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Local and Non-Local Conserved Quantities for Generalized Non-Linear Schrödinger Equations

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Abstract. It is shown how to construct infinitely many conserved quantities for the classical non-linear Schrödinger equation associated with an arbitrary Hermitian symmetric space G/K. These quantities are non-local in general, but include a series of local quantities as a special case. Their Poisson bracket algebra is studied, and is found to be a realization of the "half" Kac-Moody algebra $\mathscr{k}_R \otimes \mathbb{C}[\lambda]$, consisting of polynomials in positive powers of a complex parameter λ which have coefficients in the compact real form of $\mathscr{k}$ (the Lie algebra of K).

1. Introduction

Fordy and Kulish [1] have considered a class of non-linear partial differential equations, each associated with an Hermitian symmetric space G/K, which are of the form

$$iq_t^\alpha = q_{xx}^\alpha - q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta}, \tag{1.1}$$

where summation is implied over repeated indices. $q^\alpha(x, t)$ are fields in one space dimension whose label α denotes a root of $\mathscr{g}$ (the Lie algebra of G) such that the step operator e_α does not lie in $\mathscr{k}$ (the Lie algebra of K). R is the "curvature tensor" defined by

$$e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]]. \tag{1.2}$$

A special case of (1.1), corresponding to $G=SU(2)$, is the non-linear Schrödinger (NLS) equation

$$iq_t = q_{xx} + 2|q|^2 q. \tag{1.3}$$

Equation (1.1) will be referred to as the Generalized non-linear Schrödinger (GNLS) equation associated with G/K. The NLS equation is known to have infinitely many conserved quantities which are local [in the sense that the currents are expressed only in terms of the fields $q(x, t)$, $q^*(x, t)$ and their derivatives at a point], and are in involution (i.e. their Poisson bracket algebra is abelian). The aim of this paper is to construct the algebra of conserved quantities for the GNLS equation.

The existence of such quantities is related to the fact that the equation of motion can be expressed as a "zero curvature condition"

$$F_{xt} \equiv [\partial_x + A_x, \partial_t + A_t] = 0\,, \tag{1.4}$$

where A_x, A_t are Lie algebra valued polynomials in a parameter $\lambda \in \mathbb{C}$ (the "spectral parameter") which does not appear in the equation of motion. Equation (1.4) is the consistency condition for the coupled pair of linear equations

$$\Phi_x + A_x \Phi = 0\,, \tag{1.5a}$$

$$\Phi_t + A_t \Phi = 0\,. \tag{1.5b}$$

For the NLS equation, A_x and A_t are 2×2 matrices, and it is fairly easy to construct the group element Φ (the "monodromy matrix"). The logarithm of its diagonal elements can be expanded in powers of λ to give conserved quantities [2].

It is shown in [1] that the GNLS equation is associated with the pair

$$A_x = \lambda E + A_x^0\,, \tag{1.6a}$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2\,[A_x^0\,[A_x^0, E]]\,, \tag{1.6b}$$

where E is a special constant element which commutes with any element of $\mathscr{h}$, satisfying

$$[E, e_\alpha] = -ie_\alpha \quad \text{for all } \alpha \tag{1.7}$$

and

$$A_x^0 \equiv -q^\alpha e_\alpha + q^{\alpha *} e_{-\alpha}\,. \tag{1.8}$$

For algebras of rank greater than one, the monodromy matrix (which is a path ordered exponential) becomes difficult to work with. It is then more convenient to use the algebraic properties of the zero curvature condition (1.4), in particular its invariance under a gauge transformation

$$A_x \to a_x = \omega^{-1} A_x \omega + \omega^{-1} \omega_x\,, \tag{1.9a}$$

$$A_t \to a_t = \omega^{-1} A_t \omega + \omega^{-1} \omega_t\,, \tag{1.9b}$$

where $\omega \in G$. The new pair a_x, a_t are associated with the same equation of motion as the pair A_x, A_t. Olive and Turok [3] have used this invariance to study the Toda equation. In that case it is possible to construct ω so that $a_x, a_t \in \mathscr{h}$, the Cartan subalgebra. ω takes the form

$$\omega = \exp \sum_{n=1}^{\infty} \lambda^{-n} \omega_n \tag{1.10}$$

and is local. Then a_x and a_t are descending power series, and the zero curvature condition becomes

$$\partial_x a_t - \partial_t a_x = 0\,, \tag{1.11}$$

which implies that the coefficients of arbitrary powers of λ are conserved currents.

In attempting to apply this method to the GNLS equation, one encounters the problem that the gauge transformation which takes A_x and A_t into the Cartan subalgebra is now non-local, so that Eq. (1.11) can no longer be interpreted as a conservation law. In order to discuss non-local conserved quantities, it is necessary to investigate the Poisson bracket algebra.

Consider first the Hamiltonian of the GNLS equation. Using (1.7) and (1.8) one finds

$$\mathrm{Tr}\left([E, A_x^0]\,\partial_t A_x^0\right) = iq^\alpha q_t^{\alpha *} - iq_t^\alpha q^{\alpha *}\,. \tag{1.12}$$

If q^α, $q^{\alpha *}$ are regarded as canonical variables, then differentiation of both sides of (1.2) with respect to q^α, $q^{\alpha *}$ gives Hamilton's equations

$$q_t^\alpha = \partial H/\partial q^{\alpha *}\,, \tag{1.13a}$$

$$q_t^{\alpha *} = -\partial H/\partial q^\alpha\,, \tag{1.13b}$$

where

$$H \propto i\int \mathrm{Tr}\left([E, A_x^0]\,\partial_t A_x^0\right) \tag{1.14}$$

(the proportionality sign is used because there is actually a constraint which must be taken into account).

Now, the equation of motion can be read off from the zero curvature condition (1.4) as the coefficient of λ^0:

$$\partial_t A_x^0 = \partial_x A_t^0 + [A_x^0, A_t^0]\,, \tag{1.15}$$

where A_t^0 is the coefficient of λ^0 in (1.6b). In this way one obtains an explicit expression for H in terms of the fields q^α, $q^{\alpha *}$ and their derivatives.

