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The Cohomological Construction of Stora's Solutions

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Abstract. Details of the cohomological construction of Stora's solutions to the Wess-Zumino consistency condition are given, where the Lie algebra consists of infinitesimal diffeomorphisms and gauge transformations on a non-trivial principal bundle over an arbitrary even-dimensional base space.

1. Introduction

Anomalies are said to occur when symmetries of a classical theory are broken by quantum corrections. In the following we shall be concerned with the anomalies of the infinitesimal symmetries of a gauge theory over an arbitrary even-dimensional space-time manifold. For a detailed review and list of references we recommend the article by Alvarez-Gaumé and Ginsparg [1]. Anomalies are defined in the context of quantum theory. Quantization of a field theory over a space-time, which is not a vector space, is still an open problem. However, starting from the Wess-Zumino consistency condition [2], Stora has indicated a purely algebraic algorithm classifying infinitesimal gauge anomalies in four-dimensional Minkowski space [3]. Using cohomological methods he indicated the construction of a class of solutions to the Wess-Zumino consistency condition. In particular this class contains the Adler-Bardeen anomaly [4]. Becchi et al. [5] had shown that for any renormalizable gauge theory all solutions are of Stora's type. Later Stora [6] and Zumino [7] have produced algebraic formulas which apply to trivial bundles over arbitrary even-dimensional base spaces. Finally Langouche et al. [8] have generalized it to non-trivial bundles and also included infinitesimal diffeomorphisms. In the following we shall give the details of this proof. Our conventions are those of [9].

2. The Base Space

Let M, the base space, be an arbitrary manifold of even dimension $n=2j-2$. N.B. for our purpose we do not need a metric on M. We denote by $\mathrm{Vect}(M)$ the infinite dimensional Lie algebra of vector fields on M and by

$$\Lambda M = \bigoplus_{q=0}^{n} \Lambda^q M \tag{2.1}$$

the infinite dimensional Grassmann algebra of differential forms on M. Consider linear maps

$$D: \Lambda^q M \to \Lambda^{q+d} M$$

satisfying the Leibniz rule

$$D(\varphi \wedge \psi) = (D\varphi) \wedge \psi + (-1)^{d \cdot \deg\varphi} \varphi \wedge D\psi . \tag{2.2}$$

They are called a (graded) derivation of ΛM of degree d. The set of all derivations of ΛM is an infinite dimensional graded Lie algebra with bracket

$$[D_1, D_2] := D_1 D_2 - (-1)^{d_1 d_2} D_2 D_1 , \tag{2.3}$$

i.e. the bracket is bilinear, graded commutative:

$$[D_1, D_2] = -(-1)^{d_1 d_2} [D_2, D_1] , \tag{2.4}$$

and it satisfies the Jacobi identity:

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{d_1 d_2} [D_2, [D_1, D_3]] . \tag{2.5}$$

The inner derivatives i_v, $v \in \mathrm{Vect}(M)$ of degree minus one, the Lie derivatives L_v of degree zero and the exterior derivative of degree one form an infinite dimensional graded Lie subalgebra with brackets:

$$[i_v, i_w] = i_v i_w + i_w i_v = 0 , \tag{2.6}$$

$$[L_v, i_w] = L_v i_w - i_w L_v = i_{[v,w]} , \tag{2.7}$$

$$[i_v, d] = i_v d + d i_v = L_v , \tag{2.8}$$

$$[L_v, L_w] = L_v L_w - L_w L_v = L_{[v,w]} , \tag{2.9}$$

$$[L_v, d] = L_v d - d L_v = 0 , \tag{2.10}$$

$$[d, d] = dd + dd = 0 . \tag{2.11}$$

The Lie derivatives alone are a Lie subalgebra isomorphic to $\mathrm{Vect}(M)$. Therefore the vector fields represent the infinitesimal diffeomorphisms.

3. The Principal Bundle and Its Infinitesimal Automorphisms

Let P be a principal bundle over M with structure group G and a trivializing open covering $\{U_r\}$ of M. Let $\mathring{\mathscr{A}}$ be a fixed connection on P. By means of $\mathring{\mathscr{A}}$ all vector fields v on M can be lifted to infinitesimal bundle automorphisms, and together with the infinitesimal gauge transformations they form an infinite dimensional Lie algebra $\mathscr{E}$. Let U be one of the trivializing open subset of M. After pull back with a local section, $\mathring{\mathscr{A}}$ is represented locally by a 1-form $\mathring{A}$ on U with values in the Lie algebra $\mathfrak{g}$ of G:

