3$, hence we write

$$\left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \bigotimes^n B \right\rangle = \sum_{m=0}^{n} \binom{n}{m} \left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \left(\bigotimes^m \tilde{\zeta}_\square B\right) \otimes \left(\bigotimes^{n-m} (1-\tilde{\zeta}_\square) B\right) \right\rangle, \quad (4.1)$$

where the function $\tilde{\zeta}_\square$ was introduced in the previous section [see the definition after (3.20)]. Terms with $m < n$ are exponentially small in $L^j \eta$. To see it we write the functional derivatives in a form, which will be convenient for other estimates also. Consider a more general expression with different functions $B_1, \ldots, B_n$. Using the gauge transformation in (3.37), and the gauge invariance of the function $\mathbf{E}^{(j)}(X, U)$, we have

$$\mathbf{E}^{(j)}(X, U_j(\square_0, \exp iB)) = \mathbf{E}^{(j)}(X, \exp i\xi \mathbf{H}_j(\square_0, B)) . \quad (4.2)$$

Differentiating the composite function on the right-hand side above n times with respect to B, we obtain the identities

$$\begin{aligned} &\left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \bigotimes_{i=1}^n B_i \right\rangle \\ &= \sum_{\{1,\ldots,n\} = N(1) \cup \ldots \cup N(r)} \left\langle \frac{\delta^r}{\delta \mathbf{H}^r} \mathbf{E}^{(j)}(X, 1), \bigotimes_{p=1}^r \left\langle \frac{\delta^{n(p)}}{\delta B^{n(p)}} \mathbf{H}_j(\square_0, 0), \bigotimes_{i \in N(p)} B_i \right\rangle \right\rangle \\ &= \sum \frac{\partial^r}{\partial \tau_1 \ldots \partial \tau_r} \mathbf{E}^{(j)} \left(X, \exp i\xi \sum_{p=1}^r \tau_p \left\langle \frac{\delta^{n(p)}}{\delta B^{n(p)}} \mathbf{H}_j(\square_0, 0), \bigotimes_{i \in N(p)} B_i \right\rangle \right) \Bigg|_{\tau=0} \\ &= \sum \prod_{p=1}^r \frac{1}{2\pi i} \int \frac{d\tau_p}{\tau_p^2} \mathbf{E}^{(j)} \left(X, \exp i\xi \sum_{p=1}^r \tau_p \left\langle \frac{\delta^{n(p)}}{\delta B^{n(p)}} \mathbf{H}_j(\square_0, 0), \bigotimes_{i \in N(p)} B_i \right\rangle \right) . \end{aligned} \quad (4.3)$$

Now, let us recall the considerations connected with the function (3.13). We have noticed there that this function is analytic on the corresponding spaces of variables $\mathbf{U}$, $\mathbf{J}$, $\mathbf{A}$, and this fact is quite general, valid for an arbitrary domain X. In the last formula above we have the function $\mathbf{E}^{(j)}(X, \exp i\xi \mathbf{A})$, i.e. the function with $\mathbf{U} = 1$, $\mathbf{J} = 0$. Thus it is defined and analytic on the space of configurations $\mathbf{A}$ satisfying

$$\max\{|\mathbf{A}|_X, |P_1(\square_0)\mathbf{A}|_X, |\nabla^\xi \mathbf{A}|_X, |\Delta^\xi \mathbf{A}|_X\} < \alpha_2 . \quad (4.4)$$

The functional derivative $(\delta^{n(p)})/(\delta B^{n(p)})\mathbf{H}_j(\square_0, 0)$ is given by a sum of several perturbative expressions discussed in Sect. G [15]. Each expression corresponds to a tree graph with $n(p)$ initial points and one final point, and it has an exponential decay in a length of this graph. The derivative has an exponential decay in a length of a shortest tree graph of this type. The norm in (4.4) of the expression

$$\left\langle \frac{\delta^{n(p)}}{\delta B^{n(p)}} \mathbf{H}_j(\square_0, 0), \bigotimes_{i \in N(p)} B_i \right\rangle$$

can be estimated by $B_3 \prod_{i \in N(p)} |B_i|$, and if one of the functions B_i is localized outside the domain X, then we have the additional exponential factor $\exp(-\delta_0 \operatorname{dist}^{(\xi)}(X, \operatorname{supp} B_i))$. Consider a term in (4.1) with $m < n$. Using (4.3) and the last remarks we obtain

$$\left| \left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \left(\bigotimes^{m} \tilde{\zeta}_\square B \right) \otimes \left(\bigotimes^{n-m} (1 - \tilde{\zeta}_\square) B \right) \right\rangle \right|$$

$$\leqq \left(2n^2 B_3 \frac{1}{\alpha_2} \right)^n E_0 \exp(-\kappa d_j(X)) \exp(-\delta_0 \operatorname{dist}^{(\xi)}(X, \operatorname{supp}(1 - \tilde{\zeta}_\square)))$$

$$\times |\tilde{\zeta}_\square B|^m |(1 - \tilde{\zeta}_\square) B|^{n-m}. \tag{4.5}$$

The distance in the second exponential is bounded from below by $M(L^j \eta)^{-1}$, hence we get the exponential factor $\exp(-\delta_0 M(L^j \eta)^{-1})$. This implies that the terms in (4.1) with $m < n$ are irrelevant.

We have to consider the terms with $m = n$ only. They are localized in $\tilde{\square}^4$, hence the function $\tilde{\zeta}_\square B$ is defined on the unit lattice $T_1^{(j)}$. We denote it simply by B again, hence $B = \tilde{\zeta}_\square Q_j(\eta A)$.

At this point it is convenient to perform the differentiation with respect to $t_\square$, at $t_\square = 0$. We replace the function $\mathbf{A}$ by $(t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(B')$, and we differentiate the sum over n. It is a simple polynomial in B, and the differentiation yields

$$\sum_{n=1}^{4} \frac{1}{(n-1)!} \left\langle \frac{\delta^n}{\delta B^n} \mathbf{E}^{(j)}(X, U_j(\square_0, 1)), \delta B, \bigotimes^{n-1} B \right\rangle,$$

$$\delta B = \left\langle \frac{\delta}{\delta A} Q_j(\eta A), \eta \left\langle \frac{\delta}{\delta \mathbf{A}} A, \zeta_\square \mathbf{H}_j(B') \right\rangle \right\rangle, \tag{4.6}$$

$$A = A\left(t\tilde{\zeta}_\square \mathbf{H}_k(B'), L^{-1} \mathbf{H}_{k+1}\left(\square_0, \frac{1}{i} \log V \right) \right),$$

where

$$A(\mathbf{A}, \mathbf{B}) = \frac{1}{i\eta} \log \exp i\eta \mathbf{A} \exp i\eta B.$$

The fields A and $\langle (\delta/\delta \mathbf{A})A, \zeta_\square \mathbf{H}_k(B') \rangle$ satisfy the bounds (3.32), the second field is localized in $\operatorname{supp} \zeta_\square$, hence δB is localized in $\square$.

The tools to investigate the sum in (4.6) are provided by a set of Ward-Takahashi identities. They express gauge invariance of a considered function. Take a function $\mathbf{E}(V)$ defined and analytic on a domain of small gauge field configurations V on a unit lattice, and assume that it is gauge invariant, i.e.,

$$\mathbf{E}(V^v) = \mathbf{E}(V), \qquad V^v(b) = v(b_-) V(b) v^{-1}(b_+). \tag{4.7}$$

We assume the gauge invariance with respect to G^c-valued gauge transformations, but it is implied by the invariance with respect to G-valued transformations, and by the analyticity of the function, as it was noticed already. For a small gauge field $V=\exp iB$, and a small gauge transformation $v=\exp i\lambda$, B and λ small, we have

$$\begin{aligned} V^v(b) &= \exp i\lambda(b_-)\exp iB(b)\exp(-i\lambda(b_+)) \\ &= \exp i(\exp iad_{\lambda(b_-)}B(b))\exp i(\lambda(b_-)-\lambda(b_+)-\tfrac{1}{2}i[\lambda(b_-),\lambda(b_+)]+\dots), \\ \frac{1}{i}\log V^v(b) &= \exp iad_{\lambda(b_-)}B(b) \\ &\quad + g^{-1}(iad_{\exp iad_{\lambda(b_-)}B(b)})(\lambda(b_-)-\lambda(b_+)-\dots)+\dots \\ &= B(b)+i[\lambda(b_-),B(b)]-g^{-1}(iad_{B(b)})(\delta\lambda)(b)+\dots, \end{aligned} \tag{4.8}$$

where the dots denote terms of higher order in λ, and $g^{-1}(z)=(-z)/(e^{-z}-1)=1+\frac{1}{2}z+k_2z^2+\dots$. For more remarks about the above equalities and expressions see (32)–(41) [12]. Now differentiate the equality (4.7) with respect to λ, at $\lambda=0$. This gives the identity

$$\left\langle \frac{\delta}{\delta B}\mathbf{E}(\exp iB),\, i[\lambda(b_-),B(b)]-g^{-1}(iad_{B(b)})(\partial\lambda)(b)\right\rangle = 0 \tag{4.9}$$