It was shown in [1] that instead of considering A_t given by (1.6b), one can look for

$$A_t = \sum_{n=0}^{N} \lambda^n A_t^n\,, \tag{1.16}$$

by substituting into the zero curvature condition (1.4) and equating coefficients of λ^n to zero. The coefficient of the highest power of λ, i.e. A_t^N, is left undetermined, but must be a constant element of $\mathscr{k}$. When $A_t^N = E$, the resulting expression for A_t is local, but for a general element $A_t^N = k \in \mathscr{k}$, one finds that A_t is non-local. $A_N(k)$ will denote A_t having the leading term $\lambda^N k$.

Each possible choice of $A_N(k)$ will give rise to a different equation of motion, given by the coefficient of λ^0 in the zero curvature condition:

$$\partial_{N,k} A_x^0 = \partial_x A_N^0(k) + [A_x^0, A_N^0(k)]\,. \tag{1.17}$$

The collection of operators $\partial_{N,k}$ will be regarded as independent evolution operators defining infinitely many "times." When $N=2$ and $k=E$, $A_N(k)$ is given by (1.6b), and so $\partial_{2,E}$ is the GNLS evolution operator. For a fixed value of k one has a hierarchy of equations of motion labelled by $N \geqq 0$. When $k=E$ this will be referred to as the "GNLS hierarchy."

For each equation of motion (1.17), one can obtain its Hamiltonian in the form (1.14). The Hamiltonian for the equation arising from the pair A_x, $A_N(k)$ will be denoted by $H_N(k)$. It will be seen that $H_N(k)$ is non-local in general, but the

Hamiltonians $H_N(E)$ of the GNLS hierarchy are local. Furthermore, the entire collection of $H_N(k)$ will turn out to be conserved quantities for the GNLS equation. To show this it is necessary to construct the Poisson bracket algebra of the Hamiltonians, and this is done by considering the commutation relations of the evolution operators $\partial_{N,k}$. One first has to find closed expressions for $\partial_{N,k}$ and $A_N(k)$ (the method used in [1] of solving the zero curvature condition gives the coefficients of $A_N(k)$ recursively). This is where the gauge invariance property proves useful. It turns out that a non-local transformation of the form (1.10) can be constructed so that

$$A_x \to a_x = \lambda E\,. \tag{1.18}$$

The zero curvature condition will then be satisfied by

$$a_t = \lambda^N k \equiv a_N(k)\,, \tag{1.19}$$

where $N \geqq 0$ and $k \in \mathscr{k}$ is constant. Now the gauge transformation (1.9b) is inverted to obtain

$$A_t = \omega a_N(k)\,\omega^{-1} - \omega_{N,k}\omega^{-1} = \lambda^N \omega k\,\omega^{-1} - \omega_{N,k}\omega^{-1}\,. \tag{1.20}$$

If A_t is chosen to have only positive powers of λ, then one can equate coefficients to obtain

$$A_t = \sum_{n=0}^{N} \lambda^{N-n}(\omega k\,\omega^{-1})_n = A_N(k) \text{ (where } \omega k\,\omega^{-1} = \sum_{n=0}^{\infty} \lambda^{-n}(\omega k\,\omega^{-1})_n)\,. \tag{1.21}$$

Also, one finds that the coefficient of λ^{-1} in (1.20) gives

$$\partial_{N,k} A_x^0 = -[E, (\omega k\,\omega^{-1})_{N+1}]\,, \tag{1.22}$$

which is the equation of motion associated with $A_N(k)$ in closed form. This will be used to derive the main result of this paper;

$$[\partial_{N,k}, \partial_{M,j}]\,A_x^0 = \partial_{N+M,[k,j]} A_x^0 \tag{1.23}$$

for all $N, M \geqq 0$, $k, j \in \mathscr{k}$. In other words, the evolution operators form "half" of a Kac-Moody algebra (since N and M take only positive values). Equation (1.23) will be used, together with the Jacobi identity, to establish the final result

$$\{H_N(k), H_M(j)\} = H_{N+M}([k,j])\,, \tag{1.24}$$

which states that the Hamiltonians have the same "Kac-Moody" algebraic structure under the Poisson bracket. In particular, one has

$$\{H_N(k), H_2(E)\} = 0\,. \tag{1.25}$$

This means that the entire collection of Hamiltonians are conserved quantities for the GNLS equation.

In Sect. 2 it will be shown how the gauge transformation ω is constructed in terms of the field variables q^α, $q^{\alpha *}$. The solution of the zero curvature condition to give $A_N(k)$ and $\partial_{N,k}$ in closed form will be discussed in Sect. 3. In Sect. 4 the Hamiltonians and their Poisson bracket algebra will be considered, and it will be shown in Sect. 5 that $H_N(E)$ is local for all N. Finally, in Sect. 6, the results obtained will be compared with the work of Olive and Turok, and possible generalizations to other systems will be discussed.

2. Construction of ω

Let G/K be an Hermitian symmetric space, where $\mathscr{k}$ (the Lie algebra of K) is the centralizer of E and $\mathscr{g}$ (the Lie algebra of G) decomposes as

$$\mathscr{g}=\mathscr{k}\oplus\mathscr{m}\,. \tag{2.1}$$

The step operators of the Cartan-Weyl basis of $\mathscr{g}$ which lie in $\mathscr{k}$ are denoted by Latin letters (e_a), while those which lie in $\mathscr{m}$ are denoted by Greek letters (e_α). The set of positive roots whose step operators lie in $\mathscr{m}$ is called θ^+.

It is explained in Appendix I that E satisfies the property

$$[E, e_\alpha]=\kappa e_\alpha\,, \tag{2.2}$$

where κ is a constant for all $\alpha\in\theta^+$. E will be chosen so that

$$\kappa=-i\,. \tag{2.3}$$

Now, following [1], define

$$A_x=\lambda E+A_x^0\,, \tag{2.4}$$

where

$$A_x^0=-q^\alpha e_\alpha+q^{\alpha *}e_{-\alpha}\in\mathscr{m}\,. \tag{2.5}$$

The main object of interest is the "zero curvature condition"

$$F_{xt}\equiv[\partial_x+A_x,\,\partial_t+A_t]=0\,, \tag{2.6}$$

where A_t is a polynomial in λ with coefficients in $\mathscr{g}$. The only restriction on A_t is that the resulting equation of motion for the fields $q^\alpha(x,t)$ implied by (2.6) must be independent of λ.