$$\mathring{A} \in \Lambda^1(U, \mathfrak{g}) .$$

The elements of $\mathscr{E}$ are represented on U by pairs (Ω, v),

$$\Omega \in \Lambda^0(U, \mathfrak{g}),$$
$$v \in \mathrm{Vect}(U),$$

with commutation relations:

$$[(\Omega', 0), (\Omega, 0)] = ([\Omega', \Omega], 0), \tag{3.1}$$

$$[(0, v'), (0, v)] = (-i_{v'} i_v \mathring{F}, [v', v]), \tag{3.2}$$

$$[(0, v), (\Omega, 0)] = (L_v \Omega + [i_v \mathring{A}, \Omega], 0), \tag{3.3}$$

where

$$\mathring{F} := d\mathring{A} + \tfrac{1}{2}[\mathring{A}, \mathring{A}]. \tag{3.4}$$

The rôle of $\mathring{\mathscr{A}}$ is to ensure that the commutators can be patched together on the overlaps $U_r \cap U_s$. This is achieved by replacing the exterior derivative in the Lie derivative by a covariant exterior derivative $\mathring{D}$ with respect to $\mathring{A}$. Indeed:

$$[(0, v), (\Omega, 0)] = (\mathring{\mathscr{L}}_v \Omega, 0) \tag{3.5}$$

with

$$\mathring{\mathscr{L}}_v := i_v \mathring{D} + \mathring{D} i_v. \tag{3.6}$$

Different connections $\mathring{\mathscr{A}}$ on P yield isomorphic Lie algebras $\mathscr{E}$. Note that the gauge transformations $(\Omega, 0)$ form an ideal of $\mathscr{E}$, while the vector fields $(0, v)$ in general do not form a subalgebra. However, if P is trivial, we can choose $U = M$ and $\mathring{A} = 0$. Then $\mathscr{E}$ is the semi-direct product of $\Lambda^0(M, \mathfrak{g})$ and $\mathrm{Vect}(M)$.

The affine space of all connections on P carries an affine representation R of $\mathscr{E}$ given locally by

$$\begin{aligned} R(\Omega, v)A &= -d\Omega - [A, \Omega] + L_v A - d i_v \mathring{A} - [A, i_v \mathring{A}] \\ &= -D\Omega + i_v F + D i_v (A - \mathring{A}), \end{aligned} \tag{3.7}$$

where $A \in \Lambda^1(U, \mathfrak{g})$ is the local expression on U of a connection $\mathscr{A}$ on P, D is the covariant exterior derivative with respect to A and

$$F := dA + \tfrac{1}{2}[A, A]. \tag{3.9}$$

By definition the fixed auxiliary connection $\mathring{\mathscr{A}}$ does not transform under $\mathscr{E}$,

$$R(\Omega, v)\mathring{A} = 0. \tag{3.9}$$

4. The Wess-Zumino Consistency Condition

Next we introduce Pl the space of "local" polynomials. N.B. the word "local" here refers to quantum theory and has nothing to do with the same word in the next sentence. Pl is the infinite dimensional vector space which we describe again locally on a trivializing open subset U of M: Let $T_1, T_2, \ldots, T_d$, $d = \dim G$, be a basis of the

Lie algebra $\mathfrak{g}$ with structure constants $f_{ij}{}^{k}$, $i,j,k=1,2,\ldots,d$,

$$[T_i, T_j] =: \sum_{k=1}^{d} f_{ij}{}^{k} T_k . \tag{4.1}$$

We decompose the 1-form $A \in \Lambda^1(U,\mathfrak{g})$ with respect to this basis

$$A =: \sum_{i=1}^{d} A^i T_i , \tag{4.2}$$

where the A^i are now real-valued 1-forms on U. An element $\not{p} \in Pl$ is represented on U by an ${}^{U}G$-invariant polynomial p in the A^i and their exterior and inner derivatives. The coefficients are from ΛU, the product is the wedge product and the interior derivatives are with respect to some given (fixed) vector fields on M. ${}^{U}G$ denotes the group of gauge transformations on U. An element γ of ${}^{U}G$ is a map from U to G. Under γ both connections $\mathring{A}$ and A transform:

$$\mathring{A}' := \gamma \mathring{A} \gamma^{-1} + (\gamma^{-1})^* \zeta , \tag{4.3}$$

$$A' := \gamma A \gamma^{-1} + (\gamma^{-1})^* \zeta , \tag{4.4}$$

where ζ is the Maurer-Cartan form on G. The local invariance of the polynomials p under ${}^{U}G$ ensures that they can be patched together on the bundle. Pl is a graded vector space

$$Pl = \bigoplus_{q=0}^{n} Pl_q , \tag{4.5}$$

where q is the degree of the polynomial as differential form.