holding for all $\mathbf{g}^c$-valued functions λ, and small, $\mathbf{g}^c$-valued configurations B. It is the fundamental identity expressing the gauge invariance of the function $\mathbf{E}$. Let us notice that it was derived in exactly the same way as the Ward-Takahashi identities for the Abelian Higgs model were derived in (2.23)–(2.28) [7]. We will call the identity (4.9) also the Ward-Takahashi identity. From this we derive a whole sequence of identities by differentiations with respect to B. For our purpose it is enough to consider expressions with four derivatives at most. Let us write the corresponding sequence of identities

$$\begin{aligned} &\left\langle \frac{\delta^2}{\delta B^2}\mathbf{E}(\exp iB),\, iad_{\lambda_-}B-g^{-1}(iad_B)\partial\lambda,\, B_1\right\rangle \\ &\quad + \left\langle \frac{\delta}{\delta B}\mathbf{E}(\exp iB),\, iad_{\lambda_-}B_1-\frac{1}{2}iad_{B_1}\partial\lambda-k_2\{iad_{B_1},iad_B\}\partial\lambda-\dots\right\rangle = 0, \end{aligned} \tag{4.10}$$

where

$$\{A,B\}=AB+BA;$$

$$\begin{aligned} &\left\langle \frac{\delta^3}{\delta B^3}\mathbf{E}(\exp iB),\, iad_{\lambda_-}B-g^{-1}(iad_B)\partial\lambda,\, B_1,B_2\right\rangle \\ &\quad + \left\langle \frac{\delta^2}{\delta B^2}\mathbf{E}(\exp iB),\, iad_{\lambda_-}B_2-\frac{1}{2}iad_{B_2}\partial\lambda-k_2\{iad_{B_2},iad_B\}\partial\lambda-\dots,\, B_1\right\rangle \\ &\quad + (B_1\leftrightarrow B_2) + \left\langle \frac{\delta}{\delta B}\mathbf{E}(\exp iB),\, -k_2\{iad_{B_1},iad_{B_2}\}\partial\lambda-\dots\right\rangle = 0, \end{aligned} \tag{4.11}$$

where the symbol $(B_1 \leftrightarrow B_2)$ denotes an expression obtained from the preceding one by the exchange of B_1 with B_2;

$$\left\langle \frac{\delta^4}{\delta B^4} \mathbf{E}(\exp iB), iad_{\lambda\_}B - g^{-1}(iad_B)\partial\lambda, B_1, B_2, B_3 \right\rangle$$
$$+ \left\langle \frac{\delta^3}{\delta B^3} \mathbf{E}(\exp iB), iad_{\lambda\_}B_3 - \frac{1}{2} iad_{B_3}\partial\lambda - k_2\{iad_{B_3}, iad_B\}\partial\lambda - \dots, B_1, B_2 \right\rangle$$
$$+ (B_1 \leftrightarrow B_3) + (B_2 \leftrightarrow B_3)$$
$$+ \left\langle \frac{\delta^2}{\delta B^2} \mathbf{E}(\exp iB), -k_2\{iad_{B_2}, iad_{B_3}\}\partial\lambda, B_1 \right\rangle + (B_1 \leftrightarrow B_2) + (B_1 \leftrightarrow B_3)$$
$$+ \left\langle \frac{\delta}{\delta B} \mathbf{E}(\exp iB), \dots \right\rangle = 0. \tag{4.12}$$

We are interested in the above identities at $B=0$, because such expressions only appear in the sum (4.6). This simplifies them in an essential way. Consider at first (4.10) at $B=0$

$$\left\langle \frac{\delta^2}{\delta B^2} \mathbf{E}(1), -\partial\lambda, B_1 \right\rangle + \left\langle \frac{\delta}{\delta B} \mathbf{E}(1), iad_{\lambda\_}B_1 - \frac{1}{2} iad_{B_1}\partial\lambda \right\rangle = 0. \tag{4.13}$$

For constant λ we get $\langle (\delta/\delta B)\mathbf{E}(1), iad_\lambda B_1 \rangle = 0$, and since the configuration B_1 is arbitrary, we get $[\lambda, (\delta/\delta B)\mathbf{E}(1)] = 0$ for all $\lambda \in \mathbf{g}^c$. The group G is semisimple, hence this is possible only for the element 0 in the algebra $\mathbf{g}^c$. Thus we have the first, very important consequence of the gauge invariance

$$\frac{\delta}{\delta B} \mathbf{E}(1) = 0. \tag{4.14}$$

This equality simplifies the identities, and also the sum (4.6), we can drop the term with $n=1$. We obtain the following set of Ward-Takahashi identities

$$\left\langle \frac{\delta^2}{\delta B^2} \mathbf{E}(1), B_1, \partial\lambda \right\rangle = 0,$$
$$\left\langle \frac{\delta^3}{\delta B^3} \mathbf{E}(1), B_1, B_2, \partial\lambda \right\rangle - \left\langle \frac{\delta^2}{\delta B^2} \mathbf{E}(1), B_1, i[\lambda_-, B_2] - \frac{1}{2} i[B_2, \partial\lambda] \right\rangle - (B_1 \leftrightarrow B_2) = 0,$$
$$\left\langle \frac{\delta^4}{\delta B^4} \mathbf{E}(1), B_1, B_2, B_3, \partial\lambda \right\rangle \tag{4.15}$$
$$- \left\langle \frac{\delta^3}{\delta B^3} \mathbf{E}(1), B_1, B_2, i[\lambda_-, B_3] - \frac{1}{2} i[B_3, \partial\lambda] \right\rangle - (B_3 \leftrightarrow B_2) - (B_3 \leftrightarrow B_1)$$
$$+ \left\langle \frac{\delta^2}{\delta B^2} \mathbf{E}(1), B_1, k_2\{iad_{B_2}, iad_{B_3}\}\partial\lambda \right\rangle + (B_1 \leftrightarrow B_2) + (B_1 \leftrightarrow B_3) = 0,$$

for an arbitrary gauge function λ, and arbitrary gauge fields B_1, B_2, B_3.

Applying the above identities to the terms in (4.6) we will get expressions with derivatives of the field B. Let us analyze bounds for these derivatives. The field

$$B_\mu(x) = B(x, x+e_\mu) = \zeta_\square(x) Q_j(\eta A, \langle x, x+e_\mu \rangle) = \zeta_\square(x) Q_{j,\mu}(\eta A, x)$$

is defined on the unit lattice $T_1^{(j)}$. The operation $Q_{j,\mu}$ is translation invariant, i.e., $Q_{j,\mu}(\eta A, x+a)=Q_{j,\mu}(\eta t_a A, x)$, where $(t_a A_\nu)(x)=A_\nu(x+a)$, $a\in T_1^{(j)}$, hence we have the following formula for the derivative $\partial_\nu B_\mu$

$$\begin{aligned}(\partial_\nu B_\mu)(x)&=\tilde{\zeta}_\square(x+e_\nu)Q_{j,\mu}(\eta A, x+e_\nu)-\tilde{\zeta}_\square(x)Q_{j,\mu}(\eta A, x)\\&=\sum_{b\subset[x,x+e_\nu]}\xi(\partial^\xi\tilde{\zeta}_\square)(b)Q_{j,\mu}(\eta A, x+e_\nu)\\&\quad+\tilde{\zeta}_\square(x)\int_0^1 dt\left\langle\frac{\delta}{\delta A}Q_{j,\mu}(\eta A+\eta t(t_{e_\nu}A-A)),\eta(t_{e_\nu}A-A)\right\rangle,\\&\quad\text{where}\quad x\in T_1^{(j)},\quad\text{and for}\quad x'\in T_\xi,\\(t_{e_\nu}A-A)_\lambda(x')&=\sum_{b\subset[x',x'+e_\nu]}\xi(\partial^\xi A_\lambda)(b).\end{aligned}\tag{4.16}$$

The field A is regular on η-lattice, it satisfies the bounds (3.32), hence the derivative $\partial^\xi A_\lambda$ can be bounded by $O(1)L^j\eta$. The same remark applies to the derivative $\partial^\xi\tilde{\zeta}_\square$. More precisely, (3.32) and Proposition 5 [7] on functional derivatives of averaging operations imply

$$|(\partial_\nu B_\mu)(x)|<O(1)(\alpha_2+B_3O(1)M\alpha_0)(L^j\eta)^2<\alpha_1(L^j\eta)^2.\tag{4.17}$$

Thus the differentiation increases the power of $L^j\eta$ by 1, and improves an overall bound of an expression. For second order derivatives we have a weaker conclusion, because we do not have bounds for second order derivatives of the field A, only for Hölder norms of first order derivatives in (3.32). They imply the bound

$$|(\partial_\lambda\partial_\nu B_\mu)(x)|<\alpha_1(L^j\eta)^{2+\beta},\qquad 0\leqq\beta\leqq\beta_0<1,\tag{4.18}$$

by similar considerations as in the proof of (4.17). The inequalities (4.17), (4.18) hold for the field δB also.