Equation (2.6) is invariant under a "gauge transformation"

$$A_\mu\to a_\mu=\omega^{-1}A_\mu\omega+\omega^{-1}\partial_\mu\omega \qquad \mu=(x,t)\,, \tag{2.7}$$

where $\omega(\lambda;\,x,t)\in G$. In other words

$$[\partial_x+a_x,\,\partial_t+a_t]=0\,. \tag{2.8}$$

Equation (2.8) is associated with the same equation of motion as (2.6). However, it may be possible to find a transformation such that a_μ is independent of the fields q^α, $q^{\alpha *}$. In that case, the equation of motion is implied by the transformation (2.7) with $\mu=t$, which can be thought of as an equation of motion for ω. Such a transformation will, in fact, prove to be very useful in what follows.

Notice that A_x and A_t can be thought of as elements of the "loop algebra" $\mathscr{g}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$ (where $\mathbb{C}[\lambda,\lambda^{-1}]$ is the algebra of Laurent polynomials in the complex variable λ). It is therefore natural to consider ω as an element of the "loop group." It will be chosen to have the form

$$\omega=\exp\sum_{n=1}^{\infty}\lambda^{-n}\omega_n\,,\qquad(\omega_n\in\mathscr{g})\,. \tag{2.9}$$

This is the type of gauge transformation used in [3] in connection with the Toda equation. By expanding (2.9) as a power series in λ one obtains the identities given in Appendix II, which can then be used to write a_x (2.7) as a power series:

$$a_x = \lambda E + \sum_{n=0}^{\infty} \lambda^{-n} a_x^n = \lambda E + \{A_x^0 - [\omega_1, E]\}$$
$$+ \lambda^{-1}\{-[\omega_2, E] + 1/2[\omega_1[\omega_1, E]] - [\omega_1, A_x^0] + \partial_x \omega_1\}$$
$$+ \lambda^{-2}\{-[\omega_3, E] + 1/2[\omega_1[\omega_2, E]] + 1/2[\omega_2[\omega_1, E]]$$
$$- 1/6[\omega_1[\omega_1[\omega_1, E]]] - [\omega_2, A_x^0] + 1/2[\omega_1[\omega_1, A_x^0]]$$
$$+ \partial_x \omega_2 - 1/2[\omega_1, \partial_x \omega_1]\} + \dots . \tag{2.10}$$

It will now be shown that it is possible to construct ω so that

$$a_x = \lambda E . \tag{2.11}$$

One can see from (2.10) that $a_x^0 = 0$ if

$$[\omega_1, E] = A_x^0 . \tag{2.12}$$

Using (I.8) and (2.3), this implies

$$\omega_1^{m} = [E, A_x^0] = iq^\alpha e_\alpha + iq^{\alpha *} e_{-\alpha} , \tag{2.13}$$

where ω_1^{m} denotes the component of ω_1 in m. Now consider a_x^1. The commutation relations (I.14) can be used to equate the k and m components to zero:

$$(a_x^1)^{k} = 1/2[\omega_1^{m}[\omega_1, E]] - [\omega_1^{m}, A_x^0] + \partial_x \omega_1^{k} = 0$$
$$\text{i.e. } \partial_x \omega_1^{k} = 1/2[\omega_1^{m}, A_x^0] , \tag{2.14}$$

using (2.12), and

$$(a_x^1)^{m} = -[\omega_2, E] + 1/2[\omega_1^{k}[\omega_1, E]] - [\omega_1^{k}, A_x^0] + \partial_x \omega_1^{m} = 0 ,$$
$$\text{i.e. } [\omega_2, E] = \partial_x \omega_1^{m} - 1/2[\omega_1^{k}, A_x^0] , \tag{2.15}$$

using (2.12) again. Notice that (2.14) determines ω_1^{k} non-locally:

$$\omega_1^{k} = 1/2 \partial^{-1}[A_x^0[A_x^0, E]] , \tag{2.16}$$

whil (2.15) gives

$$\omega_2^{m} = -\partial_x A_x^0 + 1/2[E[A_x^0, \partial^{-1}[A_x^0[A_x^0, E]]]] . \tag{2.17}$$

For a general term $a_x^n (n > 1)$ one has

$$a_x^n = -[\omega_{n+1}, E] + 1/2[\omega_1[\omega_n, E]] + 1/2[\omega_n[\omega_1, E]] - [\omega_n, A_x^0]$$
$$+ \partial_x \omega_n + (\text{terms involving } \omega_{j<n}) . \tag{2.18}$$

This can be split up into k and m components and equated to zero to obtain

$$\partial_x \omega_n^{k} = -1/2[\omega_1^{m}[\omega_n^{m}, E]] - 1/2[\omega_n^{m}[\omega_1^{m}, E]] + [\omega_n^{m}, A_x^0]$$
$$+ (\text{terms involving } \omega_{j<n}) , \tag{2.19}$$

$$[\omega_{n+1}, E] = 1/2[\omega_1^{k}[\omega_n^{m}, E]] + 1/2[\omega_n^{k}[\omega_1^{m}, E]] - [\omega_n^{k}, A_x^0]$$
$$+ \partial_x \omega_n^{m} + (\text{terms involving } \omega_{j<n}) . \tag{2.20}$$

So for each n the requirement that $a_x^n = 0$ determines ω_n^{k} and ω_{n+1}^{m}.

In [3], only the condition $(a_x^n)^{m}=0$ is imposed, so that ω_n^{k} is left undetermined and can be chosen to be zero to all orders. This gauge transformation will be denoted $\tilde{\omega}$. The first few terms are obtained from (2.10) as follows:

$$[\tilde{\omega}_1, E]=A_x^0\,, \quad \text{i.e.} \quad \tilde{\omega}_1=[E, A_x^0]\,, \tag{2.21}$$

$$[\tilde{\omega}_2, E]=\partial_x\tilde{\omega}_1\,, \quad \text{i.e.} \quad \tilde{\omega}_2=-\partial_x A_x^0\,, \tag{2.22}$$

$$\begin{aligned}[\tilde{\omega}_3, E]&=-1/6[\tilde{\omega}_1[\tilde{\omega}_1[\tilde{\omega}_1, E]]]+1/2[\tilde{\omega}_1[\tilde{\omega}_1, A_x^0]]+\partial_x\tilde{\omega}_2\\ \text{i.e.} \quad \tilde{\omega}_3&=1/3[E[\tilde{\omega}_1[\tilde{\omega}_1, A_x^0]]]-\partial_{xx}\tilde{\omega}_1\,,\end{aligned} \tag{2.23}$$

and so on. Notice that, unlike ω, $\tilde{\omega}$ is local to all orders.