The affine representation R of $\mathscr{E}$ on the connections induces a linear representation W of $\mathscr{E}$ on the vector spaces Pl_q locally given by the "Ward operators":

$$W(E)p := [p(A+\alpha) - p(A)]_{\text{lin}}|_{\alpha = -R(E)A} \tag{4.6}$$

with $E = (\Omega, v)$. The subscript lin means: Keep only terms linear in α, a different way of denoting the functional derivative. The Ward operator has the following properties:

$$W(E)A = -R(E)A , \tag{4.7}$$

$$W(E)(p \wedge p') = (W(E)p) \wedge p' + p \wedge W(E)p' , \tag{4.8}$$

$$dW(E)p = W(E)dp , \tag{4.9}$$

$$i_v W(E)p = W(E) i_v p , \qquad v \in \text{Vect}(U) . \tag{4.10}$$

An anomaly $\mathfrak{A}(E)$ is a linear map from $\mathscr{E}$ to Pl_n defined only up to exact forms and variations of "local" polynomials

$$\mathfrak{A}(E) \sim \mathfrak{A}(E) + dX(E) + W(E)p , \qquad X(E) \in \Lambda^{n-1} U , \qquad p \in Pl_n . \tag{4.11}$$

The anomalies satisfy the Wess-Zumino consistency condition

$$W(E')\mathfrak{A}(E) - W(E)\mathfrak{A}(E') = \mathfrak{A}([E', E]) \text{ modulo exact forms} , \tag{4.12}$$

for all E', $E \in \mathscr{E}$.

5. Stora's Solutions

Stora's solutions are the linear maps from $\mathscr{E}$ to Pl_n given locally on U by:

$$\begin{aligned}\mathfrak{A}(\Omega, v) = & -j \int_0^1 d\tau I(\Omega, F_\tau^{j-1}) \\ & + j(j-1) \int_0^1 d\tau I(A-\mathring{A}, (\tau^2-\tau)[\Omega, A-\mathring{A}], F_\tau^{j-2}) \\ & + j(j-1) \int_0^1 d\tau I(A-\mathring{A}, (1-\tau) i_v \mathring{F}, F_\tau^{j-2}), \end{aligned} \tag{5.1}$$

where j was defined by $\dim M = n = 2j-2$, I is a symmetric invariant j-linear form on the Lie algebra $\mathfrak{g}$, and

$$F_\tau := d(\mathring{A} + \tau(A-\mathring{A})) + \tfrac{1}{2}[\mathring{A} + \tau(A-\mathring{A}), \mathring{A} + \tau(A-\mathring{A})]. \tag{5.2}$$

In principle one can of course show by brute force that (5.1) solves the consistency condition. In the following we give details of the cohomological proof [8].

6. The Proof

Let

$$A(\mathscr{E}, Pl) = \bigoplus_{\ell=0}^{\infty} A^\ell(\mathscr{E}, Pl) \tag{6.1}$$

be the space of alternating ℓ-forms on $\mathscr{E}$ with values in Pl_q. It is a doubly graded vector space with grading (ℓ, q), ℓ is often addressed as "ghost number." We make $A(\mathscr{E}, Pl)$ a simply graded associative algebra with grading $\ell + q$ by defining the following product:

$$\begin{aligned} \wedge : A^\ell(\mathscr{E}, Pl_q) \times A^{\ell'}(\mathscr{E}, Pl_{q'}) &\to A^{\ell+\ell'}(\mathscr{E}, Pl_{q+q'}), \\ (Q, Q') &\mapsto Q \wedge Q', \\ (Q \wedge Q')(E_1, \ldots, E_{\ell+\ell'}) &:= \frac{(-1)^{\ell q'}}{\ell! \ell'!} \sum_{\pi \in S_{\ell+\ell'}} \operatorname{sign} \pi \\ &\times Q(E_{\pi(1)}, \ldots, E_{\pi(\ell)}) \wedge Q'(E_{\pi(\ell+1)}, \ldots, E_{\pi(\ell+\ell')}). \end{aligned} \tag{6.2}$$