We begin the analysis of (4.6) introducing simpler notations. We drop the superscript (j) in (4.6), and denote

$$\frac{\delta^n}{\delta B^n}\mathbf{E}^{(j)}(X, U_j(\square_0,1))=\mathbf{E}^{(n)}(X),\quad\text{or simply }\mathbf{E}^{(n)},\tag{4.19}$$

hence $\mathbf{E}^{(n)}$ is a $\overset{n}{\bigotimes}\mathbf{g}$-valued function $\mathbf{E}^{(n)}_{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n)$ defined on a neighborhood of $\tilde{\square}^4$ in the unit lattice $T_1^{(j)}$, and

$$(4.6)=\sum_{n=2}^{4}\frac{1}{(n-1)!}\left\langle\mathbf{E}^{(n)},\delta B,\overset{n-1}{\bigotimes}B\right\rangle.\tag{4.20}$$

Let us consider the third term in the above sum. We apply the same method which was used in [7], see especially (3.25)–(3.32) [7]. We write

$$\begin{aligned}&\left\langle\mathbf{E}^{(4)},\delta B,\overset{3}{\bigotimes}B\right\rangle\\&=\sum_{(\mu,x),\ldots,(\mu_3,x_3)}\langle\mathbf{E}^{(4)}_{\mu,\mu_1,\mu_2,\mu_3}(x,x_1,x_2,x_3),\delta B_\mu(x),B_{\mu_1}(x_1),B_{\mu_2}(x_2),B_{\mu_3}(x)\rangle\\&\quad+\sum_{(\mu,x),\ldots,(\mu_3,x_3)}\langle\mathbf{E}^{(4)}_{\mu,\mu_1,\mu_2,\mu_3}(x,x_1,x_2,x_3),\delta B_\mu(x),B_{\mu_1}(x_1),\\&\quad\times B_{\mu_2}(x_2),(\partial B_{\mu_3})(\Gamma_{x,x_3})\rangle.\end{aligned}\tag{4.21}$$

Because of the bound (4.17) the second term on the right-hand side above should have a bound of the form (3.35). To prove it we use the identities (4.3) again. There is a problem now, connected with the facts that the last factor $(\partial B_{\mu_s})(\Gamma_{x,x_3})$ is a function of two points instead of one, and it has the bound $|(\partial B_{\mu_3})(\Gamma_{x,x_3})| < \alpha_1(L^j\eta)^2|\Gamma_{x,x_3}|$. The length $|\Gamma_{x,x_3}| = |x_3 - x|$ is in ξ-scale, so it can be very big (of the order $O(M(L^j\eta)^{-1})$ for points x, x_3 in $\tilde{\Box}^4$, but far apart. We have to use the exponential decay properties to get a proper bound. Consider two possible cases. In the first the points x, x_3 are connected with the same set $N(p)$ in (4.3). Then the exponential factor, with a length of a shortest tree graph in the exponent, yields the factor $\exp(-\delta_0|x_3 - x|)$, and the product of it with $|x_3 - x|$ is bounded by δ_0^{-1}. The second case is more complicated, the points x, x_3 are connected with different sets in a partition, and we apply the identities (4.3) with localizations at the points x, x_3 fixed, i.e. with the summations over x, x_3 left undone. A term corresponding to such a partition can be estimated by

$$\sum_{x,x_3}\left(8B_3\frac{1}{\alpha_2}\right)^4 E_0 \exp(-\kappa d_j(X) - \delta_0 \,\mathrm{dist}^{(\xi)}(X,x)$$
$$-\delta_0\,\mathrm{dist}^{(\xi)}(X,x_3))\,|B|^2\,|\delta B(x)|\,|(\partial B)(\Gamma_{x,x_3})|$$
$$< \sum_{x,x_3}\left(8B_3\frac{\alpha_1}{\alpha_2}\right)^4 E_0\exp(-\tfrac{1}{3}\kappa d_j(X) - \delta_1|x - x_0| - \delta_1|x_3 - x|)\,|x_3 - x|(L^j\eta)^5,$$

where x_0 is a fixed point in X. The exponential decay factors with δ_1 are determined by the three factors on the left-hand side, therefore the decay rate is rather poor, $\delta_1 = O(M^{-1})$, because of the factor with $\kappa d_j(X)$. Summing over x, x_3 we get finally the following estimate

$$|(\text{the second sum in (4.21)})| \leq \left(32B_3\frac{\alpha_1}{\alpha_2}c_0(\delta_1)c_1(\delta_1)\right)^4 \exp(-\tfrac{1}{3}\kappa d_j(X))\,(L^j\eta)^5, \tag{4.22}$$

hence this sum in (4.21) represents an irrelevant term. More generally, let us notice that by a similar argument we can bound the derivative $\mathbf{E}^{(n)}(X, x_1, \ldots, x_n)$ by a constant times the exponential

$$\exp(-O(1)\kappa d_j(X \cup \{x_1, \ldots, x_n\})),$$

where $d_j(X \cup \{x_1, \ldots, x_n\})$ is a length of a shortest tree graph connecting cubes $\Box$ building the domain X, and points $x_1, \ldots, x_n$. This bound is enough to prove bounds of the type (4.22), although with a worse constant.

Consider now the first sum in (4.21). The last factor $B_{\mu_3}(x)$ is constant as a function of x_3, and we have

$$B_{\mu_3}(x) = (\partial_{\mu_3}\lambda_x)(x_3), \qquad \lambda_x(x_3) = \sum_{\nu=1}^{4}(x_{3,\nu} - x_\nu)B_\nu(x). \tag{4.23}$$

The first sum can be written as $\langle \mathbf{E}^{(4)}, \delta B, B, B, \partial\lambda_x\rangle$, where we have indicated the dependence on x explicitly. We apply the third identity in (4.15) taking into acount the special role of the first variable x. It is simplest to differentiate this identity with respect to B_1, take $B_2 = B_3 = B$, $\lambda = \lambda_x$, and then multiply by $\delta B(x)$ and sum over x.

We get

$$\begin{aligned}\langle \mathbf{E}^{(4)}, \delta B, B, B, \partial\lambda_x\rangle = 2\langle \mathbf{E}^{(3)}, \delta B, B, i[\lambda_x, B] - \tfrac{1}{2} i[B, \partial\lambda_x]\rangle \\ + \langle \mathbf{E}^{(3)}, i[\lambda_x, \delta B] - \tfrac{1}{2} i[\delta B, \partial\lambda_x], B, B\rangle \\ - 2\langle \mathbf{E}^{(2)}, \delta B, k_2(iad_B)^2 \partial\lambda_x\rangle \\ - 2\langle \mathbf{E}^{(2)}, k_2\{iad_B, iad_{\delta B}\}\partial\lambda_x, B\rangle .\end{aligned} \tag{4.24}$$

Now we analyze successively all the terms on the right-hand side. For the last term we have as in (4.21),

$$\begin{aligned}&\langle \mathbf{E}^{(2)}, \{iad_B, iad_{\delta B}\}\partial\lambda_x, B\rangle \\ &= \sum_{(\mu,x),(\mu_1,x_1)} \langle \mathbf{E}^{(2)}_{\mu,\mu_1}(x,x_1), \{iad_{B_\mu(x)}, iad_{\delta B_\mu(x)}\} B_\mu(x), B_{\mu_1}(x)\rangle \\ &+ \sum_{(\mu,x),(\mu_1,x_1)} \langle \mathbf{E}^{(2)}_{\mu,\mu_1}(x,x_1), \{iad_{B_\mu(x)}, iad_{\delta B_\mu(x)}\} B_\mu(x), (\partial B_{\mu_1})(\Gamma_{x,x_1})\rangle .\end{aligned} \tag{4.25}$$

In the first sum above we write again $B_{\mu_1}(x) = (\partial_{\mu_1}\lambda_x)(x_1)$, and by the first identity in (4.15) this sum equals 0. The second sum can be bounded as in (4.22), hence it is irrelevant. The second function in the term before the last is equal to

$$\begin{aligned}k_2(iad_{B_{\mu_1}(x_1)})^2 B_{\mu_1}(x) &= k_2\, iad_{B_{\mu_1}(x_1)} i[B_{\mu_1}(x_1), B_{\mu_1}(x)] \\ &= k_2\, iad_{B_{\mu_1}(x_1)} i[(\partial B_{\mu_1})(\Gamma_{x,x_1}), B_{\mu_1}(x)] ,\end{aligned}$$

hence this term is irrelevant also. Consider the second term on the right-hand side in (4.24). The first function in it is equal to $-1/2i[\delta B_\mu(x), B_\mu(x)]$, because $\lambda_x(x) = 0$. We apply again the same procedure as in (4.21), or (4.25), and it yields

$$\begin{aligned}-\tfrac{1}{2}\langle \mathbf{E}^{(3)}, i[\delta B, B], B, B\rangle &= -\tfrac{1}{2}\langle \mathbf{E}^{(3)}, i[\delta B, B], B, \partial\lambda_x\rangle \\ &\quad + (\text{the irrelevant term}) \\ &= -\tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], i[\lambda_x, B] - \tfrac{1}{2} i[B, \partial\lambda_x]\rangle \\ &\quad -\tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\lambda_x, i[\delta B, B]] - \tfrac{1}{2} i[i[\delta B, B], \partial\lambda_x], B\rangle \\ &\quad + (\text{the irrelevant term}) \\ &= -\tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], i[\lambda_x, B]\rangle \\ &\quad + \tfrac{1}{4}\langle \mathbf{E}^{(2)}, i[i[\delta B, B], B], B\rangle \\ &\quad + (\text{the irrelevant terms}),\end{aligned} \tag{4.26}$$

where we have used again the equalities

$$i[B_{\mu_1}(x_1), (\partial\lambda_x)_{\mu_1}(x_1)] = i[B_{\mu_1}(x_1), B_{\mu_1}(x)] = i[(\partial B_{\mu_1})(\Gamma_{x,x_1}), B_{\mu_1}(x)], \qquad \lambda_x(x) = 0 .$$