3. Solution of the Zero Curvature Condition

One wishes to find A_t such that the zero curvature condition (2.6) is satisfied with A_x given by (2.4). As in [1], A_t will be assumed to be of the form

$$A_t=\sum_{n=0}^{N}\lambda^n A_t^n\,. \tag{3.1}$$

Consider the gauge transformed potentials

$$a_x=\omega^{-1}A_x\omega+\omega^{-1}\omega_x=\lambda E\,, \tag{3.2a}$$

$$a_t=\omega^{-1}A_t\omega+\omega^{-1}\omega_t\,, \tag{3.2b}$$

where ω is the gauge transformation constructed in Sect. 2. One can see from the identities (II.1a), (II.2a) that a_t is a descending power series of the form

$$a_t=\lambda^N a_t^{-N}+\ldots+a_t^0+\lambda^{-1}a_t^1+\ldots\,. \tag{3.3}$$

Now substitute (3.2a), (3.3) into the zero curvature condition

$$\partial_x a_t-\partial_t a_x+[a_x, a_t]=0\,, \tag{3.4}$$

and equate powers of λ to zero:

$$\lambda^{N+1}\colon [E, a_t^{-N}]=0 \quad \text{i.e.} \quad a_t^{-N}\in k\,, \tag{3.5}$$

$$\lambda^N\colon \partial_x a_t^{-N}+[E, a_t^{1-N}]=0\,. \tag{3.6}$$

Split this into parts in k and m to obtain the result that a_t^{-N} is a constant and $a_t^{1-N}\in k$. Continuing in this way one finds that all of the coefficients of a_t are constant elements of k. One can choose

$$a_t=\lambda^N k\,. \tag{3.7}$$

Now invert the transformation (3.2b):

$$A_t=\omega a_t\omega^{-1}-\omega_t\omega^{-1}=\lambda^N\omega k\,\omega^{-1}-\omega_t\omega^{-1}\,. \tag{3.8}$$

Since $\omega_t\omega^{-1}$ has only negative powers of λ, and A_t is chosen to have no negative powers, it follows that

$$A_t=\sum_{n=0}^{N}\lambda^n(\omega k\omega^{-1})_{N-n}\,. \tag{3.9}$$

where $(\omega k\,\omega^{-1})_m$ denotes the coefficient of μ^{-m} in $\omega(\mu)k\omega^{-1}(\mu)$. By (II.1 b), A_t given by (3.9) has $\lambda^N k$ as its highest order term. Since N and k are arbitrary, the notation $A_N(k)$ will be used for the object defined by (3.9).

Turning now to the negative powers of λ in (3.8), one can equate coefficients to obtain

$$(\omega_t \omega^{-1})_n = (\omega k\,\omega^{-1})_{N+n} \tag{3.10}$$

for all $n \geqq 1$. The derivative with respect to t corresponds to the equation of motion arising from $A_N(k)$. For each choice of N or k there will be a different equation of motion, and so the evolution operator ∂_t associated with $A_N(k)$ will be denoted $\partial_{N,k}$. The collection of these operators can be thought of as describing the evolution of the fields with respect to infinitely many independent time variables.

In this notation, (3.10) becomes

$$(\omega_{N,k}\omega^{-1})_n = (\omega k\,\omega^{-1})_{N+n} \tag{3.11}$$

for all $n \geqq 1$, $N \geqq 0$. In particular, choose $n=1$. Then, using (II.2b)

$$\partial_{N,k}\omega_1 = (\omega k\,\omega^{-1})_{N+1}\,. \tag{3.12}$$

By (2.13), this implies

$$iq^{\alpha}_{N,k} = (\omega k\,\omega^{-1})^{\alpha}_{N+1}\,, \tag{3.13a}$$

$$iq^{\alpha *}_{N,k} = (\omega k\,\omega^{-1})^{-\alpha}_{N+1}\,, \tag{3.13b}$$

where $(\omega k\,\omega^{-1})^{\pm\alpha}$ is the coefficient of $e_{\pm\alpha}$ in $\omega k\,\omega^{-1}$. Equations (3.13) give the equation of motion corresponding to the pair A_x, $A_N(k)$. [One can check them directly from the zero curvature condition (2.6), using (2.4) and (3.9).] Consistency of (3.13a) and (3.13b) requires the restriction to the compact real form of $\mathscr{g}$ [1], which means that k must be of the form

$$k^{i*} = -k^i\,, \tag{3.14a}$$

$$k^{a*} = -k^{-a}\,, \tag{3.14b}$$

(where $k = k^i h_i + k^a e_a + k^{-a} e_{-a}$).

4. Poisson Bracket Algebra

The algebra of the evolution operators will now be investigated. This will allow the construction of the Poisson bracket algebra of the Hamiltonians for the equations of motion (3.13).

Recall Eq. (3.12), and act on both sides with the evolution operator $\partial_{M,j}$ $(M \geqq 0, j \in \mathscr{k})$ to obtain

$$\begin{aligned}\partial_{M,j}\partial_{N,k}\omega_1 &= \partial_{M,j}(\omega k\,\omega^{-1})_{N+1} = ([\omega_{M,j}\omega^{-1},\ \omega k\,\omega^{-1}])_{N+1}\\ &= \sum_{p=0}^{N} [(\omega_{M,j}\omega^{-1})_{N+1-p},\ (\omega k\,\omega^{-1})_p]\\ &= \sum_{p=0}^{N} [(\omega j\,\omega^{-1})_{M+N+1-p},\ (\omega k\,\omega^{-1})_p]\,.\end{aligned} \tag{4.1}$$

(Use has been made of (3.11) and the identity

$$\partial_\mu(gXg^{-1})=[g_\mu g^{-1}, gXg^{-1}]+g\partial_\mu Xg^{-1} \text{ for all } g\in G,\ X\in \mathscr{g}.)$$

The same calculation with (M,j) and (N,k) interchanged leads to

$$\begin{aligned}[\partial_{N,k}, \partial_{M,j}]\omega_1 &= \sum_{q=0}^{N+M+1} [(\omega k\,\omega^{-1})_{N+M+1-q}, (\omega j\,\omega^{-1})_q] \\ &=([\omega k\omega^{-1}, \omega j\omega^{-1}])_{N+M+1}=(\omega[k,j]\omega^{-1})_{N+M+1} \\ &=\partial_{N+M,[k,j]}\omega_1 \end{aligned} \tag{4.2}$$

using (3.12).