It is graded commutative:

$$Q \wedge Q' = (-1)^{(\ell+q)(\ell'+q')} Q' \wedge Q. \tag{6.3}$$

We define five linear maps d, i_ξ, L_ξ, $i_{[\xi,\xi]}$ and $L_{[\xi,\xi]}$:

$$\begin{aligned} d : A^\ell(\mathscr{E}, Pl_q) &\to A^\ell(\mathscr{E}, Pl_{q+1}), \\ Q &\mapsto dQ, \\ (dQ)(E_1, \ldots, E_\ell) &:= d(Q(E_1, \ldots, E_\ell)), \end{aligned} \tag{6.4}$$

$$\begin{aligned} i_\zeta : A^\ell(\mathscr{E}, Pl_q) &\to A^{\ell+1}(\mathscr{E}, Pl_{q-1}), \\ Q &\mapsto i_\zeta Q, \\ (i_\zeta Q)(E_0, E_1, \ldots, E_\ell) &:= (-1)^q \sum_{a=0}^{\ell} (-1)^a i_{v_a} Q(E_{01}, \ldots, \hat{E}_a, \ldots, E_\ell), \end{aligned} \tag{6.5}$$

where the argument with hat ^ is omitted.

$$L_\xi : \Lambda^\ell(\mathscr{E}, Pl_q) \to \Lambda^{\ell+1}(\mathscr{E}, Pl_q),$$
$$Q \mapsto L_\xi Q,$$
$$(L_\xi Q)(E_0, \ldots, E_\ell) := (-1)^{q+1} \sum_{a=0}^{\ell} (-1)^a L_{v_a} Q(E_0, \ldots, \hat{E}_a, \ldots, E_\ell), \tag{6.6}$$
$$i_{[\xi,\xi]} : \Lambda^\ell(\mathscr{E}, Pl_q) \to \Lambda^{\ell+2}(\mathscr{E}, Pl_{q-1}),$$
$$Q \mapsto i_{[\xi,\xi]} Q,$$
$$(i_{[\xi,\xi]} Q)(E_{-1}, E_0, \ldots, E_\ell) := -2 \sum_{\substack{a,b=-1 \\ a<b}}^{\ell} (-1)^{a+b} i_{[v_a, v_b]} Q(E_{-1}, \ldots, \hat{E}_a, \ldots, \hat{E}_b, \ldots, E_\ell), \tag{6.7}$$

and finally

$$L_{[\xi,\xi]}, \Lambda^\ell(\mathscr{E}, Pl_q) \to \Lambda^{\ell+2}(\mathscr{E}, Pl_q),$$
$$Q \mapsto L_{[\xi,\xi]} Q,$$
$$(L_{[\xi,\xi]} Q)(E_{-1}, E_0, \ldots, E_\ell) := -2 \sum_{\substack{a,b=-1 \\ a<b}}^{\ell} (-1)^{a+b} L_{[v_a, v_b]} Q(E_{-1}, \ldots, \hat{E}_a, \ldots, \hat{E}_b, \ldots, E_\ell). \tag{6.8}$$

They are all derivations of $\Lambda(\mathscr{E}, Pl)$ with respect to the wedge product (6.2). Their degrees are one, zero, one, one and two, respectively, and they form a 5-dimensional graded Lie subalgebra with brackets:

$$[i_\xi, d] = i_\xi d - d i_\xi = L_\xi, \tag{6.9}$$
$$[L_\xi, i_\xi] = L_\xi i_\xi - i_\xi L_\xi = i_{[\xi,\xi]}, \tag{6.10}$$
$$[L_\xi, L_\xi] = L_\xi L_\xi + L_\xi L_\xi = L_{[\xi,\xi]}, \tag{6.11}$$
$$[i_{[\xi,\xi]}, d] = i_{[\xi,\xi]} d + d i_{[\xi,\xi]} = L_{[\xi,\xi]}. \tag{6.12}$$

All other brackets vanish. Note the different signs with respect to Eqs. (2.6)–(2.11).

We now define a sixth linear map s, which leads to the Lie algebra cohomology. In this context s is often called BRS operator,

$$s : \Lambda^\ell(\mathscr{E}, Pl_q) \to \Lambda^{\ell+1}(\mathscr{E}, Pl_q),$$
$$Q \mapsto sQ,$$
$$(sQ)(E_0, \ldots, E_\ell) := (-1)^{q+1} \sum_{a=0}^{\ell} (-1)^a W(E_a) Q(E_0, \ldots, \hat{E}_a, \ldots, E_\ell)$$
$$+ (-1)^{q+1} \sum_{\substack{a,b=0 \\ a<b}}^{\ell} (-1)^{a+b} Q([E_a, E_b], E_0, \ldots, \hat{E}_a, \ldots, \hat{E}_b, \ldots, E_\ell). \tag{6.13}$$

Again it is a derivation of $\Lambda(\mathscr{E}, Pl)$ with grading one. The operator s together with the other five form a six-dimensional graded Lie subalgebra, the additional nonvanishing brackets are:

$$[s, i_\xi] = si_\xi - i_\xi s = -\tfrac{1}{2} i_{[\xi,\xi]}, \tag{6.14}$$

$$[s, L_\xi] = sL_\xi + L_\xi s = -\tfrac{1}{2} L_{[\xi,\xi]}. \tag{6.15}$$

With these definitions we have: The solutions of the Wess-Zumino consistency condition are in one-to-one correspondence with the cohomology group of $\Lambda^1(\mathscr{E}, Pl_n)$ with respect to the co-boundary operator $d+s$.