The second term on the right-hand side of the last equality in (4.26) is treated as in (4.25), hence it is irrelevant and we obtain

$$\begin{aligned}-\tfrac{1}{2}\langle \mathbf{E}^{(3)}, i[\delta B, B], B, B\rangle = -\tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], i[\lambda_x, B]\rangle \\ + (\text{the irrelevant terms}).\end{aligned} \tag{4.27}$$

Finally consider the first term on the right-hand side of (4.24). This term with the function $-1/2i[B, \partial\lambda_x]$ is irrelevant, by the equality written after (4.26). We apply the procedure in (4.21) to the term with $i[\lambda_x, B]$

$$\begin{aligned}2\langle \mathbf{E}^{(3)}, \delta B, i[\lambda_x, B], B\rangle &= 2\langle \mathbf{E}^{(3)}, \delta B, i[\lambda_x, B], \partial\lambda_x\rangle \\ &\quad + (\text{the irrelevant term}) \\ &= 2\langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, i[\lambda_x, B]] - \tfrac{1}{2} i[i[\lambda_x, B], \partial\lambda_x]\rangle \\ &\quad + 2\langle \mathbf{E}^{(2)}, i[\lambda_x, \delta B] - \tfrac{1}{2} i[\delta B, \partial\lambda_x], i[\lambda_x, B]\rangle \\ &\quad + (\text{the irrelevant term}) \\ &= 2\langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, i[\lambda_x, B]]\rangle \\ &\quad - \langle \mathbf{E}^{(2)}, \delta B, i[i[\lambda_x, B], \partial\lambda_x]\rangle \\ &\quad - \langle \mathbf{E}^{(2)}, i[\delta B, B], i[\lambda_x, B]\rangle \\ &\quad + (\text{the irrelevant term}). \end{aligned} \tag{4.28}$$

We can still simplify expressions in the last equalities in (4.27), (4.28), replacing all fields B by their values at the point x. The difference $B(\cdot) - B(x) = (\partial B)(\Gamma_{x,\cdot})$ gives rise to irrelevant terms. Thus the equalities (4.21), (4.24), and the simplified equalities (4.27), (4.28) yield

$$\begin{aligned}\left\langle \mathbf{E}^{(4)}, \delta B, \bigotimes^{3} B\right\rangle &= 2\langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, i[\lambda_x, B(x)]]\rangle - \langle \mathbf{E}^{(2)}, \delta B, i[i[\lambda_x, B(x)], \partial\lambda_x]\rangle \\ &\quad - \tfrac{3}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], i[\lambda_x, B(x)]\rangle \\ &\quad + (\text{the irrelevant terms}). \end{aligned} \tag{4.29}$$

Let us consider now the second term in the sum (4.20). Again, we apply the procedure in (4.21), but we expand the last factor up to second order around the point x. We have the following Taylor's formula on a unit lattice, analogous to the formula (3.10) [7]

$$f(y) = f(x) + \sum_{\mu=1}^{d} (y_\mu - x_\mu)(\partial_\mu f)(x) + \sum_{\mu=1}^{d} \sum_{x' \in [\Gamma_{x,y}[_\mu} \sum_{b \in \Gamma'_{x',x}} (\partial\partial_\mu f)(b), \tag{4.30}$$

where $[\Gamma_{x,y}[_\mu$ denotes the part of the contour $\Gamma_{x,y}$ parallel to the μ-th axis, including the initial point and excluding the final point. This formula is applied to the function $B_{\mu_3}(x_3)$ at $y = x_3$. By (4.18) the last term on the right-hand side can be estimated by $\frac{1}{2}|\Gamma_{x,x_3}|^2 \alpha_1 (L^j\eta)^{2+\beta}$ with a positive β. For the expression $\langle \mathbf{E}^{(3)}, B, B, B^{(2)}\rangle$, with this last term inserted in the place of $B^{(2)}$, it implies the bound (4.22) with the power $4+\beta$ instead of 5 in the last factor. Hence this is an irrelevant term. Let us denote

$$B_{\mu_3}(x_3, x) = \sum_{\nu=1}^{4} (x_{3,\nu} - x_\nu)(\partial_\nu B_{\mu_3})(x).$$

We have

$$\begin{aligned}\left\langle \mathbf{E}^{(3)}, \delta B, \overset{2}{\bigotimes} B\right\rangle &= \langle \mathbf{E}^{(3)}, \delta B, B, \partial\lambda_x\rangle + \langle \mathbf{E}^{(3)}, \delta B, B, B(\cdot, x)\rangle \\ &\quad + (\text{the irrelevant term}) \\ &= \langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, B] - \tfrac{1}{2} i[B, \partial\lambda_x]\rangle - \tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], B\rangle \\ &\quad + \langle \mathbf{E}^{(3)}, \delta B, B(\cdot, x), \partial\lambda_x\rangle + (\text{the irrelevant term}) \\ &= \langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, B]\rangle - \tfrac{1}{2}\langle \mathbf{E}^{(2)}, \delta B, i[B, B(x)]\rangle \\ &\quad - \tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], B(\cdot, x)\rangle \\ &\quad + \langle \mathbf{E}^{2}, \delta B, i[\lambda_x, B(\cdot, x)] - \tfrac{1}{2} i[B(\cdot, x), \partial\lambda_x]\rangle \\ &\quad - \tfrac{1}{2}\langle \mathbf{E}^{(2)}, i[\delta B, B], B(\cdot, x)\rangle + (\text{the irrelevant terms}) \\ &= \langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, B(x)]\rangle + 2\langle \mathbf{E}^{(2)}, \delta B, i[\lambda_x, B(\cdot, x)]\rangle \\ &\quad - \langle \mathbf{E}^{(2)}, \delta B, i[B(\cdot, x), B(x)]\rangle \\ &\quad - \langle \mathbf{E}^{(2)}, i[\delta B, B], B(\cdot, x)\rangle + (\text{the irrelevant terms}). \end{aligned} \tag{4.31}$$

Here, in the last equality, all fields and their derivatives are taken at the point x.

Thus we have reduced the analysis of the sum in (4.20) to the expression involving the functions $\mathbf{E}^{(2)}$ only. We will investigate these functions carefully in the next section, now we will draw further consequences of the gauge invariance. The basic identity (4.9) holds for all gauge transformations, hence for constant transformations it is $\mathbf{E}(\exp iR(v)B) = \mathbf{E}(\exp iB)$, and it implies

$$\langle \mathbf{E}^{(2)}, R(v)B, R(v)B\rangle = \langle \mathbf{E}^{(2)}, B, B\rangle\,, \qquad v \in G\,. \tag{4.32}$$

The corresponding function $\mathbf{E}^{(2)}_{\mu,\nu}(x, y)$ has values in the tensor product $\mathbf{g}\otimes\mathbf{g}$, and the above identity implies

$$R(v)\otimes R(v)\mathbf{E}^{(2)}_{\mu,\nu}(x, y) = \mathbf{E}^{(2)}_{\mu,\nu}(x, y)\,, \qquad v \in G\,. \tag{4.33}$$

The infinitesimal form of this identity is a consequence of (4.11), or the second identity in (4.15). An element $\mathbf{E}$ in $\mathbf{g}\otimes\mathbf{g}$ can be identified with the matrix $\mathbf{E}_{ab}$ of components in the basis $\{\tau_a\otimes\tau_b\}$, τ_a are generators of the algebra $\mathbf{g}$, or with the bilinear form $\langle \mathbf{E}, A\otimes B\rangle$ on $\mathbf{g}$. With our assumptions on the group G the identity (4.33) holds for $\mathbf{E}$ if and only if $\mathbf{E}_{ab}$ is proportional to the identity matrix, $\mathbf{E}_{ab} = \mathbf{E}\delta_{ab}$, and then $\langle \mathbf{E}, A\otimes B\rangle = \mathbf{E}\,\mathrm{tr}\,AB$. This gives a further simplification of the expressions in (4.29), (4.31).