In particular, (2.12) enables one to write

$$[\partial_{N,k}, \partial_{M,j}]A_x^0=\partial_{N+M,[k,j]}A_x^0 \text{ for all } N,\ M\geqq 0,\ k,j\in \mathscr{k}. \tag{4.3}$$

Equation (4.3) states that the evolution operators form an algebra isomorphic to $\mathscr{k}_R\otimes\mathbb{C}[\lambda]$. $\mathscr{k}_R$ denotes the compact real form of $\mathscr{k}$ [this distinction is necessary because of the consistency condition (3.14)] and $\mathbb{C}[\lambda]$ denotes the algebra of Laurent polynomials in positive powers of λ. The algebra defined by (4.3) can be thought of as "half" of a Kac-Moody algebra [4].

Now define the Poisson bracket between two functions A and B as

$$\{A, B\}=\sum_\alpha\int dz(\partial A/\partial q^\alpha(z)\cdot\partial B/\partial q^{\alpha *}(z)-\partial B/\partial q^\alpha(z)\cdot\partial A/\partial q^{\alpha *}(z)) \tag{4.4}$$

(arguments, delta functions etc. will subsequently be suppressed for clarity). The Hamiltonian $H_N(k)$ for the equation of motion (3.13) associated with $A_N(k)$ is defined by the relation

$$\partial_{N,k}A_x^0=\{A_x^0, H_N(k)\} \tag{4.5}$$

(the Poisson bracket between an element of $\mathscr{g}$, such as A_x^0, and a function, such as $H_N(k)$, is of course well defined). Definition (4.4) is equivalent to Hamilton's equations:

$$q^\alpha_{N,k}=\partial H_N(k)/\partial q^{\alpha *}\,, \tag{4.6a}$$

$$q^{\alpha *}_{N,k}=-\partial H_N(k)/\partial q^\alpha\,. \tag{4.6b}$$

Equations (4.3) and (4.5) can be used to rewrite the Jacobi identity

$$\{A_x^0\{H_N(k), H_M(j)\}\}+\{H_N(k)\ \{H_M(j), A_x^0\}\}+\{H_M(j)\ \{A_x^0, H_N(k)\}\}=0 \tag{4.7}$$

in the form

$$\{A_x^0\{H_N(k), H_M(j)\}\}=[\partial_{N,k}, \partial_{M,j}]A_x^0=\partial_{N+M,[k,j]}A_x^0=\{A_x^0, H_{N+M}([k,j])\}, \tag{4.8}$$

which implies that

$$\{H_N(k), H_M(j)\}=H_{N+M}([k,j])+C^{k,j}_{N,M}\,, \tag{4.9}$$

where $C^{k,j}_{N,M}$ is a constant. Equation (4.9) states that the Poisson bracket algebra is the "half" Kac-Moody algebra with central extension. In fact, the central term can always be made to disappear by a suitable re-definition of the generators [5]. (In the present case, this is simply a reflection of the fact that the Hamiltonians are only defined up to a constant.) For the case $j=E$, it is easy to check using the Jacobi identity that $C^{k;E}_{N,M}$ vanishes identically. In particular, this means that

$$\{H_N(k),\, H_2(E)\}=0\,, \tag{4.10}$$

where $H_2(E)$ is the GNLS Hamiltonian. Therefore one can consider the entire collection of Hamiltonians $H_N(k)$ to be conserved quantities for the GNLS equation.

It only remains to find the explicit form of $H_N(k)$. First, put $k=E$ and $N=0$ in (3.13):

$$iq^{\alpha}_{0,E}=[\omega_1,\, E]^{\alpha}=-q^{\alpha}\,, \tag{4.11}$$

using (II.1 b) and (2.12). It is then clear from (4.6) that

$$H_0(E)=i\textstyle\int q^{\alpha}q^{\alpha*} \tag{4.12}$$

(summation implied). Now use (4.10), (4.5) to deduce

$$\partial_{N,k}\textstyle\int q^{\alpha}q^{\alpha*}=0\,. \tag{4.13}$$

It follows that

$$\textstyle\int(q^{\alpha}q^{\alpha*}_{N,k}-q^{\alpha}_{N,k}q^{\alpha*})=-2\int q^{\alpha}_{N,k}q^{\alpha*}=2\int q^{\alpha*}_{N,k}q^{\alpha}\,. \tag{4.14}$$

Next, use (3.12) to write

$$\textstyle\int\mathrm{Tr}(A^0_x\omega k\,\omega^{-1})_{N+1}=\int\mathrm{Tr}(A^0_x\partial_{N,k}\omega_1)=\int(-iq^{\alpha}q^{\alpha*}_{N,k}+iq^{\alpha}_{N,k}q^{\alpha*})\,. \tag{4.15}$$

Then use (4.14) and differentiate:

$$q^{\alpha}_{N,k}=-i/2\partial/\partial q^{\alpha*}\textstyle\int\mathrm{Tr}(A^0_x\omega k\,\omega^{-1})_{N+1}\,, \tag{4.16a}$$

$$q^{\alpha*}_{N,k}=i/2\partial/\partial q^{\alpha}\textstyle\int\mathrm{Tr}(A^0_x\omega k\,\omega^{-1})_{N+1}\,. \tag{4.16b}$$

Comparing these with (4.6), one can choose

$$H_N(k)=-i/2\textstyle\int\mathrm{Tr}(A^0_x\omega k\,\omega^{-1})_{N+1}\,. \tag{4.17}$$

5. The GNLS Hierarchy

It is clear from the construction of ω in Sect. 2 that the operators $\partial_{N,k}$ give rise, in general, to non-local equations of motion (with non-local Hamiltonians). What is, perhaps, rather surprising is that for $k=E$ the equations of motion (the GNLS hierarchy) are all local. To show this, the objects $A_N(E)$, $H_N(E)$ and $\partial_{N,E}$ will here be reconstructed in terms of local quantities.