Next we define the "(algebraic) Faddeev-Popov ghost" z. It is the element of $\Lambda^1(\mathscr{E}, Pl_0)$ given by

$$z(E) = -\Omega. \tag{6.16}$$

Its definition resembles the Maurer-Cartan form, and indeed it transforms as

$$sz = -\tfrac{1}{2}[z, z] - L_\xi z - \tfrac{1}{2} i_\xi i_\xi \mathring{F} - [i_\xi \mathring{A}, z]. \tag{6.17}$$

Furthermore Eq. (3.7) is equivalent to

$$sA = -dz - [A, z] - L_\xi A - di_\xi \mathring{A} - [A, i_\xi \mathring{A}]. \tag{6.18}$$

The following lemma [10] is known as homotopy formula. Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be two connections and $\mathbf{d}$ a co-boundary operator

$$\mathbf{d}^2 = 0. \tag{6.19}$$

Define an interpolating connection

$$\mathbf{A}_\tau := \mathbf{A}_0 + \tau(\mathbf{A}_1 - \mathbf{A}_0). \tag{6.20}$$

Let $\mathbf{F}_0$, $\mathbf{F}_1$, and $\mathbf{F}_\tau$ be the corresponding curvatures with respect to $\mathbf{d}$, e.g.

$$\mathbf{F}_0 := \mathbf{d}\mathbf{A}_0 + \tfrac{1}{2}[\mathbf{A}_0, \mathbf{A}_0]. \tag{6.21}$$

Then

$$I(\mathbf{F}_1^j) - I(\mathbf{F}_0^j) = \mathbf{d}Q, \tag{6.22}$$

where

$$Q := j \int_0^1 d\tau I(\mathbf{A}_1 - \mathbf{A}_0, \mathbf{F}_\tau^{j-1}) \tag{6.23}$$

is a Chern-Simons form.

We use this lemma by putting

$$\mathbf{A}_0 := \mathring{A}, \tag{6.24}$$

$$\mathbf{A}_1 := A + z - i_\xi(A - \mathring{A}), \tag{6.25}$$

$$\mathbf{d} := d + s \tag{6.26}$$

a straightforward calculation gives:

$$\mathbf{F}_1 = F - i_\xi F + \tfrac{1}{2} i_\xi i_\xi F. \tag{6.27}$$

The Chern-Simons form Q is of total degree $\ell+q=2j-1$ and can therefore be decomposed:

$$Q=Q^1_{2j-2}+Q^2_{2j-3}+\ldots+Q^{2j-1}_0 \tag{6.28}$$

with

$$Q^\ell_q\in A^\ell(\mathscr{E},Pl_q).$$

We are interested in the component $\ell=2$, $q=2j-2$ of Eq. (6.22):

$$\tfrac{1}{2}i_\xi i_\xi I(F^j)=dQ^2_{2j-3}+sQ^1_{2j-2}. \tag{6.29}$$

But $I(F^j)$ is a differential $2j$-form on the $2j-2$ dimensional manifold U, hence zero. Therefore $\mathfrak{A}=Q^1_{2j-2}$ represents an element of the cohomology group of $A^1(\mathscr{E},Pl_{2j-2})$. Its explicit form is

$$\begin{aligned}\mathfrak{A}=&j\int_0^1 d\tau I(z,F_\tau^{j-1})\\&+j(j-1)\int_0^1 d\tau I(A-\mathring{A},(\tau^2-\tau)[z,A-\mathring{A}],F_\tau^{j-2})\\&+j(j-1)\int_0^1 d\tau I(A-\mathring{A},(1-\tau)i_\xi\mathring{F},F_\tau^{j-2}).\end{aligned} \tag{6.30}$$

Evaluating $\mathfrak{A}$ on a Lie algebra element $E=(\Omega,v)$ we obtain the desired solution (5.1).

Finally we remark that in this general setting it is not known whether there are other solutions [11].

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