Let us write the result of the preceding analysis

$$\begin{aligned}(4.6) = &\sum_{(x,\mu),(y,\nu)} \mathbf{E}^{(2)}_{\mu,\nu}(X, x, y)\,\mathrm{tr}\,\delta B_\mu(x) B_\nu(y) \\ &+ \sum_x \sum_{\mu,\nu,\kappa,\lambda} \left(\sum_y \mathbf{E}^{(2)}_{\mu,\nu}(X, x, y)(y_\kappa - x_\kappa)(y_\lambda - x_\lambda)\right) \\ &\times \{\mathrm{tr}\,\delta B_\mu(x) i[B_\kappa(x), (\partial_\lambda B_\nu)(x)] + \tfrac{1}{3}\mathrm{tr}\,\delta B_\mu(x) i[B_\kappa(x), i[B_\lambda(x), B_\nu(x)]]\} \\ &+ \sum_x \sum_{\mu,\nu,\kappa} \left(\sum_y \mathbf{E}^{(2)}_{\mu,\nu}(X, x, y)(y_\kappa - x_\kappa)\right)\left\{\frac{1}{2}\mathrm{tr}\,\delta B_\mu(x) i[B_\kappa(x), B_\nu(x)]\right.\end{aligned}$$

$$-\frac{1}{2}\operatorname{tr}\delta B_\mu(x) i[(\partial_\kappa B_\nu)(x), B_\nu(x)] - \frac{1}{2}\operatorname{tr}\delta B_\mu(x) i[B_\mu(x), (\partial_\kappa B_\nu)(x)]$$

$$-\frac{1}{3!}\operatorname{tr}\delta B_\mu(x) i[i[B_\kappa(x), B_\nu(x)], B_\nu(x)]$$

$$\left.-\frac{1}{4}\operatorname{tr}\delta B_\mu(x) i[B_\mu(x), i[B_\kappa(x), B_\nu(x)]]\right\}$$

$$+(\text{the irrelevant terms}). \tag{4.34}$$

The sums over x above are restricted to $\operatorname{supp}\delta B \subset \square$, and the sums over y are restricted to $\operatorname{supp} B \subset \operatorname{supp}\tilde{\zeta}_\square$. The expressions in this formula are local polynomials in the fields δB, B. The first expression on the right-hand side, which is simply $\langle \mathbf{E}^{(2)}(X), \delta B, B\rangle$, is not analyzed yet. We will analyze it in the next section using momentum representation and symmetries. The second expression is already in an almost correct form; we have to use only Euclidean symmetries to reduce it to a final form. By the Euclidean symmetries the third expression should vanish.

To make use of the Euclidean symmetries we have still to make changes and resummations of the expressions in (4.34). At first we have to change the function $U_j(\square_0)$ used in the definitions of $\mathbf{E}^{(n)}(X)$. For $n=2$, and by the identity (4.14), the formula (4.3) yields

$$\mathbf{E}^{(2)}(X) = \left\langle \frac{\delta^2}{\delta \mathbf{H}^2}\mathbf{E}(X,1), H_j(\square_0), H_j(\square_0)\right\rangle. \tag{4.35}$$

If we replace $H_j(\square_0)$ by H_j with free boundary conditions, then the difference $H_j(\square_0) - H_j$ restricted to $X \times \operatorname{supp}\tilde{\zeta}_\square$, yields the factor $B_0 \exp(-\delta_0 M(L^j\eta)^{-1})$ in a bound of the corresponding expression. Thus this change increases the sum of the irrelevant terms by the expression of the form (4.34), but with very small coefficients. Next, we extend summations over y to the whole lattice Z^4. The difference between the sum over $\operatorname{supp}\tilde{\zeta}_\square$ and the sum over Z^4 is a sum over a subset of $(\tilde{\square}^3)^c \cap Z^4$. This gives again the exponentially small coefficients. Finally, the crucial step is an extension of the sum over localization domains X. In this section we consider the expressions with the localization domains X satisfying the condition $X \subset \tilde{\square}^2$, for a given cube $\square$. The expressions with localization domains, which do not satisfy this condition, are exponentially small in $L^j\eta$, either because the domains are large, or because their distances to $\square$ are large. We have analyzed it in Sect. 3. Of course we have the sum over all $X \in \mathbf{D}_j$ in (3.7), the sum connected with the representation (1.7) of $\mathbf{E}^{(j)}$. This sum was divided into two subsums for a given $\square$, and in this section we consider this subsum for which $X \in \mathbf{D}_j$, $X \subset \tilde{\square}^2$. We extend it to all $X \in \mathbf{D}_j^0$, where $\mathbf{D}_j^0$ is the class of localization domains constructed for the lattice ξZ^4. More exactly we make this extension for all explicitly written terms in (3.34), not for irrelevant terms. This means that we sum the function $\mathbf{E}^{(2)}(X)$ over $X \in \mathbf{D}_j^0$, because the other expressions do not depend on X. The difference between the two sums gives a function satisfying the following inequality

$$\left|\sum_{X\in\mathbf{D}_j^0,\, X\cap(\square^2)^c \neq \emptyset} \mathbf{E}^{(2)}_{\mu,\nu}(X,x,y)\right|$$

$$\leq O(1) E_0 \exp(-(L^j\eta)^{-1}) \exp(-\delta_1|x-y|), \quad \text{for} \quad x\in\square, \tag{4.36}$$

hence, substituting it into (4.34), we get an irrelevant expression again.

Thus, after all the changes and resummations, we obtain an expression which is equal to this in (4.34), with the function $\mathbf{E}^{(2)}_{\mu,\nu}(X,x,y)$ replaced by

$$\Pi_{\mu,\nu}(x,y)=\sum_{X\in\mathbf{D}^0_j}\mathbf{E}^{(2)}_{\mu,\nu}(X,x,y)\,, \tag{4.37}$$

and the irrelevant terms resummed over $X\in\mathbf{D}_j$, $X\subset\tilde{\Box}^2$. The function Π is called the vacuum polarization tensor. We will investigate it in the next section.

In this and the previous sections we were concerned with the term $\mathbf{E}^{(j)}(U_k(\exp iB'V^{(k)}))-\mathbf{E}^{(j)}(U_k(V^{(k)}))$ in the fluctuation field action. There is the second term under the sum over j, the term

$$\beta_j(g_{j-1})\,(A(U_j(\exp iB'V^{(k)}))-A(U_k(V^{(k)})))\,. \tag{4.38}$$

We localize it by the partition of unity connected with the partition π_j, and we apply to it the whole procedure of these two sections. As a result we obtain all the well controlled terms, and the terms in (4.34) with the function $\mathbf{E}^{(2)}$ replaced by the corresponding function calculated for the expression (4.38). By (4.35) it is equal to

$$\beta_j(g_{j-1})\left\langle\frac{\delta^2}{\delta\mathbf{H}^2}\sum_{x,\mu<\nu}\zeta_{\Box'}(x)\,[1-\mathrm{Re\,tr}(\partial\exp i\xi\mathbf{H})\,(p_{\mu\nu}(x))]|_{\mathbf{H}=0},H_j(\Box_0),H_j(\Box_0)\right\rangle$$
$$=\beta_j(g_{j-1})\,\langle\partial^{\xi}H_j(\Box_0),\zeta_{\Box'}\partial^{\xi}H_j(\Box_0)\rangle\,,\qquad \Box'\in\pi_j\,. \tag{4.39}$$

Replacing the functions $H_j(\Box_0)$ by H_j and summing over all $\Box'$ in ξZ^4 yields the following expression, as the expression corresponding to (4.37)

$$\beta_j(g_{j-1})\,\langle\partial^{\xi}H_j,\partial^{\xi}H_j\rangle=\beta_j(g_{j-1})\Delta_j\,, \tag{4.40}$$

where Δ_j is given by the explicit formula (1.66) [10]. This formula allos us to understand clearly a final form of the expression (4.34) in this case. The function $\Delta_{j,\mu\nu}(x-y)$ in the momentum representation of (1.66) has the following expansion around 0:

$$\Delta_{j,\mu\nu}(p')=\Delta_0(p')\delta_{\mu\nu}-\bar{\partial}^1_{\mu}(p')\partial^1_{\nu}(p')+(\text{terms of higher order in } p')\,, \tag{4.41}$$

see (1.29)–(1.37) [10] for an explanation of symbols used in connection with momentum representations. This implies that the first term in (4.34), written for (4.40) instead of $\mathbf{E}^{(2)}$, is represented as

$$\beta_j(g_{j-1})\tfrac{1}{2}\sum_{x,\mu,\nu}\mathrm{tr}(\partial\delta B)_{\mu\nu}(x)(\partial B)_{\mu\nu}(x)+(\text{irrelevant terms})\,. \tag{4.42}$$

In the second term we have the expression

$$\beta_j(g_{j-1})\sum_y\Delta_{j,\mu\nu}(x-y)\,(y_\kappa-x_\kappa)\,(y_\lambda-x_\lambda)=-\beta_j(g_{j-1})\left(\frac{\partial^2}{\partial p'_\kappa\partial p'_\lambda}\Delta_{j,\mu\nu}\right)(0)$$
$$=\beta_j(g_{j-1})\,(\delta_{\mu\kappa}\delta_{\nu\lambda}+\delta_{\mu\lambda}\delta_{\nu\kappa}-2\delta_{\mu\nu}\delta_{\kappa\lambda})\,. \tag{4.43}$$

Using this equality we represent the second term as

$$\beta_j(g_{j-1})\sum_{x,\mu,\nu}\mathrm{tr}\,\delta B_\mu(x)\,\{i[B_\nu(x),(\delta B)_{\mu\nu}(x)]+i[B_\mu(x),(\partial_\nu B_\nu)(x)]$$
$$+i[(\partial_\nu B_\mu)(x),B_\nu(x)]+i[B_\nu(x),i[B_\mu(x),B_\nu(x)]]\}\,. \tag{4.44}$$

In the third term we have the expression

$$\beta_j(g_{j-1})\sum_y \Delta_{k,\mu\nu}(x-y)(y_\kappa - x_\kappa) = \beta_j(g_{j-1})\left(\frac{1}{i}\frac{\partial}{\partial p'_\kappa}\Delta_{j,\mu\nu}\right)(0) = 0\,, \tag{4.45}$$

hence this term vanishes. Thus the only terms in the expansion of (4.38), which we do not control yet, more exactly for which the sum over j has not a uniform bound, are terms (4.42), (4.44). In the next section we will prove that the polarization tensor Π has a similar structure as the operator Δ_j, especially it has an expansion of the form (4.41), but with a coefficient. We define the β-function $\beta_j(g_{j-1})$ equal to this coefficient. The corresponding terms from (4.34) are equal to (4.42), (4.44) also, hence both groups of terms cancel, because (4.38) appears with the minus sign in the effective action (1.6).