Consider the gauge transformation $\tilde{\omega}$ which takes A_x into $\mathscr{k}$:

$$A_x\to\tilde{a}_x=\tilde{\omega}^{-1}A_x\tilde{\omega}+\tilde{\omega}^{-1}\tilde{\omega}_x=\lambda E+\sum_{n=1}^{\infty}\lambda^{-n}\tilde{a}^n_x\,. \tag{5.1}$$

It was shown in Sect. 2 that this is a local gauge transformation. Now, as in Sect. 3, one wishes to find $\tilde{a}_t$ such that the zero curvature condition

$$\partial_x\tilde{a}_t-\partial_t\tilde{a}_x+[\tilde{a}_x,\, \tilde{a}_t]=0 \tag{5.2}$$

is satisfied, where $\tilde{a}_t$ has the general form

$$\tilde{a}_t=\lambda^N\tilde{a}_t^{-N}+\ldots+\tilde{a}^0_t+\lambda^{-1}\tilde{a}^1_t+\ldots\,. \tag{5.3}$$

Substitute (5.1) and (5.3) into (5.2), and equate coefficients of powers of λ:

$$\lambda^{N+1}\colon [E, \tilde{a}_t^{-N}]=0 \quad \text{i.e.} \quad \tilde{a}_t^{-N}\in k\,, \tag{5.4}$$

$$\lambda^{N}\colon \partial_x\tilde{a}_t^{-N}+[E, \tilde{a}_t^{-(N-1)}]=0\,, \tag{5.5}$$

i.e. $\tilde{a}_t^{-(N-1)}\in k$ and $\tilde{a}_t^{-N}$ is a constant. Choose $\tilde{a}_t^{-N}=E$.

$$\lambda^{N-1}\colon \partial_x\tilde{a}_t^{-(N-1)}+[E, \tilde{a}_t^{-(N-2)}]+[\tilde{a}_x^1, \tilde{a}_t^{-N}]=0\,. \tag{5.6}$$

Again, split this into parts in m and k to find $\tilde{a}_t^{-(N-2)}\in k$ and

$$\partial_x\tilde{a}_t^{-(N-1)}=[\tilde{a}_t^{-N}, \tilde{a}_x^1]\,. \tag{5.7}$$

Since $\tilde{a}_t^{-N}=E$, and $\tilde{a}_x^1\in k$, this becomes

$$\tilde{a}_t^{-(N-1)}=\text{constant}\,. \tag{5.8}$$

Choose $\tilde{a}_t^{-(N-1)}=0$, and continue in the same fashion. One finds that $\tilde{a}_t$ can be chosen to have the form

$$\tilde{a}_t=\lambda^N E+\sum_{n=1}^{\infty}\lambda^{-n}\tilde{a}_t^n\,. \tag{5.9}$$

Now invert the gauge transformation:

$$A_t=\tilde{\omega}\tilde{a}_t\tilde{\omega}^{-1}-\tilde{\omega}_t\tilde{\omega}^{-1}\,, \tag{5.10}$$

and equate positive powers of λ to obtain

$$A_t=\sum_{n=0}^{N}\lambda^n(\tilde{\omega}E\tilde{\omega}^{-1})_{N-n}\,. \tag{5.11}$$

This has leading term $\lambda^N E$, and is equal to $A_N(E)$ as given by (3.9) with $k=E$ (and the constants of integration set to zero). It immediately follows that

$$\omega E\omega^{-1}=\tilde{\omega}E\tilde{\omega}^{-1} \tag{5.12}$$

to all orders. One can deduce from this that the equations of motion (3.13) and Hamiltonians (4.17) become local for $k=E$. Notice, incidentally, that the equation of motion cannot be read off from the coefficient of λ^{-1} in (5.10), since $\tilde{a}_t^1$ is non-zero. One can, however, obtain it from the zero curvature condition:

$$\begin{aligned}\partial_t A_x^0&=\partial_x A_t^0+[A_x^0, A_t^0]=([\tilde{\omega}_x\tilde{\omega}^{-1}, \tilde{\omega}E\tilde{\omega}^{-1}])_N+[A_x^0, (\tilde{\omega}E\tilde{\omega}^{-1})_N]\\&=-([\lambda E, \tilde{\omega}E\tilde{\omega}^{-1}])_N \text{ (since } [\tilde{a}_x, E]=0)\\&=-[E, (\tilde{\omega}E\tilde{\omega}^{-1})_{N+1}]\,.\end{aligned} \tag{5.13}$$

Finally, the Hamiltonians $H_N(E)$ will be calculated for $N=0, 1, 2$. One uses (2.21), (2.22), (2.23) to obtain

$$\mathrm{Tr}(A_x^0\tilde{\omega}E\tilde{\omega}^{-1})_1=\mathrm{Tr}(A_x^0[\tilde{\omega}_1, E])=\mathrm{Tr}(A_X^0A_X^0)\,, \tag{5.14}$$

$$\begin{aligned}\mathrm{Tr}(A_x^0\tilde{\omega}E\tilde{\omega}^{-1})_2&=\mathrm{Tr}(A_x^0\{[\tilde{\omega}_2, E]+1/2[\tilde{\omega}_1[\tilde{\omega}_1, E]]\})\\&=\mathrm{Tr}(A_x^0[\tilde{\omega}_2, E]) \text{ (since } [\tilde{\omega}_1[\tilde{\omega}_1, E]]\in k)\\&=\mathrm{Tr}(E[\partial_x A_x^0, A_x^0])\,,\end{aligned} \tag{5.15}$$

$$\mathrm{Tr}(A_x^0\tilde{\omega}E\tilde{\omega}^{-1})_3 = \mathrm{Tr}(A_x^0([\tilde{\omega}_3, E] + 1/6[\tilde{\omega}_1[\tilde{\omega}_1[\tilde{\omega}_1, E]]]))$$
$$\text{(only terms in } \mathscr{m} \text{ contribute)}$$
$$= -\mathrm{Tr}(A_x^0\partial_{xx}A_x^0) - 1/2\,\mathrm{Tr}([A_x^0[A_x^0, E]]^2)\,. \tag{5.16}$$

One can work these out explicitly in terms of the fields q^α, $q^{\alpha *}$ (see Appendix I) to find

$$H_0(E) = i\int q^\alpha q^{\alpha *}\,, \tag{5.17}$$

$$H_1(E) = 1/2\int q_x^\alpha q^{\alpha *} - q^\alpha q_x^{\alpha *}\,, \tag{5.18}$$

$$H_2(E) = i\int q_x^\alpha q_x^{\alpha *} + q^{\alpha *} q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \tag{5.19}$$

[integration by parts has been used in (5.19)]. Equations (5.17) and (5.18) are straightforward generalizations of the "particle number" and "momentum" of the NLS equation [2]. Equation (5.19) gives the Hamiltonian of the GNLS equation.