5. The Analysis of the Vacuum Polarization Tensor and the β-Functions

The vacuum polarization tensor is defined by the formula (4.37) of the previous section. From this it follows that it can be defined also as

$$\Pi(b,b') = \lim_{T_1^{(j)}\nearrow Z^4} \frac{\delta^2}{\delta B(b)\delta B(b')}\mathbf{E}^{(j)}(U_j(\exp iB))|_{B=0}\,. \tag{5.1}$$

This representation is basic for the further analysis, because it implies symmetries of the tensor. The function $\mathbf{E}^{(j)}(U_j(\exp iB))$ is invariant with respect to all Euclidean symmetries r of the lattice $T_1^{(j)}$,

$$\mathbf{E}^{(j)}(U_j(\exp irB)) = \mathbf{E}^{(j)}(rU_j(\exp iB)) = \mathbf{E}^{(j)}(U_j(\exp iB))\,, \tag{5.2}$$

where the fields rU, rB are defined by the identity $(rU)(rb) = U(b)$, hence

$$(rU)(b) = U(r^{-1}b)\,, \qquad (rB)(b) = B(r^{-1}b)\,. \tag{5.3}$$

The invariance (5.2) yields the following covariant transformation law for the polarization tensor:

$$\Pi(rb, rb') = \Pi(b,b')\,, \tag{5.4}$$

where Π is extended to negatively oriented bonds by the equality $\Pi(-b,b') = -\Pi(b,b')$, similarly for the second argument. We would like to get the representation (4.41) for the function

$$\Pi_{\mu\nu}(x,y) = \Pi(\langle x, x+e_\mu\rangle, \langle y, y+e_\nu\rangle)\,. \tag{5.5}$$

Unfortunately the Euclidean covariance (5.4) takes on a more complicated form for this function. We formulate it explicitly in two special cases, which we will use in the sequel. If r is a transformation defined by a permutation π: $(rx)_\mu = x_{\pi^{-1}(\mu)}$, then the equality (5.4) can be written as

$$\Pi_{\mu\nu}(rx, ry) = \Pi_{\pi^{-1}(\mu),\pi^{-1}(\nu)}(x,y) = ((r\otimes r)\Pi)_{\mu\nu}(x,y)\,. \tag{5.6}$$

If r is a reflection in a part of the components of x: $rx=\varepsilon x$, $(\varepsilon x)_\mu=\varepsilon_\mu x_\mu$, $\varepsilon_\mu=\pm 1$, $\mu=1,\dots,d$, then (5.4) can be written as

$$\Pi(\langle \varepsilon x, \varepsilon x+\varepsilon_\mu e_\mu\rangle, \langle \varepsilon y, \varepsilon y+\varepsilon_\nu e_\nu\rangle)=\Pi(\langle x, x+e_\mu\rangle, \langle y, y+e_\nu\rangle).$$

This and the definition (5.5) yield

$$\Pi_{\mu\nu}\left(\varepsilon x-\frac{1-\varepsilon_\mu}{2}e_\mu, \varepsilon y-\frac{1-\varepsilon_\nu}{2}e_\nu\right)=\varepsilon_\mu\varepsilon_\nu\Pi_{\mu\nu}(x,y). \tag{5.7}$$

The function Π is also translation invariant and symmetric, hence

$$\Pi_{\mu\nu}(x,y)=\Pi_{\mu\nu}(x-y), \qquad \Pi_{\mu\nu}(x)=\Pi_{\nu\mu}(-x). \tag{5.8}$$

The gauge invariance, expressed in the first identity (4.15), implies

$$\sum_\mu \partial_\mu^* \Pi_{\mu\nu}(x-y)=\sum_\nu \partial_\nu \Pi_{\mu\nu}(x-y)=0. \tag{5.9}$$

The representation (4.37) yields the following inequality

$$|\Pi_{\mu\nu}(x-y)|\leqq O(1)E_0\exp(-\delta_1|x-y|), \tag{5.10}$$

with a positive constant δ_1 determined by δ_0, κ, and M (e.g., $\delta_1=1/2\min\{\delta_0,\kappa M^{-1}\}$). Take the momentum representation of this tensor

$$\Pi_{\mu\nu}(p)=\sum_{x\in Z^4} e^{-ip\cdot x}\Pi_{\mu\nu}(x), \qquad \Pi_{\mu\nu}(x)=(2\pi)^{-4}\int dp e^{ix\cdot p}\Pi_{\mu\nu}(p), \tag{5.11}$$

the integration over p with components p_μ satisfying $|p_\mu|\leqq\pi$. The function $\Pi_{\mu\nu}(p)$ is periodic in variables p_μ, with the period 2π. By the inequality (5.10) it can be extended as an analytic function to complex variables $\zeta_\mu=p_\mu+iq_\mu$, $|q_\mu|<\delta_1$. This property is the basic reason why we have taken the infinite volume limit in (5.1). Let us write the symmetry properties (5.6)–(5.9) for the function $\Pi_{\mu\nu}(\zeta)$:

$$\Pi_{\mu\nu}(r\zeta)=((r\otimes r)\Pi)_{\mu\nu}(\zeta) \quad \text{if } r \text{ is a permutation}, \tag{5.12}$$

$$\Pi_{\mu\nu}(\varepsilon\zeta)=\varepsilon_\mu\varepsilon_\nu\exp\left(-i\frac{1-\varepsilon_\mu}{2}\zeta_\mu\right)\exp\left(i\frac{1-\varepsilon_\nu}{2}\zeta_\nu\right)\Pi_{\mu\nu}(\zeta), \tag{5.13}$$

$$\Pi_{\mu\nu}(\zeta)=\Pi_{\nu\mu}(-\zeta), \tag{5.14}$$

$$\sum_\mu \partial_\mu(-\zeta)\Pi_{\mu\nu}(\zeta)=\sum_\nu \partial_\nu(\zeta)\Pi_{\mu\nu}(\zeta)=0, \tag{5.15}$$

where $\partial_\mu(\zeta)=e^{i\zeta_\mu}-1$.

We will analyze this function using the above properties only. Our goal is to prove a representation of the form (4.41), more exactly of the form

$$\Pi_{\mu\nu}(p)=\beta(\delta_{\mu\nu}\Delta(p)-\overline{\partial_\mu(p)}\partial_\nu(p))+(\text{terms of higher orders in derivatives } \partial(p),\overline{\partial(p)}), \tag{5.16}$$

and to find the coefficient β. To prove bounds for the expression in (4.34) connected with the remainder in the above representation, it is important to write it explicitly as a higher order polynomial in $\partial(p),\overline{\partial(p)}$ with analytic coefficients. We obtain such a representation using some simple expansion connected with the Laurent series expansion. It was used already in [43] for a similar purpose. To use this expansion we have to choose properly new variables. Because of the periodicity properties of the function $\Pi_{\mu\nu}(\zeta)$ it is natural to choose the variables $z_\mu = e^{i\zeta_\mu}$, hence we define

$$f_{\mu\nu}(z_1, \ldots, z_d) = \Pi_{\mu\nu}\left(\frac{1}{i}\log z_1, \ldots, \frac{1}{i}\log z_d\right), \quad e^{-\delta_1} < |z_\mu| < e^{\delta_1}, \tag{5.17}$$

or

$$\Pi_{\mu\nu}(\zeta_1, \ldots, \zeta_d) = f_{\mu\nu}(e^{i\zeta_1}, \ldots, d^{i\zeta_d}).$$

The functions $f_{\mu\nu}(z)$ are analytic on the polyring $\bigtimes_\mu \{e^{-\delta_1} < |z_\mu| < e^{\delta_1}\}$, and the properties (5.12)–(5.15) imply the following ones:

$$f_{\mu\nu}(rz) = ((r \otimes r)f)_{\mu\nu}(z) \quad \text{if } r \text{ is a permutation}, \tag{5.18}$$

$$f_{\mu\nu}(z^\varepsilon) = \varepsilon_\mu \varepsilon_\nu z_\mu^{-(1-\varepsilon_\mu)/2} z_\nu^{(1-\varepsilon_\nu)/2} f_{\mu\nu}(z) \tag{5.19}$$

[here $z^\varepsilon = (z_1^{\varepsilon_1}, \ldots, z_d^{\varepsilon_d})$],

$$f_{\mu\nu}(z) = f_{\nu\mu}(z^{-1}), \tag{5.20}$$

$$\sum_\mu (z_\mu^{-1} - 1) f_{\mu\nu}(z) = \sum_\nu (z_\nu - 1) f_{\mu\nu}(z) = 0. \tag{5.21}$$