One can also check the expressions for $A_N(E)$ and $\partial_{N,E}$. For example, from (5.13)

$$\partial_{2,E}A_x^0 = -([E, \tilde{\omega}E\tilde{\omega}^{-1}])_3 = [E, \partial_{xx}A_x^0 - 1/2[\tilde{\omega}_1[\tilde{\omega}_1, A_x^0]]]\,. \tag{5.20}$$

In terms of the fields this becomes

$$iq^\alpha_{2,E} = q^\alpha_{xx} - q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta}\,, \tag{5.21 a}$$

$$-iq^{\alpha *}_{2,E} = q^{\alpha *}_{xx} - q^{\beta *} q^{\gamma *} q^\delta R^{-\alpha}_{-\beta-\gamma\delta}\,, \tag{5.21 b}$$

as expected. The calculation of $A_2(E)$ is as follows:

$$A_2(E) = \sum_{n=0}^{2} \lambda^n(\tilde{\omega}E\tilde{\omega}^{-1})_{N-n} = \lambda^2 E + \lambda[\tilde{\omega}_1, E] + [\tilde{\omega}_2, E] + 1/2[\tilde{\omega}_1[\tilde{\omega}_1, E]]$$
$$= \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]] \tag{5.22}$$

[using (2.21), (2.22)]. This is in agreement with (1.6b).

Of course, Eq. (5.11) ensures that the same results would be obtained if ω were used instead of $\tilde{\omega}$, although the calculation is more complicated.

6. Conclusions

The GNLS equation has two important special cases. As was mentioned earlier, the familiar non-linear Schrödinger equation corresponds to $\mathscr{g} = su(2)$. In that case, $\mathscr{k}$ is the one dimensional Cartan subalgebra, so that any element of $\mathscr{k}$ is a scalar multiple of E. Consequently only the local series of charges exists. The GNLS equation associated with $SU(n+1)/(U(1)\times SU(n))$ is known as the vector non-linear Schrödinger equation, and has arisen (like the NLS equation) in non-linear optics [6]. Non-local charges will exist for $n \geqq 2$. It would be interesting to find out whether such quantities could have any physical significance.

A major step in the construction of $H_N(k)$ was to find a general form for $A_N(k)$. For $k = E$, the same expression can, in fact, be found using the P-operator method of Olive and Turok [3] (the P-operator in the present case is the Casimir operator for $\mathscr{g} \otimes \mathscr{g}$ [1]) although the conditions they assume no longer hold (i.e. E is not regular).

As a generalization of the system considered here, one could begin with a trivial solution of the zero curvature condition:

$$a_x = \lambda^p \tilde{A}\,, \tag{6.1 a}$$

$$a_t = \lambda^N A\,, \tag{6.1 b}$$

where $\tilde{A}$, A are constants and $[\tilde{A}, A]=0$. One then finds A_x, A_t as series in positive powers of λ using the inverse gauge transformation

$$A_\mu = \omega a_\mu \omega^{-1} - \omega_\mu \omega^{-1}\,. \tag{6.2}$$

Those coefficients ω_n which remain undetermined by the requirement that (6.2) be consistent can be considered as dynamical fields (for the GNLS case this was ω_1^m). This will be discussed further in a subsequent paper. The evolution operators will obey the same "half" Kac-Moody algebra, but the precise form of the Hamiltonians will depend on the structure of A_x. It is anticipated that a generalization of the P-operator method will be applicable.

Appendix I

Some results are given here concerning Lie algebras and symmetric spaces. Further details can be found in, e.g., [7].

The Cartan-Weyl basis $\{h_i, e_r\colon h_i \in \mathscr{h}, r \in \Phi\}$ of a complex semi-simple Lie algebra $\mathscr{g}$, with Cartan subalgebra $\mathscr{h}$, satisfies the following relations (where Φ is the set of roots and $r \in \Phi$ can be positive or negative):

$$[h_i, h_j] = 0\,, \tag{I.1 a}$$

$$[h_i, e_r] = r_i e_r\,. \tag{I.1 b}$$

(If $H = H^i h_i \in \mathscr{h}$, where summation over i is implied, then

$$[H, e_r] = H^i r_i e_r \equiv H \cdot r e_r\,. \tag{I.1 c}$$

The dot is used to indicate summation over Cartan subalgebra indices.)

$$[e_r, e_{-r}] = r \cdot h\,. \tag{I.1 d}$$

If $r \neq -s$, then

$$[e_r, e_s] = N_{r,s} e_{r+s}\,, \tag{I.1 e}$$

where $N_{r,s}=0$ if $r+s$ is not a root. One can check the useful identities:

$$N_{r,s} = -N_{-r,-s} = N_{-s,-r} = N_{s,-r-s}\,. \tag{I.2}$$

The basis is scaled so that

$$\mathrm{Tr}(h_i h_j) = \delta_{ij}\,, \tag{I.3 a}$$

$$\mathrm{Tr}(e_r e_{-s}) = \delta_{rs}\,, \tag{I.3 b}$$

$$\mathrm{Tr}(h_i e_r) = 0\,. \tag{I.3 c}$$

From now on, $r, s, \ldots$ will denote only positive roots.

For any element $A \in \mathscr{g}$ define the "centralizer" $C(A)$ of A by

$$C(A) = \{B \in \mathscr{g}: [A, B] = 0\} . \tag{I.4}$$

An element $H \in \mathscr{h}$ is called regular if $C(H) = \mathscr{h}$.