Now, a given function $f(z)$ analytic on the ring $\{e^{-\delta_1} < |z| < e^{\delta_1}\}$ can be represented as $f(z) = g^+(z) + g^-(z^{-1})$, where $g^+(z), g^-(z)$ are analytic on the disc $\{|z| < e^{\delta_1}\}$. This representation is obtained by taking regular and singular parts of the Laurent expansion. It is unique up to an additive constant, and it can be made unique requiring some normalization conditions. These conditions depend on the property (5.19). Consider cases with one component of ε equal to -1. If the index of this component is different from μ, ν, then the transformation law in the one variable is $f(z^{-1}) = f(z)$. The normalization condition $g^+(0) = g^-(0)$ implies then $g^-(z) = g^+(z)$, and we have the representation $f(z) = g(z) + g(z^{-1})$, $g(z) = g^+(z)$. If the index is equal to μ, then the transformation law is $f(z^{-1}) = -z^{-1} f(z)$. The normalization condition $g^+(0) = 0$ implies $g^-(z) = -z^{-1} g^+(z)$, and we have $f(z) = g(z) - z g(z^{-1})$. Finally, if the index is equal to ν, then $f(z^{-1}) = -z f(z)$, and the normalization condition $g^{-1}(0) = 0$ implies $g^-(z) = -z g^+(z)$, $f(z) = g(z) - z^{-1} g(z^{-1})$. Applying these representations to the functions $f_{\mu\nu}(z)$, to each variable separately, we obtain

$$f_{\mu\nu}(z) = \sum_\varepsilon g_{\mu\nu}^\varepsilon(z^\varepsilon), \tag{5.22}$$

where the functions $g_{\mu\nu}^\varepsilon(z)$ are analytic on the polydisc $\bigtimes_\mu \{|z_\mu| < e^{\delta_1}\}$. They satisfy the normalization conditions:

$$g_{\mu\nu}^{(\varepsilon', \varepsilon_\lambda = +1, \varepsilon'')}(z', 0, z'') = g^{(\varepsilon', \varepsilon_\lambda = -1, \varepsilon'')}(z', 0, z'') \tag{5.23}$$

for the indices λ different from μ, ν, if $\mu \neq \nu$, or for all the indices λ, if $\mu = \nu$;

$$g_{\mu\nu}^{(\varepsilon', \varepsilon_\mu = +1, \varepsilon'')}(z', 0, z'') = g_{\mu\nu}^{(\varepsilon', \varepsilon_\mu = -1, \varepsilon'')}(z', 0, z'') = 0, \tag{5.24}$$

if $\mu \neq \nu$. These normalization conditions and the transformation laws (5.19) imply the equalities

$$g^{\varepsilon}_{\mu\nu}(z) = \varepsilon_\mu \varepsilon_\nu z_\mu^{-(1-\varepsilon_\mu)/2} z_\nu^{(1-\varepsilon_\nu)/2} g^{(+1,\dots,+1)}_{\mu\nu}(z). \tag{5.25}$$

Denoting $g_{\mu\nu}(z) = g^{(+1,\dots,+1)}_{\mu\nu}(z)$, we get the following representation:

$$f_{\mu\nu}(z) = \sum_{\varepsilon} \varepsilon_\mu \varepsilon_\nu z_\mu^{(1-\varepsilon_\mu)/2} z_\nu^{-(1-\varepsilon_\nu)/2} g_{\mu\nu}(z^{\varepsilon}). \tag{5.26}$$

Let us now write properties of the functions $g_{\mu\nu}(z)$ equivalent to the properties (5.18)–(5.20). The equalities (5.18) are equivalent to

$$g_{\mu\nu}(rz) = ((r \otimes r)g)_{\mu\nu}(z) \quad \text{if } r \text{ is a permutation}. \tag{5.27}$$

The equalities (5.19) are implied by the form of the representation (5.26). The equalities (5.20) are equivalent to

$$g_{\mu\nu}(z) = z_\nu^{-1} z_\mu g_{\nu\mu}(z). \tag{5.28}$$

From the definition of the function $f_{\mu\nu}(z)$, we have also the following representation:

$$f_{\mu\nu}(z) = \sum_{x \in Z^d} z^{-x} \Pi_{\mu\nu}(x), \tag{5.29}$$

hence the terms of the representation (5.22) are obtained by restricting correspondingly the range of the summation in (5.29). This implies that all these terms, and in particular the function $g_{\mu\nu}(z)$, are real functions for real variables z. It is possible also to write the gauge invariance (5.21) in terms of the functions $g_{\mu\nu}(z)$, but it is much simpler to investigate its consequences later on, for simplified representations.

The first step to get (5.16) is to expand the functions $g_{\mu\nu}(z)$ up to the third order around the point $(1, \dots, 1) = 1$,

$$\begin{aligned} g_{\mu\nu}(z) &= g_{\mu\nu}(1) + \sum_{\kappa} (z_\kappa - 1)\left(\frac{\partial}{\partial z_\kappa} g_{\mu\nu}\right)(1) + \frac{1}{2} \sum_{\kappa,\lambda} (z_\kappa - 1)(z_\lambda - 1)\left(\frac{\partial^2}{\partial z_\kappa \partial z_\lambda} g_{\mu\nu}\right)(1) \\ &\quad + \sum_{\kappa,\lambda,\varrho} (z_\kappa - 1)(z_\lambda - 1)(z_\varrho - 1)\frac{1}{2}\int_0^1 dt(1-t)\left(\frac{\partial^3}{\partial z_\kappa \partial z_\lambda \partial z_\varrho} g_{\mu\nu}\right)(1 + t(z-1)) \\ &= g_{\mu\nu}(1) + \sum_{\kappa} a_{\mu\nu,\kappa}(z_\kappa - 1) + \frac{1}{2} \sum_{\kappa,\lambda} b_{\mu\nu,\kappa\lambda}(z_\kappa - 1)(z_\lambda - 1) \\ &\quad + \sum_{\kappa,\lambda,\varrho} g_{\mu\nu,\kappa\lambda\varrho}(z)(z_\kappa - 1)(z_\lambda - 1)(z_\varrho - 1). \end{aligned} \tag{5.30}$$

The coefficients a, b and the functions g above are real for real values of z, the functions are analytic on the polydisc $\bigtimes_{\mu} \{|z_\mu| < e^{\delta_1}\}$. Let us investigate the coefficients. The properties (5.27) imply

$$\left(\frac{\partial}{\partial z_\varrho} g_{\mu\nu}\right)(z) = \sum_{\kappa,\lambda,\sigma} r_{\mu\kappa} r_{\nu\lambda} r_{\varrho\sigma} \left(\frac{\partial}{\partial z_\sigma} g_{\kappa\lambda}\right)(r^{-1}z) \tag{5.31}$$

for all permutations r, identically for higher order derivatives, hence

$$a_{\mu\nu,\varrho}=\sum_{\kappa,\lambda,\sigma} r_{\mu\kappa}r_{\nu\lambda}r_{\varrho\sigma}a_{\kappa\lambda,\sigma}, \quad \text{the same for } b_{\mu\nu,\kappa\lambda}. \tag{5.32}$$

This implies that all the coefficients $a_{\mu\nu,\kappa}$ with three different indices are equal, similarly all the coefficients $a_{\mu\mu,\nu}$ with $\mu\neq\nu$ are equal, and so on. The same conclusions hold for $b_{\mu\nu,\kappa\lambda}$.

Now we substitute the expansion (5.30) into the representation (5.26). Only the first three terms on the right-hand side of (5.30) may contribute to the basic second order operator in (5.16), the third order terms in (5.30) give rise to the higher order terms of the remainder in (5.16). They have also the desired analyticity properties. Some of the lower order terms in (5.30) generate higher order terms also. Writing explicitly only the lower order terms, and making simple algebraic transformations, we finally get the following representation:

$$\begin{aligned}
f_{\mu\nu}(z)=&\ \delta_{\mu\nu}2^{d-2}g_{\mu\mu}(1)[4-(z_\mu^{-1}-1)(z_\mu-1)]\\
&+2^{d-2}[g_{\mu\nu}(1)-2a_{\mu\nu,\mu}+2a_{\mu\nu,\nu}-4b_{\mu\nu,\mu\nu}](z_\mu^{-1}-1)(z_\nu-1)\\
&-\delta_{\mu\nu}2^{d-1}\Big[\sum_\kappa(a_{\mu\mu,\kappa}+b_{\mu\mu,\kappa\kappa})(z_\kappa^{-1}-1)(z_\kappa-1)\\
&-2(a_{\mu\mu,\mu}+b_{\mu\mu,\mu\mu})(z_\mu^{-1}-1)(z_\mu-1)\Big]\\
&+\sum_{\kappa,\lambda,\varrho}[f'_{\mu\nu,\kappa\lambda\varrho}(z)(z_\kappa^{-1}-1)(z_\lambda^{-1}-1)(z_\varrho^{-1}-1)\\
&+f'_{\mu\nu,\kappa,\lambda\varrho}(z)(z_\kappa-1)(z_\lambda^{-1}-1)(z_\varrho^{-1}-1)+\dots],
\end{aligned} \tag{5.33}$$

where the dots denote summation over other possible third order monomials in $z^{-1}-1$, $z-1$. The coefficients f' are analytic functions on the polyring in (5.17). Let us make a few comments about the above formula. For $z=1$ we get $f_{\mu\nu}(1)=\delta_{\mu\nu}2^d g_{\mu\mu}(1)$, hence $f_{\mu\nu}(1)=0$ for $\mu\neq\nu$. Differentiating the second identity in (5.21) with respect to z_μ and taking it at the point 1 yields $f_{\mu\mu}(1)=0$, hence $g_{\mu\mu}(1)=0$ by the above equality. Thus the whole term in the first line on the right-hand side of (5.33) is equal to 0. Now we investigate consequences of the gauge invariance (5.21) for the explicitly written lower order terms in (5.33). We multiply (5.33) by $z_\nu-1$, sum over ν, and we obtain an expression identically equal to 0. This expression is an analytic function in a neighborhood of the point 1. Expanding it into a power series we obtain a system of equations from the fact that the coefficients of this power series are equal to 0. We have to consider third order terms only, and the equations are

$$\begin{aligned}
&g_{\mu\nu}(1)-2a_{\mu\nu,\mu}+2a_{\mu\nu,\nu}-4b_{\mu\nu,\mu\nu}=2(a_{\mu\mu,\nu}+b_{\mu\mu,\nu\nu}) \quad \text{for} \quad \mu\neq\nu,\\
&g_{\mu\mu}(1)-2a_{\mu\mu,\mu}+2a_{\mu\mu,\mu}-4b_{\mu\mu,\mu\mu}=-2(a_{\mu\mu,\mu}+b_{\mu\mu,\mu\mu}).
\end{aligned} \tag{5.34}$$

Denoting the left-hand side of these equations by $-\beta_{\mu\nu}$, we have

$$f_{\mu\nu}(z)=-2^{d-2}\beta_{\mu\nu}(z_\mu^{-1}-1)(z_\nu-1)+\delta_{\mu\nu}2^{d-2}\sum_\kappa\beta_{\mu\kappa}(z_\kappa^{-1}-1)(z_\kappa-1)+\dots. \tag{5.35}$$

We have discussed already the consequences of the symmetries (5.32). They imply that all the coefficients $\beta_{\mu\nu}$ for $\mu\neq\nu$ are equal. Denoting the common value of $2^{d-2}\beta_{\mu\nu}$ by β, we get

$$f_{\mu\nu}(z)=\beta\Big(\delta_{\mu\nu}\sum_{\kappa}(z_\kappa^{-1}-1)(z_\kappa-1)-(z_\mu^{-1}-1)(z_\nu-1)\Big)+\dots. \tag{5.36}$$

This is the desired representation of the functions $f_{\mu\nu}(z)$. Using the relations (5.17) and substituting $z_\mu=e^{ip_\mu}$, we obtain finally

$$\Pi_{\mu\nu}(p)=\beta(\delta_{\mu\nu}\Delta(p)-\overline{\partial_\mu(p)}\partial_\nu(p))+\Pi'_{\mu\nu}(p). \tag{5.37}$$

The function $\Pi'_{\mu\nu}(p)$ has all the symmetries of the function $\Pi_{\mu\nu}(p)$ and it can be written in the form of a third order polynomial in the derivatives $\overline{\partial(p)}, \partial(p)$,

$$\Pi'_{\mu\nu}(p)=\sum_{\kappa,\lambda,\varrho}[\Pi'_{\mu\nu,\kappa\lambda\varrho}(p)\overline{\partial_\kappa(p)}\overline{\partial_\lambda(p)}\overline{\partial_\varrho(p)}+\Pi'_{\mu\nu,\kappa,\lambda\varrho}(p)\partial_\kappa(p)\overline{\partial_\lambda(p)}\overline{\partial_\varrho(p)}+\dots]. \tag{5.38}$$

The coefficients Π' can be extended to analytic functions of $\zeta=p+iq$ on the polystrip $\bigtimes_{\mu}\{|q_\mu|<\delta_1\}$.

It remains to calculate the coefficient β, i.e., to express it in terms of the tensor Π. We have

$$\beta=2^{d-2}(-g_{\mu\nu}(1)+2a_{\mu\nu,\mu}-2a_{\mu\nu,\nu}+4b_{\mu\nu,\mu\nu}) \tag{5.39}$$

for $\mu\neq\nu$, e.g., for $\mu=1$, $\nu=2$. The representation (5.26) yields

$$\begin{aligned}f_{12}(z_1,z_2,1)=2^{d-2}(&z_1z_2^{-1}g_{12}(z_1^{-1},z_2^{-1},1)-z_2^{-1}g_{12}(z_1,z_2^{-1},1)\\&-z_1g_{12}(z_1^{-1},z_2,1)+g_{12}(z_1,z_2,1)).\end{aligned} \tag{5.40}$$

Differentiating it with respect to z_1,z_2, at $z_1=z_2=1$, we obtain

$$\left(\frac{\partial^2}{\partial z_1\partial z_2}f_{12}\right)(1)=2^{d-2}(-g_{12}(1)+2a_{12,1}-2a_{12,2}+4b_{12,12})=\beta, \tag{5.41}$$

and

$$\beta=-\left(\frac{\partial^2}{\partial p_1\partial p_2}\Pi_{12}\right)(0)=-\left(\frac{\partial^2}{\partial p_\mu\partial p_\nu}\Pi_{\mu\nu}\right)(0)=\sum_x\Pi_{\mu\nu}(x)x_\mu x_\nu \tag{5.42}$$

for $\mu\neq\nu$. This is the fundamental equality defining the β-function.

Let us finish now the analysis of the previous section. The first term on the right-hand side of (4.34), after all the changes and resummations, and using (5.37), (5.38), can be written as

$$\begin{aligned}&\sum_{(x,\mu),(y,\nu)}\Pi_{\mu\nu}(x-y)\,\mathrm{tr}\,\delta B_\mu(x)B_\nu(y)\\&=\beta\tfrac{1}{2}\sum_{x,\mu,\nu}\mathrm{tr}(\partial\delta B)_{\mu\nu}(x)(\partial B)_{\mu\nu}(x)\\&\quad+\sum_{(x,\mu),(y,\nu),\kappa,\lambda,\varrho}[\Pi'_{\mu\nu,\kappa\lambda\varrho}(x-y)\,\mathrm{tr}(\partial_\kappa\partial_\lambda\partial_\varrho\delta B_\mu)(x)B_\nu(y)\\&\quad+\Pi'_{\mu\nu,\kappa,\lambda\varrho}(x-y)\,\mathrm{tr}(\partial_\lambda\partial_\varrho\delta B_\mu)(x)(\partial_\kappa B_\nu)(y)+\dots].\end{aligned} \tag{5.43}$$

The analyticity properties mentioned after (5.38) imply the corresponding exponential decay properties of the functions in the above formula. More exactly, we have

$$|\Pi'_{\mu\nu,\kappa\lambda\varrho}(x-y)|\leqq O(1)E_0\exp(\tfrac{1}{2}\delta_1|x-y|). \tag{5.44}$$

In the third order terms in (5.43) we can always shift the derivatives onto the other function, so we can write them in the form in which δB is differentiated once, and B twice. The bound (5.44), and the bounds (4.14), (4.15) for derivatives of δB, B imply that all these third order terms in (5.43) are irrelevant. Defining the function β_j as equal to the coefficient β in (5.43), we see that the first expression there is cancelled by the first expression in (4.42). The second term on the right-hand side of (4.34) is equal to (4.44), and is cancelled by this expression. The third term is equal to 0. In both cases we have used the fact that the first and second order derivatives of the function $\Pi'_{\mu\nu}(p)$ vanish at $p=0$, so we have to calculate the corresponding expressions in (4.34) for the first term on the right-hand side of (5.37). These calculations are the same as in (4.43), (4.45).

Thus we have finished the analysis of the fluctuation field effective action in (2.13) from the point of view of analyticity properties and bounds. We have transformed this action in such a way that a uniform bound is clear. The terms of the transformed action have also good analyticity properties, but they are still nonlocal in the fluctuation and the background field; more exactly they do not have localization properties connected with good bounds. In the next paper we will construct such localizations and prove the bounds. They are used to construct a cluster expansion of the integral in (2.13), and to finish the proof of the inductive assumptions for the term $\mathbf{E}^{(k+1)}$ defined by (2.13).

Finally, let us make some remarks about the functions $\mathbf{E}^{(n)}$, obtained by the changes and the resummations of the functions $\mathbf{E}^{(n)}(X)$ in (4.3), and about the β-functions. In this section we have analyzed the function $\mathbf{E}^{(2)}=\Pi$ using the symmetry properties and the analyticity properties of the momentum representation. We can do a similar analysis for the higher functions, e.g., for $\mathbf{E}^{(3)}$, $\mathbf{E}^{(4)}$, again using the symmetry properties, especially the Ward-Takahashi identities (4.15). We can represent these functions as sums of basic marginal operators with the coefficient β, and higher order operators leading to irrelevant terms. Then we can use this representation to analyze the whole expression (4.3) in the same way as the term with $\mathbf{E}^{(2)}$ was analyzed in (5.43). This would give a method alternative to that applied in the previous section. It is connected, unfortunately, with a lot of technicalities and calculations of an algebraic character, and it seems that the method presented in the last section is more clear and simpler. The β-functions are related in the simple way to the tensors Π by the formula (5.42). In fact we should write the superscript (j) at the tensor in the formula (5.42) defining the function β_j. We write β_j as explicitly dependent on g_{j-1}, although it depends also on all preceding coupling constants. The dependence on g_{j-1} is important and it determines main properties of the renormalization group equations.

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