Let $E \in \mathscr{h}$ be an element with the property that for any (positive) root r, $E \cdot r$ is either zero or takes a constant value κ. (Such an element does not always exist – for example E_8 does not possess one.) Now define the set θ^+ of roots which satisfy

$$E \cdot \alpha = \kappa \tag{I.5}$$

for all $\alpha \in \theta^+$. Denoting by Φ^+ the set of positive roots, and defining $\bar{\theta}^+ \equiv \Phi^+ - \theta^+$, then

$$E \cdot a = 0 \tag{I.6}$$

for all $a \in \bar{\theta}^+$. The Greek letters $\alpha, \beta, \gamma, \ldots$ will always denote elements of θ^+, and the Latin letters $a, b, c, \ldots$ will denote elements of $\bar{\theta}^+$.

$C(E)$ is a subalgebra spanned by $\{h_i, e_{\pm a}: h_i \in \mathscr{h}, a \in \bar{\theta}^+\}$, which will be denoted by $\mathscr{k}$. Then

$$\mathscr{g} = \mathscr{k} \oplus \mathscr{m} , \tag{I.7}$$

where $\mathscr{m}$, the orthogonal complement of $\mathscr{k}$, is a subspace spanned by $\{e_{\pm\alpha}: \alpha \in \theta^+\}$. Notice that $[E, A] \in \mathscr{m}$ for any element $A \in \mathscr{g}$. Also

$$[E[E, A]] = \kappa^2 A^{\mathscr{m}} , \tag{I.8}$$

where $A^{\mathscr{m}}$ is the component of A in $\mathscr{m}$. The Jacobi identity implies the useful special cases:

$$[[E, m]k] = [E[m, k]] \quad \text{for all } m \in \mathscr{m},\ k \in \mathscr{k} , \tag{I.9}$$

$$[[E, m_1]m_2] = [[E, m_2]m_1] \quad \text{for all } m_1, m_2 \in \mathscr{m} . \tag{I.10}$$

From the definition of E one deduces the following:

$$[e_\alpha, e_\beta] = [e_{-\alpha}, e_{-\beta}] = 0 , \tag{I.11}$$

$$[e_\alpha, e_{-\beta}] \in \mathscr{k} , \tag{I.12}$$

$$\alpha \pm a \in \theta^+ \text{ (if it is a root)} . \tag{I.13}$$

Then

$$[\mathscr{k}, \mathscr{k}] \subset \mathscr{k} , \quad [\mathscr{k}, \mathscr{m}] \subset \mathscr{m} , \quad [\mathscr{m}, \mathscr{m}] \subset \mathscr{k} , \tag{I.14}$$

i.e., g is a "symmetric algebra" and G/K is a symmetric space. The curvature tensor is defined as

$$R_{\pm\alpha\pm\beta\pm\gamma} = [e_{\pm\alpha}[e_{\pm\beta}, e_{\pm\gamma}]] . \tag{I.15}$$

The identity (I.11) implies

$$R_{\alpha\beta\gamma} = R_{\alpha-\beta-\gamma} = 0 , \tag{I.16}$$

while (I.12), (I.13) give

$$R^{-\delta}_{\alpha\beta-\gamma} = 0 , \tag{I.17}$$

etc. In fact, the symmetric spaces constructed in the way described above are "Hermitian," and the curvature tensor satisfies

$$(R^{\alpha}_{\beta\gamma-\delta})^{*}=R^{-\alpha}_{-\beta-\gamma\delta}\,. \tag{I.18}$$

Finally, it is useful to give the commutator for two general elements of $\mathscr{g}$. Writing the components as

$$A=A\cdot h+A^{a}e_{a}+A^{-a}e_{-a}+A^{\alpha}e_{\alpha}+A^{-\alpha}e_{-\alpha}, \tag{I.19}$$

then

$$\begin{aligned}[A,B]=&(A^{a}B^{-a}-A^{-a}B^{a})a\cdot h+(A^{\alpha}B^{-\alpha}-A^{-\alpha}B^{\alpha})\alpha\cdot h\\
&+(A\cdot aB^{a}-A^{a}B\cdot a+A^{b}B^{a-b}N_{-a,b}+A^{-b}B^{a+b}N_{-a,-b}\\
&+A^{\alpha}B^{-\alpha+a}N_{-a,\alpha}+A^{-\alpha}B^{\alpha+a}N_{-a,-\alpha})e_{a}\\
&+(A^{-\alpha}B\cdot a-A\cdot aB^{-a}+A^{b}B^{-a-b}N_{a,b}+A^{-b}B^{-a+b}N_{a,-b}\\
&+(A^{\alpha}B^{-\alpha-a}N_{a,\alpha}+A^{-\alpha}B^{\alpha-a}N_{a,-\alpha})e_{-a}\\
&+(A\cdot\alpha B^{\alpha}-A^{\alpha}B\cdot\alpha+A^{\alpha-\beta}B^{\beta}N_{\alpha,-\beta}+A^{\beta}B^{\alpha-\beta}N_{-\alpha,\beta})e_{\alpha}\\
&+(A^{-\alpha}B\cdot\alpha-A\cdot\alpha B^{-\alpha}+A^{\beta-\alpha}B^{-\beta}N_{-\alpha,\beta}+A^{-\beta}B^{\beta-\alpha}N_{\alpha,-\beta})e_{-\alpha}\,.\end{aligned} \tag{I.20}$$

Appendix II

If $\omega=\exp\sum\limits_{n=1}^{\infty}\lambda^{-n}\omega_n$, then one can expand in powers of λ to obtain the following identities (where A is any element of $\mathscr{g}$ and $(\,)_n$ denotes the coefficient of λ^{-n}):

$$(\omega^{-1}A\omega)_n=\sum_{r=1}^{n}(-1)^r(r!)^{-1}\sum_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_r},A]\ldots]]]\,, \tag{II.1a}$$

$$(\omega A\omega^{-1})_n=\sum_{r=1}^{n}(r!)^{-1}\sum_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_r},A]\ldots]]]\,, \tag{II.1b}$$

$$(\omega^{-1}\omega_\mu)_n=\sum_{r=1}^{n}(-1)^{r+1}(r!)^{-1}\sum_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\ldots[\omega_{k_{r-1}},\partial_\mu\omega_{k_r}]\ldots]]\,, \tag{II.2a}$$

$$(\omega_\mu\omega^{-1})_n=\sum_{r=1}^{n}(r!)^{-1}\sum_{(k_i:\Sigma k_i=n)}[\omega_{k_1}[\omega_{k_2}[\ldots[\omega_{k_{r-1}},\partial_\mu\omega_{k_r}]\ldots]]]\,. \tag{II.2b}$$

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